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VARIATIONAL APPROACH TO IMAGE SEGMENTATION

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ABSTRACT. This paper focuses on a second order functional depending on free discontinuity and free gradient-discontinuity, whose minimizers provide a variational solution to contour detection problem in image segmentation.
We briefly resume the state of the art about Blake & Zisserman functional under different types of boundary condition which are related to contour enhancement in image segmentation.
We prove a new Caccioppoli inequality suitable to study regularity of minimizers of related boundary value problems in any dimension $n \geq 1$ and deduce that there are no nontrivial local minimizers in half-space.

1. INTRODUCTION

This paper deals with free discontinuity problems related to image segmentation, focusing on the mathematical analysis of Blake & Zisserman functional.

Calculus of Variations is the framework where energy minimization and equilibrium notions find a precise language and formalizations by means of variational principles.

Image segmentation is a relevant problem both in digital image processing and in the understanding of biological vision.

There exist many different way to define the tasks of segmentation (template matching, component labelling, thresholding, boundary detection, quadtrees, texture matching, texture segmentation) and there is no universally accepted notion (optimality criteria for segmentation, analogies and differences between biological and automata perspective in segmentation): here the exposition is confined to one model for decomposing an image field, where is given a function describing the signal intensity associate to each point (typically the light intensity on a screen image). Such purpose has a clear connection with the problem of optimal partitions of a domain minimizing the length of the boundaries.

In simple words the segmentation we look for provides a cartoon of the given image satisfying some requirements: the decomposition of the image

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is performed by choosing a pattern of lines of steepest discontinuity for light intensity, and this pattern will be called segmentation of the image.
The variational formalizations of segmentation models ([7],[23],[30],[31],[32]) provided deeper understanding of image analysis, produced intriguing mathematical questions (some of them still open) and entailed global estimates for geometric quantities in visual and automatic perception at both low and high level vision.

We discuss some recent results based on the innovative notion of free discontinuity problem introduced by Ennio De Giorgi ([25]). This approach balances carefully signal smoothing and segmentation length. In such framework, modern tools of Geometric Measure Theory and recent developments about minimal surfaces and regularity of extremals in Calculus of Variations allow the study of problems coupling bulk and surface terms: in such context discontinuous (in the mathematical sense) solutions are admissible and sometimes their discontinuities are the main features of the solution.

Usual techniques for interior regularity do not apply at the boundary to bi-laplacian operator with homogeneous Dirichlet condition, since Duffin extension of bi-harmonic function may increase a lot H^2 norm ([19],[27]): this fact introduces an additional difficulty in the study of regularity through blow-up.

Nevertheless capability of dealing Dirichlet-type boundary conditions seems a relevant point in relationship to in-painting problems ([22],[23]).

Here we focus a second order functional depending on free discontinuity and free gradient-discontinuity, whose minimizers provide a variational solution to contour detection problem in image segmentation. In Section 2 we resume the present state of the art about Blake & Zisserman functional ([5],[6],[9],[10],[11],[12],[13],[14],[15],[16],[17],[18],[19]). Then we focus the case of homogeneous Dirichlet condition at flat boundary in any dimension $n \geq 1$: first we prove a Caccioppoli inequality (Theorems 3.4,3.6), suitable in further study of regularity ([20],[21]) for minimizers of related boundary value problems, then we deduce that there are no nontrivial local minimizers in half-space with bounded segmentation (Theorem 3.8); eventually we show that neither a 1-dimensional step nor an infinite wedge are local minimizers (Theorems 3.9, 3.10).

Outline of the paper

2. Research background

Image segmentation is a relevant problem both in digital image processing and in the understanding of biological vision. There are many different ways ([23],[31],[32],[33]) to define the tasks of segmentation. Several different variational formalizations ([2],[3],[10],[23],[24],[26],[33]) of segmentation models provided deeper understanding of image analysis.

This paper deals with a variational model for decomposing an image field, where is given a function describing the signal intensity associate to each point (typically the light intensity on a screen image).

The segmentation we look for provides a cartoon of the given image satisfying some requirements: the decomposition of the image is performed by choosing a pattern of lines of steepest discontinuity for light intensity, and this pattern will be called segmentation of the image. The analysis is based on the notion of free discontinuity problem introduced by Ennio De Giorgi ([25]) and balances carefully signal smoothing and segmentation length by coupling bulk and surface terms: in such context discontinuous solutions are admissible and their discontinuity is the main features of the solution ([31],[32]).

In the book [7] a variational principle for image segmentation was introduced in the context of visual reconstruction: the Blake & Zisserman functional which depends on second derivatives, free discontinuities and free gradient discontinuities of the intensity levels. In this approach contour detection in segmentation is faced as an energy minimization problem: input is a noisy image and two outputs are produced, say a boundary process map which indicates the location of boundaries (jump and creases of luminance), and a surface attribute map which indicates the smoothed (interpolated) luminance values on the surface of objects in the field.

We introduced a weak formulation of this functional and proved the existence of weak minimizers and the corresponding optimal segmentation in [9], [10], [16]. Then we showed regularity properties, energy and density estimates for optimal segmentation ([10], [11], [12], [13]). Approximation properties of the functional were studied in [4]. In [18] we derived many necessary conditions about weak extremals by performing various kind of first variations: these computations were performed by taking into account the differential geometry of free discontinuity set in any dimension $n \geq 2$ and arbitrary geometry of singular set; in particular we developed the full analysis of crack-tip and crease-tip (boundaries of free discontinuity set). Some of these results were announced in [14].

