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Abstract

We consider weak and strong survival for branching random walks on multigraphs with bounded degree. We prove that, at the strong critical value, the process dies out locally almost surely. We relate the weak critical value to a geometrical parameter of the multigraph. For a large class of multigraphs (which enlarges the class of quasi-transitive or regular graphs) we prove that, at the weak critical value, the process dies out globally almost surely. Moreover for the same class we prove that the existence of a pure weak phase is equivalent to nonamenability. The results are extended to branching random walks on weighted graphs.

Keywords: branching random walks, phase transition, multigraphs, amenability, trees.
AMS subject classification: 60K35.

1 Introduction

In recent years, much study has been devoted to various stochastic processes, such as percolation, Ising model, contact process and branching random walk, on general graphs ([3], [9], [10], [13], [14] only to mention a few, see [11] for more references). A double motivation underlies the search for settings other than the usual $\mathbb{Z}^d$: on one hand the need for structures which may serve as models for inhomogeneous crystals, biological structures or social networks and on the other hand the fact that on general graphs interesting phenomena, which are absent in $\mathbb{Z}^d$, are observed. In particular the branching random walk (BRW) has been studied on trees (see [12], [7], [8], [5], [14]) and on quasi-transitive graphs (see [15]).

In this paper we study the BRW on a connected multigraph X with bounded degree (see Section 2.1 for the formal definition). Roughly speaking a λ-BRW can be described by the following rules:

each particle dies after an exponential time with parameter 1 and breeds independently on each edge at exponential intervals with parameter λ. We start with a finite number of particles, hence the λ-BRW can be viewed as a continuous-time random walk on the countable state space of finite configurations $\eta \in \mathbb{N}^X$. On each site $x \in X$ the transitions are:

$$\begin{aligned} \eta(x) &\to \eta(x) - 1 \qquad && \text{at rate } \eta(x), \\ \eta(x) &\to \eta(x) + 1 \qquad && \text{at rate } \lambda \sum_{y \in X} n_{yx} \eta(y), \end{aligned}$$

where $\eta(x)$ is the number of particles at site x and n_{yx} is the number of edges from y to x.

The BRW has originally been introduced as a model for biological populations dynamics (although it has been argued that this model is far from being satisfactory, see for instance the discussion in [6]) and, besides being interesting in itself, has also been studied for its relationship with the contact process (the process which has the same transition rules of the BRW but state space $\{0,1\}^X$). Indeed the BRW stochastically dominates the contact process and has the property that the sum of two λ-BRWs is still a λ-BRW.

The λ-BRW on $\mathbb{Z}^d$ shows only two possible behaviors (called *phases*): if $\lambda \leq 1/2d$ there is extinction almost surely; if $\lambda > 1/2d$, for all $t_0 > 0$ we have that $\mathbb{P}(\eta_t(0) > 0 \text{ for some } t \geq t_0) > 0$ where $\eta_t(0)$ is the number of particles at 0 at time t. The main interest of the study of BRW on trees is that a third phase appears. Indeed we may identify two kinds of survival:

(*i*) *weak* (or global) *survival* – the total number of particles is positive at each time;

(*ii*) *strong* (or local) *survival* – the number of particles at one site x is not eventually 0.

Let us denote by λ_w (resp. λ_s) the infimum of the values λ such that there is weak (resp. strong) survival with positive probability. Clearly $\lambda_w \leq \lambda_s$ and we may have three distinct phases corresponding to the following intervals for λ: $[0, \lambda_w)$, (λ_w, λ_s), $(\lambda_s, +\infty)$. The middle interval may be empty; if, on the contrary, $\lambda_w < \lambda_s$ then we say that the BRW has a *pure weak phase*. In this phase the process leaves any finite subset eventually a.s., hence it survives globally by drifting to infinity (see [5] for details on the convergence to the boundary in the case of homogeneous trees).

This paper is devoted to three main issues: the identification of the critical value λ_w, the behavior of the process at the critical values $\lambda = \lambda_s$ and $\lambda = \lambda_w$ and the existence of the pure weak phase. In [14] it was proved that λ_s is related to a particular asymptotic degree of the graph. We extend this characterization to multigraphs (Theorem 3.1) and we relate λ_w to another asymptotic degree (Theorems 3.4, 3.5 and 3.9). Moreover we prove, by using generating functions techniques, that if $\lambda = \lambda_s$ the process dies out locally a.s. and that, if $\lambda = \lambda_w$, on a large class of multigraphs the process dies out globally a.s. (Theorem 3.11). The use of multigraphs is natural; indeed given a vertex on a graph the birth rate is equal for all neighbors, while on a multigraph one can consider more inhomogeneous situations (for a generalization see Section 4.2). Moreover some of our results

need that in some sense "there are only a finite number of types of vertices" (see Definition 3.6), and this requirement on one hand is naturally written by means of multigraphs and on the other hand identifies our main class of multigraphs which includes both quasi-transitive and regular graphs.

As for conditions for the existence of the pure weak phase, one is lead to investigate nonamenable graphs. Indeed, usually, nonamenable graphs are graphs where certain phenomena, absent in the amenable case, appear (see [11] for a survey). Nevertheless a statement like "nonamenability of the graph is equivalent to the existence of a pure weak phase for the BRW" has been disproved in [14]. The authors showed a nonamenable tree where the BRW has no pure weak phase and an amenable tree where there is such a phase (note that these counterexamples are both of bounded degree). Hence one hopes to prove a similar statement for a more restricted class of graphs. Work is this direction has been done in [15, Theorem 3.1] which states the equivalence between nonamenability and the existence of a pure weak phase for quasi-transitive graphs. We prove the same equivalence for our main class of multigraphs (Theorem 3.13 and Example 3).

Let us give the outline of the paper. In Section 2 we introduce the main definitions and we define some generating functions and a generalized branching process which will be useful in the sequel. Moreover we introduce two *asymptotic degrees* M_s and M_w which depend only on the geometrical structure of the multigraph.

Subsection 3.1 is devoted to the detailed study of λ_s and λ_w. We give a sufficient condition for the absence of the pure weak phase, which, in particular, implies that there is no weak phase on subexponentially growing multigraphs (Remark 3.3). We prove that on multigraphs $\lambda_s = 1/M_s$ (Theorem 3.1) and we show that for a large class of multigraphs $\lambda_w = 1/M_w$ (Theorem 3.4). Clearly for this class we have that $\lambda_w < \lambda_s$ if and only if $M_s < M_w$. We give two different sufficient conditions for a multigraph to satisfy the hypothesis of Theorem 3.4. The first condition (Theorem 3.5) is satisfied, for instance, by certain radial trees which are not quasi transitive; for these trees we show that nonamenability is equivalent to the existence of the pure weak phase (Example 3). As for the second condition (Theorem 3.9), we introduce a class of morphisms (Definition 3.6) of multigraphs and we show that it preserves λ_w, M_w and, in some cases, λ_s (Proposition 3.7). By using these morphisms, the class of $\mathcal{F}$-multigraphs is defined; this class satisfies the hypothesis of Theorem 3.4. In Subsection 3.2 we prove that on general multigraphs the λ_s-BRW dies out locally almost surely and that on $\mathcal{F}$-multigraphs the λ_w-BRW dies out globally almost surely (Theorem 3.11). The main result of Subsection 3.3 (Theorem 3.13) along with Theorem 3.9 yields, for non-oriented $\mathcal{F}$-multigraphs, the equivalence of the following conditions: (i) $\lambda_w < \lambda_s$, (ii) $M_s < M_w$ and (iii) nonamenability. In Subsection 3.4 some examples of multigraphs, which can be studied via our results, are given.

