CENTRAL LIMIT THEOREM FOR A CLASS OF ONE-DIMENSIONAL KINETIC EQUATIONS

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Abstract. We introduce a class of Boltzmann equations on the real line, which constitute extensions of the classical Kac caricature. The collisional gain operators are defined by smoothing transformations with quite general properties. By establishing a connection to the central limit problem, we are able to prove long-time convergence of the equation's solutions towards a limit distribution. If the initial condition for the Boltzmann equation belongs to the domain of normal attraction of a certain stable law ν_α, then the limit is a scale mixture of ν_α. Under some additional assumptions, explicit exponential rates for the convergence to equilibrium in Wasserstein metrics are calculated, and strong convergence of the probability densities is shown.

1. Introduction

In a variety of recent publications, intimate relations between the central limit theorem of probability theory and the celebrated Kac caricature of the Boltzmann equation from statistical physics have been revealed. The idea to represent the solutions of the Kac equation in a probabilistic way dates back at least to the works of McKean in the 60's, see e.g. McKean (1966), but has been fully formalized and employed in the derivation of analytic results only in the last decade. For instance, probabilistic methods have been used to get estimates on the quality of approximation of solutions by truncated Wild sums in Carlen et al. (2000), to study necessary and sufficient conditions for the convergence to a steady state in Gabetta and Regazzini (2006b), to study the blow-up behavior of solutions of infinite energy in Carlen et al. (2007, 2008), to obtain rates of convergence to equilibrium of the solutions both in strong and weak metrics, Gabetta and Regazzini (2006c); Dolera et al. (2007); Dolera and Regazzini (2007). The power of the probabilistic approach is illustrated, for instance, by the fact that in Dolera et al. (2007) very refined estimates for the classical central limit theorem enabled the authors to deliver the first proof of a conjecture that has been formulated by McKean about fourty years ago.

The applicability of probabilistic methods is not restricted to the classical Kac equation, but extends to the inelastic Kac model, proposed by Pulvirenti and Toscani (2004). In the inelastic model, the energy (second moment) of the solution is not conserved but dissipated, and hence infinite energy is needed initially to obtain a non-trivial long-time limit. In Bassetti et al. (2008) probabilistic methods have been used to study the speed of approach to equilibrium under the assumption that the initial condition belongs to the domain of normal attraction of a suitable stable law. In this context, indeed, the steady states are the corresponding stable laws.

In the current paper, we continue in the spirit of the aforementioned results. By means of the central limit theorem for triangular arrays, we are able to study the long time behavior of solutions of a wide class of one-dimensional Boltzmann equations, which contains (essentially) the classical and the inelastic Kac model as special cases.

To be more specific, recall that the Kac equation describes the evolution of a time-dependent probability measure $\mu(t)$ on the real axis, and is most conveniently written as an evolution equation for the characteristic function $\phi(t)$ of $\mu(t)$. The equation has the form

$$(1)\qquad \begin{cases} \partial_t \phi(t;\xi) + \phi(t;\xi) = \widehat{Q}^+[\phi(t;\cdot),\phi(t;\cdot)](\xi) & (t>0, \xi\in\mathbb{R}) \\ \phi(0;\xi) = \phi_0(\xi) \end{cases}$$

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where the collisional gain operator is given by

$$\widehat{Q}^+[\phi(t;\cdot),\phi(t;\cdot)](\xi) := \mathbb{E}[\phi(t;L\xi)\phi(t;R\xi)]. \tag{2}$$

Above, (L,R) is a random vector defined on a probability space $(\Omega,\mathcal{F},P)$ and $\mathbb{E}$ denotes the expectation with respect to P. The initial condition ϕ_0 is the characteristic function of a prescribed real random variable X_0; by abuse of notation, we shall also refer to X_0, to its probability distribution function F_0 or to its law $\mu(0)$ as the initial condition.

For the classical Kac equation, one writes $(L,R) = (\sin(\Theta),\cos(\Theta))$, with Θ uniformly distributed on $[0,2\pi)$, and hence $L^2+R^2 = 1$ a.s. The inelastic Kac equation is obtained by

$$(L,R) = (\sin(\Theta)|\sin(\Theta)|^p,\cos(\Theta)|\cos(\Theta)|^p),$$

with $p>0$, and hence $|L|^\alpha+|R|^\alpha = 1$ a.s., if $\alpha = 2/(1+p)$. It is worth recalling that the study of the respective initial value problems can be reduced to the study of the same problems under the additional assumption that the initial distribution is symmetric, i.e. the initial characteristic function is real, and $(L,R) = (|\sin(\Theta)|^{1+p},|\cos(\Theta)|^{1+p})$. See Section 2.1.

In this paper, we consider the problem (1), where the random variables L and R in the definition of the collision operator in (2) are non-negative and satisfy the condition

$$\mathbb{E}[L^\alpha+R^\alpha] = 1, \tag{3}$$

for some α in $(0,2]$. The therewith defined bilinear operators $\widehat{Q}^+$ are examples of smoothing transformation, which have been extensively studied in the context of branched random walks, see e.g. Kahane (1976); Durrett and Liggett (1983); Guivarc'h (1990); Liu (1998); Iksanov (2004) and the references therein.

Our motivation, however, originates from applications to statistical physics. These applications are discussed in Section 2.1. At this point, we just mention the two main examples.

1. Passing from the Kac condition $L^2+R^2 = 1$ to (3) with $\alpha = 2$, the model retains its crucial physical property to conserve the second moment of the solution. However, the variety of possible steady states grows considerably: depending on the law of (L,R), the latter may exhibit heavy tails.
2. For certain distributions satisfying (3) with $\alpha = 1$, equation (1) has been used to model the redistribution of wealth in simplified market economies, which conserve the society's total wealth (first moment). Whereas the condition $L+R = 1$ would correspond to deterministic trading and lead eventually to a fair but unrealistic distribution of wealth in the long time limit, the relaxed condition (3) allows trade mechanisms that involve randomness (corresponding to risky investments) and lead to a realistic, highly unequal distribution of wealth.

Our main results from Theorems 3.2 and 3.4 can be rephrased as follows:

Assume that (3) *holds with* $\alpha\in(0,2]$, *but* $\alpha\neq 1$, *and in addition that* $\mathbb{E}[L^\gamma+R^\gamma]<1$ *for some* $\gamma>\alpha$. *Let* $\mu(t)$, *for* $t\geq 0$, *be the probability measure on* $\mathbb{R}$ *such that its characteristic function* $\phi(t)$ *is the unique solution to the associated Boltzmann equation* (1). *Assume further that the initial datum* $\mu(0)$ *lies in the normal domain of attraction of some* α*-stable law* ν_α, *and that* $\mu(0)$ *is centered if* $\alpha>1$. *Then, as* $t\to+\infty$, *the probability measures* $\mu(t)$ *converge weakly to a limit distribution* μ_∞, *which is a non-trivial scale mixture of* ν_α.

The results in the case $\alpha = 1$ are more involved; see Theorems 3.3 and 3.5.

Under the previous general hypotheses, no more than weak convergence can be expected. However, slightly more restrictive assumptions on the initial condition ϕ_0 suffice to obtain exponentially fast convergence in some Wasserstein distance. Finally, if the initial condition possesses a density with finite Linnik-Fisher functional and the condition $L^r+R^r\geq 1$ holds a.s. for some $r>0$, then the probability density of $\mu(t)$ exists for every $t>0$ and converges strongly in the Lebesgue spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ as $t\to+\infty$.

The largest part of the paper deals with the proofs of weak convergence, which are obtained in application of the central limit theorem for triangular arrays. Consequently, the core element of the proof is to establish a suitable probabilistic interpretation of the solution to (1). The link to probability theory is provided by a semi-explicit solution formula: the Wild sum,

$$\phi(t) = e^{-t} \sum_{n=0}^{\infty} (1 - e^{-t})^n \hat{q}_n, \tag{4}$$

represents the solution $\phi(t)$ as a convex combination of characteristic functions $\hat{q}_n$, which are obtained by iterated application of the gain operator $\widehat{Q}^+$ to the initial condition ϕ_0 — see formula (11). Following Gabetta and Regazzini (2006b) we consider a sequence of random variables W_n such that W_n has $\hat{q}_{n-1}$ as its characteristic function, and possesses the representation

$$W_n = \sum_{j=1}^{n} \beta_{j,n} X_j, \tag{5}$$

where the X_j are independent and identically distributed random variables with common characteristic function ϕ_0. The weights $\beta_{j,n}$ are random variables themselves and are obtained in a recursive way, see (12).

The behavior of $\phi(t)$ in (4) as $t \to \infty$ is obviously determined by the behavior of the law of W_n as $n \to \infty$. It is important to note that a direct application of the central limit theorem to the study of W_n is inadmissible since the weights in (5) are not independent. However, one can apply the central limit theorem to study the conditional law of W_n, given the array of weights $\beta_{j,n}$.

