DIPARTIMENTO DI MATEMATICA
"Francesco Brioschi"
POLITECNICO DI MILANO

# About k-digit rational approximations

Citterio, M.; Pavani, R.

Collezione dei *Quaderni di Dipartimento*, numero **QDD 59**
Inserito negli *Archivi Digitali di Dipartimento* in data 23-11-2009

Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

# About $k$-digit rational approximations

Maurizio Citterio[a], Raffaella Pavani[*,b]

[a] *Università dell'Insubria, 22100 Como; Italy*
[b] *Politecnico di Milano, 20133 Milano; Italy*

---

**Abstract**

We focus on the computation of best $k$-digit rational bounds for a given irrational number. We present new results, which allow to implement a reliable very fast algorithm whose computational time increases at most linearly with $k$. Several numerical examples are reported.

*Key words:* rational bounds, irrational numbers
*2000 MSC:* 65G99, 11A55

---

## Introduction

In the framework of applications of numerical analysis to Diophantine approximations, here we address the problem of computing the best rational approximation for any given irrational number $\alpha$, with the bound of a $k$-digit representation of the involved integer numbers, where $k$ is a fixed integer. This peculiar problem is definitively interesting within the frame of finite arithmetic used by scientific computations. Indeed, the general problem of finding the best rational bounds of a given irrational number was solved many years ago using its continued fractions representation and relating convergents (see, for instance [1], [2]). In particular, it is explicitly proved that the best rational approximations of the second kind to $\alpha$ coincide with the convergents of $\alpha$ ([3, pp. 26, 27]).

Here we deal with the best rational approximations restricted to $k$-digits, concerning with numerators and denominators of approximating rationals.

[*]Corresponding author

*Email addresses:* `maurizio.citterio@uninsubria.it` (Maurizio Citterio), `raffaella.pavani@polimi.it` (Raffaella Pavani)

Thus our problem is completely different from the generally treated problem, even if it is related to it. Indeed, this problem needs some peculiar proofs in order to be solved and then to implement solution into a reliable numerical method . The final result involves the use of convergents , semiconvergents and pseudoconvergents (see Section 1 for definitions).

It can be shown (see, for instance [1]) that every best approximation of the first kind is either a convergent or a semiconvergent (provided that some non-positive indexes are allowed). Here we specialize these results to the case of $k$-digit representations and we extend them, taking into account a new type of related rationals, which we term *pseudoconvergents*. They provide the best rational approximation to $\alpha$ by fractions of integers built by exactly $k$-digits. It is worth noting that our new method requires a computational time increasing at most linearly with $k$ (but in practice "almost" constant for small $k$) and improves results recently presented in [4]

The contents of this paper are as follows. The next Section presents our notation and definitions. Section 2 presents some propositions and proofs and includes two Subsections. The first one is devoted to the construction of a numerical method for computing the best $k$-digit rational approximation with *at most* $k$-digits. Then, Subsection 2.2 presents analogous results referring to the best $k$-digit rational approximation with *exactly* $k$-digits. In Section 3 we estimate computational time needed for the actual computation of the best $k$-digit rational approximation. In Section 4 we report several numerical examples, by our algorithms (written both in MATLAB® and Mathematica®). The paper ends with a final Appendix, which reports the script of the used MATLAB® and Mathematica® programs.

## 1. Basic notions and facts

In this article, all the rational numbers are represented by reduced fractions.

Let $x$ and $x'$ be two approximations of a real number $\alpha$, we say that $x$ is finer than $x'$, and $x'$ is coarser than $x$, when $|\alpha - x| < |\alpha - x'|$.

Let $\alpha$ be a positive irrational number. We write $\alpha$ in its development in infinite continued fraction

$$\alpha = [a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} ,$$

where $a_i$ are positive integers for any $i \geq 1$, while $a_0$ is a nonnegative integer.

The convergents of $\alpha$ are the rational numbers

$$s_n = \frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$$

where, for any non-negative integer $n$,

$$p_{-2} = 0 \ , \quad p_{-1} = 1 \ , \quad p_n = a_n p_{n-1} + p_{n-2} \ ;$$

$$q_{-2} = 1 \ , \quad q_{-1} = 0 \ , \quad q_n = a_n q_{n-1} + q_{n-2} \ .$$

The semiconvergents $s_{n,m}$ of $\alpha$ are defined by

$$s_{n,m} = \frac{p_{n,m}}{q_{n,m}} = \frac{mp_n + p_{n-1}}{mq_n + q_{n-1}} \ ,$$

where $m$ is an integer with $1 \leq m < a_{n+1}$. We will use the same notation $s_{n,m}$ for any rational of the form $s_{n,m} = p_{n,m}/q_{n,m} = (mp_n + p_{n-1})/(mq_n + q_{n-1})$, for any nonnegative integer $m$; in particular, we include two special cases:

$$s_{n-1} = s_{n,0} \ , \qquad \text{for } m = 0 \ ;$$

$$s_{n,a_{n+1}} = s_{n+1} \ , \qquad \text{for } m = a_{n+1} \ .$$

The semiconvergents $s_{n,m}$ are also called subconvergents of $s_n$.

For any nonnegative integer $m$, we define the *pseudo-convergent* $\tilde{s}_{n,m}$ of the convergent $s_n = p_n/q_n$ by

$$\tilde{s}_{n,m} = \frac{\tilde{p}_{n,m}}{\tilde{q}_{n,m}} = \frac{mp_n + p_{n-1,a_n-1}}{mq_n + q_{n-1,a_n-1}} = \frac{mp_n + (a_n - 1)p_{n-1} + p_{n-2}}{mq_n + (a_n - 1)q_{n-1} + q_{n-2}} \ .$$

The pseudo-convergents $\tilde{s}_{n,m}$ of $s_n$ are infinite fractions, which lie on the same side of $s_n$ with respect to $\alpha$; their value tends to the value of $s_n$ as $m$ tends to infinity; moreover, they are all coarser than $s_n$.

If $w$ is an integer we write $d(w)$ for its number of digits.

We say that two rational numbers $a/b$, $a'/b'$ form a pair of Farey fractions when $a'b - ab' = \pm 1$. A convergent $s_n$ forms Farey pairs with $s_{n-1}$, $s_{n+1}$ as well as $s_{n,m}$ (with any $m$).

Let $k$ be a positive integer. We say that the rational number $a/b$ is the *k-best-N-approximation* (*D-approximation*) of $\alpha$ when it is the best rational approximation of $\alpha$ with at most $k$ digits at the numerator (resp. denominator), namely, for every rational $x/y$, if $|\alpha - x/y| < |\alpha - a/b|$, then $d(x) \geq k$

(resp. $d(y) \geq k$). Moreover it is called the *k-proper-N-approximation* (*D-approximation*) of $\alpha$ if it is the best rational approximation of $\alpha$ with exactly $k$ digits at the numerator (resp. denominator), namely, for every rational $x/y$, if $|\alpha - x/y| < |\alpha - a/b|$, then $d(x) \neq k$ (resp. $d(y) \neq k$). We remark that the $k$-proper-approximation can be coarser than the $k$-best-approximation, if the latter is obtained with less than $k$ digits, as it happens for instance in the case $\alpha = \pi$ and $k = 4$.

We recall the following facts which are either known or easily checked.

**Fact 1.** The non-zero convergents and the semiconvergents of $\alpha$ are exactly the reciprocals of those of $1/\alpha$.

**Fact 2.** If $a/b$ and $a'/b'$ are Farey fractions, any rational $x/y$ with $a/b < x/y < a'/b'$ has denominator $y \geq b + b'$ with $y = b + b'$ only for $x = a + a'$, and numerator $x \geq a + a'$ with $x = a + a'$ only for $y = b + b'$.

**Fact 3.** For any $i > j$, the convergent $s_i$ is finer than $s_j$ with $p_i > p_j$ and $q_i > q_j$. For any $n$ and $a_{n+1} \geq i > j \geq 0$, the number $s_{n,i}$ is finer than $s_{n,j}$ with $p_{n,i} > p_{n,j}$ and $q_{n,i} > q_{n,j}$.