The BRW studied in Section 3 may be viewed as a population which reproduces following an "edge breeding" pattern, while some authors prefer a "site breeding" pattern. In Section 4 we consider this modification of the BRW. These two versions of the BRW are essentially equivalent

on regular graphs, while in the general setting the behavior of the "site breeding" BRW can be much more easily characterized (Theorem 4.1). We show that BRWs and modified BRWs may both be seen as particular cases of BRWs on weighted graphs. Most of the results given in the previous sections still hold in this general setting.

Section 5 is devoted to a final discussion of open questions.

2 Basic definitions and preliminaries

2.1 Multigraphs

A countable (or finite) multigraph is a couple $(X, E(X))$, where X is the countable (or finite) set of vertices and $E(X) \subseteq X \times X \times \mathbb{N}_*$ is the set of (oriented) edges (where $\mathbb{N}_*$ is the set of positive natural numbers); we define the number of edges from x to y as $n_{xy} := |\{i : (x, y, i) \in E(X)\}| \equiv \max\{i : (x, y, i) \in E(X)\}$ (where $|\cdot|$ denotes cardinality). We denote by $D(x) := \{y \in X : n_{xy} > 0\}$ the set of neighbors of x and by $\deg(x) := \sum_{y \in D(x)} n_{xy}$ the degree of x. If $n_{xy} = n_{yx}$ for all $x, y \in X$ then the multigraph is called *non oriented*. A multigraph is a graph if and only if $n_{xy} = \mathbb{1}_{D(x)}(y)$.

A path from x to y of length n is a couple of sequences $(\{x = x_0, x_1, \ldots, x_n = y\}, \{k_1, \ldots, k_n\})$ such that $n_{x_i x_{i+1}} \geq k_{i+1} > 0$ for all $i = 0, 1, \ldots, n-1$. The multigraph is said to be *connected* if there exists a path (of suitable length) from x to y, for all $x, y \in X$. From now on, the multigraph will always be connected and of *bounded degree*, that is, $M(X) := \sup_{x \in X} \deg(x) < +\infty$; obviously M depends on $(X, E(X))$, nevertheless to avoid cumbersome notation the dependence on the set of edges will be tacitly understood. The same implicit assumption will be made for all quantities depending on the multigraph. Moreover if not explicitly stated, the multigraph does not need to be non oriented.

Let $\gamma^n_{x,y}$ be the number of paths of length n from x to y (and $\gamma^0_{x,y} := \delta_{x,y}$). More explicitly to each sequence $\{x = x_0, x_1, \ldots, x_n = y\}$ there corresponds a set of $\prod_{i=0}^{n-1} n_{x_i x_{i+1}}$ paths in the multigraph, whence $\gamma^n_{x,y}$ is the sum over all the sequences $\{x = x_0, x_1, \ldots, x_n = y\}$ of $\prod_{i=0}^{n-1} n_{x_i x_{i+1}}$. Moreover let T^n_x be the number of paths from x of length n, that is, $T^n_x := \sum_{y \in X} \gamma^n_{x,y}$. Finally, let $\phi^n_{x,y}$ the number of paths of length n starting from x and reaching y for the first time; to be precise, $\phi^n_{x,y}$ is the number of paths $(\{x = x_0, x_1, \ldots, x_n = y\}, \{k_1, \ldots, k_n\})$ such that $x_i \neq y$ for all $i = 1, \ldots, n-1$. By definition $\phi^0_{x,y} := 0$ for all $x, y \in X$.

For $\gamma^n_{x,y}$ and T^n_x the following recursive relations hold for all $n, m \geq 0$

$$\begin{cases} \gamma^{n+m}_{x,y} = \sum_{w \in X} \gamma^n_{x,w} \gamma^m_{w,y} \\ \\ \gamma^1_{x,y} = n_{xy} \end{cases} \qquad \begin{cases} T^{n+m}_x = \sum_{w \in X} \gamma^m_{x,w} T^n_w \\ \\ T^1_x = \deg(x) \end{cases}$$

and, for all $n \geq 1$,

$$\gamma^n_{x,y} = \sum_{i=0}^{n} \phi^i_{x,y} \gamma^{n-i}_{y,y}.$$

Given any vertex $x \in X$ and $n \in \mathbb{N}$, we define $\rho(x,y) := \min\{i : \gamma^i_{x,y} > 0\}$ and $B(x,n) := \{y \in X : \rho(x,y) \leq n\}$; note that ρ is a metric if $n_{xy} > 0$ is equivalent to $n_{yx} > 0$ for all $x, y \in X$ (for instance in the case of non-oriented multigraphs).

By using the number of paths it is possible to introduce two asymptotic degrees, namely

$$M_s(X) := \limsup_n (\gamma^n_{x,y})^{1/n}, \qquad M_w(X) := \limsup_n (T^n_x)^{1/n}.$$

It is easy to show that the above definitions do not depend on the choice of $x, y \in X$ and simple arguments of supermultiplicativity show that $M_s(X) = \lim_{n\to\infty}(\gamma^{dn}_{x,x})^{1/dn} = \sup_n(\gamma^{dn}_{x,x})^{1/dn}$ where $d := \gcd\{n : \gamma^n_{x,x} > 0\}$ is the period of the multigraph (which does not depend on the choice of x). Analogously $M_s(X) = \lim_{n\to\infty}(\gamma^{dn+i}_{x,y})^{1/(dn+i)}$, where $0 \leq i \leq d-1$ is uniquely chosen such that $\gamma^n_{x,y} > 0$ implies $n = i \pmod d$. In the rest of the paper, whenever there is no ambiguity, we will denote $M(X)$, $M_s(X)$ and $M_w(X)$ simply by M, M_s and M_w.

By definition $1 \leq M_s \leq M_w \leq M$. We note that $M_w = M$ if the multigraph is regular, that is, it has constant degree. It is well known that, for a regular non-oriented graph, $M_s < M_w$ if and only if it is nonamenable (see Section 3.3 for the definition). A simple case where $M_w = M_s$ is provided by the following proposition.

Proposition 2.1. *Let $(X, E(X))$ be a non-oriented multigraph. If $|B(x,n)|^{1/n} \to 1$ for some (equivalently for all) $x \in X$ then $M_w = M_s$.*

Proof. It is enough to prove that $M_w \leq M_s$. Note that, by the Cauchy-Schwarz inequality,

$$M_s^{2n} \geq \gamma^{2n}_{x,x} = \sum_{y\in X} \gamma^n_{x,y}\gamma^n_{y,x} = \sum_{y\in B(x,n)} (\gamma^n_{x,y})^2 \geq \frac{\left(\sum_y \gamma^n_{x,y}\right)^2}{|B(x,n)|} = \frac{(T^n_x)^2}{|B(x,n)|},$$

hence

$$M_s \geq \limsup_n \sqrt[2n]{\frac{(T^n_x)^2}{|B(x,n)|}} = \limsup_n \sqrt[n]{T^n_x} = M_w.$$

□

2.2 Generating functions

In order to find some characterizations of M_s and M_w, let us define the generating functions

$$H(x,y|\lambda) := \sum_{n=0}^{\infty} \gamma^n_{x,y}\lambda^n, \qquad \Theta(x|\lambda) := \sum_{n=0}^{\infty} T^n_x \lambda^n,$$

with radius of convergence $1/M_s$ and $1/M_w$ respectively. Of course for all $\lambda \in \mathbb{C}$ such that $|\lambda| < 1/M_w$ we have $\Theta(x|\lambda) = \sum_{y\in Y} H(x,y|\lambda)$ and the following relations hold

$$\begin{aligned} H(x,y|\lambda) &= \delta_{x,y} + \lambda \sum_{w\in X} \gamma^1_{x,w} H(w,y|\lambda) \\ &= \delta_{x,y} + \lambda \sum_{w\in X} H(x,w|\lambda)\gamma^1_{w,y}, \qquad \forall \lambda \in \mathbb{C} : |\lambda| < 1/M_s, \\ \Theta(x|\lambda) &= 1 + \lambda \sum_{w\in X} \gamma^1_{x,w}\Theta(w|\lambda), \qquad \forall \lambda \in \mathbb{C} : |\lambda| < 1/M_w. \end{aligned} \tag{2.1}$$