Representations in the form (4) with (5) are known for the (classical and inelastic) Kac equation, see Gabetta and Regazzini (2006b) and Bassetti et al. (2008). The situation here is more involved, since (3) only implies that

$$\mathbb{E}\big[\beta_{1,n}^\alpha + \beta_{2,2}^\alpha + \cdots + \beta_{n,n}^\alpha\big] = 1, \tag{6}$$

whereas for the Kac equation,

$$\beta_{1,n}^\alpha + \beta_{2,n}^\alpha + \cdots + \beta_{n,n}^\alpha = 1 \quad a.s. \tag{7}$$

In order to be able to apply the central limit theorem, one needs to prove that $\max_{1\le j\le n} |\beta_{j,n}|$ converges in probability to zero, and that $\sum_j \beta_{j,n}^\alpha$ converges (almost surely) to a random variable. Thanks to (7), the latter condition is immediately satisfied for the Kac equation, while it is not always true for the general model considered here. We stress that the generality of (6) in comparison to (7) is the origin of the richness of possible steady states in (1).

The paper is organized as follows. In Section 2, we recall some basic facts about the Boltzmann equation under consideration, present a couple of examples to which the theory applies, and derive the stochastic representation of solutions. Section 3 contains the statements of our main theorems. The results are classified into those on convergence in distribution (Section 3.1), convergence in Wasserstein metrics at quantitative rates (Section 3.2) and strong convergence of the probability densities (Section 3.3). All proofs are collected in Section 4.

2. Examples and preliminary results

One-dimensional kinetic equations of type (1)-(2), like the Kac equation and its variants, provide simplified models for a spatially homogeneous gas, in which particle move only in one spatial direction. The measure $\mu(t)$, whose characteristic function is the solution of (1), describes the probability distribution of the velocity of a molecule at time t. The basic assumption is that particles change their velocities only because of binary collisions. When two particles collide, then their velocities change from v and w, respectively, to

$$v' = L_1 v + R_1 w \quad \text{and} \quad w' = R_2 v + L_2 w \tag{8}$$

with $L_1 = L_2 = \sin(\Theta)$ and $R_1 = -R_2 = \cos(\Theta)$.

More generally, one can consider binary interaction obeying (8), where (L_1, R_1) and (L_2, R_2) are two identically distributed random vectors (not necessarily independent) with the same law of (L, R). This leads, at least formally, to equation (1).

2.1. **Examples.** The following applications are supposed to serve to motivate the study of the Boltzmann equation (1) with the condition (3). The first two examples are taken from gas dynamics, while the third originates from econophysics.

Kac like models. Instead of discussing the physical relevance of the Kac model, we simply remark that it constitutes the most sensible one-dimensional caricature of the Boltzmann equation for elastic Maxwell molecules in three dimensions. A comprehensive review on the mathematical theory of the latter is found e.g. in Bobylëv (1988). The term "elastically" refers to the fact that the kinetic energy of two interacting molecules – which is proportional to the square of the particles' velocities – is preserved in their collisions. Indeed, since $L_1 = L_2 = \sin(\Theta)$ and $R_1 = -R_2 = \cos(\Theta)$, one obtains $(v')^2 + (w')^2 = v^2 + w^2$.

We shall not detail any of the numerous results available in the extensive literature on the Kac equation, but simply summarize some basic properties that are connected with our investigations here. First, we remark that the microscopic conservation of the particles' kinetic energy implies the conservation of the average energy, which is the second moment of the solution $\mu(t)$ to (1). Moreover, it is easily proven that the average velocity, i.e. the first moment of $\mu(t)$, converges to zero exponentially fast. For $t \to +\infty$, the solution $\mu(t)$ converges weakly to a Gaussian measure that is determined by the conserved second moment.

As already mentioned in the introduction, the study of the original Kac model can be reduced to the study of a particular case of the model we are considering. Indeed, it is well-known that the solution of the Kac equation can be written as

$$\phi(t,\xi) = e^{-t} Im\big(\phi_0(\xi)\big) + \phi^*(t,\xi) \tag{9}$$

where ϕ^* is the solution to problem (1) with $Re(\phi_0)$ in the place of ϕ_0, $L = |\sin(\Theta)|$ and $R = |\cos(\Theta)|$. Hence, we can invoke Theorem 3.4, which provides another proof of the large-time convergence of solutions $\mu(t)$ to a Gaussian law. In fact, also Theorem 3.8 is applicable, which shows that the densities of $\mu(t)$ converge in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, provided $\mu(0)$ possesses a density with finite Linnik functional.

These consequences are weak in comparison to the various extremely refined convergence estimates for the solutions to the Kac equation available in the literature. See, e.g., the review Regazzini (2008). On the other hand, our proofs do not rely on any of the symmetry properties that are specific for the Kac model. Thus, our aforementioned results extend — word by word — to the wide class of problems (1)-(2) with $L^2 + R^2 = 1$ a.s.

The variant of the Kac equation, introduced in Pulvirenti and Toscani (2004), is called inelastic because the total kinetic energy of two colliding particles is not preserved in the collision mechanism, but decreases in general. Consequently, if the second moment of the initial condition $\mu(0)$ is finite, then the second moment of the solution $\mu(t)$ converges to zero exponentially fast in $t > 0$. Non-trivial long-time limits are thus necessarily obtained from initial conditions with infinite energy. In Bassetti et al. (2008), it is shown that the solution $\mu(t)$ converges weakly to a α-stable law ν_α if $\mu(0)$ belongs to the normal domain of attraction of ν_α,

As for the Kac model, the study of the inelastic Kac model too can be reduced to the framework of the present paper. Hence, Theorem 3.8 and Proposition 3.6 yield new results concerning strong convergence of densities in $L^1(\mathbb{R})$ and convergence with respect to the Wasserstein metrics.

Inelastic Maxwell molecules. We shall now consider a variant of the Kac model in which the energy is *not* conserved in the individual particle collisions, but gains and losses balance in such a way that the average kinetic energy is conserved. This is achieved by relaxing the condition $L^2 + R^2 = 1$ to $\mathbb{E}[L^2 + R^2] = 1$, which is (3) with $\alpha = 2$.

Just as the Kac equation is a caricature of the Boltzmann equation for elastic Maxwell molecules, the model at hand can be thought of as a caricature of a Boltzmann equation for *inelastic* Maxwell molecules in three dimensions. For the definition of the corresponding model, its physical justification, and a collection of relevant references, see Carrillo et al. (2008). We stress, however, that the Kac caricature of inelastic Maxwell molecules is not the same as the inelastic Kac model from the preceeding paragraph.

Conservation of the total energy can be proven for centered solution $\mu(t)$; like symmetry, also centering is propagated from $\mu(0)$ to any $\mu(t)$ by (1). The argument leading to energy conservation is given in the remarks following Theorem 3.4.

Relaxation from strict energy conservation to conservation in the mean affects the possibilities for the large-time dynamics of $\mu(t)$. It follows from Theorem 3.4 that if $\mathbb{E}[L^\gamma + R^\gamma] < 1$ for some $\gamma > 2$, then any solution $\mu(t)$, which is centered and of finite second moment initially, converges weakly to a non-trivial steady state μ_∞. However, unless $L^2 + R^2 = 1$ a.s., μ_∞ is *not* a Gaussian. In fact, (L, R) can be chosen in such a way that μ_∞ possesses only a finite number of moments. In physics, such velocity distributions are referred to as "high energy tailed", and typically appear when the molecular gas is connected to a thermal bath.

An example leading to high energy tails is the following: let (L, R) such that $P\{L = 1/2\} = P\{L = \sqrt{5}/2\} = 1/2$ and $P\{R = 1/2\} = 1$. One verifies that $\mathbb{E}[L^2 + R^2] = 1$ and $\mathbb{E}[L^4 + R^4] = 7/8 < 1$, so Theorem 3.4 guarantees the existence of a non-degenerate steady state μ_∞. Moreover, $\mathbb{E}[L^6 + R^6] = 1$, and one concludes further from Theorem 3.4 that the sixth moment of μ_∞ diverges, whereas all lower moments are finite.

Wealth distribution. Recently, an alternative interpretation of the equation (1) has become popular. The homogeneous gas of colliding molecules is replaced by a simple market with a large number of interacting agents. The current "state" of each individual is characterized by a single number, his or her wealth v. Correspondingly, the measure $\mu(t)$ represents the distribution of wealth among the agents. The collision rule (8) describes how wealth is exchanged between agents in binary trade interactions. See, e.g., Slanina (2004); Cordier et al. (2005).

Typically, it is assumed that $\mu(t)$ is supported on the positive semi-axis. In fact, the first moment of $\mu(t)$ represents the total wealth of the society and plays the same rôle as the energy in the previous discussion. In particular, it is conserved by the evolution.

In the first approaches, see e.g. Angle (1986), conservation of wealth in each trade was required, i.e. $v' + w' = v + w$. Hence, assuming $L_1 = L_2$ and $R_1 = R_2$ in (8), this yields $L + R = 1$ a.s. However, the obtained results were unsatisfactory: in the long time limit, the wealth distribution $\mu(t)$ concentrates on the society's average wealth, so that asymptotically, all agents possess the same amount of money. This also follows from our Theorem 3.3.

More realistic results have been obtained by Matthes and Toscani (2008), where trade rules (L, R) have been introduced that satisfy (3) with $\alpha = 1$, but in general $P\{L + R = 1\} < 1$. Thus wealth can be increased or diminished in individual trades, but the society's total wealth, i.e. the first moment of $\mu(t)$, remains constant in time. The proof of conservation of the mean wealth can also be found in the remarks after Theorem 3.3.

