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Bell's Inequality Violations: Relation with de Finetti's Coherence Principle and Inferential Analysis of Experimental Data

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Abstract

It is often believed that de Finetti's coherence principle in the finite case naturally leads to the Kolmogorov's probability theory of random phenomena and that such a theory unavoidably implies Bell's inequality. Thus, an alternative probability theory allowing for a violation of Bell's inequality, such as quantum probability, should violate also de Finetti's coherence principle. Firstly, we show that this is not the case: the typical violations of Bell's inequality are in theoretical agreement with de Finetti's coherence principle. Secondly, we consider the experimental data of measurements of polarization of photons, performed to verify empirically violations of Bell's inequality. We analyze them to test the null hypothesis of validity of the Kolmogorov's probability model for the observed phenomenon and we compute their p-value to quantify the experimental violation of the null hypothesis.

1 INTRODUCTION

The mixed second moments of four random variables taking values ± 1, defined on a same probability space, necessarily satisfy Bell's inequality, independently of their joint distribution. Such inequality, which can be easily proved, appeared for the first time in 1964 in a paper by J.S. Bell [1] on the Einstein - Podolsky - Rosen paradox [2]. Since then it has been at the center of a vivid interest as it is violated both theoretically, by quantum mechanics, and experimentally, by measurements on quantum systems. Thus violations of Bell's inequality are often involved with considerations and reconsiderations on the limits of classical probability and on the foundations and interpretations of quantum mechanics.

The aim of the paper is not at all to review the subject, but to discuss the problem for an audience of (classical) probabilists and statisticians, without any need of notions of quantum mechanics.

Firstly, we afford the topic from a theoretical point of view to show why and how violations of Bell's inequality would reveal the need for a theory of random phenomena more general than the Kolmogorov's one and to show that, nevertheless, such violations would not necessarily clash with de Finetti's coherence principle.

Secondly, given a random phenomenon supposed to violate Bells' inequality, we consider the problem of testing statistically, on the basis of experimental data, if a Kolmogorov's probability model has to be rejected for the observed phenomenon. The asymptotic p-value of data from a celebrated physical experiment is computed, leading to a clear rejection of the null hypothesis.

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2 VIOLATION OF BELL'S INEQUALITY VS KOLMOGOROV'S PROBABILITY THEORY

There exist various formulations of Bell's inequality. Let us state and prove a rather general form of the Clauser, Horne, Shimony, Holt version [3].

Theorem 1. *Let X_1, X_2, Y_1, Y_2 be random variables taking values ± 1 on a measurable space $(\Omega, \mathcal{F})$. Then the following inequality holds for every probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$:*

$$b = \Big|\mathbb{E}\, X_1 Y_1 + \mathbb{E}\, X_1 Y_2\Big| + \Big|\mathbb{E}\, X_2 Y_1 - \mathbb{E}\, X_2 Y_2\Big| \leq 2. \tag{1}$$

Proof. Since $|Y_\ell| = 1$, the following equalities hold

$$\Big|Y_1 - Y_2\Big| = 1 - Y_1 Y_2, \qquad \Big|Y_1 + Y_2\Big| = 1 + Y_1 Y_2.$$

Then, as $|X_k| = 1$,

$$\begin{aligned} b = \Big|\mathbb{E}\, X_1 Y_1 + \mathbb{E}\, X_1 Y_2\Big| + \Big|\mathbb{E}\, X_2 Y_1 - \mathbb{E}\, X_2 Y_2\Big| &= \Big|\mathbb{E}\, X_1 (Y_1 + Y_2)\Big| + \Big|\mathbb{E}\, X_2 (Y_1 - Y_2)\Big| \\ \leq \mathbb{E}\Big[|X_1|\,|Y_1 + Y_2| + |X_2|\,|Y_1 - Y_2|\Big] &= \mathbb{E}\Big[1 + Y_1 Y_2 + 1 - Y_1 Y_2\Big] = 2. \end{aligned}$$

□

Thus *Bell's inequality* (1) necessarily holds under the only hypothesis that four random numbers taking values ± 1 exist in a same random experiment, independently of their joint distribution. When four random numbers can be modelled with a term $(\Omega, \mathcal{F}, \mathbb{P}, X_1, X_2, Y_1, Y_2)$, as in the hypothesis of Theorem 1, we say that they admit a Kolmogorov's probability model. In any case, let b be called *Bell's parameter*.