We recall the strong formulation F of Blake & Zisserman functional (see [11]) for 2-dimensional images, say:

$$\begin{aligned} F(K_0, K_1, u) \; := \; & \int_{\Omega \setminus (K_0 \cup K_1)} \left(|D^2 u|^2 + \mu |u-g|^q \right) d\mathbf{y} \\ & + \alpha\, \mathcal{H}^1(K_0 \cap \Omega) \, + \, \beta\, \mathcal{H}^1((K_1 \setminus K_0) \cap \Omega) \, . \end{aligned} \tag{2.1}$$

to be minimized over triplets (K_0, K_1, u) : if (K_0, K_1, u) is a minimizing triplet of F then $K_0 \cup K_1$ can be interpreted as an optimal segmentation of the monochromatic image of brightness intensity g.
We label the main part E of functional F as follows

$$E(K_0, K_1, u) := \int_{\Omega \setminus (K_0 \cup K_1)} |D^2 u|^2 \, d\mathbf{y} \\ + \alpha \, \mathcal{H}^1(K_0 \cap \Omega) + \beta \, \mathcal{H}^1((K_1 \setminus K_0) \cap \Omega) \, . \tag{2.2}$$

In (2.1), (2.2), $\mathcal{H}^1$ denotes 1-dimensional Hausdorff measure and the a priori unknown triplet $(K_0,\, K_1, u)$ must fulfil: $K_0,\, K_1 \subset \mathbb{R}^2$ are Borel sets with $K_0 \cup K_1$ closed, u belongs to $C^2(\Omega \setminus (K_0 \cup K_1))$, u is approximately continuous on $\Omega \setminus K_0$, while the given data and structural assumptions about F are

$$\begin{array}{c} \Omega \subset \mathbb{R}^2 \text{ open set, } g \in L^q(\Omega) \cap L^{2q}_{loc}(\Omega), \\ q > 1, \ \mu > 0, \ 0 < \beta \le \alpha \le 2\beta < +\infty. \end{array} \tag{2.3}$$

Due to dependence on second derivatives the Blake & Zisserman functional detects both jump and crease sets and avoids the inconvenient of "staircasing" effect due to over-segmentation of steep gradients: that is the appearing of one or more spurious discontinuities in the output image u determined by Mumford & Shah model ([8],[26],[29],[31],[32]), when the datum g is a continuous ramp with steep gradient.
Existence of global minimizers of (2.1) under assumptions (2.3) was proven by regularization of solution of the weak formulation and a precise notion of essential minimizing triplet ([9],[10],[11]). When $g \notin L^{2q}_{loc}(\Omega)$ the infimum of F cannot be achieved with general g in $L^q(\Omega)$ (see [12], section 5).
In [19] we prove the existence of globally minimizing triplets of functionals (2.1),(2.2) under Dirichlet boundary condition and we show the following properties for any locally minimizing triplet: it has well defined two-sided traces and is related to a weak minimizer; it fulfils Euler equation $2{\Delta_{\mathbf{x}}}^2 u + \mu q |u-g|^{q-2}(u-g) = 0$ outside jump discontinuity set and crease discontinuity set, say in $\Omega \setminus (K_0 \cup K_1)$; it fulfils plate-like Neumann conditions on jump and crease discontinuity sets; it exhibits a link between curvature of free discontinuity and squared hessian jump and a variational balance at crack-tip and crease-tip. An integral Euler equation summarizes all the above properties in a tight formulation, by taking into account any admissible compactly supported triplet variation (Theorem 5.1 in [18]).

Blake & Zisserman functional (2.1) depends both on bulk energy and a lineic discontinuity energy; their coupling introduces both technical and substantial difficulties. Moreover the discontinuities of u and of Du are located respectively on the sets $K_0,\ K_1 \setminus K_0$ which are a priori unknown, hence the associated minimization problem turns out to be essentially non-convex: non uniqueness of minimizers may develop for some choice of data (see [5] for explicit examples of multiple minimizers of F). Nevertheless generic uniqueness of minimizers (with respect to data α, β, g) is proven in the 1 dimensional case in [6]. Since for some data the essential minimizing triplet may be not unique, it is far from obvious that the third term of a minimizing triplet is also a minimizer of the weak functional: anyway this is true as proven in [19] thanks to the analysis of associated Dirichlet problem.

Another difficulty in the mathematical analysis of the Blake & Zisserman functional is the fact that (2.1) does not control the intermediate (first) derivatives, moreover truncating competing functions does not reduce the energy, while in case of Mumford & Shah functional any (not affecting datum) truncation reduces energy.

Extremals of Blake & Zisserman functional and its main part must fulfill several Euler-type conditions of differential, integral and geometric type ([5],[18],[20]).