A typical example for trade rules is the following. Let $L_1 = L_2$ and $R_1 = R_2$ with $P\{L = 1 - p + r\} = P\{L = 1 - p - r\} = 1/2$ and $P\{R = p\} = 1$, where p in $(0, 1)$ is a relative price of an investment and r in $(0, p)$ is a risk factor. The interpretation reads as follows: each of the two interacting agents buys from the other one some risky asset at the price of the pth fraction of the respective buyer's current wealth; these investments either pay off and produce some additional wealth, or lose value, both proportional (with r) to their original price. Over-simplified as this model might be, it is able to produce (for suitable choices of p and r) steady distributions μ_∞ with only finitely many moments, as are typical wealth distributions for western countries; see Matthes and Toscani (2008) for further discussion.

An example is provided by choosing $p = 1/4$ and $r = 1/2$. One easily verifies that $\mathbb{E}[L+R] = 1$, $\mathbb{E}[L^2 + R^2] = 7/8 < 1$ and $\mathbb{E}[L^3 + R^3] = 1$. By Theorem 3.3, it follows that there exists a non-degenerate steady distribution μ_∞ that possesses all moments up to the third, whereas the third moment diverges.

2.2. **Probabilistic representation of the solution.** As already mentioned, a convenient way to represent the solution ϕ to the problem (1) is

$$\phi(t; \xi) = \sum_{n=0}^{\infty} e^{-t}(1 - e^{-t})^n \hat{q}_n(\xi) \qquad (t \geq 0, \xi \in \mathbb{R}) \tag{10}$$

where $\hat{q}_n$ is recursively defined by

$$\begin{cases} \hat{q}_0(\xi) := \phi_0(\xi) \\ \hat{q}_n(\xi) := \frac{1}{n}\sum_{j=0}^{n-1} \mathbb{E}[\hat{q}_j(L\xi)\hat{q}_{n-1-j}(R\xi)] \qquad (n = 1, 2, \dots). \end{cases} \tag{11}$$

The series in (10) is referred to as *Wild sum*, since the representation (10) has been derived in Wild (1951) for the solution of the Kac equation. In this section, we shall rephrase the Wild sum in a probabilistic way. The idea goes back to McKean (1966, 1967), where McKean relates the Wild series to a random walk on a class of binary trees, the so–called McKean trees. It is not hard to verify that each of the expressions $\hat{q}_n$ in the Wild series is indeed a characteristic function. Now, following Gabetta and Regazzini (2006b), we shall define a sequence of random variables W_n such that $\hat{q}_{n-1}(\xi) = \mathbb{E}[e^{i\xi W_n}]$.

On a sufficiently large probability space $(\Omega, \mathcal{F}, P)$, let the following be given:

- a sequence $(X_n)_{n\in\mathbb{N}}$ of independent and identically distributed random variables with common distribution function F_0;
- a sequence $\big((L_n, R_n)\big)_{n\in\mathbb{N}}$ of independent and identically distributed random vectors, distributed as (L, R);
- a sequence $(I_n)_{n\in\mathbb{N}}$ of independent integer random variables, where each I_n is uniformly distributed on the indices $\{1, 2, \dots, n\}$;
- a stochastic process $(\nu_t)_{t\geq 0}$ with values in $\mathbb{N}$ and $P\{\nu_t = n\} = e^{-t}(1-e^{-t})^{n-1}$.

We assume further that

$$(I_n)_{n\geq 1}, \quad (L_n, R_n)_{n\geq 1}, \quad (X_n)_{n\geq 1} \quad \text{and} \quad (\nu_t)_{t>0}$$

are stochastically independent. The random array of weights $[\beta_{j,n} : j = 1, \dots, n]_{n\geq 1}$ is recursively defined as follows:

$$\beta_{1,1} := 1, \qquad (\beta_{1,2}, \beta_{2,2}) := (L_1, R_1)$$

and, for any $n \geq 2$,

$$(\beta_{1,n+1}, \dots, \beta_{n+1,n+1}) := (\beta_{1,n}, \dots, \beta_{I_n-1,n}, L_n\beta_{I_n,n}, R_n\beta_{I_n,n}, \beta_{I_n+1,n}, \dots, \beta_{n,n}). \tag{12}$$

Finally set

$$W_n := \sum_{j=1}^{n} \beta_{j,n} X_j \quad \text{and} \quad V_t := W_{\nu_t} = \sum_{j=1}^{\nu_t} \beta_{j,\nu_t} X_j. \tag{13}$$

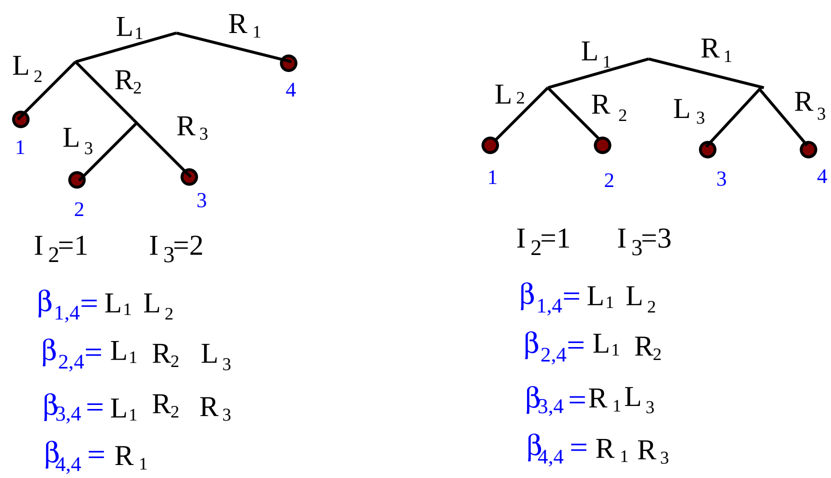

FIGURE 1. Two McKean trees, with associated weights β.

There is a direct interpretation of this construction in terms of McKean trees. For an introduction to McKean trees, see, e.g., Carlen et al. (2000). Each finite sequence $\mathcal{I}_n = (I_1, I_2, \dots, I_{n-1})$ corresponds to a McKean tree with n leaves. The tree associated to $\mathcal{I}_{n+1}$ is obtained from the tree associated to $\mathcal{I}_n$ upon replacing the I_n-th leaf (counting from the left) by a binary branching with two new leaves. The left of the new branches is labelled with L_n, and the right one with R_n. Finally, the weights $\beta_{j,n}$ are associated to the leaves of the $\mathcal{I}_n$-tree; namely, $\beta_{j,n}$ is the product of the labels assigned to the branches along the ascending path connecting the jth leaf to the root. The trees with $\mathcal{I}_4 = (1, 1, 2)$ and $\mathcal{I}_4 = (1, 2, 3)$, respectively, are displayed in Figure 1.

In the Wild construction (11), McKean trees with n leaves are obtained by joining pairs of trees with k and $n-k$ leaves, respectively, at a new common root. Wheras our construction produces the n leaved trees from the $n-1$ leaved trees replacing a leaf by a binary branching. In a way, the second construction is much more natural — or, at least, more biological! The next proposition shows that both constructions indeed lead to the same result.

In the rest of the paper expectations with respect to P will be denoted by $\mathbb{E}$.

Proposition 2.1 (Probabilistic representation)**.** *Equation* (1) *has a unique solution $\phi(t)$, which coincides with the characteristic function of V_t, i.e.*

$$\phi(t, \xi) = \mathbb{E}[e^{i\xi V_t}] = \sum_{n=0}^{\infty} e^{-t}(1-e^{-t})^n \mathbb{E}[e^{i\xi W_{n+1}}] \qquad (t > 0,\, \xi \in \mathbb{R}).$$

Proof. The respective proof for the Kac case is essentially already contained in McKean (1966). See Gabetta and Regazzini (2006b) for a more complete proof. Here, we extend the argument to the problem (1). First of all it is easy to prove, following Wild (1951) and McKean (1966), that formulas (10) and (11) produce the unique solution to problem (1). See also Sznitman (1986). Hence, comparing the Wild sum representation (10) and the definition of V_t in (13), it obviously suffices to prove that

$$\hat{q}_{\ell-1}(\xi) = \mathbb{E}[e^{i\xi W_\ell}], \tag{14}$$

which we will show by induction on $\ell \geq 1$. First, note that $\mathbb{E}[\exp(i\xi W_1)] = \mathbb{E}[\exp(i\xi X_1)] = \phi_0(\xi) = \hat{q}_0(\xi)$ and

$$\mathbb{E}[e^{i\xi W_2}] = \mathbb{E}[e^{i\xi(L_1X_1+R_1X_2)}] = \mathbb{E}[\mathbb{E}[e^{i\xi(L_1X_1+R_1X_2)}|L_1, R_1]] = \hat{q}_1(\xi),$$

which shows (14) for $\ell = 1$ and $\ell = 2$. Let $n \geq 3$, and assume that (14) holds for all $1 \leq \ell < n$; we prove (14) for $\ell = n$.