Consider now an experiment on a physical system consisting of two particles. On the first particle we can measure two quantities of modulus 1, getting the results $X_1 = \pm 1$ and $X_2 = \pm 1$. Analogously we can measure two quantities of modulus 1 on the second particle, with results $Y_1 = \pm 1$ and $Y_2 = \pm 1$. Typically, X_1, X_2, Y_1, Y_2 are random results whose distribution depends on the procedure used to prepare the pair of particles for the experiment.

From a "classical point of view", they are the values of four physical quantities, which exist independently of some eventual constraint which could prevent us from measuring all of them simultaneously. Thus the four random results admit a Kolmogorov's probability model, they do have a joint distribution and, whatever it is, their second mixed moments have to satisfy Bell's inequality.

Nevertheless, there are experiments on elementary particles where "the classical point of view" is contradicted both theoretically, by quantum physics, and experimentally, by experimental data. One of the most important examples is found by considering a pair of photons and, for each photon, its polarization along two given angles. For the purpose of this paper we do not need to know what polarization is, but it suffices to know that it can be measured only along an angle $-\pi/2 \leq \alpha < \pi/2$, that the polarization along a given angle α can take the values ± 1 and that the distribution of the results depends on the choice of α and on the preparation of the photon (its state). Moreover, only one angle α per measurement can be chosen: it is not physically possible to measure the polarization of a same photon along two different angles α and α' simultaneously.

Therefore, given a photon pair, we can choose an angle for each photon and observe a pair X_α, Y_β. Quantum mechanics states that, if the photon pair is suitably prepared (Bell state), the random results X_α and Y_β are correlated with joint distribution

$$f_{\alpha,\beta}(i,j) = \mathbb{P}(X_\alpha = i, Y_\beta = j) = \begin{cases} \frac{1}{2}\sin^2(\beta - \alpha), & \text{if } ij = +1, \\ \frac{1}{2}\cos^2(\beta - \alpha), & \text{if } ij = -1, \end{cases} \tag{2}$$

so that both X_α and Y_β take the values ± 1 with equal probability, $\mathbb{E}\, X_\alpha = \mathbb{E}\, Y_\beta = 0$, and

$$\mathbb{E}\, X_\alpha Y_\beta = \mathrm{Cov}(X_\alpha, Y_\beta) = -\cos 2(\beta - \alpha). \tag{3}$$

In this situation, let us fix four angles $\alpha_1, \alpha_2, \beta_1, \beta_2$ and let us consider

$$\begin{aligned} X_1 &= \text{polarization of the first photon along } \alpha_1, \\ X_2 &= \text{polarization of the first photon along } \alpha_2, \\ Y_1 &= \text{polarization of the second photon along } \beta_1, \\ Y_2 &= \text{polarization of the second photon along } \beta_2. \end{aligned}$$

For every repetition of the experiment, that is for every preparation of a photon pair, we can get a pair (X_k, Y_ℓ), $k, \ell = 1, 2$. If the photon pair is prepared in the Bell state and if we choose

$$\alpha_1 = \pi/8, \quad \alpha_2 = -\pi/8, \quad \beta_1 = 0, \quad \beta_2 = \pi/4, \tag{4}$$

then Bell's inequality is violated:

$$\begin{aligned} f_{\alpha_k,\beta_\ell}(i,j) &= \frac{1}{4}\left(1 - ij\frac{\sqrt{2}}{2}\right), \qquad \mathbb{E}\, X_k Y_\ell = -\frac{\sqrt{2}}{2}, \qquad (k,\ell) \neq (2,2), \\ f_{\alpha_2,\beta_2}(i,j) &= \frac{1}{4}\left(1 + ij\frac{\sqrt{2}}{2}\right), \qquad \mathbb{E}\, X_2 Y_2 = \frac{\sqrt{2}}{2}, \end{aligned} \tag{5}$$

so that

$$b = 2\sqrt{2}$$

Therefore, the bivariate joint distributions (5) foreseen by quantum mechanics are incompatible with the existence of a quadrivariate joint distribution for X_1, X_2, Y_1, Y_2, that is with the existence of the four polarizations X_1, X_2, Y_1, Y_2 at every replicate of the experiment, independently of the pair that is actually measured.