We define

$$\Phi(x,y|\lambda) := \sum_{n=1}^{\infty} \phi^n_{x,y}\lambda^n;$$

it is easy to see that $\Phi(x,x|\lambda) = \lambda \sum_{y\in X, y\neq x} \gamma^1_{x,y}\Phi(y,x|\lambda) + \lambda\gamma^1_{x,x}$ and if $x,y,w \in X$ are distinct vertices such that every path from x to y contains w then $\Phi(x,y|\lambda) = \Phi(x,w|\lambda)\Phi(w,y|\lambda)$. Moreover

$$H(x,y|\lambda) = \Phi(x,y|\lambda)H(y,y|\lambda) + \delta_{x,y}, \quad \forall \lambda : |\lambda| < 1/M_s.$$

Since the radius of the series $H(x,x|\cdot)$ does not depend on the choice of $x \in X$ and since

$$H(x,x|\lambda) = \frac{1}{1-\Phi(x,x|\lambda)}, \qquad \forall \lambda \in \mathbb{C} : |\lambda| < 1/M_s, \tag{2.2}$$

we have that $1/M_s = \max\{\lambda \geq 0 : \Phi(x,x|\lambda) \leq 1\}$ for all $x \in X$ (remember that $\Phi(x,x|\cdot)$ is left-continuous on $[0, 1/M_s]$ and that $1/(1-\Phi(x,x|\lambda))$ has no analytic prolongation in $1/M_s$).

The computation of M_w is not easy in general, but in the case of finite multigraphs there is a simple characterization of M_w. In the following proposition, $\mathbf{Id}$ is the identity matrix.

Proposition 2.2. *Let $(X, E(X))$ be an irreducible, finite multigraph with adjacency matrix $N := (n_{xy})_{x,y\in X}$, then*

$$1/M_w = 1/M_s = \min\{\lambda > 0 : \det(\lambda N - \mathbf{Id}) = 0\}.$$

Proof. We use the same notation N for the matrix and the linear operator. By the Perron-Frobenius Theorem there exists an eigenvalue $\sigma_0 > 0$ of N such that any other eigenvalue σ satisfies $|\sigma| < \sigma_0$ and the same holds for N^t. Moreover $\dim(\mathrm{Ker}(N^t - \sigma_0\mathbf{Id})) = 1$ and it is possible to choose the eigenvector v in such a way that $v > 0$. It is clear that any vector $w < 0$ cannot possibly belong to $\mathrm{Rg}(N - \sigma_0\mathbf{Id}) \equiv \mathrm{Ker}(N^t - \sigma_0\mathrm{Id})^\perp$ since $\langle w, v\rangle < 0$. Then the equation (2.1) (which holds for $|\lambda| < 1/M_w$), can be written as

$$(\lambda N - \mathbf{Id})\Theta(\lambda) = -\mathbf{1} = -\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{2.3}$$

and has no solutions if $\lambda = 1/\sigma_0$. On the other hand equation (2.3) defines a holomorphic (vector) function $\Theta'(\lambda) = -(\lambda N - \mathbf{Id})^{-1}\mathbf{1}$ on $\{\lambda \in \mathbb{C} : |\lambda| < 1/\sigma_0\}$. Note that Θ' coincides with Θ on $\{\lambda \in \mathbb{C} : |\lambda| < \min\{1/\sigma_0, 1/M_w\}\}$, hence $1/\sigma_0 \leq 1/M_w$. If $1/\sigma_0 < 1/M_w$ then there would be an analytic prolongation of Θ' to $1/\sigma_0$ and by continuity eq. (2.3) would hold for $\lambda = 1/\sigma_0$. □

2.3 Generalized branching process

In the classical branching process (see for instance [4]) there is a unique offspring distribution according to which each individual breeds. We consider a generalized branching process where each father may have different types of children and each of them breeds according to a specific distribution which depends on its type and on the father. To be more specific, let $\mathbb{T} = (\bigcup_{i=0}^{\infty} \mathbb{N}_*^{2i}, E(\mathbb{T}))$ where $\mathbb{N}_*^0 := \{o\}$ where o is the root of the tree $\mathbb{T}$. Identifying as usual $\mathbb{N}_*^{2n} \times \mathbb{N}_*^2$ with $\mathbb{N}_*^{2n+2}$, the set of edges is

$$E(\mathbb{T}) := \{(x, y) \in \mathbb{T} : \exists k \in \mathbb{N}_*^2, y = (x, k)\} \cup \{(o, k) : k \in \mathbb{N}_*^2\}.$$

Roughly speaking $y = (x, i, j)$ means that y is the j-th son of type i of its father x (whereas (i, j) is the j-th son of type i of o) and the oriented edges are drawn from fathers to sons. Moreover $\bigcup_{i=0}^{n} \mathbb{N}_*^{2i}$ represents the genealogic tree of the progeny of o up to the n-th generation. We provide each individual x with a distribution μ_x such that if $x = (v, i, j)$ and $y = (v, i, k)$ then $\mu_x \equiv \mu_y$ (that is, the offspring distribution depends only on the father and on the type). Now, each distribution is defined on the countable space $\mathcal{E} := \{f \in \mathbb{N}^{\mathbb{N}_*} : S(f) < +\infty\}$ where $S(f) = \sum_{i=1}^{\infty} f(i)$. To be more precise it is possible to construct a canonical probability space $(\Omega, \mathcal{A}, \mathbb{P})$ supporting the generalized branching process and such that $\mathbb{P}$ satisfies

$$\mu_x(f) = \mathbb{P}\left(\bigcap_{i=1}^{\infty} \{x \text{ has } f(i) \text{ sons of type } i\}\right), \qquad \forall f \in \mathcal{E}.$$

Moreover for every $x \in \mathbb{T}$ let ν_x be the distribution of the total number of children of x, that is, $\nu_x(k) = \mu_x(\{f : S(f) = k\})$ for all $k \in \mathbb{N}$. Take a family of independent $\mathcal{E}$-valued random variables $\{Z_x\}_{x \in \mathbb{T}}$ such that Z_x has distribution μ_x.

Let us recursively construct this generalized branching process $\{B_n\}_{n \geq 0}$:

$$B_0 = \{o\}, \qquad B_{n+1} = \{(v, i, j) : v \in B_n, 1 \leq j \leq Z_v(i)\},$$

where B_n is the n-th generation, and its member v has exactly $Z_v(i)$ children of type i. Extinction is ($B_n = \emptyset$ eventually).