Recall that the weights $\beta_{j,n}$ are products of random variables L_i and R_i. By the recursive definition in (12), one can define a random index $K_n < n$ such that all products $\beta_{j,n}$ with $j \leq K_n$ contain L_1 as a factor, while the remaining products $\beta_{j,n}$ with $K_n + 1 \leq j \leq n$ contain R_1. (In terms of McKean trees, K_n is the number of leaves in the left sub-tree, and $n - K_n$ the number of leaves in the right one.) By induction it is easy to see that

$$P\{K_n = i\} = \frac{1}{n-1} \qquad i = 1, \ldots, n-1;$$

c.f. Lemma 2.1 in Carlen et al. (2000). Now,

$$A_{K_n} := \sum_{j=1}^{K_n} \frac{\beta_{j,n}}{L_1} X_j, \quad B_{K_n} := \sum_{j=K_n+1}^{n} \frac{\beta_{j,n}}{R_1} X_j \quad \text{and} \quad (L_1, R_1)$$

are conditionally independent given K_n. By the recursive definition of the weights $\beta_{j,n}$ in (12), the following is easily deduced: the conditional distribution of A_{K_n}, given $\{K_n = k\}$, is the same as the (unconditional) distribution of $\sum_{j=1}^{k} \beta_{j,k} X_j$, which clearly is the same distribution as that of W_k. Analogously, the conditional distribution of B_{K_n}, given $\{K_n = k\}$, equals the distribution of $\sum_{j=1}^{n-k} \beta_{j,n-k} X_j$, which further equals the distribution of W_{n-k}. Hence,

$$\begin{aligned}\mathbb{E}[e^{i\xi W_n}] &= \frac{1}{n-1}\sum_{k=1}^{n-1} \mathbb{E}\big[e^{i\xi(L_1 A_k + R_1 B_k)}\big|\{K_n = k\}\big] \\ &= \frac{1}{n-1}\sum_{k=1}^{n-1} \mathbb{E}\big[\mathbb{E}[e^{i\xi L_1 W_k}|L_1, R_1]\mathbb{E}[e^{i\xi R_1 W_{n-k}}|L_1, R_1]\big] \\ &= \frac{1}{n-1}\sum_{k=1}^{n-1} \mathbb{E}[\hat{q}_{k-1}(L_1\xi)\hat{q}_{n-k-1}(R_1\xi)] = \frac{1}{n-1}\sum_{j=0}^{n-2} \mathbb{E}[\hat{q}_{n-2-j}(L_1\xi)\hat{q}_j(R_1\xi)]\end{aligned}$$

which is $\hat{q}_{n-1}$ by the recursive definition in (11). □

3. Convergence results

In order to state our results we need to review some elementary facts about the central limit theorem for stable distributions. Let us recall that a probability distribution is said to be *a centered stable law* of exponent α (with $0 < \alpha \leq 2$) if its characteristic function is of the form

$$\hat{g}_\alpha(\xi) = \begin{cases} \exp\{-k|\xi|^\alpha(1 - i\eta\tan(\pi\alpha/2)\,\mathrm{sign}\,\xi)\} & \text{if } \alpha \in (0,1)\cup(1,2) \\ \exp\{-k|\xi|(1 + 2i\eta/\pi\log|\xi|\,\mathrm{sign}\,\xi)\} & \text{if } \alpha = 1 \\ \exp\{-\sigma^2|\xi|^2/2\} & \text{if } \alpha = 2. \end{cases} \tag{15}$$

where $k > 0$ and $|\eta| \leq 1$.

By definition, a distribution function F belongs to the *domain of normal attraction* of a stable law of exponent α if for any sequence of independent and identically distributed real-valued random variables $(X_n)_{n\geq 1}$ with common distribution function F, there exists a sequence of real numbers $(c_n)_{n\geq 1}$ such that the law of

$$\frac{1}{n^{1/\alpha}}\sum_{i=1}^{n} X_i - c_n$$

converges weakly to a stable law of exponent $\alpha \in (0, 2]$.

It is well-known that, provided $\alpha \neq 2$, F belong to the domain of normal attraction of an α-stable law if and only if F satisfies

$$\begin{aligned}\lim_{x\to+\infty} x^\alpha(1 - F(x)) &= c^+ < +\infty, \\ \lim_{x\to-\infty} |x|^\alpha F(x) &= c^- < +\infty.\end{aligned} \tag{16}$$

Typically, one also requires that $c^+ + c^- > 0$ in order to exclude convergence to the probability measure concentrated in $x = 0$, but here we shall include the situation $c^+ = c^- = 0$ as a special case. The parameters k and η of the associated stable law in (15) are identified from c^+ and c^- by

$$k = (c^+ + c^-)\frac{\pi}{2\Gamma(\alpha)\sin(\pi\alpha/2)}, \qquad \eta = \frac{c^+ - c^-}{c^+ + c^-}, \tag{17}$$

with the convention that $\eta = 0$ if $c^+ + c^- = 0$. In contrast, if $\alpha = 2$, F belongs to the domain of normal attraction of a Gaussian law if and only if it has finite variance σ^2.

For more information on stable laws and central limit theorem see, for example, Chapter 2 of Ibragimov and Linnik (1971) and Chapter 17 of Fristedt and Gray (1997).

3.1. Convergence in distribution.

We return to our investigation of solutions to the initial value problem (1)-(2). For definiteness, let the two non-negative random variables L and R, which define the dynamics in (2), be fixed from now on. We assume that they satisfy

$$\mathbb{E}[L^\alpha + R^\alpha] = 1 \tag{18}$$

for some number $\alpha \in (0, 2]$. We introduce the convex function $\mathcal{S} : [0,\infty) \to [-1,\infty]$ by

$$\mathcal{S}(s) = \mathbb{E}[L^s + R^s] - 1,$$

where we adopt the convention $0^0 = 0$. From (18) it follows that $\mathcal{S}(\alpha) = 0$. Recall that F_0 is the probability distribution function of the initial condition X_0 for (1), and its characteristic function is ϕ_0.

The main results presented below show that if F_0 belongs to the domain of normal attraction of an α-stable law, then the solution $\phi(t;\cdot)$ to the problem (1)-(2) converges, as $t \to +\infty$, to the characteristic function of a mixture of stable distributions of exponent α. The mixing distribution is given by the law of the limit for $n \to \infty$ of the random variables

$$M_n^{(\alpha)} = \sum_{j=1}^{n} \beta_{j,n}^{\alpha},$$

which are defined in terms of the random weights defined in (12). The content of the following lemma is that $M_n^{(\alpha)}$ converges almost surely to a random variable $M_\infty^{(\alpha)}$.

Lemma 3.1. *Under condition* (18),

$$\mathbb{E}[M_n^{(\alpha)}] = \mathbb{E}[M_{\nu_t}^{(\alpha)}] = 1 \quad \textit{for all } n \geq 1 \textit{ and } t > 0, \tag{19}$$

and $M_n^{(\alpha)}$ converges almost surely to a non-negative random variable $M_\infty^{(\alpha)}$.

In particular,

- *if $L^\alpha + R^\alpha = 1$ a.s., then $\mathcal{S}(s) \geq 0$ for every $s < \alpha$ and $\mathcal{S}(s) \leq 0$ for every $s > \alpha$. Moreover, $M_n^{(\alpha)} = M_\infty^{(\alpha)} = 1$ almost surely;*
- *if $P\{L^\alpha + R^\alpha = 1\} < 1$ and if $\mathcal{S}(\gamma) < 0$ for some $0 < \gamma < \alpha$, then $M_\infty^{(\alpha)} = 0$ almost surely;*
- *if $P\{L^\alpha + R^\alpha = 1\} < 1$ and if $\mathcal{S}(\gamma) < 0$ for some $\gamma > \alpha$, then $M_\infty^{(\alpha)}$ is a non-degenerate random variable with $\mathbb{E}[M_\infty^{(\alpha)}] = 1$ and $\mathbb{E}[(M_\infty^{(\alpha)})^{\frac{\gamma}{\alpha}}] < +\infty$. Moreover, the characteristic function ψ of $M_\infty^{(\alpha)}$ is the unique solution of*

$$\psi(\xi) = \mathbb{E}[\psi(\xi L^\alpha)\psi(\xi R^\alpha)] \qquad (\xi \in \mathbb{R}) \tag{20}$$

with $-i\psi'(0) = 1$. Finally, for any $p > \alpha$, the moment $\mathbb{E}[(M_\infty^{(\alpha)})^{\frac{p}{\alpha}}]$ is finite if and only if $\mathcal{S}(p) < 0$.

We are eventually in the position to formulate our main results. The first statement concerns the case where $\alpha \neq 1$ and $\alpha \neq 2$.

Theorem 3.2. *Assume that* (18) *holds with $\alpha \in (0,1) \cup (1,2)$ and that $\mathcal{S}(\gamma) < 0$ for some $\gamma > 0$. Moreover, let condition* (16) *be satisfied for $F = F_0$ and let X_0 be centered if $\alpha > 1$. Then V_t converges in distribution, as $t \to +\infty$, to a random variable V_∞ with the following characteristic function*

$$\phi_\infty(\xi) = \mathbb{E}[\exp(i\xi V_\infty)] = \mathbb{E}[\exp\{-|\xi|^\alpha k M_\infty^{(\alpha)}(1 - i\eta \tan(\pi\alpha/2)\,\mathrm{sign}\,\xi)\}] \qquad (\xi \in \mathbb{R}), \tag{21}$$

where the parameters k and η are defined in (17)*. In particular, V_∞ is a non-degenerate random variable if $c^+ + c^- > 0$ and $\gamma > \alpha$, whereas $V_\infty = 0$ a.s. if $c^+ = c^- = 0$, or if $\gamma < \alpha$. Moreover, if $L^\alpha + R^\alpha = 1$ a.s., then the distribution of V_∞ is an α-stable law. Finally, if V_∞ is non-degenerate, then $\mathbb{E}[|V_\infty|^p] < +\infty$ if and only if $p < \alpha$.*

If $c^- = c^+$ then the limit distribution is a mixture of symmetric stable distributions. For instance this is true if F_0 is the distribution function of a symmetric random variable.