We could simply conclude that, actually, there are only four different random experiments and four different Kolmogorov's probability models, one for each pair (X_k, Y_ℓ). However, we would get four bivariate distributions f_{α_k,β_ℓ} without a clear relation among them, notwithstanding they originate from the same random situation (the same preparation of a photon pair).

What quantum probability does in such a case is to introduce a unique non-commutative probability model, where a unique mathematical object describes the preparation of the physical system and generates all the distributions of all the possible measurements on the system.

Anyway, let us point out that there exist also models inside Kolmogorov's theory which can reproduce violations of Bell's inequality thanks to some devices. We mention just N. Gisin and B. Gisin [4], who reproduce Bell's correlations (3) by introducing a random efficiency of the measurement apparatus, correlated with the outcome, and L. Accardi *et al.* [5], who reproduce Bell's correlations (3) by employing the chameleon effect [6], that is by introducing a deterministic evolution of each particle, depending on the choice of the measurement angle α (resp. β).

To conclude the section, let us stress that, in order to violate Bell's inequality, it is fundamental that X_α and its distribution do not depend on β and, analogously, that Y_β and its distribution do not depend on α. Otherwise, chosen the angles (4), we would get 8 random variables instead of 4, and so there would be no reason for Bell's inequality to hold. This assumption (called locality or separability in physical literature) forbids any influence of the measurement over one photon on the measurement over the other one.

3 VIOLATION OF BELL'S INEQUALITY VS DE FINETTI'S COHERENCE PRINCIPLE

De Finetti's subjective approach to probability theory introduces the notion of probability and clarifies its meaning by means of the paradigm of bets and the notion of coherent evaluation [7].

Given a family $\mathcal{E}$ of events which could occur or not in a random experiment, the probability of $E \in \mathcal{E}$ is the price $\mathbb{P}(E)$ of a bet on E with payoff 1_E, that is 1 if E is observed to occur, 0 if E is observed not to occur. Chosen n events $E_1, \dots, E_n \in \mathcal{E}$, a finite combination of bets on them, with amounts $c_i \, \mathbb{P}(E_i)$, $c_i \neq 0$, determines the random total gain for the bank

$$G = \sum_{i=1}^{n} c_i \Big(\mathbb{P}(E_i) - 1_{E_i} \Big). \tag{6}$$

De Finetti's coherence principle states that the prices $\mathbb{P} : \mathcal{E} \to \mathbb{R}$ have to be fixed so that there is no combination of bets with surely positive (or surely negative) gain. That is, for every finite class $\{E_1, \dots, E_n\}$ of events in $\mathcal{E}$ and every non vanishing $c_1, \dots, c_n$, a probability $\mathbb{P}$ must give

$$\min G \leq 0 \leq \max G,$$

where the minimum and the maximum gain are computed with respect to the possible logical values of $E_1, \dots, E_n$. The coherence principle does not fix $\mathbb{P}$, but it implies some properties which $\mathbb{P}$ has to enjoy:

(a) $0 \leq \mathbb{P}(E) \leq 1$ for every $E \in \mathcal{E}$,

(b) the probability of the certain event Ω is $\mathbb{P}(\Omega) = 1$ and the probability of the impossible event $\emptyset$ is $\mathbb{P}(\emptyset) = 0$,

(c) chosen n events $E_1, \dots, E_n \in \mathcal{E}$ such that there exists the logic sum $\bigvee_{i=1}^n E_i \in \mathcal{E}$ and there exist the logic products $E_i \wedge E_j = \emptyset$ for every $i \neq j$, it holds $\mathbb{P}\left(\bigvee_{i=1}^n E_i\right) = \sum_{i=1}^n \mathbb{P}(E_i)$.