We proved an Almansi decomposition property in 2 dimensional disk B with a straight cut Γ up to the center: this provided very useful heuristic to obtain asymptotic expansion in $B \setminus \Gamma$ of functions which are bi-harmonic outside the cut and have a jump on Γ ([1],[20]). Then ([20]) we computed all eigenfunctions of eigenvalue 0 for operator $\Delta_{\mathbf{x}}{}^2$ in a disk with a cut together with natural Neumann boundary conditions on Γ related to necessary conditions for extremality; as a consequence we deduced an essential asymptotic expansion convergent in $H^2(B_R \setminus \Gamma)$ for any bi-harmonic function in $B_R \setminus \Gamma$ which is $H^2(B_R \setminus \Gamma)$ orthogonal to smooth functions.
In [20] we deduced a complete description of all functions v which are defined almost everywhere in a disk $B \subset \mathbb{R}^2$, are bi-harmonic in $B \setminus \Gamma$ (where Γ is a closed radius of the disk) and fulfill all necessary conditions for locally minimizing triplets of E in B. These properties are so many that, at a first glance, this set must be very small (if not empty!); nevertheless at the end of the analysis we are able to exhibit functions fulfilling all of them. A key result is the computation (performed in [20]) of the leading terms and the asymptotic expansion of any local minimizer of E with jump discontinuity on a half-line: the evaluation is done by imposing all Euler conditions. These leading terms are holomorphic branches of multivalued function, have exact homogeneity $3/2$ in r and are energy-invariant with respect to natural dilations. If equipartition of energy around the origin (among the volume integral and the segmentation length) is imposed in addition to the whole list of Euler conditions for minimality, then the coefficients of the main part of a local minimizer are fixed and we can evaluate them explicitly. Eventually (see [20]) we exhibit a nontrivial function, with jump discontinuity along the negative real axis:

$$\pm \sqrt{\frac{\alpha}{193\,\pi}}\, r^{3/2} \left(\sqrt{21}\, \omega(\,\vartheta) \pm w(\vartheta) \right), \qquad -\pi < \vartheta < \pi\,, \tag{2.4}$$

more explicitly, by expanding the *modes* ω and w,

$$\pm \sqrt{\frac{\alpha}{193\,\pi}}\; r^{3/2} \left(\sqrt{21} \left(\sin\frac{\theta}{2} - \frac{5}{3} \sin\left(\frac{3}{2}\theta\right) \right) \pm \left(\cos\frac{\theta}{2} - \frac{7}{3} \cos\left(\frac{3}{2}\theta\right) \right) \right),$$

so that (2.4) satisfies all extremality conditions proven for functional E in $\mathbb{R}^2$: hence such function is a natural candidate to be a local minimizer. Such function has jump set on the negative real axis and empty jump discontinuity set of the gradient. All these facts led us to formulate the following statement.

Conjecture 2.1. - *Assume $0 < \beta \le \alpha \le 2\beta < +\infty$. Then triplet*

$$(K_0 = \text{ closed negative real axis}\, ,\ K_1 = \emptyset\, ,\ \text{ function (2.4) })$$

is a locally minimizing triplet for E in $\mathbb{R}^2$, and there are no other nontrivial locally minimizing triplets, up to (possibly independent in each mode ω and w) sign change, rigid motions of $\mathbb{R}^2$ co-ordinates and/or addition of affine functions.

3. Caccioppoli inequality and some consequences.

In this section we prove a Caccioppoli type inequality for local minimizers of main part of Blake & Zisserman functional in $\mathbb{R}^n$ under homogeneous Dirichlet boundary conditions. Usually Caccioppoli inequality is a preliminary step to a further study of regularity through blow-up techniques, here we apply the inequality to show that local minimizers in $\mathbb{R}^n$, with finite energy and bounded singular set, are trivial.
At the end of this section we prove that neither a 1-dimensional step nor an infinite wedge are local minimizers.

The general strong formulation of Blake & Zisserman functional F for monochromatic images and its main part E in any dimension $n \ge 1$ requires the minimization of, respectively, ([11]):

$$\text{(3.1)}\quad F(K_0, K_1, u) \ := \int_{\Omega\setminus(K_0\cup K_1)} \big(|D^2u|^p + \mu|u-g|^q\big)d\mathbf{x} + \alpha\mathcal{H}^{n-1}(K_0\cap\Omega) + \beta\mathcal{H}^{n-1}((K_1\setminus K_0)\cap\Omega)\ ,$$

$$\text{(3.2)}\quad E(K_0, K_1, u) \ := \int_{\Omega\setminus(K_0\cup K_1)} |D^2u|^p\, d\mathbf{x} + \alpha\mathcal{H}^{n-1}(K_0\cap\Omega) + \beta\mathcal{H}^{n-1}((K_1\setminus K_0)\cap\Omega),$$

where $\Omega \subset \mathbb{R}^n$ is an open set, $n \ge 1$, $p \ge 2$ and $p' = p/(p-1)$, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\alpha, \beta, \mu, q \in \mathbb{R}$, with given

$$\text{(3.3)}\qquad q > 1\ ,\ \mu > 0\ ,\ 0 < \beta \le \alpha \le 2\beta\ ,\ g \in L^q(\Omega)\cap L^{nq}_{loc}(\Omega)\ ,$$

while the minimization is done among admissible triplets, say (K_0, K_1, u) such that K_0, $K_1 \subset \mathbb{R}^n$ are Borel sets fulfilling $K_0 \cup K_1$ closed, $u \in C^2(\Omega\setminus(K_0\cup K_1))$ and u is approximately continuous (see [2],[28]) on $\Omega\setminus K_0$.
We emphasize that (3.1) does not achieve a minimum if $g \notin L^{nq}_{loc}(\Omega)$, due to counterexample of Section 5 in [12].
In the following F_A and E_A denotes localized version of F, E : replace Ω by the open set A in (3.1),(3.2).

Definition 3.1. (Locally minimizing triplet of strong functionals F and E) An admissible triplet (K_0, K_1, u) is a locally minimizing triplet of functional F defined by (3.1) if

$$F_A(K_0, K_1, u) < +\infty \tag{3.4}$$

$$F_A(K_0, K_1, u) \leq F_A(T_0, T_1, v) \tag{3.5}$$

for every open subset $A \subset\subset \Omega$ and for every admissible triplet (T_0, T_1, v) such that

$$\mathrm{spt}(v-u) \quad \text{and} \quad (T_0 \cup T_1) \triangle (K_0 \cup K_1) \quad \text{are subsets of } A.$$

An admissible triplet (K_0, K_1, u) is a locally minimizing triplet of the functional E defined by (3.2) if (3.4),(3.5) hold true with E_A in place of F_A.