If $\alpha < 1$ and $X_0 \geq 0$, then clearly $c^- = 0$ and the limit distribution is a mixture of positive stable distributions. Recall that a positive stable distribution is characterized by its Laplace transform $s \mapsto \exp(-ks^\alpha)$; hence, in this case,

$$\mathbb{E}[\exp(-sV_\infty)] = \mathbb{E}[\exp\{-s^\alpha \bar{k} M_\infty^{(\alpha)}\}] \quad \text{for all } s > 0, \text{ with } \bar{k} = c^+ \textstyle\int_0^{+\infty} \frac{(1-e^{-y})}{y^{\alpha+1}} dy.$$

A consequence of Theorem 3.2 is that if $\mathbb{E}[|X_0|^\alpha] < \infty$, then the limit V_∞ is zero almost surely, since $c^+ = c^- = 0$. The situation is different in the cases $\alpha = 1$ and $\alpha = 2$, where V_∞ is non-trivial provided that the first respectively second moment of X_0 is finite.

Theorem 3.3. *Assume that* (18) *holds with* $\alpha = 1$ *and that* $\mathcal{S}(\gamma) < 0$ *for some* $\gamma > 0$. *If the initial condition possesses a finite first moment* $m_0 = \mathbb{E}[X_0]$, *then* V_t *converges in distribution, as* $t \to +\infty$, *to* $V_\infty := m_0 M_\infty^{(1)}$. *In particular,* V_∞ *is non-degenerate if* $\gamma > 1$ *and* $m_0 \neq 0$, *whereas* $V_\infty = 0$ *if* $\gamma < 1$. *Moreover, if* $L + R = 1$ *a.s., then* $V_\infty = m_0$ *a.s. Finally, if* V_∞ *is non-degenerate and* $p > 1$, *then* $\mathbb{E}[|V_\infty|^p] < +\infty$ *if and only if* $\mathcal{S}(p) < 0$.

We remark that under the hypotheses of the previous theorem, the first moment of the solution is preserved in time. Indeed one has,

$$\mathbb{E}[V_t] = \mathbb{E}\big[\mathbb{E}\big[\textstyle\sum_{j=1}^{\nu_t} \beta_{j,\nu_t} X_j \big| \nu_t,\, \beta_{1,\nu_t}, \dots, \beta_{\nu_t,\nu_t}\big]\big] = m_0 \mathbb{E}[M_{\nu_t}^{(1)}] = m_0,$$

where the last equality follows from (19).

Theorem 3.3 above is the most natural generalization of the results in Matthes and Toscani (2008), where the additional condition $\mathbb{E}[|X_0|^{1+\epsilon}] < \infty$ for some $\epsilon > 0$ has been assumed. The respective statement for $\alpha = 2$ reads as follows.

Theorem 3.4. *Assume that* (18) *holds with* $\alpha = 2$ *and that* $\mathcal{S}(\gamma) < 0$ *for some* $\gamma > 0$. *If* $\mathbb{E}[X_0] = 0$ *and* $\sigma^2 = \mathbb{E}[X_0^2] < +\infty$, *then* V_t *converges in distribution, as* $t \to +\infty$, *to a random variable* V_∞ *with characteristic function*

$$\phi_\infty(\xi) = \mathbb{E}[\exp(i\xi V_\infty)] = \mathbb{E}\left[\exp(-\xi^2 \frac{\sigma^2}{2} M_\infty^{(2)})\right] \qquad (\xi \in \mathbb{R}). \tag{22}$$

In particular, V_∞ *is a non-degenerate random variable if* $\gamma > 2$ *and* $\sigma^2 > 0$, *whereas* $V_\infty = 0$ *a.s. if* $\gamma < 2$. *Moreover, if* $L^2 + R^2 = 1$ *a.s., then* V_∞ *is a Gaussian random variable. Finally, if* V_∞ *is non-degenerate and* $p > 2$, *then* $\mathbb{E}[|V_\infty|^p] < +\infty$ *if and only if* $\mathcal{S}(p) < 0$.

Some additional properties of the solution V_t should be mentioned: centering is obviously propagated from the initial condition X_0 to the solution V_t at all later times $t \geq 0$. Moreover, under the hypotheses of the theorem, the second moment of the solution is preserved in time. Indeed, taking into account that the X_i are independent and centered,

$$\mathbb{E}[V_t^2] = \mathbb{E}\big[\mathbb{E}\big[\textstyle\sum_{j,k=1}^{\nu_t} \beta_{j,\nu_t}\beta_{k,\nu_t} X_j X_k \big| \nu_t,\, \beta_{1,\nu_t}, \dots, \beta_{\nu_t,\nu_t}\big]\big] = \sigma^2 \mathbb{E}[M_{\nu_t}^{(2)}] = \sigma^2,$$

where we have used (19) in the last step.

The technically most difficult result concerns the situation $\alpha = 1$ for an initial condition of infinite first moment. Weak convergence to a limit can still be proven if $\mathbb{E}[|X_0|] = \infty$, but the law of X_0 belongs to the domain of normal attraction of a 1-stable distribution. However, a suitable time-dependent centering needs to be applied to the random variables V_t.

Theorem 3.5. *Assume that* (18) *holds with* $\alpha = 1$ *and that* $\mathcal{S}(\gamma) < 0$ *for some* $\gamma > 0$. *Moreover, let the condition* (16) *be satisfied for* $F = F_0$. *Then the random variable*

$$V_t^* := V_t - \sum_{j=1}^{\nu_t} q_{j,\nu_t}, \quad \textit{where} \quad q_{j,n} := \int_{\mathbb{R}} \sin(\beta_{j,n} x) dF_0(x), \tag{23}$$

converges in distribution to a limit V_∞^* *with characteristic function*

$$\phi_\infty(\xi) = \mathbb{E}[\exp(i\xi V_\infty^*)] = \mathbb{E}[\exp\{-|\xi| k M_\infty^{(1)} (1 + 2i\eta/\pi \log|\xi| \operatorname{sign}\xi)\}] \qquad (\xi \in \mathbb{R}) \tag{24}$$

where the parameters k *and* η *are defined in* (17). *In particular,* V_∞ *is a non-degenerate random variable if* $c^+ + c^- > 0$ *and* $\gamma > 1$, *whereas* $V_\infty = 0$ *a.s. if* $c^+ = c^- = 0$, *or if* $\gamma < 1$. *Moreover, if* $L + R = 1$ *a.s., then the distribution of* V_∞ *is a* 1*-stable law. Finally, if* V_∞ *is non-degenerate, then* $\mathbb{E}[|V_\infty|^p] < +\infty$ *if and only if* $p < 1$.

3.2. **Rates of convergence in Wasserstein metrics.** Recall that the Wasserstein distance of order $\gamma > 0$ between two random variables X and Y is defined by

$$\mathcal{W}_\gamma(X, Y) := \inf_{(X', Y')} (\mathbb{E}|X' - Y'|^\gamma)^{1/\max(\gamma,1)}. \tag{25}$$

The infimum is taken over all pairs (X', Y') of real random variables whose marginal distribution functions are the same as those of X and Y, respectively. In general, the infimum in (25) may be infinite; a sufficient (but not necessary) condition for finite distance is that both $\mathbb{E}[|X|^\gamma] < \infty$ and $\mathbb{E}[|Y|^\gamma] < \infty$. For more information on Wasserstein distances see, for example, Rachev (1991).

Recall that the V_t are random variables whose characteristic functions $\phi(t)$ solve the initial value problem (1) for the Boltzmann equation for $t \geq 0$, and V_∞ is the limit in distribution of V_t as $t \to \infty$.

Proposition 3.6. *Assume* (18) *and* $\mathcal{S}(\gamma) < 0$, *for some* γ *with* $1 \leq \alpha < \gamma \leq 2$ *or* $\alpha < \gamma \leq 1$. *Assume further that* (16) *holds if* $\alpha \neq 1$, *or that* $\mathbb{E}[|X_0|^\gamma] < +\infty$ *if* $\alpha = 1$, *respectively. Then*

$$\mathcal{W}_\gamma(V_t, V_\infty) \leq A\mathcal{W}_\gamma(X_0, V_\infty)e^{-Bt|\mathcal{S}(\gamma)|}, \tag{26}$$

with $A = B = 1$ *if* $\gamma \leq 1$, *or* $A = 2^{1/\gamma}$ *and* $B = 1/\gamma$ *otherwise.*

Clearly, the content of Proposition 3.6 is void unless

$$\mathcal{W}_\gamma(X_0, V_\infty) < \infty. \tag{27}$$

In the case $\alpha = 1$, the hypothesis $\mathbb{E}|X_0|^\gamma < +\infty$ guarantees (27). In all other cases, (27) is a non-trivial requirement since, by Theorem 3.2, either $V_\infty = 0$ or $\mathbb{E}[|V_\infty|^\alpha] = +\infty$. The following Lemma provides a sufficient criterion for (27), tailored to the situation at hand.

