THE EVALUATION OF AMERICAN OPTIONS IN A STOCHASTIC VOLATILITY MODEL WITH JUMPS: A FINITE ELEMENT METHOD APPROACH

Luca Vincenzo Ballestra
Dipartimento di Scienze Sociali "D. Serrani",
Universita' Politecnica delle Marche,
Piazza Martelli 8, 60121 Ancona, Italy,
Ph. N. +39-071-2207251, FAX N. +39-071-2207150,
E-mail: l.v.ballestra@univpm.it

Carlo Sgarra *
Dipartimento di Matematica "F. Brioschi",
Politecnico di Milano,
Piazza Leonardo Da Vinci 32, 20133 Milano, Italy,
Ph. N. +39-02-23994570, FAX N. +39-02-23994621,
E-mail: carlo.sgarra@polimi.it

Abstract

In the present paper we consider the problem of pricing American options in the framework of a well-known stochastic volatility model with jumps, the Bates model. According to this model the asset price is assumed to follow a jump-diffusion equation in which the jump term consists of a Lévy process of compound Poisson type, while the volatility is modeled as a CIR-type process correlated with the asset price. In this model the American option valuation is reduced to a final-free-boundary-value partial integro-differential problem. Using a Richardson extrapolation technique this problem is reduced to a partial integro-differential problems with fixed boundary. Then the transformed problem is solved using an ad-hoc finite element method which efficiently combines an operator splitting technique with a non-uniform mesh of right-angled triangles. Numerical experiments are presented showing that the option pricing algorithm developed in this paper is very accurate and fast. In particular it is significantly faster than the numerical method recently proposed by C. Chiarella et al. (2008).

*Corresponding Author

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1 Introduction

A huge effort has been made in the last few years in order to overcome the intrinsic limitations and drawbacks of the Black-Scholes model. Although it has been a great success as a first attempt to provide an evaluation for financial derivatives, it was soon clear that its description of the financial market behavior is not satisfactory. In particular the empirical probability distribution of log-asset returns often exhibits features that are not taken into account by the Black-Scholes model: heavy tails, volatility clustering, aggregational Gaussianity are some peculiarities that cannot be explained on the basis of the log-normal assumption on which the Black-Scholes model stands. The volatility smile is another relevant phenomenon that cannot be explained on the basis of a Black-Scholes description. Several different approaches have been exploited in order to give a more satisfactory description of financial markets, but the main contributions in this direction can be grouped in two different classes of models, the models with stochastic volatility (or stochastic volatility models) and the models with jumps. An extended literature is available on both these approaches: in particular, they give a more realistic description of the prices evolution in financial markets, however, if considered separately, they perform significantly well only in some situations. For example, while models with jumps can successfully reproduce the volatility smiles on short term maturity ranges, stochastic volatility models give a better description of the same phenomenon on long maturity terms. This has naturally led to the introduction of more complex, but more realistic models in which both features of stochastic volatility and jumps can be present. The three more popular models in which the integration of jumps and stochastic volatility has been performed are the BNS model introduced by Barndorff-Nielsen and Shepard (2001a, 2001b), the model introduced by Bates (1996), and the time-changed Lévy models introduced by Carr, Madan, Geman and Yor (2003). While in the former the volatility dynamics is driven by a positive Lévy process correlated with the jump process which drives the log-asset price, in the latter the volatility dynamics is governed by a time-changed Lévy process. In the present work we shall concentrate on the second model we have just mentioned, the Bates model, in which a Merton jump-diffusion model is combined with a stochastic volatility model of the Heston type. As Cont and Tankov (2004) have pointed out, the time changed Lévy models can fit observed option prices much better than the BNS model: in the BNS model, in fact, the implied volatility patterns are restricted by the requirement that the same correlation parameter characterize the returns both for short and long maturities; on the other hand the capability of the Bates model to calibrate to realized market prices are comparable to those of the time changed Lévy processes, "Thus the Bates model appears to be at the same time the simplest and the most flexible of the models" (R.Cont and P. Tankov (2004)

pag. 495).

In the financial literature several numerical approaches have been proposed for pricing options under both stochastic volatility models and models with jumps. In particular, in the case of models with jumps, we mention the paper by Matache, Nitsche and Schwab (2005), who propose a finite element method to price options on Lévy driven assets, and the paper by R. Cont, E. Voltchkova (2005), where a finite-difference scheme is introduced. Moreover, concerning stochastic volatility models, finite element approximation techniques are developed by Achdou and Tchou (2002) and by Hilber, Matache and Schwab (2005), while finite-difference schemes are proposed by Ikonen and Toivanen (2004), by Ito and Toivanen (2008), and by Clarke and Parrot (1999).

As far as numerical techniques for the Bates model are concerned, only few results are available. In particular, a finite element method for European Call and Put options is developed in E. Miglio and C. Sgarra (2008), in order to establish a basis for pricing more complex exotic products. Instead, concerning the case of American options, a numerical approach to the Bates model has recently been presented by Chiarella et al. (2008). In particular, in this work, using the method of lines the problem of pricing American options is reduced to a system of second-order ordinary differential equations. However, due to the early exercise feature, the differential problem obtained is a free boundary problem and must be solved using a complex iteration procedure.

In this paper we deal with the problem of pricing American options on an underlying described by the Bates model. We start by noticing that this problem poses severe difficulties: first of all the early exercise feature strongly complicates the problem from the mathematical standpoint and makes it impracticable to use a method based on the knowledge of the characteristic function (see for example Carr and Madan (1998)). Furthermore a fully implicit time discretization scheme, that would guarantee unconditional stability, cannot be applied. In fact such an approach, due to presence of the jump integral, leads to a dense linear system of equations, whose numerical approximation is very time consuming and requires large memory storage; finally the fact that the option payoff has discontinuous derivative may produce severe losses of accuracy if a finite difference/finite element discretization is employed.

In this paper the American option price is computed by Richardson extrapolation of the prices of two Bermudan options. Such an approach requires to solve only partial differential problems with fixed boundary and hence is computationally simpler than that followed in Chiarella et al. (2008). The Bermudan option pricing problems obtained are solved using a finite element method based on a non-uniform (stretched) mesh of right-angled triangles. This kind of discretization, used in conjunction with an operator splitting technique, allows to obtain linear systems of equations that can be solved very efficiently.

Numerical experiments are presented showing that the numerical method proposed in this paper is accurate and computationally very fast. In fact, if the simulations are carried out on a computer with a Pentium Dual Core E 2140 Processor 1.6 GHz 2 GB Ram, the American option price is obtained with relative error of order 10^{-4} and 10^{-5} in a time equal or smaller than $24\,s$.

The experiments performed also reveal that the numerical method proposed in this paper is approximately a thousand of times faster than that presented in Chiarella et al. (2008).

We point out that the numerical techniques employed in this paper (the Richardson extrapolation, the operator splitting, the finite element method), considered separately, are not new. Nevertheless, to the best of our knowledge, an approach that puts all these techniques together has never been proposed in mathematical finance to solve complex models and it appears to be interesting in itself. In fact it is thanks to the proper combination of all the above techniques that an accurate and computationally fast approximation of the Bates model can be obtained.

The paper is organized as follows: in Section 2 we recall the basic facts about the Bates model, while in Section 3 we present the free-boundary partial integro-differential problem that must be solved in order to price American options; in Section 4 we describe the finite element method used to solve the problem presented in Section 3. In Section 4 we expose and discuss the numerical results obtained.

2 The Bates model

The Bates model combines a stochastic volatility dynamics with jumps in the asset price. While the former is assumed to follow a CIR-type evolution, in which a mean-reversion behavior is present, the latter are assumed to be described by means of a compound Poisson process, i.e. a Lèvy process with finite activity.