Definition 3.2. (Weak formulation of Blake & Zisserman functional [10]) Under the assumptions (3.3), with $\Omega \subset \mathbb{R}^n$ open set, we define the weak functional $\mathcal{F} : X(\Omega) \to [0, +\infty]$ by

$$\mathcal{F}(v) := \int_\Omega (|\nabla^2 v|^p + \mu |v-g|^q)\, d\mathbf{x} + \alpha \mathcal{H}^{n-1}(S_v) + \beta \mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v) \tag{3.6}$$

where $X(\Omega) := GSBV^2(\Omega) \cap L^q(\Omega)$.
About functions (in $GSBV^2$) whose second derivatives are special measures in the sense by De Giorgi we refer to [10],[21]: here and in the following, ∇v denotes the absolutely continuous part of distributional gradient Dv and S_v denotes the singular set of v; notice that $\nabla^2 v$ may be nonsymmetric.

We consider also the localization $\mathcal{F}_A$ of $\mathcal{F}$ on any Borel set $A \subseteq \Omega$:

$$\mathcal{F}_A(v) := \int_A (|\nabla^2 v|^p + \mu |v-g|^q)\, d\mathbf{x} + \alpha \mathcal{H}^{n-1}(S_v \cap A) + \beta \mathcal{H}^{n-1}((S_{\nabla v} \setminus S_v) \cap A)\,.$$

We remark that the subset of $GSBV^2(\Omega)$ where $\mathcal{F}$ is finite is a vector space, while $GSBV^2(\Omega)$ is not a vector space.

Definition 3.3. (Local minimizer for weak formulation) We say that u is a local minimizer of the functional $\mathcal{F}$ in Ω if

$$u \in GSBV^2(A), \qquad \mathcal{F}_A(u) < +\infty, \qquad \mathcal{F}_A(u) \leq \mathcal{F}_A(u + \varphi) \tag{3.7}$$

for every open subset $A \subset\subset \Omega$ and for every $\varphi \in GSBV^2(\Omega)$ with compact support in A.

We introduce also the weak form of functional E defined in (3.2)

$$\mathcal{E}(v) := \int_\Omega |\nabla^2 v|^p \, d\mathbf{x} + \alpha \mathcal{H}^{n-1}(S_v) + \beta \mathcal{H}^{n-1}(S_{\nabla v} \setminus S_v)\,. \tag{3.8}$$

We say that u is a local minimizer of the functional $\mathcal{E}$ in Ω if, by denoting $\mathcal{E}_A$ the localization of $\mathcal{E}$,

$$u \in GSBV^2(A), \qquad \mathcal{E}_A(u) < +\infty, \qquad \mathcal{E}_A(u) \leq \mathcal{E}_A(u + \varphi) \tag{3.9}$$

for every open subset $A \subset\subset \Omega$ and for every $\varphi \in GSBV^2(\Omega)$ with compact support in A.

Theorem 3.4. (Caccioppoli inequality for local weak minimizer with homogeneous Dirichlet conditions) *Assume* $0<\beta\leq\alpha\leq 2\beta$, $p\geq 2$, $n\geq 1$, $R>0$ *and* u *is a local minimizer of* $\mathcal{E}$ *in* $B_R\subset\mathbb{R}^n$ *among* $v\in GSBV^2(B_R)$ *fulfilling* $v\equiv 0$ *in* $B_R^-=B_R\cap\{x_n<0\}$.
Then for every $0<\varrho<R$, *we have*

$$\int_{B_\varrho^+}|\nabla^2 u|^p\,d\mathbf{x}\leq\frac{C}{(R-\varrho)^p}\int_{B_R^+\setminus B_\varrho^+}|\nabla u|^p d\mathbf{x}+\frac{C}{(R-\varrho)^{2p}}\int_{B_R^+\setminus B_\varrho^+}|u|^p d\mathbf{x} \tag{3.10}$$

where $C=C(n,p)$ *is a constant independent of* u, R *and* ϱ.

Proof. If the right-hand side of (3.10) is infinite then the thesis is trivial. Hence we are left to prove it only when right-hand side is finite, so we may assume $u\,,\nabla u\in L^p(B_R^+\setminus B_\varrho^+)$. Let $\varphi\in C_0^\infty(B_R)$ such that

$$0\leq\varphi\leq 1\,,\quad \varphi\equiv 1\text{ in }B_\varrho\,,\quad |D\varphi|\leq\frac{c_1}{R-\varrho}\,,\quad |D^2\varphi|\leq\frac{c_1}{(R-\varrho)^2}\,.$$

For $|\varepsilon|\leq 1$ we set $u_\varepsilon=u+\varepsilon\varphi^{2p}u$. Then $S_{u_\varepsilon}=S_u$, $S_{\nabla u_\varepsilon}=S_{\nabla u}$ and

$$\nabla u_\varepsilon=(1+\varepsilon\varphi^{2p})\nabla u+2p\varepsilon u\varphi^{2p-1}D\varphi,$$

$$\nabla^2 u_\varepsilon=(1+\varepsilon\varphi^{2p})\nabla^2 u$$

$$+2p\varepsilon(\varphi^{2p-1}D\varphi\nabla u+\nabla u\varphi^{2p-1}D\varphi+(2p-1)u\varphi^{2p-2}D\varphi D\varphi+u\varphi^{2p-1}D^2\varphi).$$