Lemma 2.3. *Let $G_x(z)$ be the generating function of ν_x and suppose that there exists $\delta \in [0, 1)$ such that $G_x(\delta) \leq \delta$ for all $x \in \mathbb{T}$. Then $\mathbb{P}(B_n = \emptyset$ eventually$) \leq \delta$.*

Proof. Denote by A_n^x the event of extinction before the n-th generation of the progeny of x. Let $q_n^x := \mathbb{P}(A_n^x)$, clearly q_n^x depends only on the father and the type of x; we claim that $q_n^x \leq \delta$ for all

$x \in \mathbb{T}$. We proceed by induction on n. Obviously, for each $x \in \mathbb{T}$, $q_0^x = \nu_x(0) = G_x(0) \leq \delta$. By induction, using the hypothesis of independence,

$$q_{n+1}^x = \mathbb{P}\left(\bigcup_{i=0}^{\infty} \bigcup_{f:S(f)=i} \bigcap_{j=1}^{\infty} \bigcap_{k=1}^{f(j)} A_n^{(x,j,k)}\right) = \sum_{i=0}^{\infty} \sum_{f:S(f)=i} \mu_x(f) \prod_{j=1}^{\infty} \left(q_n^{(x,j,1)}\right)^{f(j)}$$
$$\leq \sum_{i=0}^{\infty} \sum_{f:S(f)=i} \mu_x(f)\delta^{S(f)} = \sum_{i=0}^{\infty} \delta^i \sum_{f:S(f)=i} \mu_x(f) = \sum_{i=0}^{\infty} \delta^i \nu_x(i) = G_x(\delta) \leq \delta.$$

Now, $q_n^o \uparrow \mathbb{P}(B_n = \emptyset$ eventually) and $\delta \geq \lim_n q_n^o$ and this yields the conclusion. □

This lemma trivially applies when each distribution μ_x is drawn from a finite set of distributions such that the corresponding ν_x represents a supercritical branching process. In this case we have a finite number of fixed points in $[0, 1)$ for the generating functions and δ may be taken as the maximum among them (indeed this is what we do in Theorem 3.4).

3 Main results

3.1 The critical values

The critical values λ_s and λ_w separate different behaviors of the λ-BRW. Namely, given non-extinction, if $\lambda > \lambda_s$, then the conditional expected value of the number of particles per site tends to infinity, while if $\lambda > \lambda_w$ the conditional probability that the total number of particles on the graph tends to infinity is equal to 1. Indeed the BRW (starting with a finite number of particles) is a continuous-time random walk on the countable state space of finite configurations $\eta \in \mathbb{N}^X$, with a trap state in $\underline{0}$ (the configuration with no particles). Hence all the states but $\underline{0}$ are transient and the process which does not hit $\underline{0}$ leaves $A_k = \{\eta \in \mathbb{N}^X : \sum_{x \in X} \eta(x) \leq k\}$ eventually for all $k \in \mathbb{N}$. Indeed, the probability of reaching $\underline{0}$ starting from any configuration in A_k is bounded from below by a positive constant (the reproduction rate is bounded from above), hence the claim follows.

Here we investigate the critical values λ_s and λ_w and their relationship with M_s and M_w. Since the critical values do not depend on the number of particles at $t = 0$ (nor on their location), we suppose that the initial state is one particle at a fixed vertex $o \in X$. To each particle p (present at some time at a site x) there corresponds a (unique) *reproduction trail* starting from the initial particle located at o at time 0 reconstructing the genealogy of p. Roughly speaking, the (space-time) reproduction trail corresponding to p is a path $(\{x_0 = o, x_1, \ldots, x_{n-1}, x_n = x\}, \{k_1, \ldots, k_n\})$ along with a sequence $(t_0, \ldots, t_{n-1})$ where t_0 is the epoch when the original particle in o generated the ancestor of p in x_1 (through the edge (o, x_1, k_1)) and, for $i = 1, \ldots, n-1$, t_i is the epoch when the ancestor in x_i generated the one in x_{i+1} (through the edge (x_i, x_{i+1}, k_{i+1})). Clearly, putting $t_{-1} = 0$, then for all $i = 0, \ldots, n-1$, $t_i - t_{i-1}$ is the realization of an exponential random variable with rate λ (it is tacitly understood that each ancestor is alive when breeding). Such a trail is said

to have length n. For a detailed construction we refer the reader to [14, Section 3] (where what we call reproduction trail is an infection trail). Note that the expected number of trails along a path of length n is λ^n, hence to each sequence $\{x_0, x_1, \ldots, x_n\}$ there corresponds a number $\lambda^n \prod_{i=0}^{n-1} n_{x_i x_{i+1}}$ of expected trails.

In [14, Lemma 3.1] it was proved that $\lambda_s = 1/M_s$ for any graph. We use a different approach to extend this result to multigraphs; this approach allows us to study the critical behavior when $\lambda = \lambda_s$ (see Theorem 3.11).

Theorem 3.1. *For each multigraph $(X, E(X))$ we have that $\lambda_s = 1/M_s$.*

Proof. Let us consider a path $\Pi := (\{o = x_0, x_1, \ldots, x_n = o\}, \{k_1, \ldots, k_n\})$ and let us define its number of cycles $\mathbb{L}(\Pi) := |\{i = 1, \ldots, n : x_i = o\}|$; the expected number of trails along such a path is λ^n (hence to each sequence $\{x_0, x_1, \ldots, x_n\}$ there corresponds a number $\lambda^n \prod_{i=0}^{n-1} n_{x_i x_{i+1}}$ of expected trails). Disregarding the original time scale, to the BRW there corresponds a Galton-Watson branching process: given any particle p in o (corresponding to a trail with n cycles), define its children as all the particles whose trail is a prolongation of the trail of p and is associated with a spatial path with $n+1$ cycles. Hence a particle is of the k-th generation if and only if the corresponding trail has k cycles; moreover it has one (and only one) parent in the $(k-1)$-th generation. Since each particle behaves independently of the others then the process is markovian. Thus the BRW survives if and only if this branching process does. The expected number of children of the branching process is the sum over n of the expected number of trails of length n and one cycle, that is $\sum_{n=1}^{\infty} \phi_{o,o}^n \lambda^n = \Phi(o, o|\lambda)$. Thus we have a.s. local extinction if and only if $\Phi(o, o|\lambda) \leq 1$, that is, $\lambda \leq 1/M_s$ (see eq. (2.2) and the remark thereafter). □

We now focus our attention on the weak critical value.

Lemma 3.2. *For every multigraph we have that $\lambda_w \geq 1/M_w$.*

Proof. Since the average number of trails on a fixed path of length n starting from $(o, 0)$ is λ^n, the average number of all the trails on any path from $(o, 0)$ is $\sum_{n=0}^{\infty} \lambda^n T_o^n$. If $\lambda M_w < 1$ then this sum is finite, hence the number of reproduction trails is a.s. finite and there is no weak survival. □

Remark 3.3. *For every multigraph, if $M_w = M_s$ there is no pure weak survival (since $1/M_w \leq \lambda_w \leq \lambda_s = 1/M_s$). Examples are subexponentially growing graphs (recall Proposition 2.1) such as $\mathbb{Z}^d$ or d-dimensional combs (see [1] for the definition).*

Let us consider now the question whether $\lambda_w = 1/M_w$. The following theorem states that this equality holds if the multigraph satisfies this geometrical condition:

$$\forall \varepsilon > 0 \, \exists \bar{n} = \bar{n}(\varepsilon) : \sup_{n \leq \bar{n}} \sqrt[n]{T_x^n} \geq M_w - \varepsilon, \, \forall x \in X. \tag{U}$$

Note that, by definition of M_w, for all fixed $\varepsilon > 0$ and $x \in X$, there exists n_x such that $\sqrt[n_x]{T_x^{n_x}} \geq M_w - \varepsilon$. Condition (U) is a request of uniformity in x.