In the Bates model the asset price evolution is then given by:

$$S_t = S_0 e^{X_t}, \tag{1}$$

where the log-returns X and its volatility Y satisfy the following stochastic differential equations:

$$dX_t = (\alpha - \frac{1}{2}Y_t)dt + \sqrt{Y_t}dW_t^1 + dZ_t, \quad X_0 = 0. \tag{2}$$

$$dY_t = \xi(\eta - Y_t)dt + \theta\sqrt{Y_t}dW_t^2, \quad Y_0 = y_0, \tag{3}$$

with $y_0 > 0$. Let's assume for the parameters the following restrictions:

$$\alpha \in \mathbb{R}, \quad -1 \leq \rho \leq 1, \quad \xi \geq 0, \quad \eta \geq 0, \quad \theta \geq 0 \tag{4}$$

Moreover, if the following condition is satisfied:

$$\theta^2 < 2\xi\eta, \tag{5}$$

the volatility process Y never hits zero (see Feller (1951)).

For a Lévy process the cumulant function $\kappa(z)$ is defined as follows:

$$\kappa(z) := \frac{1}{t} \log E\left[e^{izZ_t}\right]. \tag{6}$$

Since the Lévy-Kintchine representation holds, the cumulant function can be written in the following way:

$$\kappa(z) = -\frac{1}{2}Az^2 + i\beta z + \int_{-\infty}^{+\infty} (e^{izx} - 1 - izxh(x))U(dx), \tag{7}$$

where β is the drift of the process, A the quadratic variation component, and h the truncation function. $U(dx)$ is the Lévy measure of Z. The choice of the truncation function h and the drift coefficient β are strongly interconnected. A usual choice of h is the following $h(x) := \mathbf{1}_{|x|\leq 1}$. In our case $A = 0$. We'll assume moreover $E[Z_1^2] < \infty$, this implying that the cumulant function of the process Z will be of the following type:

$$\kappa(z) = i\zeta z + \int_{-\infty}^{+\infty} (e^{izx} - 1 - izx)U(dx) \tag{8}$$

where $\zeta = E[Z_1]$, and its relation with β is given by: $\zeta = \beta + \int_{|x|\geq 1} xU(dx)$. We'll denote by $\mu(dx, dt)$ the jump measure of Z and by $\nu(dt, dx)$ its predictable compensator. We'll have moreover $\nu(dx, dt) = U(dx)dt$. By the Lévy-Ito decomposition specified for compound Poisson processes, we can write then:

$$Z_t = \zeta t + \int_0^t \int_{-\infty}^{+\infty} x(\mu - \nu)(dx, ds). \tag{9}$$

Lemma 1 *The dynamics of the asset price process is given by*

$$dS_t = (\alpha + \kappa(1))S_{t-}dt + S_{t-}\sqrt{Y_t}dW_t^1 + \int_{-\infty}^{+\infty} S_{t-}(e^x - 1)(\mu - \nu)(dx, dt). \tag{10}$$

In particular if

$$\alpha + \kappa(1) = 0 \tag{11}$$

the process S is a local martingale.

Proof: This follows immediately from Itô's formula for general semimartingales applied to the asset price model with the dynamics described by (1), (2) and from the Lévy-Ito representation formula for compound Poisson processes (9). □

Remark 2 *In the original model proposed by Bates (1996), the process Z is a compound Poisson process,*

$$Z_t = \sum_{i=1}^{N_t} J_i, \tag{12}$$

where N is a standard Poisson process with intensity $\lambda > 0$ and $(J_i)_{i\geq 1}$ are iid $N(\gamma, \delta^2)$, with $\gamma = \ln(1+\kappa(1)/\lambda) - \delta^2/2$. The corresponding cumulant function is in that case

$$\kappa(z) = \lambda(e^{\gamma z + \delta^2 z^2/2} - 1). \tag{13}$$

Remark 3 *If $Z = 0$ then we obtain Heston's stochastic volatility model from Heston (1993). If $\theta = 0$ and $Y_t = \eta$ we obtain Merton's jump-diffusion model with lognormal jumps in the asset price, Merton (1976) . Consequently we might consider the Bates model as an extension of a Merton jump-diffusion model with stochastic volatility, or as an extension of a Heston volatility model with jumps in the returns.*

As in other affine stochastic volatility models with and without jumps, it is possible to obtain the characteristic function of the log-price in closed form. This characteristic function has been calculated by D. Bates (1996); a detailed computation is provided also in Cont and Tankov (2004); when the jumps are lognormally distributed, it is given by the following expression:

$$\begin{aligned}\Phi_t(u) &= \exp\left[t\lambda(e^{-\delta^2 u^2/2 + i(\ln(1+\kappa(1)/\lambda) - \delta^2/2)u} - 1\right] \\ &\left[\cosh\frac{\varepsilon t}{2} + \frac{\xi - i\rho\theta u}{\varepsilon}\sinh\frac{\varepsilon t}{2}\right]^{-2\xi\eta/\theta^2} \times \exp\left[-\frac{(u^2+iu)y_0}{\varepsilon\coth\frac{\varepsilon t}{2} + \xi - i\rho\theta u}\right]\end{aligned} \tag{14}$$

where:

$$\varepsilon = \sqrt{\theta^2(u^2+iu) + (\xi - i\rho\theta u)^2} \tag{15}$$

Once the characteristic function of the log-price process is known in a closed form, the valuation problem for vanilla options can be easily solved by an FFT-related technique like that provided in the paper by Carr and Madan (1998). Nevertheless we point out that such an approach cannot be extended in a straightforward way to price American options. Although some attempts have been made in this direction, an efficient generalization of the FFT technique to American option valuation in a Bates model framework is not yet available.

3 The partial integro-differential approach

Following the usual derivation based on Ito's lemma and no arbitrage requirement, and introducing the market price of risk π associated to the volatility dynamics, and the dividend yield q, we obtain the following partial integro-differential equation for the price of a European Call option $C(S, y, t)$ on an underlying described by the Bates model :

$$\begin{aligned}&\frac{\partial C}{\partial t} + (r - q - \kappa(1))S\frac{\partial C}{\partial S} + \frac{1}{2}yS^2\frac{\partial^2 C}{\partial S^2} + [\xi(\eta - y) - \pi]\frac{\partial C}{\partial y} + \frac{1}{2}\theta^2 y\frac{\partial^2 C}{\partial y^2} + \\ &+\rho\theta yS\frac{\partial^2 C}{\partial y \partial S} + \lambda\int_{-\infty}^{+\infty}\left[C(Se^x, y, t) - C(S, y, t)\right]W(dx) = rC\end{aligned} \tag{16}$$

where:

$$W(dx) = \frac{1}{\delta\sqrt{2\pi}} \exp\left[-\frac{(x-\gamma)^2}{2\delta^2}\right], \tag{17}$$

with the following final condition at $t = T$:

$$C(S_T, y_T, T) = \max\left[S_T - K, 0\right] \tag{18}$$

and the following boundary conditions in S:

$$C(0, y, t) = 0, C(S, y, t) = S - K, \text{ as } S \to +\infty \tag{19}$$

The Gaussian density (17) with respect to which the integration in (16) is performed follows by our assumption of lognormal jumps. We just recall from Remark 1 that γ, δ^2 are related to the cumulant function introduced before by the following relation: $\kappa(1) = \lambda\left(e^{\gamma-\delta^2/2} - 1\right)$.

In the European case the variables S, Y, t can assume values in the following domains: $S \in [0, +\infty)$, $t \in [0, +\infty)$, $y \in [0, +\infty)$.