**Fact 4.** The sequence of convergents with even index $\{s_{2n}\}$ is monotonically increasing, while the sequence of convergents with odd index $\{s_{2n+1}\}$ is monotonically decreasing.

$$s_0 < s_2 < \cdots < s_{2n} < s_{2n+2} < \cdots < \alpha < \cdots < s_{2n+1} < s_{2n-1} < \cdots < s_3 < s_1 \ .$$

For any fixed $n$, the sequences $\{s_{n,i}\}$ are monotonic:

$$s_{2n} = s_{2n+1,0} < \cdots < s_{2n+1,m} < s_{2n+1,m+1} < \cdots < s_{2n+1,a_{2n+2}} = s_{2n+2} \ ;$$

$$s_{2n+1} = s_{2n,a_{2n+1}} < \cdots < s_{2n,m} < s_{2n,m-1} < \cdots < s_{2n,0} = s_{2n-1} \ .$$

## 2. The construction of the $k$-digit rational approximations

Firstly we prove a property of rational approximations which is essential for our construction.

**Proposition 1.** *For any positive integer $k$, both the $k$-best-N-approximation and the $k$-best-D-approximation are either convergents or semiconvergents, while the $k$-proper-approximations are either convergents or semiconvergents or pseudo-convergents.*

*Proof.* Let us prove the proposition for the $k$-best-D-approximation; with the obvious modification, the same proof holds for the $k$-best-N-approximation.

Set the positive integer $k$ and suppose that $a/b$ is the required $k$-best-D-approximation of $\alpha$, assuming the hypothesis that $a/b$ is not a convergent of $\alpha$. Since the sequence $\{s_n\}$ converges to $\alpha$, there exists $n_*$, the greatest integer $n$ such that $|\alpha - s_n| > |\alpha - a/b|$. Consider the fractions $s_{n_*,m} = (mp_{n_*} + p_{n_*-1})/(mq_{n_*} + q_{n_*-1})$, where $m$ is an integer with $m \geq 0$, and let $m_*$ be the greatest integer $m$ such that $|\alpha - s_{n_*,m}| > |\alpha - a/b|$; such an integer $m_*$ exists because the inequality is not fulfilled for $m = a_{n+1}$ and it is fulfilled for $m = 0$ (if necessary we can assume $s_{-1} = \infty$). We claim that $a/b = s_{n_*,m_*+1}$. As both $s_{n_*}$ and $s_{n_*,m_*}$ are worse approximations of $\alpha$ than $a/b$, it follows that $a/b$ (together with $\alpha$) belongs to the open interval whose extremes are $s_{n_*}$ and $s_{n_*,m_*}$. If $a/b \neq s_{n_*,m_*+1}$, then $s_{n_*,m_*+1}$ is finer than $a/b$ and $a/b$ belongs to an open interval whose extremes are $s_{n_*,m_*+1}$ and, on the other side, either $s_{n_*}$ or $s_{n_*,m_*}$. Observe that $s_{n_*,m_*+1}$ forms Farey pairs with both $s_{n_*}$ and $s_{n_*,m_*}$. From Fact 2 it follows that $s_{n_*,m_*+1}$ has a denominator smaller than $b$ (and a fortiori smaller than $10^k$), while it provides a better approximation of $\alpha$, which is against the hypothesis that $a/b$ is the $k$-best-D-approximation of $\alpha$. Thus $a/b = s_{n_*,m_*+1}$. Since we have supposed that $a/b$ is not a convergent, we have $s_{n_*,m_*+1} \neq s_{n_*+1}$, namely $1 \leq m_* + 1 < a_{n_*+1}$ and $a/b$ turns out to be the semiconvergent $s_{n_*,m_*+1}$.

Now we are going to prove the second part of the proposition, namely our claim concerning the $k$-proper-approximation. We distinguish two cases:

*a.* The $k$-best-D-approximation has exactly $k$ digits at the denominator. Then it coincides with the $k$-proper-D-approximation which therefore is a convergent or a semiconvergent for the first part of this proof.

*b.* The $k$-best-D-approximation has less than $k$ digits at the denominator. Then there are no convergents $s_n$ with $d(q_n) = k$. Let's make the following positions:

   *i.* $n_k$ is the greatest integer such that $d(q_{n_k}) < k$;
   *ii.* $m_k$ is the greatest integer such that $d(q_{n_k,m_k}) = k$;
   *iii.* $M_k$ is the greatest integer such that $d(\tilde{q}_{n_k,M_k}) = k$.

   Observe that the above defined integers exist from Fact 3 and because, for any $m$, $n$ integers, $d(m) < k$ and $d(n) < k$ imply $d(m+n) \leq k$. Consider the semi-convergent $s_{n_k,m_k}$ and the pseudo-convergent $\tilde{s}_{n_k,M_k}$. We claim that the finer of them is the required $k$-proper-D-approximation. Indeed, any rational $a/b$ finer than them belongs, together with $\alpha$ and $s_{n_k}$, to the open interval whose extremes are $s_{n_k,m_k}$ and $\tilde{s}_{n_k,M_k}$; we point out that they both form a Farey pair with $s_{n_k}$ and remember the Fact 2; if $a/b$ is in the open interval whose extremes are $s_{n_k}$ and $s_{n_k,m_k}$, then it is such that $b \geq q_{n_k} + q_{n_k,m_k} \geq (m_k+1)q_{n_k} + q_{n_k-1}$; while, if $a/b$ is in the open interval whose extremes are $s_{n_k}$ and $\tilde{s}_{n_k,M_k}$ then $b \geq q_{n_k} + \tilde{q}_{n_k,M_k} \geq (M_k+1)q_{n_k} + q_{n_k-1,a_{n_k}-1}$. In both cases $d(b) > k$, proving our statement. □

*2.1. Approximations with at most k-digits rational numbers*

The construction of the $k$-best-N-approximation can be described in the following way.

Given $\alpha$ positive irrational, set $k$ positive integer. Develop $\alpha$ in continued fraction, $\alpha = [a_0; a_1, a_2, a_3, \ldots]$. Let $n_k + 1$ be the smallest integer such that $p_{n_k+1}$ has more than $k$ digits and let $m_k$ be the greatest integer such that $m_k p_{n_k} + p_{n_k-1}$ has no more than $k$ digits, i.e.

$$m_k = \left\lfloor \frac{10^k - 1 - p_{n_k-1}}{p_{n_k}} \right\rfloor .$$

Consider the convergent $s_{n_k}$ and the (semi)-convergent

$$s_{n_k,m_k} = \frac{m_k p_{n_k} + p_{n_k-1}}{m_k q_{n_k} + q_{n_k-1}} ;$$

observe that the number $s_{n_k,m_k}$ is the convergent $s_{n_k-1}$ in the case $m_k = 0$, otherwise it is a subconvergent of $s_{n_k}$.

**Proposition 2.** *The better approximation of $\alpha$ between $s_{n_k}$ and $s_{n_k,m_k}$ is the best approximation of $\alpha$ among all the rational numbers whose numerator has no more than $k$ digits.*

*Moreover, if $n_k$ is odd, then $s_{n_k,m_k} < \alpha < s_{n_k}$ , where $s_{n_k,m_k}$ is the best rational lower bound with $k$ digits and $s_{n_k}$ is the best rational upper bound with $k$ digits of $\alpha$; if $n_k$ is even, then $s_{n_k} < \alpha < s_{n_k,m_k}$, where $s_{n_k,m_k}$ is the best rational upper bound with $k$ digits and $s_{n_k}$ is the best rational lower bound with $k$ digits of $\alpha$.*

*Proof.* We know by Proposition 1 that the best approximation with $k$ digits is either a convergent or a semiconvergent of $\alpha$. The rational $s_{n_k+1}$ is the first convergent whose numerator has more than $k$ digits, and, taking Fact 3 and Fact 4 into account, we must investigate in two directions:

1. in the sequence of convergents whose index has the parity of $n_k + 1$, the best approximation could be either the convergent $s_{n_k-1}$ or a semiconvergents $s_{n_k,m}$ between $s_{n_k-1}$ and $s_{n_k+1}$, with no more than $k$ digits; writing $s_{n_k,0} = s_{n_k-1}$, the finest of these numbers is the one with $m = m_k$ (Fact 3);
2. in the sequence of convergents whose index has the parity of $n_k$, the best approximation could be either the convergent $s_{n_k}$ or a semiconvergents $s_{n_k+1,m}$ between $s_{n_k}$ and $s_{n_k+2}$; but any number $s_{n_k+1,m}$ has a numerator $mp_{n_k+1} + p_{n_k} > p_{n_k+1} > 10^k$, whenever $m \neq 0$, thus we have here to consider only the convergent $s_{n_k+1,0} = s_{n_k}$.