Lemma 3.7. *Assume, in addition to the hypotheses of Proposition 3.6, that* $\gamma < 2\alpha$ *and that* F_0 *satisfies hypothesis* (16) *in the more restrictive sense that there exists a constant* $K > 0$ *and some* $0 < \epsilon < 1$ *with*

$$|1 - c^+x^{-\alpha} - F_0(x)| < Kx^{-(\alpha+\epsilon)} \quad \textit{for } x > 0, \tag{28}$$

$$|F_0(x) - c^-(-x)^{-\alpha}| < K(-x)^{-(\alpha+\epsilon)} \quad \textit{for } x < 0. \tag{29}$$

Provided that $\gamma < \alpha/(1-\epsilon)$ *it follows* $\mathcal{W}_\gamma(X_0, V_\infty) < \infty$, *and then estimate* (26) *is non-trivial.*

3.3. **Strong convergence of densities.** As already mentioned in the introduction, under suitable hypotheses, the probability densities of $\mu(t)$ exist and converge strongly in the Lebesgue spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.

Theorem 3.8. *For given* $\alpha \in (0,1) \cup (1,2]$, *let the hypotheses of Theorem 3.2 or Theorem 3.4 hold with* $\gamma > \alpha$. *Assume further that* (16) *holds with* $c^- + c^+ > 0$ *if* $\alpha < 2$, *so that the* V_t *converges in distribution, as* $t \to +\infty$, *to a non-degenerate limit* V_∞. *Moreover assume also that*

- *(H1)* $L^r + R^r \geq 1$ *a.s. for some* $r > 0$,
- *(H2)* X_0 *possesses a density* f_0 *with finite Linnik-Fisher functional, i.e.* $h := \sqrt{f_0} \in H^1(\mathbb{R})$, *or equivalently, its Fourier transform* $\widehat{h}$ *satisfies*

$$\int_{\mathbb{R}} |\xi|^2 \left|\widehat{h}(\xi)\right|^2 d\xi < +\infty.$$

Then, the random variable V_t *possesses a density* $f(t)$ *for all* $t \geq 0$, V_∞ *has a density* f_∞, *and the* $f(t)$ *converges, as* $t \to +\infty$, *to* f_∞ *in any* $L^p(\mathbb{R})$ *with* $1 \leq p \leq 2$, *that is*

$$\lim_{t\to+\infty} \int_{\mathbb{R}} |f(t;v) - f_\infty(v)|^p dv = 0.$$

Remark 1. *Some comments on the hypotheses (H1) and (H2) are in order.*

- *In view of* $\mathcal{S}(\alpha) = 1$, *condition (H1) can be satisfied only if* $r < \alpha$. *Notice that (H1) becomes the weaker the smaller* $r > 0$ *is; in fact, the sets* $\{(x,y)|x^r + y^r \geq 1\} \subset \mathbb{R}^2$ *exhaust the first quadrant as* $r \searrow 0$.
- *The smoothness condition (H2) is not quite as restrictive as it may seem. For instance, recall that the convolution of any probability density* f_0 *with an arbitrary "mollifier" of finite Linnik-Fisher functional (e.g. a Gaussian) produces again a probability density of finite Linnik-Fisher functional.*

4. Proofs

We continue to assume that the law of the random vector (L, R) is given and satisfies (18) with $\alpha \in (0, 2]$, implying $\mathcal{S}(\alpha) = 0$.

4.1. **Properties of the weights $\beta_{j,n}$ (Lemma 3.1).** In this subsection we shall prove a generalization of a useful result obtained in Gabetta and Regazzini (2006a). Set

$$G_n = (I_1, \dots, I_{n-1}, L_1, R_1, \dots, L_{n-1}, R_{n-1}). \tag{30}$$

and denote by $\mathcal{G}_n$ the σ-algebra generated by G_n.

Proposition 4.1. *If $\mathbb{E}[L^s + R^s] < +\infty$ for some $s > 0$, then*

$$\mathbb{E}[M_n^{(s)}] = \mathbb{E}[\sum_{j=1}^n \beta_{j,n}^s] = \frac{\Gamma(n + \mathcal{S}(s))}{\Gamma(n)\Gamma(\mathcal{S}(s)+1)}$$

and

$$\mathbb{E}[M_{\nu_t}^{(s)}] = \sum_{n \geq 1} e^{-t}(1-e^{-t})^{n-1}\mathbb{E}[M_n^{(s)}] = e^{t\mathcal{S}(s)}.$$

If in addition $\mathcal{S}(s) = 0$, for some $s > 0$, then $M_n^{(s)}$ is a martingale with respect to $(\mathcal{G}_n)_{n\geq 1}$.

Proof. Recall that $(\beta_{1,1}, \beta_{1,2}, \beta_{2,2}, \dots, \beta_{n,n})$ is $\mathcal{G}_n$–measurable, see (12). We first prove that $\mathbb{E}[M_{n+1}^{(s)}|\mathcal{G}_n] = M_n^{(s)}(1 + \mathcal{S}(s)/n)$, which implies that $M_n^{(s)}$ is a $(\mathcal{G}_n)_n$–martingale whenever $\mathcal{S}(s) = 0$, since $M_n^{(s)} \geq 0$ and, as we will see, $\mathbb{E}[M_n^{(s)}] < +\infty$ for every $n \geq 1$. To prove the claim write

$$\begin{aligned}
\mathbb{E}[M_{n+1}^{(s)}|\mathcal{G}_n] &= \mathbb{E}\Big[\sum_{i=1}^n \mathbb{I}\{I_n = i\}\sum_{j=1}^{n+1}\beta_{j,n+1}^s\Big|\mathcal{G}_n\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^n \mathbb{I}\{I_n = i\}\Big(\sum_{j=1,\dots,n+1, j\neq i,i+1}\beta_{j,n+1}^s + \beta_{i,n+1}^s + \beta_{i+1,n+1}^s\Big)\Big|\mathcal{G}_n\Big] \\
&= \mathbb{E}\Big[\sum_{i=1}^n \mathbb{I}\{I_n = i\}\Big(\sum_{j=1}^n \beta_{j,n}^s + \beta_{i,n}^s(L_n^s + R_n^s - 1)\Big)\Big|\mathcal{G}_n\Big] \\
&= M_n^{(s)} + \mathcal{S}(s)\mathbb{E}\Big[\sum_{i=1}^n \mathbb{I}\{I_n = i\}\beta_{i,n}^s|\mathcal{G}_n\Big] \\
&= M_n^{(s)} + \mathcal{S}(s)\sum_{i=1}^n \beta_{j,n}^s \mathbb{E}[\mathbb{I}\{I_n = i\}] = M_n^{(s)}(1 + \mathcal{S}(s)/n).
\end{aligned}$$

Taking the expectation of both sides one gets

$$\mathbb{E}[M_{n+1}^{(s)}] = \mathbb{E}[M_n^{(s)}](1 + \frac{1}{n}\mathcal{S}(s)).$$

Since $\mathbb{E}[M_2^{(s)}] = \mathcal{S}(s) + 1$ it follows easily that

$$\mathbb{E}[M_n^{(s)}] = \prod_{i=1}^{n-1}(1 + \mathcal{S}(s)/i) = \frac{\Gamma(n + \mathcal{S}(s))}{\Gamma(n)\Gamma(\mathcal{S}(s)+1)}.$$

To conclude the proof use formula 5.2.13.30 in Prudnikov et al. (1986). □

Lemma 4.2. *If $\mathcal{S}(\gamma) < 0$ for some $\gamma > 0$, then*

$$\beta_{(n)} := \max_{1 \leq j \leq n} \beta_{j,n}$$

converges to zero in probability as $n \to +\infty$.

Proof. Observe that, for every $\epsilon > 0$,

$$P\{\beta_{(n)} > \epsilon\} \leq P\Big\{\sum_{j=1}^n \beta_{j,n}^\gamma \geq \epsilon^\gamma\Big\}$$

and hence, by Markov's inequality and Proposition 4.1,

$$P\{\beta_{(n)} > \epsilon\} \leq \frac{1}{\epsilon^\gamma}\mathbb{E}[M_n^{(\gamma)}] = \frac{1}{\epsilon^\gamma}\frac{\Gamma(n + \mathcal{S}(\gamma))}{\Gamma(n)\Gamma(\mathcal{S}(\gamma)+1)} \leq C\frac{1}{\epsilon^\gamma}n^{\mathcal{S}(\gamma)}.$$

The last expression tends to zero as $n \to \infty$ because $\mathcal{S}(\gamma) < 0$. □

Proof of Lemma 3.1. Since $\mathcal{S}(\alpha) = 0$, the random variables $M_n^{(\alpha)}$ form a positive martingale with respect to $(\mathcal{G}_n)_n$ by Proposition 4.1. By the martingale convergence theorem, see e.g. Theorem 19 in Chapter 24 of Fristedt and Gray (1997), it converges a.s. to a positive random variable $M_\infty^{(\alpha)}$ with $\mathbb{E}[M_\infty^{(\alpha)}] \le \mathbb{E}[M_1^{(\alpha)}] = 1$. The goal of the following is to determine the law of $M_\infty^{(\alpha)}$ in the different cases we consider.