These consequences are therefore necessary conditions for coherence, but, typically, they are not sufficient to guarantee that a function $\mathbb{P} : \mathcal{E} \to \mathbb{R}$ is coherent. Anyway this happens if the family of events $\mathcal{E}$ is a field of subsets of a given nonempty space Ω. Thus, a Kolmogorov's probability model satisfies the coherence principle, but de Finetti approach to probability theory leads only to finite additivity, not necessarily to σ-additivity. Nevertheless, in the case of a finite field $\mathcal{E}$, additivity and σ-additivity are equivalent, so that de Finetti's and Kolmogorov's approaches lead to the same mathematical model.

However, the assumption that $\mathcal{E}$ is a field, which is closely related with the notion of event, can not be always taken for granted, even in the finite case.

Basing the introduction of probability on the bet paradigm, de Finetti discusses deeply the notion of event and its essential feature of verifiability. By the end of a random experiment, every event should be assigned the value "true" (occurred) or "false" (not occurred), so to determine the gain of the gambler and of the bank.

In [7] verifiability is discussed in relation with accuracy, time, cost and number of partial verifications. It is also discussed in relation with subordination: the verifiability of an event E can be subordinated to the occurrence of another event H, so that a bet on E is won if H and E are true, lost if H is true and E false, annulled if H is false. For example, considered a piece of wood (given or randomly chosen from a given pile), the two events could be

$$\begin{aligned} E &= \text{"The piece of wood burns in the fire in less then 15 minutes"}, \\ H &= \text{"The piece of wood is thrown in the fire"}. \end{aligned}$$

Typically, H is "An observation to verify E is executed".

Furthermore, de Finetti discusses verifiability in relation with what he calls "uncertainty" and "complementarity" in quantum mechanics. Indeed, similarly to subordination, an event E regarding elementary particles typically is not simply true or false at the end of the experiment, but it can be also uncertain, meaningless, if an appropriate measurement has not been performed. Moreover, two events E_1 and E_2 can be complementary in the sense that they can not be jointly verified, in a same experiment, so that one of them necessarily remains meaningless.

De Finetti discusses the implications of "uncertainty" and "complementarity" on logic operations among events, shows similar behaviors outside quantum mechanics (for example, the behavior of a same object subjected to one or the other of two different destructive experiments), but he does not analyze the consequences with the coherence principle.

Of course, a bank could be asked to assign the prices $\mathbb{P}$ to a family $\mathcal{E}$ of events not necessarily compatible. Note that if two events E_1 and E_2 in $\mathcal{E}$ are not compatible, then there is no event $E_1 \wedge E_2$ as there is no way to verify it; thus $\mathcal{E}$ is not an algebra and, consequently, a coherent probability on $\mathcal{E}$ is not a Kolmogorovian probability. When $\mathcal{E}$ contains non compatible events, not only $\mathcal{E}$ is not a field, but we also have that some combinations of bets on events in $\mathcal{E}$ are not admissible: if a combination mixes two or more "complementary" events, at least one can not be verified, it remains meaningless and the corresponding bet has to be nullified and so the whole combination. Therefore we should specify the coherence principle as follows: for every finite class $\{E_1, \dots, E_n\}$ of *compatible* events in $\mathcal{E}$ and every non vanishing $c_1, \dots, c_n$, a probability $\mathbb{P}$ must give

$$\min G \leq 0 \leq \max G, \tag{7}$$

where the minimum and maximum gain are computed with respect to the possible logical values of $E_1, \dots, E_n$.

For example, consider again the piece of wood mentioned above and the bets on

$$\begin{aligned} E_1 &= \text{"The piece of wood burns in the fire in less then 15 minutes"}, \\ E_2 &= \text{"The piece of wood reaches the bottom of the swimming pool in less then 15 hours"}. \end{aligned}$$

These events can not be checked simultaneously, in a same random experiment, for the same piece of wood, so that there can be no combination of bets on E_1 and E_2 for the same piece of wood, and thus the coherence principle implies no relation between $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$.