Since $s_{n_k,m_k} < \alpha < s_{n_k}$ in the case $n_k$ is odd, while $s_{n_k} < \alpha < s_{n_k,m_k}$ in the case $n_k$ is even, we have found both the best $k$-digit lower bound and the best $k$-digit upper bound of $\alpha$. The finer approximation of $\alpha$ between $s_{n_k}$ and $s_{n_k,m_k}$ is now the best approximation of $\alpha$ among all the rational numbers whose numerator has no more than $k$ digits. □

We observe that in the case $a_{n_k+1} = 1$, we have $m_k = 0$ and then the best $k$-digit approximation of $\alpha$ is the convergent $s_{n_k}$, since it is finer than $s_{n_k,m_k} = s_{n_k-1}$. For example, when $\alpha$ is the golden ratio $\varphi = (1 + \sqrt{5})/2 = [1; 1, 1, 1, \dots]$, the best $k$-digit approximation is, for any $k$, the convergent $s_{n_k}$.

*Remark* 1. The construction of the $k$-best-D-approximation can be described as above with the obvious modifications and the proof used for Proposition 2 gives us the analogous results. Thus from Fact 2 we obtain the following:

**Corollary.** *The rational $a/b$ is the $k$-best-N-approximation of $\alpha$ if and only if $b/a$ is the $k$-best-D-approximation of $1/\alpha$.*

*Remark* 2. The given construction works for any positive integer $k$ greater or equal to the number of digits of $\lfloor \alpha \rfloor = a_0$. Clearly the $k$-best-N-approximation of any positive number $\alpha$ is $10^k - 1$, whenever $k$ is lower than the number of digits of $\lfloor \alpha \rfloor$.

*Remark* 3. The construction of the $k$-best-N-approximation of $\alpha$ works also in the case $\alpha = x/y$ is a positive rational number; we should limit our computation to those integers $k$ lower than the number of digits of $x$. Note that in the case $\alpha = x/y$ is a positive rational number, then both $s_{n_k}$ and $s_{n_k,m_k}$ could be the $k$-best-N-approximation of $\alpha$.

*2.2. The best approximation with exactly k-digits rational number*

Fixed $k$ and given a positive irrational number $\alpha$, we obtained its $k$-best-N-approximation working on the numbers $s_{n_k}$ and $s_{n_k,m_k}$ defined in the previous subsection. Clearly, if both $s_{n_k}$ and $s_{n_k,m_k}$ have exactly $k$ digits at the numerator we already have the $k$-proper-N-approximation (together with $k$-proper-lower bound and $k$-proper-upper bound) of $\alpha$. Otherwise, using the same notation introduced above, we have to consider the two cases of the following propositions.

**Proposition 3.** *If the numerator of $s_{n_k,m_k}$ has less than $k$ digits, then $s_{n_k}$ is both the $k$-best-N-approximation and the $k$-proper-N-approximation of $\alpha$.*

*Proof.* Since $p_{n_k,m_k+1} \geq 10^k$, if $p_{n_k,m_k} < 10^{k-1}$ we have $p_{n_k} = p_{n_k,m_k+1} - p_{n_k,m_k} > 9 \cdot 10^{k-1}$; in particular $s_{n_k}$ has exactly $k$ digits at the numerator. Since $10^{k-1} > p_{n_k,m_k} \geq m_k p_{n_k} \geq 9 \cdot m_k \cdot 10^{k-1}$, we have $m_k = 0$ and $s_{n_k,m_k} = s_{n_k-1}$ is coarser than $s_{n_k}$. □

Let $M_k$ be the greatest integer such that $M_k p_n + p_{n-1,a_n-1}$ has no more than $k$ digits, i.e.

$$M_k = \left\lfloor \frac{10^k - 1 - (a_{n_k} - 1)p_{n_k-1} - p_{n_k-2}}{p_{n_k}} \right\rfloor ,$$

and consider the pseudo-convergent

$$\tilde{s}_{n_k,M_k} = \frac{\tilde{p}_{n_k,M_k}}{\tilde{q}_{n_k,M_k}} = \frac{M_k p_{n_k} + p_{n_k-1,a_{n_k}-1}}{M_k q_{n_k} + q_{n_k-1,a_{n_k}-1}} = \frac{M_k p_{n_k} + (a_{n_k} - 1)p_{n_k-1} + p_{n_k-2}}{M_k q_{n_k} + (a_{n_k} - 1)q_{n_k-1} + q_{n_k-2}} .$$

**Proposition 4.** *If the numerator of $s_{n_k}$ has less than $k$ digits, the better approximation of $\alpha$ between $\tilde{s}_{n_k,M_k}$ and $s_{n_k,m_k}$ is the best approximation of $\alpha$ among all the rational numbers whose numerator has exactly $k$ digits.*

*Moreover, if $n_k$ is odd, then $s_{n_k,m_k} < \alpha < \tilde{s}_{n_k,M_k}$ , where $s_{n_k,m_k}$ is the best rational lower bound with exactly $k$ digits and $\tilde{s}_{n_k,M_k}$ is the best rational upper bound with exactly $k$ digits of $\alpha$; if $n_k$ is even, then $\tilde{s}_{n_k,M_k} < \alpha < s_{n_k,m_k}$, where $s_{n_k,m_k}$ is the best rational upper bound with exactly $k$ digits and $\tilde{s}_{n_k,M_k}$ is the best rational lower bound with exactly $k$ digits of $\alpha$.*

*Proof.* Since the pseudo-convergent $\tilde{s}_{n_k,M_k}$ lies on the same side of the convergent $s_{n_k}$ (with respect to $\alpha$) and its numerator has exactly $k$ digits, the proof follows from Proposition 1. □

**Remark.** Given a positive irrational number $\alpha$ and an integer $k$ with $10^k \geq \lfloor \alpha \rfloor$, the algorithm described in this section supplies $\alpha$ with the $k$-best-N-approximation, the $k$-proper-N-approximation, the best rational lower and upper bounds with at most $k$ digits at the numerator and, in all the cases but the one considered in Proposition 3, the best rational lower and upper bounds with exactly $k$ digits at the numerator. This means that the current version of our algorithm, when the numerator of $s_{n_k,m_k}$ has less than $k$ digits, gives us directly the $k$-proper-N-approximation of $\alpha$, without providing the best rational lower and upper bounds with exactly $k$ digits at the numerator.

## 3. Computation time

Proposition 2 allows us to obtain the best $k$-digit rational approximation by a significantly reduced computation time. Indeed, for any $k$, once the first convergent $s_{n_k+1}$ with numerator (denominator) greater or equal to $10^k$ is found, then the computation time of the best $k$-digits rational approximation remains constant for any $k$. Now a question arises: is it possible to estimate the number of operations we have to perform in order to solve our problem? The answer is that, independently of $\alpha$, this number grows at most linearly with $k$.

**Proposition 5.** *Let $k$ be a positive integer, $\alpha$ a positive irrational number. The number $K_0$ of coefficients $a_i$ in the development of $\alpha$ in continued fraction $\alpha = [a_0; a_1, a_2, a_3, \dots]$ which have to be computed to solve the $k$-best-N-approximation problem is independent of $\alpha$ and satisfies the following inequality:*

$$K_0 < lk + 3 \ ,$$

*where the coefficient of $k$ is the number $l = \frac{\ln 10}{\ln \varphi} \simeq 4.785$, with $\varphi = \frac{1+\sqrt{5}}{2}$.*

*Proof.* Using the notations previously introduced, let $n_k + 1$ the least integer such that $p_{n_k+1} \geq 10^k$. In order to solve our problem we need to compute the coefficients $a_n$ and the related convergents $s_n$, with $n \leq n_k + 1$.