For American Call options, some constraints should be imposed in order to avoid arbitrage. In particular we must require that the option price never falls under the payoff, and we must prescribe a "value matching" and a "smooth pasting" conditions at the free boundary. Therefore, in the American case, the boundary conditions in the asset domain become:

$$C(0, y, t) = 0, C(S_f(y,t), y, t) = S_f(y,t) - K \tag{20}$$

where $S_f(y,t)$ denotes the early exercise boundary at time t and volatility level y. Moreover we impose the following "smooth pasting" conditions (see Chiarella et al. (2008)):

$$\lim_{S \to S_f(y,t)} \frac{\partial C}{\partial S} = 1, \lim_{S \to S_f(y,t)} \frac{\partial C}{\partial y} = 0 \tag{21}$$

Remark 4 *Looking at the partial differential problem (16), (19), we may note that no boundary condition has been prescribed at $y = 0$ and $y \to +\infty$. In fact the partial differential equation (16) is singular at $y = 0$ and $y \to +\infty$ and it is not clear which boundary conditions to apply. The most insightful result on this subject has been obtained by Feller (1951), who shows that at $y = 0$ a boundary condition should be imposed only if the condition (5) is not satisfied. However to the best of our knowledge a thorough investigation on the boundary conditions at $y = 0$ and $y \to +\infty$ is still lacking. In this paper following a common approach (see for instance Chiarella et al. (2008)) we will circumvent the problem by extrapolating the solution at $y = 0$ and $y \to +\infty$ from the numerical solution obtained in the interior of the (S, y) computational domain (see Section 4).*

Remark 5 *The market price of risk π related to the stochastic volatility dynamics can be obtained in different ways in the frame of general equilibrium models; consumption-based models give a risk premium proportional to y. In the following, for the sake of simplicity, we'll assume without any loss of generality that the market price of risk associated to the volatility is zero.*

Remark 6 *The market price of risk associated to the jumps has been considered equal to zero. In practice we have assumed that passing from the objective measure to the risk-neutral measure leaves the jump probability distribution unchanged. This assumption, although questionable from a financial viewpoint, is usually made when pricing options on an underlying described by models with jumps.*

We can easily obtain in a similar way the formulation of the valuation problem for an American Put option: the partial integro-differential equation will be as follows:

$$\frac{\partial P}{\partial t} + (r - q - \kappa(1))S\frac{\partial P}{\partial S} + \frac{1}{2}yS^2\frac{\partial^2 P}{\partial S^2} + [\xi(\eta - y) - \pi]\frac{\partial P}{\partial y} + \frac{1}{2}\theta^2 y\frac{\partial^2 P}{\partial y^2} + \quad (22)$$

$$+\rho\theta yS\frac{\partial^2 P}{\partial y \partial S} + \lambda\int_{-\infty}^{+\infty}\left[P(Se^x, y, t) - P(S, y, t)\right]W(dx) = rC. \quad (23)$$

The final condition is now:

$$P(S_T, y_T, T) = \max\left[K - S_T, 0\right], \quad (24)$$

while the boundary conditions in S, y are:

$$P(S_f(y,t), y, t) = K - S_f(y,t), P(S, y, t) = 0, \text{ as } S \to \infty \quad (25)$$

$$\lim_{S \to S_f(y,t)}\frac{\partial P}{\partial S} = -1, \lim_{S \to S_f(y,t)}\frac{\partial P}{\partial y} = 0 \quad (26)$$

4 The Numerical Method

For ease of exposition we will limit our attention to the case of American Call options. However the reader will note that the numerical method presented in this Section can be used with little modifications also to price American Put options (an American Put option will be considered in one of the test-cases presented in Section 5). For the sake of simplicity the dividends are assumed to be paid continuously in time and at constant dividend yield, but generalizations to more complex dividend structures are straightforward. Our starting point will then be the following partial integro-differential equation:

$$\frac{\partial C}{\partial t} + \frac{1}{2}yS^2\frac{\partial^2 C}{\partial S^2} + \frac{1}{2}\theta^2 y\frac{\partial^2 C}{\partial y^2} + \rho\theta yS\frac{\partial^2 C}{\partial y \partial S} + (r - q - \lambda\kappa(1))S\frac{\partial C}{\partial S}$$
$$+[\xi(\eta - y)]\frac{\partial C}{\partial y} + \lambda\int_{-\infty}^{+\infty}C(Se^x, y, t)W(dx) = (r + \lambda)C, \quad (27)$$

where $W(dx)$ is given by (17).
Equation (27) can be rewritten as follows:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial}{\partial S}\left(yS^2\frac{\partial C}{\partial S}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\theta^2 y\frac{\partial C}{\partial y}\right) + \frac{1}{2}\rho\theta\frac{\partial}{\partial S}\left(yS\frac{\partial C}{\partial y}\right)$$
$$+\frac{1}{2}\rho\theta\frac{\partial}{\partial y}\left(yS\frac{\partial C}{\partial S}\right) + \left(r - q - \lambda\kappa(1) - y - \frac{1}{2}\rho\theta\right)S\frac{\partial C}{\partial S}$$
$$+\left[\xi(\eta - y) - \frac{1}{2}\theta^2 - \frac{1}{2}\rho\theta y\right]\frac{\partial C}{\partial y} + \lambda\int_{-\infty}^{+\infty} C(Se^x, y, t)W(dx) = (r+\lambda)C. \quad (28)$$

It is convenient to rewrite equation (28) in the compact form:

$$\frac{\partial C}{\partial t} + \mathcal{L}C = (r+\lambda)C, \quad (29)$$

where

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_6, \quad (30)$$

and

$$\mathcal{L}_1 C = \frac{1}{2}\frac{\partial}{\partial S}\left(yS^2\frac{\partial C}{\partial S}\right), \quad (31)$$

$$\mathcal{L}_2 C = \frac{1}{2}\frac{\partial}{\partial y}\left(\theta^2 y\frac{\partial C}{\partial y}\right), \quad (32)$$

$$\mathcal{L}_3 C = \frac{1}{2}\rho\theta\frac{\partial}{\partial S}\left(yS\frac{\partial C}{\partial y}\right) + \frac{1}{2}\rho\theta\frac{\partial}{\partial y}\left(yS\frac{\partial C}{\partial S}\right), \quad (33)$$

$$\mathcal{L}_4 C = \left(r - q - \lambda\kappa(1) - y - \frac{1}{2}\rho\theta\right)S\frac{\partial C}{\partial S}, \quad (34)$$

$$\mathcal{L}_5 C = \left[\xi(\eta - y) - \frac{1}{2}\theta^2 - \frac{1}{2}\rho\theta y\right]\frac{\partial C}{\partial y}, \quad (35)$$

$$\mathcal{L}_6 C = \lambda\int_{-\infty}^{+\infty} C(Se^x, y, t)W(dx). \quad (36)$$

For the sake of clarity this Section is divided in 4 subsections: in Subsection 4.1 equation (29) is discretized in time. In particular it is shown how to take into account the early exercise feature. In Section 4.2 the finite element approximation of the differential operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ and $\mathcal{L}_5$ is carried out. Finally in Subsection 4.3 the discretization of the integral operator $\mathcal{L}_6$ is performed.

4.1 Time discretization

Following a common approach (see Lord et al. (2008) and references therein), the American option price is approximated by Richardson extrapolation of the prices of two Bermudan options. This is a very simple technique to take into account the early exercise feature, as the final-free-boundary-value partial integro-differential problem typical of American option pricing is reduced to a final-boundary-value partial integro-differential problem with fixed boundary.

In the interval $[0,T]$ let us consider N_t+1 equally spaced time levels $t_0=0$, t_1, t_2, ..., $t_{N_t}=T$. Moreover let us define $\Delta t = t_k - t_{k-1}$, $k=1,2,\ldots,N_t$. Let $C_{N_t}(S,y,t)$ denote the price of a Bermudan option with maturity T and exercise dates t_k, $k=0,1,\ldots,N_t$. The Bermudan option price $C_{N_t}(S,y,t)$ is obtained using the following recursion procedure: first of all set $k := N_t$ and define

$$\Psi_{N_t}(S,y,t_k) = \max\left[S-K,0\right]. \tag{37}$$

Then solve the final-boundary value partial differential problem

$$\frac{\partial C_{N_t}}{\partial t} + \mathcal{L}C_{N_t} = (r+\lambda)C_{N_t}, \qquad t\in[t_{k-1},t_k), \tag{38}$$

$$C_{N_t}(0,y,t)=0, \quad C_{N_t}(S,y,t) = S-E \ \text{ as } S\to+\infty, \tag{39}$$

$$C_{N_t}(S,y,t_k) = \Psi_{N_t}(S,y,t_k). \tag{40}$$

Then define

$$\Psi_{N_t}(S,y,t_{k-1}) = \max\left[C_{N_t}(S,y,t_{k-1}), S-K\right], \tag{41}$$

and set

$$k := k-1. \tag{42}$$

Repeat the cycle (38)-(42) and stop when $k:=0$.