Theorem 3.4. *If $(X, E(X))$ is a multigraph such that condition (U) holds, then $\lambda_w = 1/M_w$.*

Proof. Fix $\varepsilon > 0$ and λ such that $\lambda(M_w - \varepsilon) > 1$. We associate to the BRW a generalized branching process where the type of each particle is the site where it is born (although in Section 2.3 the type was indexed by $\mathbb{N}$ this is not a restriction since X is at most countable). For all $x \in X$ define n_x to be the smallest positive integer such that $\sqrt[n_x]{T_x^{n_x}} \geq M_w - \varepsilon$. In this generalized branching process the "children" of the initial particle (which represents the root of the tree of the process) are all the particles associated with trails of length n_o starting from o. Each of these trails ends on a specific vertex in $B(o, n_o)$, which represents the type of the children generated there. The offspring distribution μ_o is supported on $\mathcal{E}_o := \{f \in \mathcal{E} : f(x) = 0, \forall x \notin B(o, n_o)\}$ and satisfies

$$\mu_o(f) = \mathbb{P}\left(\bigcap_{y \in B(o,n_o)} \{K_y = f(y)\}\right), \qquad \forall f \in \mathcal{E}_o,$$

where $\mathbb{P}$ is the probability on the space where the BRW is defined and K_y is the (random) number of trails of length n_o starting at o and ending at y. The corresponding ν_o is supercritical in the sense that

$$G_o'(1) \equiv \sum_{n=0}^{\infty} n\nu_o(n) = \lambda^{n_o} T_o^{n_o} > 1.$$

This means that G_o has a fixed point $\delta_o < 1$.

Analogously, we repeat this construction for any particle at any site x. The children of such a particle p are the particles associated with trails which are prolongations of the trail of p and the difference between the lengths of the prolongation and of the trail of p is n_x. Clearly the offspring distribution μ_x is supported on $\mathcal{E}_x := \{f \in \mathcal{E} : f(z) = 0, \forall z \notin B(x, n_x)\}$ and is defined as

$$\mu_x(f) = \mathbb{P}\left(\bigcap_{y \in B(x,n_x)} \{K_y = f(y)\}\right), \qquad \forall f \in \mathcal{E}_x,$$

where K_y is the (random) number of prolongations, ending at y, of the trail of p, such that the difference between the lengths of the prolongation and of the trail of p is n_x. By Markov property, these laws do not depend on the particle, but only on the site x, hence the definition is well posed. More precisely, μ_x depends only on the submultigraph $B(x, n_x)$. We call G_x the generating function of ν_x.

These generating functions G_x are taken from a finite set of G's; indeed in a bounded degree multigraph the set of the equivalence classes up to isometries of the balls of radius at most $\bar{n}$ is finite. Since all these generating functions are convex, we may apply Lemma 2.3 with $\delta = \max\{\delta_x : x \in X\}$ obtaining that the generalized branching process is supercritical. Since for each $x \in X$ we consider

only the particles generated along a path of length n_x (starting from x) the generalized branching process is dominated by the total number of particles of the original BRW, hence this last one is supercritical as well. Since ε was arbitrary, we deduce that $\lambda_w \leq 1/M_w$. Lemma 3.2 yields the conclusion. $\square$

A large family of multigraphs for which condition (U) holds is described by the following theorem. Roughly speaking, the geometrical request therein is a sort of "translational quasi-invariance" which allows to compare all T_x^n with a single $T_{x_0}^n$.

Theorem 3.5. *Let $(X, E(X))$ be a multigraph; let us suppose that there exists $x_0 \in X$, $Y \subseteq X$ and $n_0 \in \mathbb{N}$ such that*

(1) for all $x \in X$ we have that $B(x, n_0) \cap Y \neq \emptyset$;

(2) for all $y \in Y$ there exists an injective map $\varphi_y : X \to X$, such that $\varphi_y(x_0) = y$ and $n_{\varphi_y(x)\varphi_y(z)} \geq n_{xz}$ for all $x, z \in X$.

Then condition (U) holds and $\lambda_w = 1/M_w$.

Proof. We fix $\varepsilon > 0$. For any given $x \in X$, condition (1) implies the existence of $y \in Y$ such that $\rho(x, y) \leq n_0$, hence $T_y^n \leq T_x^{n+n_0}$ for all $n \in \mathbb{N}$. Using condition (2), we have that $T_y^n \geq T_{x_0}^n$ for all $n \in \mathbb{N}$, which in turn implies $T_x^{n+n_0} \geq T_{x_0}^n$. Since $\limsup_{n\to\infty}(T_{x_0}^n)^{1/(n+n_0)} = M_w$, we may find $n_1 \in \mathbb{N}$ such that $(T_{x_0}^{n_1})^{1/(n_1+n_0)} \geq M_w - \varepsilon$, whence $\overline{n}(\varepsilon) := n_1 + n_0$ satisfies the hypothesis of Theorem 3.4. $\square$

For a nontrivial example of trees satisfying the hypotheses of the previous theorem see Example 3. Another important class of multigraphs satisfying condition (U) is described by the following definition (see also Theorem 3.9).

Definition 3.6. *Let $(X, E(X))$ and $(Y, E(Y))$ be two multigraphs. A map $\varphi : X \to Y$ is called a local isomorphism from X onto Y if and only if for all $x \in X$, $y \in Y$ we have $\sum_{z\in X:\varphi(z)=y} n_{xz}^X = n_{\varphi(x)y}^Y$.*
We say that a multigraph (resp. a graph) $(X, E(X))$ is an $\mathcal{F}$-multigraph (resp. an $\mathcal{F}$-graph) if it is locally isomorphic to a finite multigraph (resp. a finite graph) .

Note that a local isomorphism from X to Y does not implies the existence of a local isomorphism from Y to X. Moreover it is easy to show that, for any local isomorphism, $\varphi(D_X(x)) = D_Y(\varphi(x))$ and that, for all $n \in \mathbb{N}$ and for all $x, y \in X$,

$$\sum_{z\in X:\varphi(z)=y} \gamma_{x,z}^n = \widetilde{\gamma}_{\varphi(x),y}^n, \qquad T_x^n(X) = T_{\varphi(x)}^n(Y), \tag{3.4}$$

where $\widetilde{\gamma}$ refers to paths in Y. The second equation in (3.4) is implied by the first one, which may be proved by induction using the properties of φ.

The following proposition shows how M_w, λ_w and λ_s (or equivalently M_s) are affected by the action of a local isomorphism.

Proposition 3.7. *Let $(X, E(X))$ and $(Y, E(Y))$ be two connected multigraphs and suppose that there exists a local isomorphism φ from X onto Y. The following assertions hold*

1. $\lambda_w(X) = \lambda_w(Y)$.
2. $\lambda_s(X) \geq \lambda_s(Y)$. *If there exists $y \in Y$ such that $|\varphi^{-1}(y)| < +\infty$ then $\lambda_s(X) = \lambda_s(Y)$.*
3. $M_w(X) = M_w(Y)$. *Moreover Y satisfies condition (U) if and only if X satisfies it.*

Proof. 1. Let η_t be a λ-BRW process on X starting with one particle at site x. One may easily show that

$$\xi_t(y) := \sum_{x \in \varphi^{-1}(y)} \eta_t(x)$$

is a λ-BRW process on Y starting with one particle at site $\varphi(x)$. It is clear that η_t survives globally if and only if ξ_t does; this implies $\lambda_w(X) = \lambda_w(Y)$.

2. If η_t survives locally then ξ_t does; hence $\lambda_s(X) \geq \lambda_s(Y)$. On the other hand, given that $|\varphi^{-1}(y)| < +\infty$, if we start the process η_t with one particle at a site $x \in \varphi^{-1}(y)$ and ξ_t survives locally (in y) the same must be true for η_t at some $z \in \varphi^{-1}(y)$ and hence at $x \in X$.

3. It is a simple consequence of the equality $T_x^n(X) = T_{\varphi(x)}^n(Y)$ which holds for all $x \in X$ and $n \in \mathbb{N}$.

□

According to the previous proposition, if $(X, E(X))$ is locally isomorphic to a multigraph $(Y, E(Y))$ which satisfies the hypotheses of Theorem 3.5, then the same conclusions of this theorem hold for $(X, E(X))$. In particular if $(Y, E(Y))$ is a finite multigraph then $M_w(Y) = M_s(Y)$ and $\lambda_w(Y) = \lambda_s(Y)$.