We denote by $F_n$ the Fibonacci numbers, with $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$. Since $p_0 \geq F_0$, $p_1 \geq F_1$ and $a_n \geq 1$, for any $n \geq 1$, we obtain inductively that

$$p_{n+1} = a_{n+1} p_n + p_{n-1} \geq p_n + p_{n-1} \geq F_n + F_{n-1} = F_{n+1} \ .$$

Since $\varphi^2 = \varphi + 1$, and consequently $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$, from $\varphi < F_3$ and $\varphi^2 = \varphi + 1 < F_3 + F_2 = F_4$, we obtain inductively that

$$\varphi^{n+1} = \varphi^n + \varphi^{n-1} < F_{n+2} + F_{n+1} = F_{n+3} \ .$$

From the equation $\varphi^x = 10^k$, whose solution is $x = lk$, we derive eventually that

$$10^k \leq \varphi^{\lceil lk \rceil} < F_{\lceil lk \rceil + 2} \leq p_{\lceil lk \rceil + 2} \ .$$

Since $K_0 = n_k + 1$, we have

$$K_0 \leq \lceil lk \rceil + 2 < lk + 3 \ . \qquad \square$$

**Corollary.** *The total amount of operations necessary to compute the best $k$-digits rational approximation of $\alpha$ is at most linear in $k$, with absolute constants independent of $k$ and $\alpha$.*

In the case $\alpha$ is a number greater than 1, namely $a_0 \geq 1$, we can state that $p_n \geq F_{n+2}$ and the inequality $K_0 < lk + 3$ expressed in Proposition 5 can be (slightly) improved in $K_0 \leq \lceil lk \rceil$. We have observed that no better improvement can be expected, since for $\alpha = \varphi = (1 + \sqrt{5})/2$, we have $K_0 = \lceil lk \rceil$ for almost all the integers $k$.

## 4. Numerical tests

The algorithm described in previous sections has been implemented in both MATLAB® and Mathematica® programs; we report here some numerical results.

In the following example, the best $k$-digit rational approximations of some numbers are reported in tables for increasing values of $k$, separating $k$-best-N-approximations and $k$-proper-N-approximations. Following the notation used in Propositions 2–4, we call $n_k$ the greatest integer such that the numerator of the convergent $s_{n_k}$ has at most $k$ digits; $m_k$ is the greatest integer such that the number $s_{n_k,m_k}$ has no more than $k$ digit in the numerator. When $n_k$ is odd, then $s_{n_k,m_k}$ is the best rational lower bound ($lb$) with $k$ digits and $s_{n_k}$ (written in boldface) is the best rational upper bound ($ub$) with at most $k$ digits; if $n_k$ is even, we interchange the roles. In the case that the numerator of $s_{n_k}$ has less than $k$ digits, in a separate table we compute the greatest integer $M_k$ such that the pseudo-convergent $\tilde{s}_{n_k,M_k}$ has no more than $k$ digit in the numerator. When $n_k$ is odd, then $s_{n_k,m_k}$ is the best rational lower bound (*plb*) with exactly $k$ digits and $\tilde{s}_{n_k,M_k}$ (written in sans serif) is the best rational upper bound (*pub*) with exactly $k$ digits; if $n_k$ is even, we interchange the roles. In the column "*bra*" we write the $k$-best-N-approximation by choosing the finer approximation between *lb* and *ub*. Similarly in the separated table, "*pbra*" means $k$-proper-N-approximation, the finer approximation between *plb* and *pub*.

We remark that, using this algorithm, a standard double precision suffices in order to compute both the best rational lower bound and the best rational upper bound, both in the case of at most $k$-digits and in the case of exactly $k$-digits. An higher working precision is required *only* when we need the computation of error. For this reason we report in the columns "*a.er.*" the absolute error of the $k$-digit-approximations. In MATLAB®, even if the machine precision is $\mathcal{O}(10^{-16})$, we use the extended accuracy available by the Symbolic Toolbox.

For any of the following example, we report the "total computation time", that is the *total* time this algorithm requires to compute *all* the data of the example, referring to the implementation with Mathematica® on a processor Intel® Core 2 Quad Q6600. Once convergents are computed (Mathematica® has a convenient built-in function), the computation time is expected to remain unaltered for any $k$. In MATLAB®, for reported examples we found that the computation time is at most $\mathcal{O}(10^{-4})$ sec for any single $k$; instead, if convergents are included, then for any $k$ we found that the computation time is always at most $\mathcal{O}(10^{-1})$.

**Example 1.** The number $\pi$.
Total computation time: 0.14 seconds.

Table 1: best approximations of $\pi$ by rationals with at most $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $lb < \pi < ub$ | *bra* | *a.er.* |
|---|---|---|---|---|---|
| 1 | 0 | 2 | $\mathbf{3} < \pi < \frac{7}{2}$ | ***lb*** | $1.42\times10^{-1}$ |
| 2 | 1 | 4 | $\frac{91}{29} < \pi < \mathbf{\frac{22}{7}}$ | ***ub*** | $1.26\times10^{-3}$ |
| 3 | 3 | 1 | $\frac{688}{219} < \pi < \mathbf{\frac{355}{113}}$ | ***ub*** | $2.67\times10^{-7}$ |
| 4 | 3 | 27 | $\frac{9918}{3157} < \pi < \mathbf{\frac{355}{113}}$ | ***ub*** | $2.67\times10^{-7}$ |
| 5 | 3 | 280 | $\frac{99733}{31746} < \pi < \mathbf{\frac{355}{113}}$ | lb | $1.20\times10^{-8}$ |

*(table continued on next page)*

*(table continued from previous page)*

| $k$ | $n_k$ | $m_k$ | $lb < \pi < ub$ | $bra$ | $a.er.$ |
|---|---|---|---|---|---|
| 6 | 8 | 0 | $\mathbf{\frac{833719}{265381}} < \pi < \frac{312689}{99532}$ | ***lb*** | $8.72{\times}10^{-12}$ |
| 7 | 11 | 1 | $\frac{9692294}{3085153} < \pi < \mathbf{\frac{5419351}{1725033}}$ | ***ub*** | $2.21{\times}10^{-14}$ |
| 8 | 12 | 1 | $\mathbf{\frac{80143857}{25510582}} < \pi < \frac{85563208}{27235615}$ | ***lb*** | $5.79{\times}10^{-16}$ |
| 9 | 15 | 1 | $\frac{657408909}{209259755} < \pi < \mathbf{\frac{411557987}{131002976}}$ | lb | $1.71{\times}10^{-17}$ |
| 10 | 18 | 1 | $\mathbf{\frac{6167950454}{1963319607}} < \pi < \frac{8717442233}{2774848045}$ | ***lb*** | $7.63{\times}10^{-20}$ |
| 11 | 20 | 4 | $\mathbf{\frac{21053343141}{6701487259}} < \pi < \frac{99098765251}{31544116688}$ | ***lb*** | $2.62{\times}10^{-22}$ |
| 12 | 20 | 46 | $\mathbf{\frac{21053343141}{6701487259}} < \pi < \frac{983339177173}{313006581566}$ | ub | $2.15{\times}10^{-22}$ |
| 13 | 24 | 0 | $\mathbf{\frac{8958937768937}{2851718461558}} < \pi < \frac{5371151992734}{1709690779483}$ | ***lb*** | $7.72{\times}10^{-27}$ |
| 14 | 24 | 10 | $\mathbf{\frac{8958937768937}{2851718461558}} < \pi < \frac{94960529682104}{30226875395063}$ | ub | $3.88{\times}10^{-27}$ |
| 15 | 26 | 2 | $\mathbf{\frac{428224593349304}{136308121570117}} < \pi < \frac{996204405225397}{317101710843087}$ | ***lb*** | $3.81{\times}10^{-30}$ |
| 16 | 28 | 0 | $\mathbf{\frac{6134899525417045}{1952799169684491}} < \pi < \frac{5706674932067741}{1816491048114374}$ | ***lb*** | $4.86{\times}10^{-32}$ |
| 17 | 30 | 1 | $\mathbf{\frac{66627445592888887}{21208174623389167}} < \pi < \frac{96873718626624808}{30835862350241505}$ | ***lb*** | $3.36{\times}10^{-34}$ |
| 18 | 31 | 2 | $\frac{926649338775027373}{294961645557763847} < \pi < \mathbf{\frac{430010946591069243}{136876735467187340}}$ | ***ub*** | $8.66{\times}10^{-36}$ |
| 19 | 32 | 3 | $\mathbf{\frac{2646693125139304345}{842468587426513207}} < \pi < \frac{8370090322008982278}{2664282497746726961}$ | ***lb*** | $1.41{\times}10^{-38}$ |
| 20 | 32 | 37 | $\mathbf{\frac{2646693125139304345}{842468587426513207}} < \pi < \frac{9835765657674533008}{3130821447024817599 9}$ | ***lb*** | $1.41{\times}10^{-38}$ |