The Bermudan option price $C_{N_t}(S,y,t)$ tends to become a fair approximation of the American option price $C(S,y,t)$ as the number of exercise dates N_t increases. In this paper the accuracy of $C_{N_t}(S,y,t)$ is enhanced by Richardson extrapolation:

$$C(S,y,t) \simeq 2C_{2N_t}(S,y,t) - C_{N_t}(S,y,t), \tag{43}$$

which is second-order accurate in time (see for instance Chung et al. (2008)).

Now let us describe the numerical method to solve the final-boundary value partial differential problem (38)-(40). Let $U^k(S,y)$ denote a function approximating $C_{N_t}(S,y,t_k)$, $k=0,1,\ldots,N_t-1$, and let $\Psi^k(S,y)$ denote a function approximating $\Psi_{N_t}(S,y,t_k)$, $k=1,2,\ldots,N_t$. Note that the subscript N_t has been removed from $U^k(S,y)$ and $\Psi^k(S,y)$ to keep the notation simple. According to (37) we set $\Psi^{N_t}(S,y)=\max[S-K,0]$.

The final-boundary value partial differential problem (38)-(40) is discretized in time using the following one-step implicit/explicit finite difference scheme:

$$\frac{\Psi^k-U^{k-1}}{\Delta t}+\mathcal{L}_1U^{k-1}+\mathcal{L}_2U^{k-1}+\mathcal{L}_3\Psi^k+\mathcal{L}_4\Psi^k+\mathcal{L}_5\Psi^k+\mathcal{L}_6U^{k-1} = (r+\lambda)U^{k-1}, \quad (44)$$

$$U^{k-1}(0,y)=0, \quad U^{k-1}(S,y)=S-E \text{ as } S\to+\infty, \quad (45)$$

which allows to compute $U^{k-1}(S,y)$ given $\Psi^k(S,y)$. Note that the operator $\mathcal{L}_1$, which contains the second-order derivative with respect to S, and the operator $\mathcal{L}_2$, which contains the second-order derivative with respect to y, are treated implicitly. This choice is crucial to achieve stability also for large values of Δt. On the contrary the hyperbolic operators $\mathcal{L}_3$, $\mathcal{L}_4$ and $\mathcal{L}_5$ are treated explicitly, in order to enhance the speed of the numerical calculations. This choice could cause some numerical instability for large values of Δt when the weight of the hyperbolic terms $\mathcal{L}_3$, $\mathcal{L}_4$ and $\mathcal{L}_5$ is bigger than that of the diffusive terms $\mathcal{L}_1$ and $\mathcal{L}_2$. Nevertheless we have performed several numerical experiments where we have considered various types of Call and Put option contracts and we have used different reasonable values of the parameters of the Bates model. In all these simulations the numerical scheme (44) has always proven to be stable, also for large values of Δt, which indicates that the Bates model is mainly dominated by diffusion.

Finally we note that in equation (44) the integral operator $\mathcal{L}_6$ is treated implicitly, in order to enhance stability. In the following we will show that this choice, thanks to an appropriate discretization of the jump integral term, does not significantly increase the computer time necessary for the calculations.

In order to efficiently solve the boundary value partial differential problem (44)-(45) let us apply the following operator splitting technique:

$$\frac{\Psi^k-V^{k-1}}{\Delta t}+\mathcal{L}_1V^{k-1}+\mathcal{L}_3\Psi^k+\mathcal{L}_4\Psi^k+\mathcal{L}_5\Psi^k=rV^{k-1}, \quad (46)$$

$$V^{k-1}(0,y)=0, \quad V^{k-1}(S,y)=S-E \text{ as } S\to+\infty, \quad (47)$$

$$\frac{V^{k-1}-W^{k-1}}{\Delta t}+\mathcal{L}_2W^{k-1}=0, \quad (48)$$

$$W^{k-1}(0,y)=0, \quad W^{k-1}(S,y)=S-E \text{ as } S\to+\infty, \quad (49)$$

$$\frac{W^{k-1} - U^{k-1}}{\Delta t} + \mathcal{L}_6 U^{k-1} = \lambda U^{k-1}, \tag{50}$$

$$U^{k-1}(0, y) = 0, \quad U^{k-1}(S, y) = S - E \text{ as } S \to +\infty. \tag{51}$$

That is first of all the function $V^{k-1}(S, y)$ is obtained solving (46)-(47), then the function $W^{k-1}(S, y)$ is obtained solving (48)-(49), and finally the function $U^{k-1}(S, y)$ is obtained solving (50)-(51). As will be clear in Subsection 4.2 and Subsection 4.3 the splitting scheme (46)-(51) will allow us to obtain linear systems of equations that can be solved very quickly .

Remark 7 *The scheme (46)-(51) is only first-order accurate in time (see W.H. Hundsdorfer and J.G. Verwer (2003)). Nevertheless, the $O(\Delta t)$ component of the error is suppressed thanks to the Richardson extrapolation (43), so that the overall numerical method is second-order accurate in time.*

For the sake of simplicity, in the following, with reference to the boundary value partial differential problems (46)-(47), (48)-(49), (50)-(51), we will write $\Psi^k(S, y)$, $V^{k-1}(S, y)$, $W^{k-1}(S, y)$, $U^{k-1}(S, y)$ instead of $\Psi(S, y)$, $V(S, y)$, $W(S, y)$, $U(S, y)$, respectively.

4.2 The finite element method

The partial differential problems (46)-(47) and (48)-(49) are solved using a finite element method based on a non-uniform (stretched) mesh of right-angled triangles (see Figure 1). This approach, used in conjunction with the operator splitting technique (46)-(51), gives the following computational advantages:
1) the problems (46)-(47) and (48)-(49) are reduced to tridiagonal systems of linear equations, which can be solved very quickly;
2) the problem (50)-(51) can be solved very efficiently by exploiting the fact that $W^{k-1}(S, y)$ is obtained on a rectangular grid (see the next subsection);
3) the accuracy of the numerical solution can be improved by refining the mesh in certain regions of the $(S - y)$ plane. In particular, along the $S-$direction, it is crucial to use a refined mesh in a neighborhood of the strike price, where the derivative of the option payoff is discontinuous. Moreover, along the $y-$direction, it is convenient to use a larger number of nodes in a neighborhood of y_0, where the possible realizations of the stochastic variance are more likely to occur. Finally it should also be noticed that on a non-uniform mesh the finite element method is particularly suitable and allows to reach a higher level of accuracy than the finite difference method (see for example Hirsch (1998), Quarteroni and Valli (1994)).

The finite element approximation is carried out as follows. First of all the spatial domain of the partial differential equations (46) and (48), which consists of the $[0, +\infty) \times [0, +\infty)$ quarter of plane, is replaced with the finite domain $\Omega = [0, S_{\max}] \times [0, y_{\max}]$, where $S_{\max}$ and $y_{\max}$ are chosen sufficiently large (such

that the possible realizations of S and y are contained in Ω with probability close to one). In the interval $[0, S_{\max}]$ let us consider N_S nodes $S_1 = 0$, S_2, S_3, ..., $S_{N_S} = S_{\max}$, and, in the interval $[0, y_{\max}]$, let us consider N_y nodes $y_1 = 0$, y_2, y_3, ..., $y_{N_y} = y_{\max}$. Moreover let us define $\Delta S_i = S_{i+1} - S_i$, $i = 1, 2, \ldots, N_S - 1$, and $\Delta y_j = y_{j+1} - y_j$, $j = 1, 2, \ldots, N_y - 1$. Let $T^1_{i,j}$ denote the right-angled triangle with vertexes (S_i, y_j), (S_{i+1}, y_j), (S_{i+1}, y_{j+1}) and let $T^2_{i,j}$ denote the right-angled triangle with vertexes (S_i, y_j), (S_{i+1}, y_{j+1}), (S_i, y_{j+1}), $i = 1, 2, \ldots, N_s - 1$, $j = 1, 2, \ldots, N_y - 1$ (see Figure 1). The finite element mesh $\mathcal{T}$ is defined as the union of all the triangles $T^1_{i,j}, T^2_{i,j}$, $i = 1, 2, \ldots, N_S - 1$, $j = 1, 2, \ldots, N_y - 1$. Moreover let $\Omega_{i,j}$ denote the set of all the triangles with a vertex at (S_i, y_j), that is $\Omega_{i,j} = \left\{T^1_{i-1,j-1}, T^2_{i,j-1}, T^1_{i,j}, T^2_{i,j}, T^1_{i-1,j}, T^2_{i-1,j-1}\right\}$, $i = 2, 3, \ldots, N_S - 1$, $j = 2, 3, \ldots, N_y - 1$ (see Figure 1).