First, suppose that $L^\alpha + R^\alpha = 1$ a.s. It follows that $L^\alpha \le 1$ and $R^\alpha \le 1$ a.s., and hence $\mathcal{S}(s) \le \mathcal{S}(\alpha) = 0$ for all $s > \alpha$. Moreover, it is plain to check that $M_n^{(\alpha)} = 1$ a.s. for every n, and hence $M_\infty^{(\alpha)} = 1$ a.s.

Next, assume that $\mathcal{S}(\gamma) < 0$ for $\gamma < \alpha$. Minkowski's inequality and Proposition 4.1 give

$$\mathbb{E}\big[(M_n^{(\alpha)})^{\gamma/\alpha}\big] \le \mathbb{E}[\sum_{j=1}^n \beta_{j,n}^\gamma] = \frac{\Gamma(n + \mathcal{S}(\gamma))}{\Gamma(n)\Gamma(\mathcal{S}(\gamma)+1)} \le Cn^{\mathcal{S}(\gamma)}.$$

Hence, $M_n^{(\alpha)}$ converges a.s. to 0.

It remains to treat the case with $\mathcal{S}(\gamma) < 0$ and $\gamma > \alpha$. Since $\mathcal{S}(\cdot)$ is a convex function satisfying $\mathcal{S}(\alpha) = 0$ and $\mathcal{S}(\gamma) < 0$ with $\gamma > \alpha$, it is clear that $\mathcal{S}'(\alpha) < 0$; also, we can assume without loss of generality that $\gamma < 2\alpha$. Further, by hypothesis,

$$\mathbb{E}[(L^\alpha + R^\alpha)^{1+(\gamma/\alpha-1)}] \le 2^{\gamma/\alpha-1}\mathbb{E}[L^\gamma + R^\gamma] < +\infty.$$

Hence, one can resort to Theorem 2(a) of Durrett and Liggett (1983) — see also Corollaries 1.1, 1.4 and 1.5 in Liu (1998) — which provides existence and uniqueness of a probability distribution $\nu_\infty \ne \delta_0$ on $\mathbb{R}^+$, whose characteristic function ψ is a solution of equation (20), with $\int_{\mathbb{R}^+} x\nu_\infty(dx) = 1$. Moreover, Theorem 2.1 in Liu (2000) ensures that $\int_{\mathbb{R}^+} x^{\gamma/\alpha}\nu_\infty(dx) < +\infty$ and, more generally, that $\int_{\mathbb{R}^+} x^{p/\alpha}\nu_\infty(dx) < +\infty$ for some $p > \alpha$ if and only if $\mathcal{S}(p) < 0$.

Consequently, our goal is to prove that the law of $M_\infty^{(\alpha)}$ is ν_∞. In what follows, enlarge the space $(\Omega, \mathcal{F}, P)$ in order to contain all the random elements needed. In particular, let $(M_j)_{j\ge1}$ be a sequence of independent random variables with common characteristic function ψ, such that $(M_j)_{j\ge1}$ and $(G_n)_{n\ge1}$, defined in (30), are independent. Recalling that ψ is a solution of (20) it follows that, for every $n \ge 2$,

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{i\xi\sum_{j=1}^n \beta_{j,n}^\alpha M_j\Big\}\Big] \\
&= \sum_{k=1}^{n-1}\frac{1}{n-1}\mathbb{E}\Big[\exp\Big\{i\xi\Big(\sum_{j=1}^{k-1}\beta_{j,n-1}^\alpha M_j + \beta_{k,n-1}^\alpha \underbrace{(L_{n-1}^\alpha M_k + R_{n-1}^\alpha M_{k+1})}_{=^d M_k} + \sum_{j=k+1}^{n}\beta_{j,n-1}^\alpha M_j\Big)\Big\}\Big] \\
&= \mathbb{E}\Big[\exp\Big\{i\xi\sum_{j=1}^{n-1}\beta_{j,n-1}^\alpha M_j\Big\}\Big].
\end{aligned}$$

By induction on $n \ge 2$, this shows that $\sum_{j=1}^n M_j\beta_{j,n}^\alpha$ has the same law as M_1, which is ν_∞. Hence

$$\mathcal{W}_{\gamma/\alpha}^{\gamma/\alpha}(M_n^{(\alpha)}, M_1) \le \mathbb{E}\Big[\Big|\sum_{j=1}^n \beta_{j,n}^\alpha - \sum_{j=1}^n M_j\beta_{j,n}^\alpha\Big|^{\gamma/\alpha}\Big] = \mathbb{E}\Big[\mathbb{E}\Big[\Big|\sum_{j=1}^n (1-M_j)\beta_{j,n}^\alpha\Big|^{\gamma/\alpha}\Big|\mathcal{G}_n\Big]\Big].$$

We shall now employ the following result from von Bahr and Esseen (1965). *Let $1 < \eta \le 2$, and assume that $Z_1, \dots, Z_n$ are independent, centered random variables and $\mathbb{E}|Z_j|^\eta < +\infty$. Then*

$$\mathbb{E}\Big|\sum_{j=1}^n Z_j\Big|^\eta \le 2\sum_{j=1}^n \mathbb{E}|Z_j|^\eta. \tag{31}$$

We apply this result with $\eta = \gamma/\alpha$ and $Z_j = \beta_{j,n}^{\alpha}(1 - M_j)$, showing that

$$\mathbb{E}\Big[\Big|\sum_{j=1}^{n}(1-M_j)\beta_{j,n}^{\alpha}\Big|^{\gamma/\alpha}\Big|\mathcal{G}_n\Big] \leq 2\sum_{j=1}^{n}\beta_{j,n}^{\gamma}\mathbb{E}\Big[\Big|1-M_j\Big|^{\gamma/\alpha}\Big|\mathcal{G}_n\Big] = 2\sum_{j=1}^{n}\beta_{j,n}^{\gamma}\mathbb{E}|1-M_1|^{\gamma/\alpha}$$

almost surely. In consequence,

$$\mathcal{W}_{\gamma/\alpha}^{\gamma/\alpha}(M_n^{(\alpha)}, M_1) \leq 2\mathbb{E}\Big[\sum_{j=1}^{n}\beta_{j,n}^{\gamma}\Big]\mathbb{E}\Big[|1-M_1|^{\gamma/\alpha}\Big]. \tag{32}$$

By means of Proposition 4.1, one obtains

$$\mathcal{W}_{\gamma/\alpha}^{\gamma/\alpha}(M_n^{(\alpha)}, M_1) \leq C' n^{\mathcal{S}(\gamma)}.$$

This proves that the law of $M_n^{(\alpha)}$ converges with respect to the $\mathcal{W}_{\gamma/\alpha}$ metric – and then also weakly – to the law of M_1. Hence, $M_\infty^{(\alpha)}$ has law ν_∞. The fact that $M_\infty^{(\alpha)}$ is non-degenerate, provided $L^\alpha + R^\alpha = 1$ does *not* hold a.s., follows immediately. □

4.2. **Proof of convergence for $\alpha \neq 1$ (Theorems 3.2 and 3.4).** Denote by $\mathcal{B}$ the σ-algebra generated by $\{\beta_{j,n} : n \geq 1, j = 1, \dots, n\}$. The proof of Theorems 3.2 and 3.4 is essentially an application of the central limit theorem to the conditional law of

$$W_n := \sum_{j=1}^{n}\beta_{j,n}X_j$$

given $\mathcal{B}$. Set

$$Q_{j,n}(x) := F_0\big(\beta_{j,n}^{-1}x\big),$$

where, by convention, $F_0(\cdot/0) := \mathbb{I}_{[0,+\infty)}(\cdot)$. In this subsection we will use the functions:

$$\begin{aligned}
\zeta_n(x) &:= \mathbb{I}\{x<0\}\sum_{j=1}^{n}Q_{j,n}(x) + \mathbb{I}\{x>0\}\sum_{j=1}^{n}(1-Q_{j,n}(x)) \qquad (x \in \mathbb{R})\\
\sigma_n^2(\epsilon) &:= \sum_{j=1}^{n}\Big\{\int_{(-\epsilon,+\epsilon]}x^2\,dQ_{j,n}(x) - \Big(\int_{(-\epsilon,+\epsilon]}x\,dQ_{j,n}(x)\Big)^2\Big\} \qquad (\epsilon > 0)\\
\eta_n &:= \sum_{j=1}^{n}\Big\{1 - Q_{j,n}(1) - Q_{j,n}(-1) + \int_{(-1,1]}x\,dQ_{j,n}(x)\Big\}.
\end{aligned}$$

In terms of $Q_{j,n}$, the conditional distribution function F_n of W_n given $\mathcal{B}$ is the convolution,

$$F_n = Q_{1,n} * \cdots * Q_{n,n}.$$

To start with, we show that the $Q_{j,n}$s satisfy the uniform asymptotic negligibility (UAN) assumption (34) below.