Table 2: best approximations of $\pi$ by rationals with exactly $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $M_k$ | $plb < \pi < pub$ | $bra$ | $a.er.$ |
|---|---|---|---|---|---|---|
| 4 | 3 | 27 | 28 | $\frac{9918}{3157} < \pi < \frac{9962}{3171}$ | plb | $2.54{\times}10^{-6}$ |
| 5 | 3 | 280 | 281 | $\frac{99733}{31746} < \pi < \frac{99777}{31760}$ | plb | $1.20{\times}10^{-8}$ |
| 12 | 20 | 46 | 47 | $\frac{995675078081}{316933220780} < \pi < \frac{983339177173}{313006581566}$ | pub | $2.15{\times}10^{-22}$ |
| 14 | 24 | 10 | 10 | $\frac{93177163465573}{29659212297655} < \pi < \frac{94960529682104}{30226875395063}$ | pub | $3.88{\times}10^{-27}$ |
| 20 | 32 | 37 | 36 | $\frac{9749763468356319152 2}{3103446099931380131 9} < \pi < \frac{9835765657674533000 8}{3130821447024817599 9}$ | pub | $2.38{\times}10^{-38}$ |

**Example 2.** The Neper number e.
Total computation time: 0.14 seconds.

Table 3: best approximations of e by rationals with at most $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $lb < \mathrm{e} < ub$ | $bra$ | $a.er.$ |
|---|---|---|---|---|---|
| 1 | 2 | 0 | $\mathbf{\frac{8}{3}} < \mathrm{e} < 3$ | ***lb*** | $5.16{\times}10^{-2}$ |
| 2 | 5 | 0 | $\frac{19}{7} < \mathrm{e} < \mathbf{\frac{87}{32}}$ | ***ub*** | $4.68{\times}10^{-4}$ |
| 3 | 7 | 4 | $\frac{878}{323} < \mathrm{e} < \mathbf{\frac{193}{71}}$ | lb | $1.56{\times}10^{-5}$ |
| 4 | 10 | 3 | $\mathbf{\frac{2721}{1001}} < \mathrm{e} < \frac{9620}{3539}$ | ***lb*** | $1.10{\times}10^{-7}$ |
| 5 | 13 | 1 | $\frac{75117}{27634} < \mathrm{e} < \mathbf{\frac{49171}{18089}}$ | ***ub*** | $2.77{\times}10^{-10}$ |
| 6 | 15 | 0 | $\frac{517656}{190435} < \mathrm{e} < \mathbf{\frac{566827}{208524}}$ | ***ub*** | $1.15{\times}10^{-11}$ |
| 7 | 16 | 8 | $\mathbf{\frac{1084483}{398959}} < \mathrm{e} < \frac{9242691}{3400196}$ | ub | $2.55{\times}10^{-13}$ |
| 8 | 19 | 3 | $\frac{99402293}{36568060} < \mathrm{e} < \mathbf{\frac{28245729}{10391023}}$ | ***ub*** | $6.16{\times}10^{-16}$ |
| 9 | 22 | 0 | $\mathbf{\frac{848456353}{312129649}} < \mathrm{e} < \frac{438351041}{161260336}$ | ***lb*** | $6.03{\times}10^{-19}$ |
| 10 | 22 | 11 | $\mathbf{\frac{848456353}{312129649}} < \mathrm{e} < \frac{9771370924}{3594686475}$ | ub | $2.89{\times}10^{-19}$ |
| 11 | 25 | 2 | $\frac{72613632504}{26713062547} < \mathrm{e} < \mathbf{\frac{28875761731}{10622799089}}$ | ***ub*** | $4.66{\times}10^{-22}$ |
| 12 | 27 | 0 | $\frac{534625820200}{196677847971} < \mathrm{e} < \mathbf{\frac{563501581931}{207300647060}}$ | ***ub*** | $1.16{\times}10^{-23}$ |
| 13 | 28 | 8 | $\mathbf{\frac{1098127402131}{403978495031}} < \mathrm{e} < \frac{9348520798979}{3439128607308}$ | ***lb*** | $2.91{\times}10^{-25}$ |
| 14 | 31 | 1 | $\frac{69774403677915}{25668568633102} < \mathrm{e} < \mathbf{\frac{46150226651233}{16977719590391}}$ | ***ub*** | $1.51{\times}10^{-28}$ |
| 15 | 31 | 21 | $\frac{992778936702575}{365222960440922} < \mathrm{e} < \mathbf{\frac{46150226651233}{16977719590391}}$ | lb | $1.06{\times}10^{-29}$ |
| 16 | 34 | 4 | $\mathbf{\frac{2124008553358849}{781379079653017}} < \mathrm{e} < \frac{9581113603440437}{3524694718233772}$ | ***lb*** | $6.55{\times}10^{-32}$ |
| 17 | 36 | 0 | $\mathbf{\frac{54185293223976266}{19933655390947129}} < \mathrm{e} < \frac{52061284670617417}{19152276311294112}$ | ***lb*** | $1.26{\times}10^{-33}$ |
| 18 | 37 | 8 | $\frac{904157916380725730}{332621109008877057} < \mathrm{e} < \mathbf{\frac{106246577894593683}{39085931702241241}}$ | ***ub*** | $2.42{\times}10^{-35}$ |

*(table continued on next page)*

(table continued from previous page)

| $k$ | $n_k$ | $m_k$ | $lb$ $<$ e $<$ $ub$ | $bra$ | $a.er.$ |
|---|---|---|---|---|---|
| 19 | 40 | 1 | $\frac{\mathbf{5739439214861417731}}{\mathbf{2111421691000680031}} <$ e $< \frac{8662282111239423438}{3186675502352140667}$ | ***lb*** | $7.73{\times}10^{-39}$ |
| 20 | 40 | 16 | $\frac{\mathbf{5739439214861417731}}{\mathbf{2111421691000680031}} <$ e $< \frac{9475387033416068 9403}{34858000867362341132}$ | ub | $5.86{\times}10^{-39}$ |

Table 4: best approximations of e by rationals with exactly $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $M_k$ | $plb$ $<$ e $<$ $pub$ | $pbra$ | $a.er.$ |
|---|---|---|---|---|---|---|
| 10 | 22 | 11 | 11 | $\frac{9743125195}{3584295452} <$ e $< \frac{9771370924}{3594686475}$ | pub | $2.89{\times}10^{-19}$ |
| 15 | 31 | 21 | 21 | $\frac{992778936702575}{365222960440922} <$ e $< \frac{991680809300444}{364818981945891}$ | plb | $1.06{\times}10^{-29}$ |
| 20 | 40 | 16 | 16 | $\frac{94647623756266095720}{34818914935660099891} <$ e $< \frac{94753870334160689403}{34858000867362341132}$ | pub | $5.86{\times}10^{-39}$ |

**Example 3.** The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.
Total computation time: 0.28 seconds.

Table 5: best approximations of $\varphi$ by rationals with at most $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $lb$ $< \varphi <$ $ub$ | $bra$ | $a.er.$ |
|---|---|---|---|---|---|
| 1 | 4 | 0 | $\frac{\mathbf{8}}{\mathbf{5}} < \varphi < \frac{5}{3}$ | ***lb*** | $1.80{\times}10^{-2}$ |
| 2 | 9 | 0 | $\frac{55}{34} < \varphi < \frac{\mathbf{89}}{\mathbf{55}}$ | ***ub*** | $1.48{\times}10^{-4}$ |
| 3 | 14 | 0 | $\frac{\mathbf{987}}{\mathbf{610}} < \varphi < \frac{610}{377}$ | ***lb*** | $1.20{\times}10^{-6}$ |
| 4 | 18 | 0 | $\frac{\mathbf{6765}}{\mathbf{4181}} < \varphi < \frac{4181}{2584}$ | ***lb*** | $2.56{\times}10^{-8}$ |
| 5 | 23 | 0 | $\frac{46368}{28657} < \varphi < \frac{\mathbf{75025}}{\mathbf{46368}}$ | ***ub*** | $2.08{\times}10^{-10}$ |
| 6 | 28 | 0 | $\frac{\mathbf{832040}}{\mathbf{514229}} < \varphi < \frac{514229}{317811}$ | ***lb*** | $1.69{\times}10^{-12}$ |
| 7 | 33 | 0 | $\frac{5702887}{3524578} < \varphi < \frac{\mathbf{9227465}}{\mathbf{5702887}}$ | ***ub*** | $1.38{\times}10^{-14}$ |
| 8 | 37 | 0 | $\frac{39088169}{24157817} < \varphi < \frac{\mathbf{63245986}}{\mathbf{39088169}}$ | ***ub*** | $2.93{\times}10^{-16}$ |
| 9 | 42 | 0 | $\frac{\mathbf{701408733}}{\mathbf{433494437}} < \varphi < \frac{433494437}{267914296}$ | ***lb*** | $2.38{\times}10^{-18}$ |
| 10 | 47 | 0 | $\frac{4807526976}{2971215073} < \varphi < \frac{\mathbf{7778742049}}{\mathbf{4807526976}}$ | ***ub*** | $1.93{\times}10^{-20}$ |