We note that both quasi-transitive graphs and regular graphs are $\mathcal{F}$-multigraphs. Indeed if X is a quasi-transitive graph, one takes Y as the quotient space with respect to the action of the automorphism group, φ as the quotient map and $n_{yy'} := \left|\varphi^{-1}(y' \cap D_X(x)\right|$ where $\varphi(x) = y$ (this definition does not depend on the choice of x); regular graphs of degree k may be mapped on the one-point multigraph with k loops (and $M_w = k$). Nevertheless this class contains graphs which are neither regular nor quasi transitive (see Examples 1, 2 and 3); moreover the "regularity" of $\mathcal{F}$-multigraphs is only "local", indeed one can easily construct examples of quite irregular $\mathcal{F}$-graphs.

The following lemma gives a sufficient condition for a graph to be an $\mathcal{F}$-graph.

Lemma 3.8. *Let us consider a graph $(X, E(X))$ such that for all $x, y \in X$ with $\deg(x) = \deg(y)$ we have that*

$$|\{z \in D(x) : \deg(z) = j\}| = |\{z \in D(y) : \deg(z) = j\}|, \qquad \forall j = 1, \dots, M.$$

Then $(X, E(X))$ is an $\mathcal{F}$-graph.

Proof. Take $Y := \{i \in \mathbb{N} : \exists x \in X, \deg(x) = i\}$, $n_{ij} := |\{z \in D(x) : \deg(z) = j\}|$ for some $x \in X$ such that $\deg(x) = i$ (the definition does not depend on x) and $\varphi := \deg$. □

In analogy to Theorem 3.5, the following theorem describes another class of multigraphs satisfying condition (U).

Theorem 3.9. *If $(X, E(X))$ is an $\mathcal{F}$-multigraph, then it satisfies condition (U) and $\lambda_w = 1/M_w$.*

Proof. Let $(X, E(X))$ be locally isomorphic to the finite multigraph $(Y, E(Y))$. We note that, being finite, Y satisfies condition (U), whence, by Proposition 3.7, also X does. □

Remark 3.10. It is natural to wonder how M_s, M_w, λ_s and λ_w are affected by local modifications of the multigraphs $(X, E(X))$ (such as, for instance, attaching a complete finite, graph to a vertex of X or removing a set of vertices and/or edges).

If $(X, E(X))$, $(Y, E(Y))$ are two multigraphs and $\psi : Y \to X$ is an injective map such that $n_{\psi(x)\psi(y)} \geq \widetilde{n}_{xy}$ for all $x, y \in Y$ (where $\widetilde{n}$ refers to Y) then $\lambda_w(X) \leq \lambda_w(Y)$, $\lambda_s(X) \leq \lambda_s(Y)$, $M_w(X) \geq M_w(Y)$, $M_s(X) \geq M_s(Y)$.

In certain cases it is easy to show that the existence of a pure weak phase on X implies the existence of a pure weak phase on some submultigraph; indeed if Y is a finite subset of X such that $X \setminus Y$ is divided into a finite number of connected multigraphs $X_1, \dots, X_n$ (which is certainly true if $n_{xy} > 0$ is equivalent to $n_{yx} > 0$ for all $x, y \in X \setminus Y$), then for every $\lambda \in (\lambda_w(X), \lambda_s(X))$ the λ-BRW leaves eventually a.s. the subset Y. Hence it survives (globally but not locally) at least on one connected component; this means that, although $\lambda_s(X_i) \geq \lambda_s(X)$, $\lambda_w(X_i) \geq \lambda_w(X)$ for all $i = 1, \dots, n$, there exists i_0 such that $\lambda_w(X_{i_0}) = \lambda_w(X)$. The existence of a pure weak phase on X_{i_0} follows from $\lambda_s(X_{i_0}) \geq \lambda_s(X) > \lambda_w(X) = \lambda_w(X_{i_0})$.

Moreover if there exists a subset Y as above such that $\lambda_w(X_i) > \lambda_w(X)$ for all i, then there is no pure weak phase for the BRW on X. Take for instance a graph $(X', E(X'))$ and $k \in \mathbb{N}$ such that $1/k < \lambda_w(X')$. Attach a complete graph of degree k to a vertex of X', we obtain a new graph X such that $\lambda_s(X) = \lambda_w(X) \leq 1/k < \lambda_w(X')$; hence even if the BRW on X' has a pure weak phase, the BRW on X has none.

3.2 Critical behaviors

It is well known that any critical branching process dies out a.s. In Theorem 3.1 we associated to the BRW a (discrete time) branching process in such a way that the progeny of the BRW in a fixed site x coincides with the total progeny of the branching process. This allows us to study the strong critical behavior of the BRW (and the weak critical behavior for a BRW on a $\mathcal{F}$-multigraph).

Theorem 3.11. *For each multigraph $(X, E(X))$ if $\lambda = \lambda_s$ then the λ-BRW dies out locally almost surely. For each $\mathcal{F}$-multigraph if $\lambda = \lambda_w$ then the λ-BRW on X dies out globally almost surely.*

Proof. Recall that (see the proof of Theorem 3.1) the λ-BRW survives if and only if the branching process with expected number of children $\Phi(o, o|\lambda)$ does. Since $\Phi(o, o|1/M_s) \leq 1$ and $\lambda_s = 1/M_s$ then there is a.s. local extinction at λ_s.

Now suppose that X is locally isomorphic to the finite multigraph Y. Since the global behavior of the λ-BRW η_t on X is the same as the corresponding behavior of the induced λ-BRW ξ_t on Y (see the proof of Proposition 3.7), then Theorem 3.4 and Proposition 3.7 imply $\lambda_w(X) = 1/M_w(X) = \lambda_w(Y) = \lambda_s(Y)$. But we just proved that each λ_s-BRW dies out locally a.s.; moreover, since Y is finite, ξ_t dies out globally a.s., hence the same holds for η_t. □

3.3 Nonamenability and weak phase

In this section we consider only non-oriented multigraphs. A multigraph $(X, E(X))$ is *nonamenable* if

$$\inf\left\{\frac{|\partial_E(S)|}{|S|} : S \subseteq X, |S| < \infty\right\} =: \iota_X > 0,$$

where $\partial_E(S)$ is the set of edges $(x, y, i) \in E(X)$ such that $x \in S$ and $y \notin S$.

We define $N : l^2(X) \to l^2(X)$ by $Nf(x) := \sum_{y \in X} n_{xy} f(y)$ which is a bounded, linear operator with $\|N\| \leq M$. It is well known that on a regular, non-oriented graph (where $M = M_w$) the existence of the weak phase is equivalent to nonamenability (see [15, Theorem 2.4]). Indeed on regular, non-oriented graphs $M_s < M_w$ is equivalent to nonamenability: one easily proves that $M_s = \|N\|$ (see Lemma 3.12 and the reference therein); moreover $\|N\| = M\|P\|$ where P is the transition operator associated to the simple random walk and Gerl proved that $\|P\| < 1$ is equivalent to nonamenability (see [2]). Hence using Theorem 3.1 and Theorem 3.9 we obtain an alternative proof of [15, Theorem 2.4].

Now we show that, for non-oriented $\mathcal{F}$-multigraphs, nonamenability is equivalent to the existence of a pure weak phase which, in turn, is equivalent to $M_s < M_w$.

Lemma 3.12. *N is self adjoint and $\|N\| = \rho(N) = M_s$ where $\rho(N) = \lim_{n\to\infty} \|N^n\|^{1/n}$ is the spectral radius of N.*

Proof. The self-adjointness of N is easy and $\|N\| = \rho(N)$ is a standard property which follows from the Spectral Theorem for any normal (hence self-adjoint) operator. To prove that $M_s = \|N\|$ one proceeds essentially as in [15, Lemma 2.2]. □

The following theorem implies the analogous results for regular and quasi-transitive graphs.