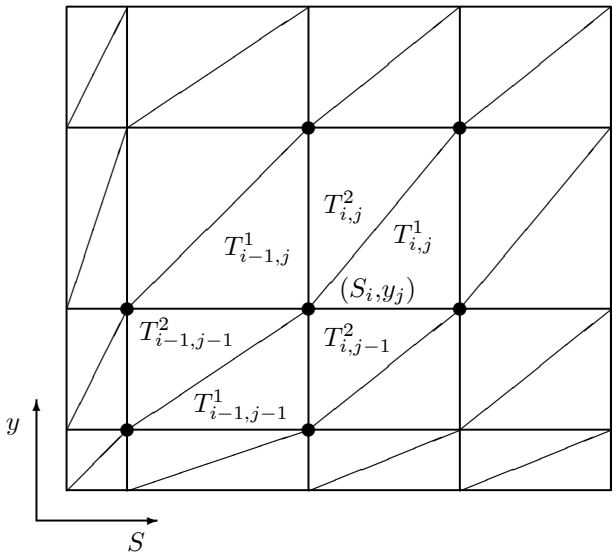

Figure 1: Finite element mesh

We want to approximate $V(S, y)$ and $W(S, y)$ using piecewise continuous linear functions. Precisely let $\phi_{i,j}(S, y)$, $i = 1, 2, \ldots, N_s$, $j = 1, 2, \ldots, N_y$, denote a set of trial functions defined as follows: $\phi_{i,j}(S, y)$ is continuous on Ω, piecewise linear on each triangle of $\mathcal{T}$, equal to one at the node (S_i, y_j) and vanishes at all the other nodes, $i = 1, 2, \ldots, N_s$, $j = 1, 2, \ldots, N_y$. Note that the functions $\phi_{i,j}(S, y)$, $i = 1, 2, \ldots, N_s$, $j = 1, 2, \ldots, N_y$, are the so-called *hat* functions that are often employed in finite element analysis (see Quarteroni and Valli (1994), Strang and Fix (1993)).

The functions $W(S, y)$, $V(S, y)$ and $\Psi(S, y)$ are approximated as follows:

$$\Psi(S,y) \simeq \sum_{i=1}^{N_S}\sum_{j=1}^{N_y} \Psi_{i,j}\phi_{i,j}(S,y), \tag{52}$$

$$V(S,y) \simeq \sum_{i=1}^{N_S}\sum_{j=1}^{N_y} V_{i,j}\phi_{i,j}(S,y), \tag{53}$$

$$W(S,y) \simeq \sum_{i=1}^{N_S}\sum_{j=1}^{N_y} W_{i,j}\phi_{i,j}(S,y). \tag{54}$$

First of all let us show how to solve problem (46)-(47). We recall that $\Psi_{i,j}$, $i = 1, 2, \ldots, N_s$, $j = 1, 2, \ldots, N_y$, must be considered as known quantities, since they are computed at the previous iteration of the cycle (38)-(42).

We multiply equation (46) by $\phi_{i,j}(S,y)$ and integrate over $\Omega_{i,j}$, $i = 2, 3, \ldots, N_S - 1$, $j = 2, 3, \ldots, N_y - 1$, obtaining:

$$(1 + r\Delta t)\int_{\Omega_{i,j}} V\phi_{i,j}dSdy - \Delta t\int_{\Omega_{i,j}} (\mathcal{L}_1 V)\phi_{i,j}dSdy = \int_{\Omega_{i,j}} \Psi\phi_{i,j}dSdy$$

$$+\Delta t\int_{\Omega_{i,j}} (\mathcal{L}_3\Psi)\phi_{i,j}dSdy + \Delta t\int_{\Omega_{i,j}} (\mathcal{L}_4\Psi)\phi_{i,j}dSdy + \Delta t\int_{\Omega_{i,j}} (\mathcal{L}_5\Psi)\phi_{i,j}dSdy,$$

$$i = 2, 3, \ldots, N_S - 1, j = 2, 3, \ldots, N_y - 1. \tag{55}$$

Note that in (55) we are only considering the trial functions centered at the nodes that are internal points of Ω. In fact at the boundary nodes $V(S,y)$ will be obtained by imposing suitable boundary conditions (see below).
Substituting (52) and (53) in (55), using relations (31), (33), (34), (35), and applying standard finite element techniques (see for instance Strang and Fix (1993)) the integrals appearing in (55) are approximated as follows:

$$\int_{\Omega_{i,j}} V\phi_{i,j}dSdy \simeq \frac{1}{6}\left|\Omega\right|_{i,j} V_{i,j}, \tag{56}$$

$$\int_{\Omega_{i,j}} \Psi\phi_{i,j}dSdy \simeq \frac{1}{6}\left|\Omega\right|_{i,j} \Psi_{i,j}, \tag{57}$$

$$\int_{\Omega_{i,j}} (\mathcal{L}_1 V)\phi_{i,j}dSdy \simeq \frac{1}{12}\left[a_{1,i,j}(V_{i+1,j} - V_{i,j}) - a_{2,i,j}(V_{i,j} - V_{i-1,j})\right], \tag{58}$$

$$\int_{\Omega_{i,j}} (\mathcal{L}_3\Psi)\phi_{i,j}dSdy \simeq \frac{1}{12}\left[a_{3,i,j}(\Psi_{i-1,j-1}-\Psi_{i,j-1}) + a_{4,i,j}(\Psi_{i,j}-\Psi_{i+1,j})\right.$$
$$+a_{5,i,j}(\Psi_{i+1,j+1}-\Psi_{i,j+1}) + a_{6,i,j}(\Psi_{i,j}-\Psi_{i-1,j}) + a_{4,i,j}(\Psi_{i,j}-\Psi_{i,j-1})$$
$$+a_{7,i,j}(\Psi_{i+1,j+1}-\Psi_{i+1,j}) + a_{6,i,j}(\Psi_{i,j}-\Psi_{i,j+1})$$
$$\left.+a_{8,i,j}(\Psi_{i-1,j-1}-\Psi_{i-1,j})\right], \quad (59)$$

$$\int_{\Omega_{i,j}} (\mathcal{L}_4\Psi)\phi_{i,j}dSdy \simeq \frac{1}{6}a_{9,i,j}\left[(\Delta y_{j-1}+\Delta y_j)(\Psi_{i+1,j}-\Psi_{i-1,j})\right.$$
$$\left.+\Delta y_j(\Psi_{i+1,j+1}-\Psi_{i,j+1}) + \Delta y_{j-1}(\Psi_{i,j-1}-\Psi_{i-1,j-1})\right], \quad (60)$$

$$\int_{\Omega_{i,j}} (\mathcal{L}_5\Psi)\phi_{i,j}dSdy \simeq \frac{1}{6}a_{10,i,j}\left[(\Delta S_{i-1}+\Delta S_i)(\Psi_{i,j+1}-\Psi_{i,j-1})\right.$$
$$\left.+\Delta S_i(\Psi_{i+1,j+1}-\Psi_{i+1,j}) + \Delta S_{j-1}(\Psi_{i-1,j}-\Psi_{i-1,j-1})\right], \quad (61)$$

where

$$|\Omega|_{i,j} = 2\Delta S_i\Delta y_j + 2\Delta S_{i-1}\Delta y_{j-1} + \Delta S_i\Delta y_{j-1} + \Delta S_{i-1}\Delta y_j, \quad (62)$$