(table continued on next page)

*(table continued from previous page)*

| $k$ | $n_k$ | $m_k$ | $lb < \varphi < ub$ | *bra* | *a.er.* |
|---|---|---|---|---|---|
| 11 | 52 | 0 | $\frac{\mathbf{86267571272}}{\mathbf{53316291173}} < \varphi < \frac{53316291173}{32951280099}$ | ***lb*** | $1.57{\times}10^{-22}$ |
| 12 | 57 | 0 | $\frac{591286729879}{365435296162} < \varphi < \frac{\mathbf{956722026041}}{\mathbf{591286729879}}$ | ***ub*** | $1.28{\times}10^{-24}$ |
| 13 | 61 | 0 | $\frac{4052739537881}{2504730781961} < \varphi < \frac{\mathbf{6557470319842}}{\mathbf{4052739537881}}$ | ***ub*** | $2.72{\times}10^{-26}$ |
| 14 | 66 | 0 | $\frac{\mathbf{72723460248141}}{\mathbf{44945570212853}} < \varphi < \frac{44945570212853}{27777890035288}$ | ***lb*** | $2.21{\times}10^{-28}$ |
| 15 | 71 | 0 | $\frac{498454011879264}{308061521170129} < \varphi < \frac{\mathbf{806515533049393}}{\mathbf{498454011879264}}$ | ***ub*** | $1.80{\times}10^{-30}$ |
| 16 | 76 | 0 | $\frac{\mathbf{8944394323791464}}{\mathbf{5527939700884757}} < \varphi < \frac{5527939700884757}{3416454622906707}$ | ***lb*** | $1.46{\times}10^{-32}$ |
| 17 | 81 | 0 | $\frac{61305790721611591}{37889062373143906} < \varphi < \frac{\mathbf{99194853094755497}}{\mathbf{61305790721611591}}$ | ***ub*** | $1.19{\times}10^{-34}$ |
| 18 | 85 | 0 | $\frac{420196140727489673}{259695496911122585} < \varphi < \frac{\mathbf{679891637638612258}}{\mathbf{420196140727489673}}$ | ***ub*** | $2.53{\times}10^{-36}$ |
| 19 | 90 | 0 | $\frac{\mathbf{7540113804746346429}}{\mathbf{4660046610375530309}} < \varphi < \frac{4660046610375530309}{2880067194370816120}$ | ***lb*** | $2.06{\times}10^{-38}$ |
| 20 | 95 | 0 | $\frac{51680708854858323072}{31940434634990099905} < \varphi < \frac{\mathbf{83621143489848422977}}{\mathbf{51680708854858323072}}$ | ***ub*** | $1.67{\times}10^{-40}$ |

**Example 4.** The number $\pi^e$.
Total computation time: 0.17 seconds.

Table 6: best approximations of $\pi^e$ by rationals with at most $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $lb < \pi^e < ub$ | *bra* | *a.er.* |
|---|---|---|---|---|---|
| 1 | $-1$ | 9 | $9 < \pi^e < -$ | | 13.46 |
| 2 | 1 | 1 | $\frac{67}{3} < \pi^e < \frac{\mathbf{45}}{\mathbf{2}}$ | ***ub*** | $4.08{\times}10^{-2}$ |
| 3 | 5 | 0 | $\frac{539}{24} < \pi^e < \frac{\mathbf{831}}{\mathbf{37}}$ | ***ub*** | $3.02{\times}10^{-4}$ |
| 4 | 8 | 0 | $\frac{\mathbf{7973}}{\mathbf{355}} < \pi^e < \frac{2201}{98}$ | ***lb*** | $2.79{\times}10^{-6}$ |
| 5 | 11 | 1 | $\frac{70387}{3134} < \pi^e < \frac{\mathbf{44267}}{\mathbf{1971}}$ | ***ub*** | $6.96{\times}10^{-8}$ |
| 6 | 12 | 6 | $\frac{\mathbf{158921}}{\mathbf{7076}} < \pi^e < \frac{997793}{44427}$ | ub | $1.04{\times}10^{-9}$ |
| 7 | 13 | 6 | $\frac{9006257}{401006} < \pi^e < \frac{\mathbf{1474556}}{\mathbf{65655}}$ | ***ub*** | $1.53{\times}10^{-11}$ |

*(table continued on next page)*

*(table continued from previous page)*

| $k$ | $n_k$ | $m_k$ | $lb < \pi^e < ub$ | $bra$ | $a.er.$ |
|---|---|---|---|---|---|
| 8 | 14 | 4 | $\frac{\mathbf{22277261}}{\mathbf{991901}} < \pi^e < \frac{90583600}{4033259}$ | ***lb*** | $3.97{\times}10^{-14}$ |
| 9 | 16 | 0 | $\frac{\mathbf{580683342}}{\mathbf{25855081}} < \pi^e < \frac{558406081}{24863180}$ | ***lb*** | $6.95{\times}10^{-16}$ |
| 10 | 18 | 1 | $\frac{\mathbf{6276130457}}{\mathbf{279446386}} < \pi^e < \frac{7415219880}{330164647}$ | ***lb*** | $2.62{\times}10^{-18}$ |
| 11 | 21 | 0 | $\frac{32519741708}{1447950191} < \pi^e < \frac{\mathbf{91283094667}}{\mathbf{4064404187}}$ | ***ub*** | $2.59{\times}10^{-20}$ |
| 12 | 23 | 4 | $\frac{984146560543}{43819388638} < \pi^e < \frac{\mathbf{215085931042}}{\mathbf{9576758565}}$ | ***ub*** | $2.13{\times}10^{-22}$ |
| 13 | 23 | 45 | $\frac{9802669733265}{436466489803} < \pi^e < \frac{\mathbf{215085931042}}{\mathbf{9576758565}}$ | lb | $2.62{\times}10^{-23}$ |
| 14 | 28 | 1 | $\frac{\mathbf{55035754735501}}{\mathbf{2450481688835}} < \pi^e < \frac{88100224763010}{3922686053849}$ | ***lb*** | $2.02{\times}10^{-26}$ |
| 15 | 31 | 0 | $\frac{473350507911517}{21076057875694} < \pi^e < \frac{\mathbf{891665261087533}}{\mathbf{39701634062553}}$ | ***ub*** | $3.43{\times}10^{-28}$ |
| 16 | 33 | 1 | $\frac{6351728337083733}{282812401815541} < \pi^e < \frac{\mathbf{4986712568084683}}{\mathbf{222034709877294}}$ | ***ub*** | $2.81{\times}10^{-30}$ |
| 17 | 35 | 1 | $\frac{67557294923098979}{3008006612281316} < \pi^e < \frac{\mathbf{36272003745591831}}{\mathbf{1615020661079305}}$ | ***ub*** | $1.83{\times}10^{-32}$ |
| 18 | 36 | 1 | $\frac{\mathbf{756725366089343768}}{\mathbf{33693399172788111}} < \pi^e < \frac{792997369834935599}{35308419833867416}$ | ***lb*** | $8.21{\times}10^{-35}$ |
| 19 | 38 | 0 | $\frac{\mathbf{8360251030728373279}}{\mathbf{372242411561748526}} < \pi^e < \frac{7603525664639029511}{338549012388960415}$ | ***lb*** | $2.40{\times}10^{-36}$ |
| 20 | 39 | 3 | $\frac{81332334209015701486}{3621343918099120927} < \pi^e < \frac{\mathbf{24324027726095776069}}{\mathbf{1083033835512457467}}$ | ***ub*** | $7.59{\times}10^{-38}$ |

Table 7: best approximations of $\pi^e$ by rationals with exactly $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $M_k$ | $plb < \pi^e < pub$ | $pbra$ | $a.er.$ |
|---|---|---|---|---|---|---|
| 13 | 23 | 45 | 46 | $\frac{9802669733265}{436466489803} < \pi^e < \frac{9985235922599}{444595298177}$ | lb | $2.62{\times}10^{-23}$ |

**Example 5.** In this example we consider a number $\alpha$ with $k$-proper-N-approximations which are pseudo-convergents. The number is

$$\alpha = \frac{32400000000179999999995 - \sqrt{5}}{64799999999999999999990} \simeq 0.50000000002777777774\ ,$$

whose development in continued fractions is

$$\alpha = [0; 1, 1, 9000000000, \{1\}],$$

that is $a_0 = 0$, $a_1 = a_2 = 1$, $a_3 = 9000000000$ and $a_i = 1$ for any integer $i$ greater than 3. Observe that for $k > 9$ we have $n_k > 3$ and $a_{n_k} = 1$; similarly to the case of the golden ratio $\varphi$, the best $k$-digit approximations of $\alpha$ is the convergent $s_{n_k}$, for any $k > 9$.
Total computation time: 0.20 seconds.