Now we set $\nabla^2 u_\varepsilon=A+\varepsilon B$, where $A=\nabla^2 u$ and

$$B=\varphi^{2p}A+2p(\varphi^{2p-1}D\varphi\nabla u+\nabla u\varphi^{2p-1}D\varphi+(2p-1)u\varphi^{2p-2}D\varphi D\varphi+u\varphi^{2p-1}D^2\varphi)\,.$$

For $a,b,c\in\mathbb{R}$, $a>0$, $\varepsilon\in[-1,1]$, we exploit Taylor expansion with Lagrange remainder of $\psi(\varepsilon)=(\,a+b\,\varepsilon+c\,\varepsilon^2)^{p/2}$, notice that $p\geq 2$ entails $\psi\in C^1$ in a neighborhood of $\varepsilon=0$:

$$\psi(\varepsilon)\;=\;\psi(0)\,+\,\psi'(\widetilde{\varepsilon})\,\varepsilon=\psi(0)\,+\,\psi'(0)\,\varepsilon\,+\,\sigma(\widetilde{\varepsilon})\varepsilon,\quad 0<|\widetilde{\varepsilon}|<|\varepsilon|\,,$$

$$\sigma(\widetilde{\varepsilon})\;=\;\psi'(\widetilde{\varepsilon})-\psi'(0)\,,\qquad \lim_{\varepsilon\to 0}\sigma(\widetilde{\varepsilon})=0\qquad\text{and}$$

$$\psi(0)=a^{p/2}\,,\quad \psi'(\widetilde{\varepsilon})=\frac{p}{2}\left(\,a+\,b\,\widetilde{\varepsilon}\,+\,c\,\widetilde{\varepsilon}^{\,2}\,\right)^{\frac{p}{2}-1}(\,b+2\,c\,\widetilde{\varepsilon}\,)\,,\quad \psi'(0)\;=\;\frac{p}{2}\,a^{\frac{p}{2}-1}\,b\,.$$

We choose $a=(A:A)$, $b=(A:B+B:A)$, $c=(B:B)$, so that $a+b\varepsilon+c\varepsilon^2\geq 0$ for all $\varepsilon\in\mathbb{R}$, while $\widetilde{\varepsilon}$ depends on $a=a(\mathbf{x}),b=b(\mathbf{x}),c=c(\mathbf{x})$ and ε, say $\widetilde{\varepsilon}=\widetilde{\varepsilon}(\mathbf{x},\varepsilon)$ and, by summarizing $\psi\in C^1([-1,1])$ and:

$$\psi(\varepsilon)=(a+b\varepsilon+c\varepsilon^2)^{p/2}=(A:A)^{p/2}+\frac{p}{2}(A:A)^{p/2-1}(A:B+B:A)\;\varepsilon+\sigma\big(\widetilde{\varepsilon}(\mathbf{x},\varepsilon)\big)\,\varepsilon,$$

where

$$\sigma\big(\widetilde{\varepsilon}(\mathbf{x},\varepsilon)\big)\;=\;\psi'(\widetilde{\varepsilon}(\mathbf{x},\varepsilon))-\psi'(0)\;=$$

$$=\frac{p}{2}\big(A:A+(A:B+B:A)\widetilde{\varepsilon}+(B:B)\,\widetilde{\varepsilon}^{\,2}\big)^{p/2-1}((A:B+B:A)+\,2\,\widetilde{\varepsilon}\,B:B)$$

$$-\frac{p}{2}|A|^{p-2}(A:B+B:A)$$

and $\sigma\big(\widetilde{\varepsilon}(\mathbf{x},\varepsilon)\big)$ tends to 0 as $\varepsilon \to 0$ uniformly in $\mathbf{x}$, since $|\widetilde{\varepsilon}| < |\varepsilon| \le 1,\ \psi' \in C^0$ and $A\!:\!A,\ A\!:\!B,\ B\!:\!A,\ B\!:\!B,$ belong to $L^{p/2}(B_R)$.
For a.e. $\mathbf{x} \in B_R$ and suitable $\widetilde{\varepsilon} = \widetilde{\varepsilon}(\mathbf{x},\varepsilon),\ 0 < |\widetilde{\varepsilon}| < |\varepsilon|$ we get

$$\begin{aligned}
&|A+\varepsilon B|^p = \big((A+\varepsilon B) : (A+\varepsilon B)\big)^{p/2}\\
&= \big(A : A \,+\, (A : B + B : A)\varepsilon \,+\, B : B\,\varepsilon^2\,\big)^{p/2}\\
&= \big(\,a \,+\, b\,\varepsilon \,+\, c\,\varepsilon^2\,\big)^{p/2}\\
&=\, a(\mathbf{x}) \,+\, \frac{p}{2}\,a(\mathbf{x})^{\frac{p}{2}-1}\,b(\mathbf{x})\,\varepsilon \,+\, \sigma\big(\widetilde{\varepsilon}(\mathbf{x},\varepsilon)\big)\,\varepsilon\\
&= (A : A)^{p/2} + \frac{p}{2}(A : A)^{p/2-1}\,(A\!:\!B + B\!:\!A)\ \varepsilon \,+\, \sigma\big(\widetilde{\varepsilon}(\mathbf{x},\varepsilon)\big)\,\varepsilon\\
&= (A : A)^{p/2} + \varepsilon\,\frac{p}{2}\,\Big((A : A)^{p/2-1}\Big)\big(2\varphi^{2p}(A : A) + A\!:\!(B-\varphi^{2p}A) + (B-\varphi^{2p}A)\!:\!A\big)\\
&\quad + o(\varepsilon)\\
&= |A|^p \,+\, \varepsilon\,\frac{p}{2}\big[2\varphi^{2p}|A|^p + |A|^{p-2}\,\big(A\!:\!(B-\varphi^{2p}A) + (B-\varphi^{2p}A)\!:\!A\big)\big] + o(\varepsilon)\,.
\end{aligned}$$