$$a_{1,i,j} = (y_jS_i^2 + y_{j-1}S_i^2 + y_jS_{i+1}^2)\frac{\Delta y_{j-1}}{\Delta S_i} + (y_jS_i^2 + y_jS_{i+1}^2 + y_{j+1}S_{i+1}^2)\frac{\Delta y_j}{\Delta S_i}, \quad (63)$$

$$a_{2,i,j} = (y_jS_i^2 + y_jS_{i-1}^2 + y_{j-1}S_{i-1}^2)\frac{\Delta y_{j-1}}{\Delta S_{i-1}}$$
$$+(y_jS_i^2 + y_{j+1}S_i^2 + y_jS_{i-1}^2)\frac{\Delta y_j}{\Delta S_{i-1}}, \quad (64)$$

$$a_{3,i,j} = \rho\theta(y_jS_i + y_{j-1}S_i + y_{j-1}S_{i-1}), \quad (65)$$

$$a_{4,i,j} = \rho\theta(y_jS_i + y_{j-1}S_i + y_jS_{i+1}), \quad (66)$$

$$a_{5,i,j} = \rho\theta(y_jS_i + y_{j+1}S_{i+1} + y_{j+1}S_i), \quad (67)$$

$$a_{6,i,j} = \rho\theta(y_j S_i + y_{j+1} S_i + y_j S_{i-1}), \tag{68}$$

$$a_{7,i,j} = \rho\theta(y_j S_i + y_j S_{i+1} + y_{j+1} S_{i+1}), \tag{69}$$

$$a_{8,i,j} = \rho\theta(y_j S_i + y_j S_{i-1} + y_{j-1} S_{i-1}), \tag{70}$$

$$a_{9,i,j} = \left(r - q - \lambda\kappa(1) - y_j - \frac{1}{2}\rho\theta\right) S_i, \tag{71}$$

$$a_{10,i,j} = \xi(\eta - y_j) - \frac{1}{2}\theta^2 - \frac{1}{2}\rho\theta y_j. \tag{72}$$

In order to satisfy the boundary conditions (47) we set

$$V_{1,j} = 0, \quad V_{N_S,j} = S_{N_S} - E, \quad j = 2, 3, \ldots, N_y - 1. \tag{73}$$

Equations (55) with the substitutions (56)-(61) and the boundary conditions (73) constitute a set of $N_y - 2$ linear systems. More precisely we have one system of N_S equations in the unknowns $V_{1,2}$, $V_{2,2}$, ..., $V_{N_S,2}$, one system of N_S equations in the unknowns $V_{1,3}$, $V_{2,3}$, ..., $V_{N_S,3}$, ..., and one system of N_S equations in the unknowns V_{1,N_y-1}, V_{2,N_y-1}, ..., V_{N_S,N_y-1}. Each one of these systems is in tridiagonal form and hence can be solved very quickly using the well-known algorithm of Thomas, see Quarteroni et al. (2000) .

Now it remains to compute $V_{i,1}$ and V_{i,N_y}, $i = 1, 2, \ldots, N_S$. However, since the partial differential equation (28) is singular at $y = 0$ and $y = +\infty$, it is not clear which boundary conditions to apply at $y = 0$ and $y = y_{\max}$. In this paper, following Chiarella et al. (2008), $V_{i,1}$ and V_{i,N_y}, $i = 2, 3, \ldots, N_S - 1$, are obtained by linear extrapolation of the already computed solution on adjacent nodes:

$$V_{i,1} = V_{i,2} - \frac{V_{i,3} - V_{i,2}}{\Delta y_2}\Delta y_1, \quad i = 2, 3, \ldots, N_S - 1, \tag{74}$$

$$V_{i,N_y} = V_{i,N_y-1} + \frac{V_{i,N_y-1} - V_{i,N_y-2}}{\Delta y_{N_y-2}}\Delta y_{N_y-1}, \quad i = 2, 3, \ldots, N_S - 1. \tag{75}$$

Using relations (74) and (75) in a sense we let the partial differential equation (46) itself impose the boundary conditions at $y = 0$ and $y = y_{\max}$. Finally, according to (47) we set

$$V_{1,j} = 0, \quad V_{N_S,j} = S_{N_S} - E, \quad j = 1, N_y. \tag{76}$$

Now let us solve problem (48)-(49). We multiply equation (48) by $\phi_{i,j}(S, y)$ and integrate over $\Omega_{i,j}$, $i = 2, 3, \ldots, N_S - 1$, $j = 2, 3, \ldots, N_y - 1$. Note that, following the same approach used to discretize equation (46), we are only considering the trial functions centered at the nodes that are internal points of Ω. We obtain:

$$\int_{\Omega_{i,j}} W\phi_{i,j} dSdy - \Delta t \int_{\Omega_{i,j}} \mathcal{L}_2 W\phi_{i,j} dSdy = \int_{\Omega_{i,j}} V\phi_{i,j} dSdy,$$
$$i = 2, 3, \ldots, N_S - 1, j = 2, 3, \ldots, N_y - 1. \tag{77}$$

Substituting (54) in (77), using relation (32), and applying standard finite element techniques the integrals appearing at the left hand side of equation (77) are approximated as follows:

$$\int_{\Omega_{i,j}} W\phi_{i,j} dSdy \simeq \frac{1}{6} |\Omega|_{i,j} W_{i,j}, \tag{78}$$

$$\int_{\Omega_{i,j}} (\mathcal{L}_2 W)\phi_{i,j} dSdy \simeq \frac{1}{12} \left[a_{11,i,j}(W_{i,j+1} - W_{i,j}) - a_{12,i,j}(W_{i,j} - W_{i,j-1}) \right], \tag{79}$$

where

$$a_{11,i,j} = \theta^2 \left[(2y_j + y_{j+1}) \frac{\Delta S_{i-1}}{\Delta y_j} + (y_j + 2y_{j+1}) \frac{\Delta S_i}{\Delta y_j} \right], \tag{80}$$

$$a_{12,i,j} = \theta^2 \left[(y_j + 2y_{j-1}) \frac{\Delta S_{i-1}}{\Delta y_{j-1}} + (2y_j + y_{j-1}) \frac{\Delta S_i}{\Delta y_{j-1}} \right]. \tag{81}$$

Moreover the integral appearing at the right hand side of equation (77) is computed according to relation (56).

As done for problem (46)-(47) we use extrapolated boundary conditions at $y = 0$ and $y = y_{\max}$:

$$W_{i,1} = W_{i,2} - \frac{W_{i,3} - W_{i,2}}{\Delta y_2} \Delta y_1, \quad i = 2, 3, \ldots, N_S - 1, \tag{82}$$

$$W_{i,N_y} = W_{i,N_y-1} + \frac{W_{i,N_y-1} - W_{i,N_y-2}}{\Delta W_{N_y-2}} \Delta y_{N_y-1}, \quad i = 2, 3, \ldots, N_S - 1. \tag{83}$$

Equations (77) with the substitutions (56), (78), (79), and the boundary conditions (82), (83) constitute a set of $N_S - 2$ linear systems. More precisely we have one system of $N_y - 2$ linear equations in the unknowns $W_{2,2}$, $W_{2,3}$, ..., W_{2,N_y-1}, one system of $N_y - 2$ linear equations in the unknowns $W_{3,2}$, $W_{3,3}$, ..., W_{3,N_y-1}, ..., and one system of $N_y - 2$ linear equations in the unknowns $W_{N_S-1,2}$, $W_{N_S-1,3}$, ..., W_{N_S-1,N_y-1}. Each one of these systems is in tridiagonal form and is efficiently solved using Thomas's algorithm.