Polarization measurements produce just a similar situation. Given one photon pair, we can not bet on any combination of events regarding X_1, X_2, Y_1, Y_2, but only on events regarding a chosen pair X_k, Y_ℓ. Therefore, as each f_{α_k,β_ℓ} in (5) is a regular bivariate distribution, this choice does not violate de Finetti's coherence principle, even if it violates Bell's inequality and it is incompatible with a joint Kolmogorov's probability model for X_1, X_2, Y_1, Y_2.

Let us stress that, when applying coherence principle, the logical relations between the events in $\mathcal{E}$ are fundamental and play a double role because, firstly, they establish which combinations of bets are admissible and, secondly, for these last they determine $\min G$ and $\max G$ in (7).

Of course, these logical relations go far beyond the set structure of $\mathcal{E}$. Take, for example, four jointly observable random variables X_1, X_2, Y_1, Y_2 taking values ± 1, and suppose that the bank decides to allow bets only on events regarding pairs X_k, Y_ℓ. Even if from a set-theoretical point of view the family $\mathcal{E}$ is the same as with polarization measurements, in this case the prices (5) are not coherent. Indeed, all the events

$$E_1 = (X_1 = Y_1), \qquad E_2 = (X_1 = Y_2), \qquad E_3 = (X_2 = Y_1), \qquad E_4 = (X_2 = -Y_2),$$

belong to $\mathcal{E}$, according to (5) they all have probability $\mathbb{P}(E_i) = (2 - \sqrt{2})/4$, but, since they can be jointly verified, a gambler can bet $c_i = 1$ on each E_i, producing the bank gain

$$G = 2 - \sqrt{2} - \sum_{i=1}^{4} 1_{E_i} < 0,$$

as at least one event must occur: if E_1, E_2 and E_3 are false, then E_4 is necessarily true.

On the contrary, in the case of polarization measurements, these events are "complementary" and so the logical relations among them do not simply change the minimum and maximum gain G for this combination of bets, but they just forbid to consider this combination of bets, thus preserving the coherence principle also for prices (5).

4 INFERENTIAL ANALYSIS OF EXPERIMENTAL BELL'S INEQUALITY VIOLATIONS

Typically, Bell test experiments are performed to estimate the Bell parameter b and so to conclude that no Kolmogorov's probability model can describe the observed phenomenon.

Let us consider again a photon pair and the measurements of polarizations X_1, X_2, Y_1, Y_2. We have four pairs of random variables taking values ± 1,

$$(X_k, Y_\ell) \sim f_{k\ell}, \qquad\qquad k, \ell = 1, 2.$$

According to quantum mechanics, if the experimenter is able to prepare the photon pair in the Bell state, then the distributions $f_{11}, f_{12}, f_{21}, f_{22}$ are given by (5), they do violate Bell's inequality and they can not be the bivariate marginals of a joint distribution $f_{(X_1,X_2,Y_1,Y_2)}$. Since one could doubt quantum mechanics, or maybe one could simply doubt the photons initial state, now the distributions $f_{11}, f_{12}, f_{21}, f_{22}$ are considered unknown.

We want to verify if X_1, X_2, Y_1, Y_2 admit a Kolmogorov's probability model, looking for a strong conclusion against it by means of a hypothesis testing procedure. Therefore we introduce the hypotheses

$$\begin{aligned} H_0 &: f_{11}, f_{12}, f_{21}, f_{22} \text{ are bivariate marginals of a unique joint distribution } f_{(X_1,X_2,Y_1,Y_2)}, \\ H_1 &: f_{11}, f_{12}, f_{21}, f_{22} \text{ do not admit a unique joint distribution } f_{(X_1,X_2,Y_1,Y_2)}. \end{aligned}$$

Because of Theorem 1, we know that $H_0 \Rightarrow b \le 2$ and that $b > 2 \Rightarrow H_1$.