Theorem 3.13. *Let $(X, E(X))$ be a non-oriented $\mathcal{F}$-multigraph. Then $\lambda_w < \lambda_s$ if and only if $(X, E(X))$ is nonamenable.*

Before proving this statement, we need a technical result concerning the Dirichlet norm of l^2 functions. Given $f \in l^2(X)$, define

$$\|f\|_{D(2)} = \left(\sum_{x,y \in X} n_{xy} |f(x) - f(y)|^2 \right)^{1/2}.$$

Lemma 3.14. *Let $(X, E(X))$ be a nonamenable multigraph. Then there exists $c > 0$ such that, for all $f \in l^2(X)$,*

$$\|f\|_{D(2)} \geq c\|f\|_2.$$

Proof. The proof is analogous to the one of [15, Theorem 2.6] (one has to deal carefully with the presence of n_{xy}), hence we omit it. □

Proof of Theorem 3.13. We follow the proof of [15, Theorem 3.1]. Let $(X; E(X))$ be nonamenable, $N = (n_{xy})_{x,y \in X}$ be its adjacency matrix and $\widetilde{N} = (\tilde{n}_{xy})_{x,y \in Y}$ be the adjacency matrix of the finite multigraph $(Y, E(Y))$ which $(X, E(X))$ is locally isomorphic to. We must prove that $\|N\| < M_w$. By definition of local isomorphism we have that $\tilde{n}_{\varphi(x)\varphi(y)} = \sum_{z:\varphi(z)=\varphi(y)} n_{xz}$. By the Perron-Frobenius theorem $\widetilde{N}$ has largest positive eigenvalue M_w with associated positive eigenvector $(a_1, \ldots, a_k)$ (k being the cardinality of Y). Then

$$M_w a_{\varphi(x)} = \sum_{y' \in Y} \widetilde{n}_{\varphi(x)y'} a_{y'} = \sum_{y' \in Y} \sum_{y \in \varphi^{-1}(y')} n_{xy} a_{\varphi(y)} = \sum_{y \in X} n_{xy} a_{\varphi(y)}. \tag{3.5}$$

Take $f \in l^2(X)$. Applying equation (3.5) and the fact that $(X, E(X))$ is non-oriented

$$M_w^2 \|f\|_2^2 = M_w^2 \sum_{y \in X} (f(y))^2 = \sum_{x \in X} \left(\sum_{z \in X} n_{xz} a_{\varphi(z)} \right) \left(\sum_{y \in X} n_{xy} \frac{(f(y))^2}{a_{\varphi(y)}} \right).$$

Hence

$$M_w^2 \|f\|_2^2 - \|Nf\|_2^2 = \sum_{x \in X} \sum_{z,y \in X} n_{xz} n_{xy} \left[\frac{a_{\varphi(z)}}{a_{\varphi(y)}} (f(y))^2 - f(z)f(y) \right] \geq \frac{1}{2} (\min a_i)^2 \|g\|_{D(2)},$$

where $g(x) = f(x)/a_{\varphi(x)}$ is considered as a map on the multigraph $G^2 = (X, \bar{E}(X))$ with adjacency matrix $\bar{N}$ defined by $\bar{n}_{xy} = \sum_{z \in X} n_{zx} n_{zy}$ and $\bar{E}(X) := \{(y, z, i) : 1 \leq i \leq \bar{n}_{xy}\}$. Applying

Lemma 3.14 to each connected component of G^2 (note that each of them is nonamenable) and noting that $\|g\|_2^2 \geq D^2\|f\|_2^2$ for $D^{-1} = \max(a_i)$, we have that for some $C > 0$

$$M_w^2\|f\|_2^2 - \|Nf\|_2^2 \geq C\|f\|_2^2,$$

whence $\|N\| \leq \sqrt{M_w^2 - C} < M_w$.

Suppose now that $(X, E(X))$ is amenable and fix $\varepsilon > 0$. Then for some finite set $S \subset X$, $|\partial_E S|/|S| < \varepsilon$. Define $f(x) = a_{\varphi(x)}\mathbb{1}_S(x)$. If $x \in S$ and $D(x) \cap S^c = \emptyset$, then, by (3.5), $Nf(x) = M_w f(x)$. Hence

$$\|Nf\|_2^2 \geq M_w^2\|f\|_2^2 - 2M_w\varepsilon|S|(\max a_i)^2,$$

and

$$\frac{\|Nf\|_2^2}{\|f\|_2^2} \geq M_w^2 - 2\varepsilon M_w\left(\frac{\max a_i}{\min a_i}\right)^2.$$

By taking ε arbitrarily small we prove that $\|N\| \geq M_w$, whence $M_s = M_w$ (recall that $\|N\| = M_s \leq M_w$). □

3.4 Examples

The first two explicit examples listed hereafter show that the class of $\mathcal{F}$-multigraphs is larger than the union of regular and quasi-transitive multigraphs. Both these examples are modifications of regular graphs: Example 1 is obtained by attaching an edge to each vertex, Example 2 by drawing a "bridge with intermediate station" between some of the vertices.

Example 1. Take a square and attach to every vertex a branch of a homogeneous tree of degree 3, obtaining a regular graph (of degree 3) which is not quasi transitive. If we attach now to each vertex a new edge with a new endpoint we obtain a non-oriented, nonamenable $\mathcal{F}$-graph $(X, E(X))$ which is neither regular nor quasi transitive. It is easily seen (by Lemma 3.8) to be locally isomorphic to a multigraph with adjacency matrix

$$N = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}.$$

According to Theorem 3.13, the BRW on this graph has a pure weak phase.

Example 2. Take an infinite graph $(X, E(X))$ with set of vertices $X = \{x_1, x_2, \ldots\}$. If $Y = \{y_1, y_2, \ldots\}$ is another countable set, disjoint from X, we may consider the graph with set of vertices $Z := X \cup Y$ and

$$E(Z) := E(X) \cup \bigcup_{i=1}^{\infty}\{(x_{2i-1}, y_i), (y_i, x_{2i-1}), (x_{2i}, y_i), (y_i, x_{2i})\};$$

roughly speaking we join x_{2i-1} and x_{2i} by a bridge and we cut this bridge into two edges by using a new vertex y_i. If the graph X is nonamenable then it is possible to show that the (multi)graph

Z is nonamenable as well. By choosing $(X, E(X))$ regular (with $\deg \equiv k$) we obtain an $\mathcal{F}$-graph which (by Lemma 3.8) is locally isomorphic to a multigraph with adjacency matrix

$$N = \begin{pmatrix} k & 1 \\ 2 & 0 \end{pmatrix}.$$

Again, by choosing accurately $(X, E(X))$ and ordering wisely its vertices we may obtain a graph which is neither quasi transitive nor regular.

The following trees are natural examples of graphs which are not quasi transitive and, nevertheless, are not "too irregular". We show that, for these trees, nonamenability is equivalent to the existence of a pure weak phase and the proof is not a direct application of Theorem 3.13.