Once that $W_{i,j}$, $i = 2, 3, \ldots, N_S - 1$, $j = 2, 3, \ldots, N_y - 1$ have been obtained, we compute $W_{i,1}$ and W_{i,N_y} using relations (82) and (83), $i = 2, 3, \ldots, N_S - 1$. Finally, according to (49), we set

$$W_{1,j} = 0, \quad W_{N_S,j} = S_{N_S} - E, \quad j = 1, 2, \ldots, N_y. \tag{84}$$

4.3 Numerical approximation of the integral operator

Let us show how to solve the partial differential problem (50)-(51). First of all equation (50) is collocated at the the nodes (S_i, y_j), $i = 2, 3, \ldots, N_S - 1$, $j = 2, 3, \ldots, N_y - 1$:

$$\frac{W_{i,j} - U_{i,j}}{\Delta t} + (\mathcal{L}_6 U)_{i,j} = \lambda U_{i,j}, \quad i = 2, 3, \ldots, N_S - 1,\ j = 2, 3, \ldots, N_y - 1, \tag{85}$$

where

$$(\mathcal{L}_6 U)_{i,j} = \lambda \int_{-\infty}^{+\infty} U(S_i e^x, y_j) W(dx), \quad i = 2, 3, \ldots, N_S - 1,\ j = 2, 3, \ldots, N_y - 1, \tag{86}$$

Moreover, in order to satisfy the boundary conditions (51), we set

$$U_{1,j} = 0, \quad U_{N_S,j} = S_{\max} - E, \quad j = 2, 3, \ldots, N_y - 1. \tag{87}$$

Let us define

$$x_{h,i} = \log\left(\frac{S_h}{S_i}\right), \quad h = 2, 3, \ldots, N_S,\ i = 2, 3, \ldots, N_S - 1. \tag{88}$$

Using (17) the integral (86) is calculated as follows:

$$(\mathcal{L}_6 U)_{i,j} = \frac{1}{\delta\sqrt{2\pi}} \sum_{h=1}^{N_S} I_{h,i,j}, \quad i = 2, 3, \ldots, N_S - 1,\ j = 2, 3, \ldots, N_y - 1, \tag{89}$$

where

$$I_{1,i,j} = \int_{-\infty}^{x_{2,i}} U(S_i e^x, y_j) \exp\left[-\frac{(x-\gamma)^2}{2\delta^2}\right] dx, \tag{90}$$

$$I_{h,i,j} = \int_{x_{h,i}}^{x_{h+1,i}} U(S_i e^x, y_j) \exp\left[-\frac{(x-\gamma)^2}{2\delta^2}\right] dx,\ h = 2, 3, \ldots, N_S - 1, \tag{91}$$

$$I_{N_S,i,j} = \int_{x_{N_S,i}}^{+\infty} U(S_i e^x, y_j) \exp\left[-\frac{(x-\gamma)^2}{2\delta^2}\right] dx. \tag{92}$$

In order to evaluate the integrals (90)-(92) $U(S, y_j)$ is approximated using a piecewise linear function in the $S-$variable. Precisely having defined $U_{i,j} = U(S_i, y_j)$, $i = 1, 2, \ldots, N_S$, $j = 1, 2, \ldots, N_y$, $U(S, y_j)$ is approximated as follows:

$$U(S, y_j) \simeq \begin{cases} U_{h,j} + \dfrac{U_{h+1,j} - U_{h,j}}{\Delta S_h}(S - S_h), \quad S \in [S_h, S_{h+1}],\ h = 1, 2, \ldots, N_s - 1, \\ S - E, \quad S \geq S_{\max}. \end{cases} \tag{93}$$

Substituting relations (93) in (90)-(92) and using the fact that, according to (88), $S_i e^{x_{h,i}} = S_h$, $h = 2, 3, \ldots, N_S$, $i = 2, 3, \ldots, N_S - 1$, we obtain:

$$I_{1,i,j} \simeq \int_{-\infty}^{x_{2,i}} \left(U_{1,j} + \frac{U_{2,j} - U_{1,j}}{\Delta S_1} S_i e^x\right) \exp\left[-\frac{(x-\gamma)^2}{2\delta^2}\right] dx, \tag{94}$$

$$I_{h,i,j} \simeq \int_{x_{h,i}}^{x_{h+1,i}} \left[U_{1,j} + \frac{U_{h+1,j} - U_{h,j}}{\Delta S_h} S_i (e^x - e^{x_{h,i}})\right] \exp\left[-\frac{(x-\gamma)^2}{2\delta^2}\right] dx, \quad h = 2, 3, \ldots, N_S - 1, \tag{95}$$

$$I_{N_S,i,j} \simeq \int_{x_{N_S,i}}^{+\infty} (S_i e^x - E) \exp\left[-\frac{(x-\gamma)^2}{2\delta^2}\right] dx. \tag{96}$$

Now the integrals (94)-(96) are elementary integrals and can be performed analytically (the calculation is left to the reader). Equations (85), with the substitutions (89), (94)-(96), and the boundary conditions (87) constitute a set of $N_y - 2$ systems of linear equations. Precisely we have one system of N_S linear equations in the unknowns $U_{1,2}$, $U_{2,2}$, ..., $U_{N_S,2}$, one system of N_S linear equations in the unknowns $U_{1,3}$, $U_{2,3}$, ..., $U_{N_S,3}$, ..., and one system of N_S linear equations in the unknowns U_{1,N_y-1}, U_{2,N_y-1}, ..., U_{N_S,N_y-1}. Each one of these

systems has the same matrix, which we term A. Therefore solving the $N_y - 2$ linear systems obtained requires to perform the numerical inversion of only one matrix. This is crucial to the efficiency of the overall numerical algorithm, as the matrix A contains $N_S \times N_S$ non-zero elements and its numerical inversion is rather expensive. Note that we have obtained $N_y - 2$ linear systems with the same matrix A because we have used a mesh of right-angled triangles, and thanks to the fact that the integral operator $\mathcal{L}_6$ does not involve integration along the $y-$variable.

The numerical approximation of the partial differential problem (50)-(51) concludes the $k-$th step of the cycle (38)-(42). At this step the approximate values $U_{i,j}^{k-1}$ are computed starting from the knowledge of $\Psi_{i,j}^{k}$, $i = 1, 2, \ldots, N_S$, $j = 1, 2, \ldots, N_y$. Once that the $k-$th iteration is performed, the $(k-1)$-th iteration can be started by prescribing, according to relation (41):

$$\Psi_{i,j}^{k-1} = \max\left[U_{i,j}^{k-1}, S_i - K\right], \quad i = 1, 2, \ldots, N_S,\ j = 1, 2, \ldots, N_y. \tag{97}$$

A theoretical analysis about the convergence of the numerical method developed in this section appears to be a very difficult task, due to the different kinds of a approximations involved (the Richardson extrapolation, the operator splitting technique, the finite element method). Hence the accuracy of the method proposed is tested by numerical simulation. This is done in the next section.

5 Numerical results

The simulations are carried out on a computer with a Pentium Dual Core E 2140 Processor 1.6 GHz 2 GB Ram, and the numerical code is written using FORTRAN 90.

The non-uniform finite element mesh is constructed as follows: along the S-direction we want to have a mesh which is finer in a neighborhood of $S = E$, where the derivative of the payoff function is discontinuous. Instead, along the y-direction, we want to have a mesh which is finer in a neighborhood of $y = y_0$, where the possible realization of the variance process are more likely to occur. Therefore, using an approach similar to that followed in Ito and Toivanen (2004), the mesh refinement is done setting:

$$S_i = E\left\{1 + \frac{1}{\zeta_S}\sinh\left[\frac{i-1}{N_S - 1}\left(c_{2,S} - c_{1,S}\right) + c_{1,S}\right]\right\}, \quad i = 1, 2, \ldots, N_S, \tag{98}$$

where

$$c_{1,S} = \operatorname{asinh}(-\zeta_S), \quad c_{2,S} = \operatorname{asinh}\left(\frac{S_{\max} - E}{E}\zeta_S\right), \tag{99}$$

and

$$y_j = y_0 \left\{ 1 + \frac{1}{\zeta_y} \sinh \left[\frac{j-1}{N_y - 1} \left(c_{2,y} - c_{1,y} \right) + c_{1,y} \right] \right\}, \quad j = 1, 2, \ldots, N_y, \quad (100)$$

where

$$c_{1,y} = \operatorname{asinh}(-\zeta_y), \quad c_{2,y} = \operatorname{asinh}\left(\frac{y_{\max} - y_0}{y_0} \zeta_y \right). \quad (101)$$

According to relations (98)-(101) the amount of mesh refinement in the S-direction near $S = E$ is proportional to the parameter ζ_S, whereas the amount of mesh refinement in the y-direction near $y = y_0$ is proportional to the parameter ζ_y (the limit case $\zeta_S \to 0$ and $\zeta_y \to 0$ corresponds to a uniform mesh). In all the simulations presented in this section we use $\zeta_S = 10$ and $\zeta_y = 1$. By several numerical experiments we have found that these values allow to obtain very accurate results.