We emphasize that $a(\mathbf{x})\,,\ \frac{p}{2}\,a(\mathbf{x})^{\frac{p}{2}-1}\,b(\mathbf{x})\,\varepsilon$ and $\sigma\big(\widetilde{\varepsilon}(\mathbf{x},\varepsilon)\big)\,\varepsilon$ belong to $L^1(U)$ uniformly in $\varepsilon \in [-1,1]$, for any measurable set $U \subset B_R$, since $p \ge 2$,

$$\sigma(\varepsilon)\,\varepsilon \,=\, |A+\varepsilon B|^p - |A|^p - \varepsilon(p/2)|A|^{p-2}(A : B + B : A)$$

and $A, B \in L^p(B_R),\ A+\varepsilon B \in L^p(B_R),\ \big(|A|^{p-2}(A\!:\!B + B\!:\!A)\big) \in L^1(B_R))$.
By minimality of u we deduce the vanishing of ε coefficient, hence:

$$\int_{B_R} 2\varphi^{2p}|A|^p\,d\mathbf{x} \,=\, -\int_{B_R} |A|^{p-2}\big(A\!:\!(B-\varphi^{2p}A) + (B-\varphi^{2p}A)\!:\!A\big)\,d\mathbf{x}. \tag{3.11}$$

By using the equation (3.11), $\mathrm{spt}(D\varphi), \mathrm{spt}(D^2\varphi) \subset \overline{B_R \setminus B_\varrho}$ and Hölder inequality (with $p' = p/(p-1)$ and p) there is $c_2 = c_2(n,p)$ s.t.

$$\begin{aligned}
&\int_{B_R} \varphi^{2p}|\nabla^2 u|^p\,d\mathbf{x}\\
&\le c_2\int_{B_R} |\nabla^2 u|^{p-1}\,\big(\varphi^{2p-1}|\nabla u||D\varphi| + \varphi^{2p-2}|u||D\varphi|^2 + \ \varphi^{2p-1}|uD^2\varphi|\big)\ d\mathbf{x}\\
&\le c_2\int_{B_R} \big(\varphi^{2p-2}|\nabla^2 u|^{p-1}\big)\,\big(|\varphi\nabla u||D\varphi| + |u||D\varphi|^2 + \ |\varphi u D^2\varphi|\big)\ d\mathbf{x}\\
&\le c_3\left(\int_{B_R} \varphi^{2p}\,|\nabla^2 u|^p\,d\mathbf{x}\right)^{\frac{p-1}{p}}\\
&\qquad\qquad \times\left\{\frac{1}{R-\varrho}\left(\int_{B_R\setminus B_\varrho} |\nabla u|^p\,d\mathbf{x}\right)^{\frac{1}{p}} + \frac{1}{(R-\varrho)^2}\left(\int_{B_R\setminus B_\varrho} |u|^p\,d\mathbf{x}\right)^{\frac{1}{p}}\right\},
\end{aligned}$$

hence by inclusion of balls and Young inequality (with p' and p) we get

$$\int_{B_\varrho} |\nabla^2 u|^p \, d\mathbf{x} \le \int_{B_R} \varphi^{2p} |\nabla^2 u|^p \, d\mathbf{x} \le$$
$$\le \frac{C}{(R-\varrho)^p} \int_{B_R \setminus B_\varrho} |\nabla u|^p \, d\mathbf{x} + \frac{C}{(R-\varrho)^{2p}} \int_{B_R \setminus B_\varrho} |u|^p \, d\mathbf{x},$$

and the inequality is proven.

□

Remark 3.5. *In the particular case $p = 2$, $n \ge 2$ the inequality (3.10) was proven in [15], Theorem 3.2.*

Theorem 3.6. (Caccioppoli inequality for locally minimizing triplets of strong functional with homogeneous Dirichlet conditions) *Assume $0 < \beta \le \alpha \le 2\beta$, $n \ge 1$, $p \ge 2$, $R > 0$ and (K_0, K_1, u) is a locally minimizing triplet of E in $B_R \subset \mathbb{R}^n$ among (T_0, T_1, v) fulfilling $v \equiv 0$ in $B_R^- = B_R \cap \{x_n < 0\}$.*
Then for every $0 < \varrho < R$, we have

$$\int_{B_\varrho^+ \setminus (K_0 \cup K_1)} |D^2 u|^p \, d\mathbf{x} \le \tag{3.12}$$
$$\frac{C}{(R-\varrho)^p} \int_{B_R^+ \setminus (K_0 \cup K_1 \cup B_\varrho^+)} |Du|^p d\mathbf{x} + \frac{C}{(R-\varrho)^{2p}} \int_{B_R^+ \setminus (K_0 \cup K_1 \cup B_\varrho^+)} |u|^p d\mathbf{x}$$

where $C = C(n,p)$ is a constant independent of u, R and ϱ.