Example 3. Given a sequence of positive natural numbers $\{m_k\}_{k\geq 1}$ we construct a non-oriented, rooted tree $\mathbb{T}$ (with root o) such that if $x \in \mathbb{T}$ satisfies $\rho(o, x) = k$ then it has m_{k+1} neighbors y such that $\rho(0, y) = k + 1$. We call this radial graph $T_{\{m_k\}}$-tree. If the sequence is periodical of period d, then Theorem 3.5 applies with $x_0 = o$, $n_0 = d$, $Y := \cup_{n\in\mathbb{N}} B(o, nd)$ and φ_y (where $y \in Y$) maps isomorphically the tree $\mathbb{T}$ onto the subtree branching from y. We call T_i the $T_{\{m'_k\}}$-tree obtained by means of this construction where $m'_k := m_{k+i-1}$. Roughly speaking we construct $T_1, \ldots, T_d$, by using cyclic permutations of the sequence $\{m_1, \ldots, m_d\}$. Obviously $\mathbb{T} = T_1$. Since T_i may be mapped into T_j for all $i, j = 1, \ldots, d$ (in the sense of Remark 3.10) then $\lambda_w(T_i)$, $\lambda_s(T_i)$, $M_w(T_i)$, and $M_s(T_i)$ do not depend on i.

Let us consider the finite cyclic graph $\widetilde{Y} := \{y_1, \ldots, y_d\}$ where $n_{y_i y_{i+1}} = n_{y_{i+1} y_i} = 1$ for all $i = 1, \ldots d$ (with the identification $y_{d+1} = y_1$). To each vertex y_i we attach $m_i - 1$ copies of T_{i+1} (again with the identification $T_{d+1} = T_1$), each of them by using a two-way edge. We denote this connected, non-oriented $\mathcal{F}$-graph by $(X, E(X))$; indeed it may be mapped onto the finite multigraph Y' where $Y' = \widetilde{Y}$ and $n'_{y_i y_{i+1}} = m_i$, $n'_{y_{i+1} y_i} = 1$ for all $i = 1, \ldots d$. Note that X is neither quasi transitive nor regular, unless $m_i = 1$ for all i. X is nonamenable if and only if T_1 is nonamenable, that is, if and only if there exists i such that $m_i \geq 2$. In this case, according to Theorem 3.13, $\lambda_w(X) < \lambda_s(X)$, hence by Remark 3.10 (considering $X \setminus \widetilde{Y}$) there exists i such that $\lambda_w(T_i) < \lambda_s(T_i)$. This means that for all i we have $\lambda_w(T_i) < \lambda_s(T_i)$ and there is a pure weak phase on T_i. On the other hand, if $m_i \equiv 1$ for all $i = 1, \ldots, d$, then there is no pure weak phase (Remark 3.3).

4 Modified BRW and BRW on weighted graphs

4.1 Modified BRW

In this section we consider an irreducible random walk (X, P). In the case of simple random walks some of the results of this section may be found also in [15]. We study the modified BRW where

each particle at site x dies at rate 1 and breeds at rate λ and sends the offspring randomly according to the probability distribution $p(x,\cdot)$.

We denote by $p^{(n)}(x,y)$ the n-step transition probabilities from x to y ($n \geq 0$) and by $f^{(n)}(x,y)$ the probability that the random walk starting from x hits y for the first time after n steps ($n \geq 1$). Then we define the corresponding generating functions $G(x,y|z) = \sum_{n\geq 0} p^{(n)}(x,y)z^n$ and $F(x,y|z) = \sum_{n\geq 1} f^{(n)}(x,y)z^n$, where $x \in X$, $z \in \mathbb{C}$ (further details can be found in [16, Chapter I.1.B], where F is called U).

The expected number of trails along a path $\Pi = \{x_0, \ldots, x_n\}$ is equal to $\lambda^n \prod_{i=0}^{n-1} p(x_i, x_{i+1})$. Hence the expected number of trails along paths starting from x and reaching y for the first time is equal to $F(x,y|\lambda)$. If x is equal to y we call them first generation trails in x. Since $G(x,x|\lambda) = 1/(1 - F(x,x|\lambda))$ and the radius R of G does not depend on the choice of x, we have that $R = \max\{\lambda : F(x,x|\lambda) \leq 1\}$.

Theorem 4.1. *For the modified BRW $\lambda_w = 1$ and if $\lambda = 1$ there is global extinction almost surely. Moreover $\lambda_s = R$ and if $\lambda = R$ there is local extinction almost surely.*

Proof. The total number of particles T_t is a branching process with rate λ, whence the claim for λ_w follows. As for the second claim, the proof is the same as in Theorems 3.1 and 3.11 using F instead of Φ. □

The following Corollary is the analog of Theorem 3.13 (see [2] for the definition of *strongly reversible random walk*).

Corollary 4.2. *For the modified BRW, the existence of a pure weak phase is equivalent to $R > 1$. If P is a strongly reversible random walk then the existence of the pure weak phase is equivalent to nonamenability.*

Proof. The result is a simple consequence of Theorem 4.1 and the main theorem of [2]. □

4.2 BRW on weighted graphs

Our methods apply, with minor modifications, to more general BRWs, which generalize simultaneously BRWs on multigraphs and modified BRWs.

Let us consider (X,N) where X is a countable (or finite) set and $N = (n_{xy})_{x,y\in X}$ is a matrix of nonnegative *weights* (that is, $n_{xy} \geq 0$) such that $\sup_{x\in X} \sum_{y\in X} n_{xy} = M < \infty$. We suppose that N is irreducible in the sense that $(X, E(X))$, where $E(X) := \{(x,y) \in X \times X : n_{xy} > 0\}$, is a connected graph. We call (X,N) a *weighted graph*.

The λ-BRW is defined by setting the reproduction rate on every edge (x,y) as λn_{xy}; hence, to each path $\{x_0, \ldots, x_n\}$ there corresponds a weight $\prod_{i=0}^{n-1} n_{x_i x_{i+1}}$. We define $\gamma^n_{x,y}$, T^n_x, $\phi^n_{x,y}$, M_s and M_w as in Section 2.1.

It is clear that the BRW on multigraphs and the modified (according to an irreducible random walk) BRW may be viewed as BRWs on weighted graphs. Moreover the expected number of trails along a path $\{x_0, x_1, \ldots, x_n\}$ is $\lambda^n \prod_{i=0}^{n-1} n_{x_i x_{i+1}}$. Substituting the word "multigraphs" with "weighted graphs" all the results of Sections 2 and 3 still hold (with the exception of Theorems 3.4 and 3.5) with unimportant modifications. In particular, extending Definition 3.6 verbatim to weighted graphs, one can prove Theorem 3.9, since in this case n_{xy} may take just a finite number of values and it is possible to apply Lemma 2.3 as we did in Theorem 3.4. Clearly, one extends the results on the critical behaviors (Theorem 3.11) to this case. For *regular* weighted graphs (that is, $\sum_{y \in X} n_{xy} = M$ for all $x \in X$) one proves results analogous to the ones of Section 4.1.

5 Open questions

As we stated in Section 1, this paper is motivated by three main issues: the identification of the critical value λ_w, the behavior of the process when $\lambda = \lambda_s$ or $\lambda = \lambda_w$ and the existence of the pure weak phase.

To complete the first point one should verify whether the equality $\lambda_w = 1/M_w$ holds for every multigraph or if M_w characterizes the critical value λ_w only on a restricted class of multigraphs.

As for the second one, the open question is the following: is it possible to construct a multigraph where if $\lambda = \lambda_w$ the process does not die out globally? In particular, is it possible to find a multigraph where $\lambda_s = \lambda_w$ but the λ_w-BRW does not die out globally (it certainly does locally)?

Finally, dealing with the existence of a pure weak phase, it is well known that there is no equivalence, in general, with nonamenability. We proved that this equivalence holds, for instance, for the class of non-oriented $\mathcal{F}$-multigraphs; we do not know what can be said in the case of oriented $\mathcal{F}$-multigraphs. To be precise: is there a nonamenable, oriented $\mathcal{F}$-multigraph, where the BRW has no weak phase? On the other hand, is it possible to find an amenable, oriented $\mathcal{F}$-multigraph where $\lambda_s = \lambda_w$?

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