In order to test a violation of Bell's inequality, typically N independent photon pairs are prepared, all of them in the same state, possibly in the Bell state, and for each photon a polarization measurement is performed, along an angle chosen according to (4). This produces a sample of independent bivariate random variables

$$
\begin{aligned}
&(X_1^{(i)}, Y_1^{(i)}), \quad i = 1, \ldots, n_{11}, && \text{i.i.d.} \sim f_{11},\\
&(X_1^{(i)}, Y_2^{(i)}), \quad i = n_{11} + j, \quad j = 1, \ldots, n_{12}, && \text{i.i.d.} \sim f_{12},\\
&(X_2^{(i)}, Y_1^{(i)}), \quad i = n_{11} + n_{12} + j, \quad j = 1, \ldots, n_{21}, && \text{i.i.d.} \sim f_{21},\\
&(X_2^{(i)}, Y_2^{(i)}), \quad i = n_{11} + n_{12} + n_{21} + j, \quad j = 1, \ldots, n_{22}, && \text{i.i.d.} \sim f_{22},
\end{aligned}
$$

where $n_{11}+n_{12}+n_{21}+n_{22} = N$. Let us stress that only one polarization per photon is measured so that, independently of violations of Bell's inequality, there is no violation of de Finetti's coherence principle and this is an ordinary sample which admits a Kolmogorov's probability model. The range of variation of the quadruple of distributions $(f_{11}, f_{12}, f_{21}, f_{22})$ specifies the statistical model for the whole sample, so that H_0 and H_1 can be seen as hypotheses on the sample, and standard inferential methods can be applied.

The typical point estimator of b is the statistic

$$
B = \left| \frac{1}{n_{11}} \sum_i X_1^{(i)} Y_1^{(i)} + \frac{1}{n_{12}} \sum_i X_1^{(i)} Y_2^{(i)} \right| + \left| \frac{1}{n_{21}} \sum_i X_2^{(i)} Y_1^{(i)} - \frac{1}{n_{22}} \sum_i X_2^{(i)} Y_2^{(i)} \right|. \tag{8}
$$

Following the usual procedure, we want to test H_0 vs H_1 on the basis of B and thus we introduce the critical region

$$
B > s,
$$

where $s > 2$. Of course, the size of the test is $\alpha = \sup_{H_0} \mathbb{P}(B > s)$ and, given a realization of the sample with estimate $\widehat{b}$ of b, the p-value of the data is

$$
p = \sup_{H_0} \mathbb{P}(B > \widehat{b}).
$$

In order to compute this p-value, let us introduce the probabilities

$$
p_{k\ell} = \mathbb{P}(X_k = Y_\ell).
$$

Then

$$
\mathbb{E}[X_k Y_\ell] = 2p_{k\ell} - 1, \quad \mathrm{Var}[X_k Y_\ell] = 4\, p_{k\ell}(1 - p_{k\ell}), \quad b = |2(p_{11} + p_{12}) - 2| + |2(p_{21} - p_{22})|.
$$

Furthermore, the distribution of B, and thus the probability $\mathbb{P}(B > \widehat{b})$, depends only on $\mathbf{p} = (p_{11}, p_{12}, p_{21}, p_{22})$. The possible values of $\mathbf{p}$ depend on the statistical model for the sample, that is on the possible values of the quadruple of distributions $(f_{11}, f_{12}, f_{21}, f_{22})$. In particular, if one parameterizes the quadrivariate distributions $f_{(X_1, X_2, Y_1, Y_2)}$, it turns out that the values of $\mathbf{p}$ compatible with the null hypothesis H_0 are

$$
\begin{aligned}
p_{11} &= \theta_1 + \theta_2 + \theta_3 + \theta_4, \qquad & p_{12} &= \theta_1 + \theta_2 + \theta_5 + \theta_6,\\
p_{21} &= \theta_1 + \theta_3 + \theta_6 + \theta_7, \qquad & p_{22} &= \theta_1 + \theta_4 + \theta_5 + \theta_7,
\end{aligned}
$$

with

$$
\theta = (\theta_1, \ldots, \theta_7) \in \Theta_0 = \left\{ \vartheta_i \geq 0 \ \forall i, \quad \sum_{i=1}^{7} \vartheta_i \leq 1 \right\}.
$$