Proof. The inequality (3.12) is a straightforward consequence of Caccioppoli inequality (3.10) which is fulfilled by local weak minimizers, since the third term of any locally minimizing triplet of E is a local minimizer of weak functional $\mathcal{E}$ with the same boundary condition (by Theorem 1.1 in [19]) □

Remark 3.7. *Theorems 3.4 and 3.6 hold true also when B_R^+ is substituted by $\mathfrak{B}^-$, where $\mathfrak{B}^-$, is the part contained in B_R of the hypo-graph of a C^2 function $\psi = \psi(x_1, \ldots, x_{n-1})$ with $\psi(\mathbf{0}) = 0$, $D\psi(\mathbf{0}) = \mathbf{0}$.*

Theorem 3.8. *Assume (K_0, K_1, u) is a locally minimizing triplet of E among the ones such that $u \equiv 0$ in $\mathbb{R}^n_- = \mathbb{R}^n \cap \{x_n < 0\}$, the set $K_0 \cup K_1$ is bounded and*

$$\int_{\mathbb{R}^n \setminus (K_0 \cup K_1)} |D^2 u|^p \, d\mathbf{x} < +\infty \, . \tag{3.13}$$

Then $u \equiv 0$ in $\mathbb{R}^n$.

Proof. Assume $(K_0 \cup K_1) \subset B_{\widetilde{\varrho}}$ and $\varrho > \widetilde{\varrho}$, so that $D^2 u \in L^p(B_{2\varrho} \setminus B_\varrho)$ and hence $u \in W^{2,2}(B_{2\varrho} \setminus B_\varrho)$, moreover $\operatorname{spt} u \subset \overline{\mathbb{R}^n_+}$.

By using Theorem 3.4, and applying Poincaré inequality to u, we get

$$\begin{aligned}\int_{B_\varrho\setminus(K_0\cup K_1)} & |D^2u|^p\, d\mathbf{x} \le \\ & \le \frac{C}{\varrho^p}\int_{B_{2\varrho}\setminus(B_\varrho\cup K_0\cup K_1)} |Du|^p\, d\mathbf{x} + \frac{C}{\varrho^{2p}}\int_{B_{2\varrho}\setminus(B_\varrho\cup K_0\cup K_1)} |u|^p\, d\mathbf{x} \\ & \le C'\int_{B_{2\varrho}\setminus B_\varrho} |D^2u|^p\, d\mathbf{x}.\end{aligned}$$

By hole-filling we obtain

$$(1+C')\int_{B_\varrho\setminus(K_0\cup K_1)} |D^2u|^p\, d\mathbf{x} \le C'\int_{B_{2\varrho}\setminus(K_0\cup K_1)} |D^2u|^p\, d\mathbf{x},$$

so that

$$\int_{B_\varrho} |\nabla^2u|^p\, d\mathbf{x} \ \le\ \tau\int_{B_{2\varrho}} |\nabla^2u|^p\, d\mathbf{x} \qquad \text{with} \ \ \tau = \frac{C'}{1+C'} < 1,$$

and, for every $k \in \mathbf{N}$,

$$\int_{B_\varrho} |\nabla^2u|^p\, d\mathbf{x} \le \tau^k\int_{B_{2^k\varrho}} |\nabla^2u|^p\, d\mathbf{x}.$$

By the assumption (3.13) and the arbitrariness of k we conclude that

$$\int_{B_\varrho} |\nabla^2u|^p\, dx = 0.$$

By the arbitrariness of ϱ, $D^2u \equiv \mathbb{O}$ in $\mathbb{R}^n \setminus B_{\widetilde{\varrho}}$, hence u is affine in $\mathbb{R}^n \setminus B_{\widetilde{\varrho}}$.
Then we get $u \equiv 0$ in $\mathbb{R}^n \setminus B_{\widetilde{\varrho}}$ by $u \equiv 0$ in $\mathbb{R}^n_-$.
By local minimality with respect to compactly supported variations, $u \equiv 0$ in $\mathbb{R}^n$. □

Theorem 3.9. *For any $\mathbf{x} \in \mathbb{R}^n$ we set $\mathbf{x} = (\mathbf{x}', x_n)$ with $\mathbf{x}' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. Then, for any $c \neq 0$, the triplet $(\{x_n = 0\}, \emptyset, c\,H(x_n))$ is not a locally minimizing triplets (K_0, K_1, v) of E in $\mathbb{R}^n$ among the ones whose third term v has support contained in $\{\mathbf{x} \in \mathbb{R}^n : x_n \ge 0\}$.*
Here H denotes the Heaviside function.

Proof. Set $u(\mathbf{x}) = cH(x_n)$. Assume by contradiction $(\{x_n = 0\}, \emptyset, u(\mathbf{x}))$ is a locally minimizing triplet of E in $\mathbb{R}^n$ among the ones whose third term has support contained in $\{\mathbf{x} : x_n \ge 0\}$, hence u is a local minimizer of $\mathcal{E}$ under the same support condition.
If $n = 1$ we define, for any $\delta > 0$,

$$u_1(x) = \begin{cases} 2\,c\,x^2/\delta^2 & \text{if } 0 \le x < \delta/2 \\[2ex] c\,\left(1 - 2(x-\delta)^2/\delta^2\right) & \text{if } \delta/2 \le x < \delta, \\[2ex] c\,H(x) & \text{if either } x < 0 \text{ or } x > \delta. \end{cases}$$