For comparison purposes, our first test-case (Test Case 1) is the same test-case presented in Chiarella et al. (2008). Precisely let us consider an American Call option with strike price $E = 100$, maturity $T = 0.5\, year$, interest rate $r = 0.03\, year^{-1}$, and dividend yield $q = 0.05\, year^{-1}$. The parameters of the Bates model are chosen as follows: $\xi = 2\, year^{-1}$, $\eta = 0.04\, year^{-1}$, $\theta = 0.4\, year^{-1}$, $\lambda = 5\, year^{-1}$, $\delta = 0.1$, and $\gamma = -\frac{\delta^2}{2}$. As far as the correlation coefficient ρ is concerned, we use both $\rho = 0.5$ (Test Case 1.a) and $\rho = -0.5$ (Test Case 1.b). As done in Ciarella et al. (2008), for the initial datum S_0 we consider five different values $S_{0,l} = 80 + 10(l-1)$, $l = 1, 2, 3, 4, 5$. Moreover we set $y_0 = 0.04\, year^{-1}$.

The mesh-size parameters are chosen as follows: $S_{\max} = 300$, $y_{\max} = 0.2\, year^{-1}$, $N_S = 250$, $N_y = 200$. Finally we set $N_t = 50$.

Let $C_{ap}(0, S, y)$ denote the approximate value of the American Call option price at time $t = 0$ obtained using the finite element method presented in Section 4. Moreover let RMSRD denote the average relative error on $C_{ap}(0, S, y)$. RMSRD is then computed as follows:

$$\text{RMSRD} = \sqrt{\frac{1}{5} \sum_{l=1}^{5} \left(\frac{C_{ap}(0, S_{0,l}, y_0) - C(0, S_{0,l}, y_0)}{C(0, S_{0,l}, y_0)} \right)^2}. \quad (102)$$

Accurate estimations of $C(0, S_{0,l}, y_0)$, $l = 1, 2, 3, 4, 5$, necessary to evaluate (102), have been obtained by Chiarella et al. (2008) using a (very time consuming) finite difference approximation on an extremely fine mesh. Finally let RUNTIME denote the computer time necessary to calculate $C_{ap}(0, S, y)$.

In Table 1 and Table 2 we report the values of $C_{ap}(0, S_{0,l}, y_0)$, $C(0, S_{0,l}, y_0)$, $l = 1, 2, 3, 4, 5$, RMSRD and RUNTIME experienced in Test Case 1.a and Test

Case 1.b respectively. Looking at these tables we may note that both in Test Case 1.a and in Test Case 1.b the numerical method proposed in this paper is very accurate and fast. In fact the American option price is always computed with at least 3 correct decimal digits in only $12\,s$. Moreover the relative error RMSRD is very small (of order 10^{-4}).

Table 1: Test Case 1.a

S_0	$C(0,S_0,y_0)$	$C_{ap}(0,S_0,y_0)$
80	1.4843	1.4844
90	3.7145	3.7153
100	7.7027	7.7040
110	13.6722	13.6734
120	21.3653	21.3663
RMSRD $= 1.34 \times 10^{-4}$		
RUNTIME $= 12\,s$		

Table 2: Test Case 1.b

S_0	$C(0,S_0,y_0)$	$C_{ap}(0,S_0,y_0)$
80	1.1359	1.1359
90	3.3532	3.3538
100	7.5970	7.5983
110	13.8830	13.8846
120	21.7186	21.7201
RMSRD $= 1.26 \times 10^{-4}$		
RUNTIME $= 12\,s$		

Now let us compare the numerical method proposed in this paper with that presented in Chiarella et al. (2008). The most significant results obtained there in Test Case 1 are the following: in Test Case 1.a the relative error is RMSRD $= 1.77 \times 10^{-4}$ and the computer time is RUNTIME $= 12120\,s$, whereas in Test Case 1.b the relative error is RMSRD $= 1.93 \times 10^{-4}$ and the computer time is RUNTIME $= 12122\,s$. Then we have: 1) the errors obtained by Chiarella et al. (2008) are slightly bigger than those reported in Table 1 and Table 2; 2) the computer times experienced by those authors are (approximately) a thousand of times bigger than those reported in Table 1 and Table 2. It should also be noted that the numerical simulations presented in that paper are carried out on a cluster of computers which is faster than our Pentium Dual Core processor. Putting all these things together we can conclude that in Test Case 1.a and in Test Case 1.b the numerical method proposed in this paper is at least a thousand of times faster than the approach followed in Chiarella et al. (2008).

As a second test-case (Test Case 2) we consider an American Put option with strike price $E = 100$, maturity $T = 5\,year$, interest rate $r = 0.0319\,year^{-1}$,

and dividend yield $q = 0$. The parameters and data in common with the Heston model are chosen as in Broadie and Kaya (2006): $\xi = 6.21\, year^{-1}$, $\eta = 0.019\, year^{-1}$, $\theta = 0.61\, year^{-1}$, $\rho = -0.7$, $y_0 = 0.010201\, year^{-1}$. Furthermore we set $\lambda = 0.5\, year^{-1}$, $\delta = 0.2$, and $\gamma = -\frac{\delta^2}{2}$. As done in Test Case 1 for the initial datum S_0 we consider five different values $S_{0,l} = 80 + 10(l-1)$, $l = 1, 2, 3, 4, 5$.

The mesh-size parameters are chosen as follows: $S_{\max} = 300$, $y_{\max} = 0.2\, year^{-1}$, $N_S = 250$, $N_y = 200$. Moreover we set $N_t = 100$.

Let $P_{ap}(0, S_0, y_0)$ denote the approximate value of the American Put option price obtained using the numerical method proposed in this paper. In order to evaluate the error on $P_{ap}(0, S_0, y_0)$, an estimation of the true American Put option price, denoted $P(0, S_0, y_0)$, is obtained by performing a very accurate (and also very expensive) simulation with the following mesh-size parameters: $S_{\max} = 500$, $y_{\max} = 0.4\, year^{-1}$, $N_S = 500$, $N_v = 500$, $N_t = 1000$. As done in Test Case 1 the average relative error on $P_{ap}(0, S_0, y_0)$, denoted RMSRD, is computed using relation (102) in which $C(0, S_0, y_0)$ and $C_{ap}(0, S_0, y_0)$ are replaced with $P(0, S_0, y_0)$ and $P_{ap}(0, S_0, y_0)$. The computer time necessary to obtain $P_{ap}(0, S_0, y_0)$ is still denoted by RMSRD.

In Table 3 are shown the values of $P_{ap}(0, S_{0,l}, y_0)$, $P(0, S_{0,l}, y_0)$, $l = 1, 2, 3, 4, 5$, RMSRD and RUNTIME experienced in Test Case 2. We may note that the numerical method proposed in this paper is very accurate and fast. In fact the American option price is always computed with at least 4 correct decimal digits (average relative error of order 10^{-5}) in only $24\, s$.

Finally we make notice that in Test Case 2 the initial datum y_0 is a very small value. In addition we have $\theta^2 > 2\xi\eta$, so that the variance process is allowed to hit the origin. Moreover. As a consequence the realizations of Y_t are contained with probability close to one in a region close to $y = 0$, where the partial differential equation (16) is singular. Also despite this fact the numerical method proposed in this paper allows to obtain very accurate results.

Table 3: Test Case 2

S_0	$P(0, S_0, y_0)$	$P_{ap}(0, S_0, y_0)$
80	21.3053	21.3030
90	15.6365	15.6364
100	11.5887	11.5890
110	8.6680	8.6685
120	6.5464	6.5466
RMSRD $= 5.77 \times 10^{-5}$		
RUNTIME $= 24\, s$		

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