Thus, under the null hypothesis, $\mathbb{P}(B > \widehat{b})$ is a function of θ and

$$p = \sup_{H_0} \mathbb{P}(B > \widehat{b}) = \sup_{\theta \in \Theta_0} \mathbb{P}(B > \widehat{b}).$$

Then the asymptotic p-value can be easily computed. Every addendum in (8) is asymptotically normal thanks to the Central Limit Theorem,

$$\frac{1}{n_{k\ell}} \sum_i X_k^{(i)} Y_\ell^{(i)} \sim AN\left(2p_{k\ell} - 1, \frac{4\,p_{k\ell}(1 - p_{k\ell})}{n_{k\ell}}\right);$$

thus, if $2(p_{11} + p_{12}) - 2 \neq 0$ and $2(p_{21} - p_{22}) \neq 0$, the Delta Method gives the asymptotic normality also of the estimator B,

$$B \sim AN\left(b, \sum_{k,\ell=1}^{2} \frac{4\,p_{k\ell}(1 - p_{k\ell})}{n_{k\ell}}\right); \tag{9}$$

therefore the asymptotic value of $\mathbb{P}(B > \widehat{b})$ is immediately got as a function of θ and the asymptotic p-value is got by a numerical computation of $\sup_{\theta \in \Theta_0}$.

Let us compute the asymptotic p-value of the data from the experiment performed on the 1st of May 1998 in Innsbruck by Gregor Weihs *et al.* (scan blue experiment) [8].

For the first time they could avoid any possible influence of β on X_α and of α on Y_β, which is a fundamental condition to violate Bell's inequality, as discussed at the end of Sect. 2. Indeed, the two photons of each pair were spatially separated, before of the polarization measurements, and, moreover, the angles α and β were selected randomly (according to (4)) at the very last moment, so to exclude any mutual influence within the realm of Einstein locality. The two photons of each pair were sent to two different experimental stations, each one registering the photon arrival time, the corresponding angle α (resp. β) of measurement and the corresponding result X_α (resp. Y_β). Because of the low efficiency of the apparata, a lot of photons were lost, and the arrival times are fundamental to couple the data of the same pairs: two photons belong to the same pair if they arrive "simultaneously". Following physical analysis of these data, we calculate coincidences with a time window of 4 ns (which, actually, is even smaller of the 6 ns window used by Weihs *et al.*) and we assume that grouping the results of the measurements on the basis of (α, β) gives independent random samples.

We analyze the data from the experiments scanblue1 – scanblue20 [9,10]. These experiments were performed spanning a lot of angles α and β, but not exactly the angles (4), for which quantum mechanics foresees $b = 2\sqrt{2} \simeq 2.8284$ in the case of photon pairs in the Bell state. Anyway we can analyze the data of polarization measurements performed along the angles

$$\alpha_1 = 3\pi/20, \quad \alpha_2 = -\pi/10, \quad \beta_1 = 0, \quad \beta_2 = \pi/4,$$

which also give a good theoretical Bell parameter, $b \simeq 2.7936$, enough bigger than 2. Experimental data give $n_{11} = 941$, $n_{12} = 941$, $n_{21} = 1203$, $n_{22} = 1014$,

$$\widehat{b} = 2.5254, \qquad \text{asymptotic p-value} = 7.4857 \cdot 10^{-8}.$$

Analogous analysis of the other data from the same experiments, related to the polarization measurements performed along the angles $\alpha_1 = \pi/10, 3\pi/20$ and $\alpha_2 = -\pi/10, -3\pi/20$, gives even smaller p-values.

Thus the statistical tests clearly lead to a rejection of the null hypothesis: given a photon pair, there is no Kolmogorov's joint probability model for X_1, X_2, Y_1, Y_2.

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