Table 8: best approximations of $\alpha$ by rationals with at most $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $lb < \alpha < ub$ | *bra* | *a.er.* |
|---|---|---|---|---|---|
| 1 | 2 | 8 | $\mathbf{\frac{1}{2}} < \alpha < \frac{9}{17}$ | ***lb*** | $2.78\times10^{-11}$ |
| 2 | 2 | 98 | $\mathbf{\frac{1}{2}} < \alpha < \frac{99}{197}$ | ***lb*** | $2.78\times10^{-11}$ |
| 3 | 2 | 998 | $\mathbf{\frac{1}{2}} < \alpha < \frac{999}{1997}$ | ***lb*** | $2.78\times10^{-11}$ |
| 4 | 2 | 9998 | $\mathbf{\frac{1}{2}} < \alpha < \frac{9999}{19997}$ | ***lb*** | $2.78\times10^{-11}$ |
| 5 | 2 | 99998 | $\mathbf{\frac{1}{2}} < \alpha < \frac{99999}{199997}$ | ***lb*** | $2.78\times10^{-11}$ |
| 6 | 2 | 999998 | $\mathbf{\frac{1}{2}} < \alpha < \frac{999999}{1999997}$ | ***lb*** | $2.78\times10^{-11}$ |
| 7 | 2 | 9999998 | $\mathbf{\frac{1}{2}} < \alpha < \frac{9999999}{19999997}$ | ***lb*** | $2.78\times10^{-11}$ |
| 8 | 2 | 99999998 | $\mathbf{\frac{1}{2}} < \alpha < \frac{99999999}{199999997}$ | ***lb*** | $2.78\times10^{-11}$ |
| 9 | 2 | 999999998 | $\mathbf{\frac{1}{2}} < \alpha < \frac{999999999}{1999999997}$ | ***lb*** | $2.78\times10^{-11}$ |
| 10 | 4 | 0 | $\mathbf{\frac{9000000002}{18000000003}} < \alpha < \frac{9000000001}{18000000001}$ | ***lb*** | $1.18\times10^{-21}$ |

Table 9: best approximations of $\alpha$ by rationals with exactly $k$-digit numerators

| $k$ | $n_k$ | $m_k$ | $M_k$ | $plb < \alpha < pub$ | *pbra* | *a.er.* |
|---|---|---|---|---|---|---|
| 1 | 2 | 8 | – | $\mathbf{\frac{1}{2}} < \alpha < \frac{9}{17}$ | ***lb*** | $2.78\times10^{-11}$ |
| 2 | 2 | 98 | 99 | $\frac{99}{199} < \alpha < \frac{99}{197}$ | lb | $2.51\times10^{-3}$ |
| 3 | 2 | 998 | 999 | $\frac{999}{1999} < \alpha < \frac{999}{1997}$ | lb | $2.50\times10^{-4}$ |
| 4 | 2 | 9998 | 9999 | $\frac{9999}{19999} < \alpha < \frac{9999}{19997}$ | lb | $2.50\times10^{-5}$ |
| 5 | 2 | 99998 | 99999 | $\frac{99999}{199999} < \alpha < \frac{99999}{199997}$ | ub | $2.50\times10^{-6}$ |
| 6 | 2 | 999998 | 999999 | $\frac{999999}{1999999} < \alpha < \frac{999999}{1999997}$ | ub | $2.50\times10^{-7}$ |

*(table continued on next page)*

*(table continued from previous page)*

| $k$ | $n_k$ | $m_k$ | $M_k$ | $plb < \alpha < pub$ | $pbra$ | $a.er.$ |
|---|---|---|---|---|---|---|
| 7 | 2 | 9999998 | 9999999 | $\frac{9999999}{19999999} < \alpha < \frac{9999999}{19999997}$ | ub | $2.50{\times}10^{-8}$ |
| 8 | 2 | 99999998 | 99999999 | $\frac{99999999}{199999999} < \alpha < \frac{99999999}{199999997}$ | ub | $2.47{\times}10^{-9}$ |
| 9 | 2 | 999999998 | 999999999 | $\frac{999999999}{1999999999} < \alpha < \frac{999999999}{1999999997}$ | ub | $2.22{\times}10^{-10}$ |
| 10 | 4 | 0 | – | $\frac{\mathbf{9000000002}}{\mathbf{18000000003}} < \alpha < \frac{9000000001}{18000000001}$ | ***lb*** | $1.18{\times}10^{-21}$ |

**Example 6.** We might consider the rational number

$$\alpha' = \frac{9000000001}{18000000001} = [0; 1, 1, 9000000000] \ ,$$

whose set of convergents is $\{s_0 = 0, s_1 = 1, s_2 = 1/2, s_3 = \alpha'\}$. For $1 \leq k \leq 9$, the number $\alpha'$ has the same approximations of the irrational number $\alpha$ introduced in Example 5. The 10-best-N-approximation of $\alpha'$ is clearly $\alpha'$, while $k$-digit approximations do not seem to be interesting for $k > d(\alpha') = 10$.

## Acknowledgments

The authors gratefully acknowledge the many helpful suggestions of Professor Rodolfo Talamo during the preparation of this paper.

## References

[1] A.Ya. Khintchine. *Continued fractions.* P. Noordhoff Ltd., Groningen, 1963. (trans. from 3rd Russian ed. by Peter Wynn).

[2] S. Lang. *Introduction to Diophantine approximations.* Springer-Verlag, New York, second edition, 1995.

[3] A. M. Rockett and P. Szüsz. *Continued Fractions.* World Scientific Publishing, Singapore, 1992.

[4] S. K. Sen, R. P. Agarwal, and R. Pavani. Best $k$-digit rational bounds for irrational numbers: pre- and super-computer era. *Math. Comput. Modelling*, 49(7-8):1465–1482, 2009.

[5] S. K. Sen, R. P. Agarwal, and G. A. Shaykhian. Best $k$-digit rational approximations -true versus convergent-, decimal-based ones: quality, cost, scope. *Advanced Studies in Contemporary Mathematics*, Kyungshang, Memoirs of the Jangjeon Mathematical Society, 19(1):59–96, 2009.

## Appendix: the programs

Here we report the scripts of used programs just for convenience of the reader. We point out that they are not optimized. Authors welcome any comments by interested users.