Then $|{u_1}''(x)| = 4|c|/\delta^2$ if $0 \le x \le \delta$, $S_{u_1} = S_{\dot u_1} = \emptyset$, hence for any $R > \delta$ we have

$$\mathcal{E}_{B_R}(u_1) \; - \; \mathcal{E}_{B_R}(u) \; = \; 4^p \frac{c^p}{\delta^{2p-1}} - \alpha \; < \; 0 \qquad \text{if} \qquad \delta^{2p-1} > 4^p \frac{c^p}{\alpha} \, ,$$

which contradicts the minimality of u.
If $n > 1$ we define, for any $\delta > 0$

$$\bar u(\mathbf{x}) \; = \; \begin{cases} u_1(x_n) & \text{if } 0 \le x_n \le \delta \text{ and } |\mathbf{x}'| < L \\ \\ u(\mathbf{x}) & \text{elsewhere} \, . \end{cases}$$

Then $|\nabla^2 \bar u(\mathbf{x})| \; = \; |{u_1}''(x_n)| \; = \; 4|c|/\delta^2$ if $0 \le x_n \le \delta$ and $|\mathbf{x}'| < L$, while $S_{\bar u} = \{\mathbf{x} : \; |\mathbf{x}'| = L, \; 0 \le x_n \le \delta\}$ and $S_{\nabla \bar u} \setminus S_{\bar u} = \emptyset$.
Hence, for any R with $R^2 > L^2 + \delta^2$, we have

$$\mathcal{E}_{B_R}(\bar u) - \mathcal{E}_{B_R}(u) \; = \; \frac{4^p c^p}{\delta^{2p-1}} \omega_{n-1} L^{n-1} + \alpha(n-1)\omega_{n-1} L^{n-2}\delta - \alpha \omega_{n-1} L^{n-1} \; < \; 0,$$

where ω_{n-1} denotes the volume of the unit ball in $\mathbb{R}^{n-1}$, so that $(n-1)\omega_{n-1}$ is the surface area of its boundary, provided we choose

$$\delta^{2p-1} > 2 \cdot 4^p \frac{c^p}{\alpha} \qquad \text{and} \qquad L > 2(n-1)\delta.$$

This contradicts the minimality of u. □

Theorem 3.10. *For any* $\mathbf{x} \in \mathbb{R}^n$ *we set* $\mathbf{x} = (\mathbf{x}', x_n)$ *with* $\mathbf{x}' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}$. *Then, for any* $c \neq 0$, *the triplet* $(\emptyset, \{x_n = 0\}, c\,(x_n)^+)$ *is not a locally minimizing triplet of* E *in* $\mathbb{R}^n$ *among the ones whose third term* v *has support contained in* $\{\mathbf{x} \in \mathbb{R}^n : \, x_n \ge 0\}$.
Here $(x_n)^+ = \max(x_n, 0)$.

Proof. Set $v(\mathbf{x}) = c\,(x_n)^+$. Assume by contradiction that $(\emptyset, \{x_n = 0\}, v(\mathbf{x}))$ is a locally minimizing triplet of E in $\mathbb{R}^n$ among the ones whose third term has support contained in $\{\mathbf{x} : \, x_n \ge 0\}$, hence v is a local minimizer of $\mathcal{E}$ under the same support condition.

If $n = 1$ we define, for any $\delta > 0$,

$$v_1(x) \; = \begin{cases} \dfrac{3\,c\,x^2}{\delta} - \dfrac{3\,c\,x^3}{\delta^2} + \dfrac{c\,x^4}{\delta^3} & \text{if } 0 \le x \le \delta \\ \\ c\,x^+ & \qquad\qquad \text{elsewhere.} \end{cases}$$

Then $|v_1''(x)| = \left| \dfrac{6\,c}{\delta} - \dfrac{18\,c\,x}{\delta^2} + \dfrac{12\,c\,x^2}{\delta^3} \right| \le \dfrac{6\,c}{\delta}$ if $0 < x < \delta$, $S_{v_1} = S_{\dot v_1} = \emptyset$, hence

$$\forall R > \delta \qquad \mathcal{E}_{B_R^+}(v_1) \; - \; \mathcal{E}_{B_R^+}(v) \; \le \; \frac{6^p c^p}{\delta^{p-1}} - \beta \; < \; 0 \qquad \text{if } \delta^{p-1} \; > \; \frac{6^p c^p}{\beta}$$

which contradicts the minimality of v.

If $n > 1$ we define, for any $\delta > 0$,

$$\bar{v}(\mathbf{x}', x_n) = \begin{cases} v_1(x_n) & \text{if } \ 0 \le x_n \le \delta, \ |\mathbf{x}'| \le L \\ \\ v(\mathbf{x}) & \text{elsewhere.} \end{cases}$$

Then $|\nabla^2 \bar{v}(\mathbf{x})| = |{v_1}''(x_n)| \le 6\, c/\delta$ if $0 < x_n < \delta$ and $|\mathbf{x}'| < L$, while $S_{\bar{v}} = \{\mathbf{x} : \ |\mathbf{x}'| = L, \ 0 \le x_n \le \delta\}$ and $S_{\nabla \bar{v}} \setminus S_{\bar{v}} = \emptyset$.
By choosing $\delta^{p-1} > 2 \cdot 6^p c^p / \beta$, $L > 2(n-1)\, \alpha\delta\, / \beta$ and $R^2 > L^2 + \delta^2$, we get

$$\mathcal{E}_{B_R^+}(\bar{v}) - \mathcal{E}_{B_R^+}(v) \le \frac{6^p c^p}{\delta^{p-1}} \omega_{n-1} L^{n-1} + \alpha\delta(n-1)\omega_{n-1} L^{n-2} - \beta\omega_{n-1} L^{n-1} \ < \ 0$$

and this contradicts the minimality of v. □

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