**A)** Script of MATLAB® program. We recall that the computation of convergents is adapted from [5].

```
% Algorithm CPT
% to compute the best k-digit rational approximation to an irrational number
% Warning:  this version of the program works with k<10.
      % input :  r-> number to be approximated,
      % N -> number of convergents to be calculated,
      % Nd -> digits of extended precision,
      % Kfin -> final value of k to be considered
      % Authors :  M. Citterio, R. Pavani
r=pi,
%r=(1+sqrt(5))/2,
%r=exp(1),
Kfin=14,
Nd=50;
r=vpa(r,Nd)
flagp=0;
flagpb=0;
for k=1:Kfin,
      N=5*k+2,
      k, tic,
% continued fraction representation
% by Sen-Agarwal-Shaykhian
p1=vpa(r,Nd);
a0=floor(r);
qq=p1;
for i=1:N,
      qq=1/frac(qq);
      a(i)=qq-frac(qq);
end
%disp('sequence of the simple continued fraction of')
%disp(r)
% [a0; a'] % continued fraction representation
p0=a0;
q0=1;
p(1)=a(1)*a0+1;
q(1)=a(1);
p(2)=a(2)*p(1)+p0;
q(2)=a(2)*q(1)+q0;
```

```
for kk=2:N-1,
      p(kk+1)=a(kk+1)*p(kk)+p(kk-1);
      q(kk+1)=a(kk+1)*q(kk)+q(kk-1);
end
% disp('convergents for the continued fraction expansion of'),
% disp(r)
%disp([p0 q0])
%disp([p' q'])
% best k-digit rational approximation
% assumption:  p(-2)=0,p(-1)=1,q(-2)=1,q(-1)=0
P=[0 1 p0 p];
Q=[1 0 q0 q];
a=[0 0 a0 a];
 i=3;
       while double(P(i))< 10^k;i=i+1; end
       i=i-1;
       cst=((10^k)-1-P(i-1))/P(i);
       m=floor(cst);
       dst=((10^k)-1-(a(i)-1)*P(i-1)-P(i-2))/P(i);
       MM=floor(dst);
       num1=m*P(i)+P(i-1); den1=m*Q(i)+Q(i-1);
       num2=P(i); den2=Q(i);
       num3=(MM*P(i)+(a(i)-1)*P(i-1)+P(i-2));
       den3=(MM*Q(i)+(a(i)-1)*Q(i-1)+Q(i-2));
       AA=num1/den1;
       BB=num2/den2;
       CC=num3/den3;
      if rem((i-3), 2) == 0 % i is even
        LB=BB; UB=AA; PLB=CC; PUB=AA; flag=1;
      else
        LB=AA; UB=BB; PLB=AA; PUB=CC; flag=2;
      end
      if double(r-LB) < double(UB-r),
        bra=LB; flagb=1;
      else
        bra=UB; flagb=2;
      end
   if double(P(i))<10^(k-1),
        if abs(double(r-PLB)) < abs(double(PUB-r)),
             PBRA=PLB; flagpb=1;
        else
            PBRA=PUB; flagpb=2;
        end,
 % disp(' n_k m_k M_k ')
 % disp( [ i-3 m MM ])
      disp('++++++++++++++++ at most k digits')
      if flag==1 & flagb==1,
        disp('lower bound'), disp( bra),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num2),double(den2))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
      elseif flag==1 & flagb==2,
        disp('upper bound'), disp( bra),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num2),double(den2))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
      elseif flag==2 & flagb==1,
        disp('lower bound'), disp( bra),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num2),double(den2))
```

```
      else %flag==2 & flagb==2,
        disp('upper bound'), disp( bra),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num2),double(den2))
      end,
      erra=double(abs(r-bra));
      disp('abs.  err.  with at most k digits'), disp(erra)
      disp(' exactly k digits')
      if flag==1 & flagpb==1,
        disp('lower bound'), disp( PBRA),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num3),double(den3))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
      elseif flag==1 & flagpb==2,
        disp('upper bound'), disp( PBRA),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num3),double(den3))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
      elseif flag==2 & flagpb==1,
        disp('lower bound'), disp( PBRA),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num3),double(den3))
      else %flag==2 & flagpb==2,
        disp('upper bound'), disp( PBRA),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num3),double(den3))
      end,
     perra=double(abs(r-PBRA));
       disp('abs.  err.  with exactly k digits'), disp(perra)
 else
 % disp(' n_k m_k M_k ')
 % disp([ i-3 m MM])
     disp('--------------- at most k digits')
     if flag==1 & flagb==1,
        disp('lower bound'), disp( bra),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num2),double(den2))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
      elseif flag==1 & flagb==2,
        disp('upper bound'), disp( bra),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num2),double(den2))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
      elseif flag==2 & flagb==1,
        disp('lower bound'), disp( bra),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num2),double(den2))
      else
        disp('upper bound'), disp( bra),
        disp('lower bound'), fprintf('%17.0f%17.0f\n',double(num1),double(den1))
        disp('upper bound'), fprintf('%17.0f%17.0f\n',double(num2),double(den2))
      end,
     erra=double(abs(r-bra));
     disp('abs.  err.  with at most k digits'), disp(erra)
  end
       toc,
      disp('***********')
      disp(' ')
      clear AA BB CC LB UB PLB PUB m MM

end,
```

## B) Script of Mathematica® program.

```
(* Algorithm CPT *)
(* Authors :  M. Citterio, R. Pavani *)
(* This algorithm supplies a positive irrational number with the best rational
approximation with at most k digits at the numerator (the k-best-N-approximation),
the best rational approximation with exactly k digits at the numerator
(the k-proper-N-approximation), the best rational lower and upper bounds with at most
k digits at the numerator and, in all the cases but the one in which the k-proper-N-
-approximation is obtained without extra computation, the best rational lower and
upper bounds with exactly k digits at the numerator.  The number k is any integer from
the number of digits of the integer part of irrational number to a convenient stop.*)
(* Inputs:  *)
r = Pi       (* positive irrational number to be approximated *);
kTOk = 20    (* final value of k *);
(* Definitions:  *)
a[n_] := ContinuedFraction[r, n + 1][[n + 1]];
Conv[n_] := Convergents[r, n + 1][[n + 1]];
p[-2] := 0; p[-1] := 1; p[n_] := Numerator[Conv[n]];
q[-2] := 1; q[-1] := 0; q[n_] := Denominator[Conv[n]];
sp[n_, m_] := m p[n] + p[n - 1];
sq[n_, m_] := m q[n] + q[n - 1];
sConv[n_, m_] := sp[n_, m_]/sq[n_, m_] ;
pConv[n_, m_] := (m p[n] + sp[n - 1, a[n] - 1])/(m q[n] + sq[n - 1, a[n] - 1]);
(* The program:  *)
If[r < 1, ALPHA = 1, ALPHA = r]; kFROMk=Floor[Log10[ALPHA]+1];
Timing[n = 0;
Print[" $\alpha$ = ", r, " $\simeq$ ", N[r, 50] ];
Do[i = n; While[p[i] < 10^k, i++]; n = i - 1;
   m = Floor[(10^k - 1 - p[n - 1])/p[n]];
   AA = sConv[n, m]; BB = Conv[n];
   If[OddQ[n],
      {LB = AA, UB = BB, PLB = AA, PUB = CC},
      {LB = BB, UB = AA, PLB = CC, PUB = AA}];
   If[r - LB < UB - r, BRA = LB, BRA = UB];
   If[p[n] < 10^(k - 1),
      {M = Floor[(10^k - 1 - (a[n] - 1) p[n - 1] - p[n - 2])/p[n]],
         CC = pConv[n, M],
         If[r - PLB < PUB - r, PBRA = PLB, PBRA = PUB],
         Print["$k$ = ", k, " ; $n_k$ = ", n," , $m_k$ = ", m," , $M_k$ = ", M,
               " ; $s_{n_k,m_k}$ = ", AA," ; $s_{n_k}$ = ", BB," ; $\tilde{s}_{n_k,M_k}$ = ", CC," ."],
         Print["  At most ", k , "-digits best approximations: ",
               LB, " < $\alpha$ < ", UB, " ; bra = ", BRA, " ."],
         Print["  Exactly ", k , "-digits best approximations: ",
               PLB, " < $\alpha$ < ", PUB, " ; proper-bra = ", PBRA, " ."],
         Print["  $|\alpha - s_{n_k,m_k}|$ = ", N[Abs[r - AA], 50], " ;"],
         Print["  $|\alpha - s_{n_k}|$ = ", N[Abs[r - BB], 50], " ;"],
         Print["  $|\alpha - \tilde{s}_{n_k,M_k}|$ = ", N[Abs[r - CC], 50], " ."]},
      {Print["$k$ = ", k, " ; $n_k$ = ", n," , $m_k$ = ", m,
               " ; $s_{n_k,m_k}$ = ", AA," ; $s_{n_k}$ = ", BB," ."],
         Print["  ", k , "-digits best approximations: ",
               LB, " < $\alpha$ < ", UB, " ; bra = ", BRA, " ."],
         Print["  $|\alpha - s_{n_k,m_k}|$ = ", N[Abs[r - AA], 50], " ;"],
         Print["  $|\alpha - s_{n_k}|$ = ", N[Abs[r - BB], 50], " ."]}];
   Print[];
   Clear[AA, BB, CC, LB, UB, PLB, PUB, m, M, BRA, PBRA],
   {k, kFROMk, kTOk}]]
```