Lemma 4.3. *Let the assumptions of Theorem 3.4 or Theorem 3.5 be in force. Then, for every divergent sequence (n') of integer numbers, there exists a divergent subsequence $(n'') \subset (n')$ and a set Ω_0 of probability one such that*

$$\begin{aligned}
&\lim_{n''\to+\infty} M_{n''}^{(\alpha)}(\omega) = M_\infty^{(\alpha)}(\omega) < \infty,\\
&\lim_{n''\to+\infty} \beta_{(n'')}(\omega) = 0,
\end{aligned} \tag{33}$$

holds for every $\omega \in \Omega_0$. Moreover, for every $\omega \in \Omega_0$ and for every $\epsilon > 0$

$$\lim_{n''\to+\infty}\max_{1\leq j\leq n''}\Big\{1 - Q_{j,n''}(\epsilon) + Q_{j,n''}(-\epsilon)\Big\} = 0. \tag{34}$$

Proof. The existence of a sub-sequence (n'') and a set Ω_0 satisfying (33) is a direct consequence of Lemmata 4.2 and 3.1. To prove (34) note that, for $0 < \alpha \leq 2$ and $\alpha \neq 1$,

$$\max_{1 \leq j \leq n''} \Big(1 - Q_{j,n''}(\epsilon) + Q_{j,n''}(-\epsilon)\Big) \leq 1 - F_0\Big(\frac{\epsilon}{\beta_{(n'')}}\Big) + F_0\Big(\frac{-\epsilon}{\beta_{(n'')}}\Big).$$

The claim hence follows from (33). □

Lemma 4.4. *Let the assumptions of Theorem 3.2 be in force. Then for every divergent sequence (n') of integer numbers there exists a divergent subsequence $(n'') \subset (n')$ and a measurable set Ω_0 of probability one such that*

$$\lim_{n'' \to +\infty} \mathbb{E}[e^{i\xi W_{n''}} | \mathcal{B}](\omega) = \exp\{-|\xi|^\alpha k M_\infty^{(\alpha)}(\omega)(1 - i\eta \tan(\pi\alpha/2) \operatorname{sign} \xi)\} \quad (\xi \in \mathbb{R}) \tag{35}$$

for every ω in Ω_0.

Proof. Let (n'') and Ω_0 be the same as in Lemma 4.3. To prove (35), we apply the central limit theorem for every $\omega \in \Omega_0$ to the conditional law of $W_{n''}$ given $\mathcal{B}$.

For every ω in Ω_0, we know that $F_{n''}$ is a convolution of probability distribution functions satisfying the asymptotic negligibility assumption (34). Here, we shall use the general version of the central limit theorem as presented e.g., in Theorem 30 in Section 16.9 and in Proposition 11 in Section 17.3 of Fristedt and Gray (1997). According to these results, the claim (35) follows if, for every $\omega \in \Omega_0$,

$$\lim_{n'' \to +\infty} \zeta_{n''}(x) = \frac{c^+ M_\infty^{(\alpha)}}{x^\alpha} \qquad (x > 0), \tag{36}$$

$$\lim_{n'' \to +\infty} \zeta_{n''}(x) = \frac{c^- M_\infty^{(\alpha)}}{|x|^\alpha} \qquad (x < 0), \tag{37}$$

$$\lim_{\epsilon \to 0^+} \limsup_{n'' \to +\infty} \sigma_{n''}^2(\epsilon) = 0, \tag{38}$$

$$\lim_{n'' \to +\infty} \eta_{n''} = \frac{1}{1-\alpha} M_\infty^{(\alpha)}(c^+ - c^-) \tag{39}$$

are simultaneously satisfied.

In what follows we assume that $P\{L = 0\} = P\{R = 0\} = 0$, which yields that $\beta_{j,n} > 0$ almost surely. The general case can be treated with minor modifications.

In order to prove (36), fix some $x > 0$, and observe that

$$\zeta_{n''}(x) = \sum_{j=1}^{n''} [1 - F_0(\beta_{j,n''}^{-1} x)] = \sum_{j=1}^{n''} [1 - F_0(\beta_{j,n''}^{-1} x)](\beta_{j,n''}^{-1} x)^\alpha \frac{\beta_{j,n''}^\alpha}{x^\alpha}.$$

Since $\lim_{y \to +\infty}(1 - F_0(y))y^\alpha = c^+$ by assumption (16), for every $\epsilon > 0$ there exists a $Y = Y(\epsilon)$ such that if $y > Y$, then $c^+ - \epsilon \leq (1 - F_0(y))y^\alpha \leq c^+ + \epsilon$. Hence if $x > \beta_{(n'')} Y$, then

$$x^{-\alpha}(c^+ - \epsilon) M_{n''}^{(\alpha)} \leq \sum_{j=1}^{n''} (1 - F_0(\beta_{j,n''}^{-1} x)) \leq x^{-\alpha}(c^+ + \epsilon) M_{n''}^{(\alpha)}. \tag{40}$$

In view of (33), the claim (36) follows immediately. Relation (37) is proved in a completely analogous way.

In order to prove (38), it is clearly sufficient to show that for every $\epsilon > 0$

$$\limsup_{n'' \to +\infty} \sum_{j=1}^{n''} \int_{(-\epsilon, +\epsilon]} x^2 dQ_{j,n''}(x) \leq C M_\infty^{(\alpha)} \epsilon^{2-\alpha} \tag{41}$$

with some constant C independent of ϵ. Recalling the definition of $Q_{j,n}$, an integration by parts gives

$$\int_{(0,\epsilon]} x^2 dF_0\big(\beta_{j,n}^{-1} x\big) = -\epsilon^2 \big[1 - F_0\big(\beta_{j,n}^{-1} \epsilon\big)\big] + 2 \int_0^\epsilon x \big[1 - F_0\big(\beta_{j,n}^{-1} x\big)\big]\, dx,$$

and similarly for the integral from $-\epsilon$ to zero. With

$$K := \sup_{x>0} x^\alpha [1 - F(x)] + \sup_{x<0} (-x)^\alpha F(x), \tag{42}$$

which is finite by hypothesis (16), it follows that

$$\int_{(-\epsilon,+\epsilon]} x^2 dF_0\big(\beta_{j,n}^{-1}x\big) \leq 2K\epsilon^2(\beta_{j,n}^{-1}\epsilon)^{-\alpha} + 4K\beta_{j,n}^\alpha \int_0^\epsilon x^{1-\alpha}\, dx \leq 2K\Big(1 + \frac{2}{2-\alpha}\Big)\beta_{j,n}^\alpha \epsilon^{2-\alpha}.$$

To conclude (41), it suffices to recall that $\sum_j \beta_{j,n''}^\alpha = M_{n''}^{(\alpha)} \to M_\infty^{(\alpha)}$ by (33).

In order to prove (39), we need to distinguish if $0 < \alpha < 1$, or if $1 < \alpha < 2$. In the former case, integration by parts in the definition of $\eta_{n''}$ reveals

$$\eta_{n''} = \int_{-1}^1 \zeta_{n''}(x) dx.$$

Having already shown (36) and (37), we know that the integrand converges pointwise with respect to x. The dominated convergence theorem applies since, by hypothesis (16),

$$|\zeta_{n''}(x)| \leq K|x|^{-\alpha} \sup_{n''} M_{n''}^{(\alpha)} \tag{43}$$

with the constant K defined in (42); observe that $|x|^{-\alpha}$ is integrable on $(-1, 1]$ since we have assumed $0 < \alpha < 1$. Consequently,

$$\lim_{n''\to\infty} \eta_{n''} = c^- M_\infty^{(\alpha)} \int_{-1}^0 |x|^{-\alpha}\, dx + c^+ M_\infty^{(\alpha)} \int_0^1 |x|^{-\alpha}\, dx = \frac{c^+ - c^-}{1-\alpha} M_\infty^{(\alpha)}.$$

It remains to check (39) for $1 < \alpha < 2$. Since $\int_{\mathbb{R}} x\, dQ_{j,n''}(x) = 0$, one can write

$$\eta_{n''} = \eta_{n''} - \sum_{j=1}^{n''} \int_{\mathbb{R}} x\, dQ_{j,n''}(x) = -\sum_{j=1}^{n''} \int_{(-\infty,-1]} (1+x) dQ_{j,n''}(x) - \sum_{j=1}^{n''} \int_{(1,+\infty)} (x-1) dQ_{j,n''}(x).$$

Similar as for $0 < \alpha < 1$, integration by parts reveals that

$$\eta_{n''} = \int_{\{|x|>1\}} \zeta_{n''}(x)\, dx. \tag{44}$$

From this point on, the argument is the same as in the previous case: (36) and (37) provide pointwise convergence of the integrand; hypothesis (16) leads to (43), which guarantees that the dominated convergence theorem applies, since $|x|^{-\alpha}$ is integrable on the set $\{|x| > 1\}$. It is straightforward to verify that the integral of the pointwise limit indeed yields the right-hand side of (39). □

Proof of Theorem 3.2. By Lemma 4.4 and the dominated convergence theorem, every divergent sequence (n') of integer numbers contains a divergent subsequence $(n'') \subset (n')$ for which

$$\lim_{n''\to+\infty} \mathbb{E}[\exp\{i\xi W_{n''}\}] = \mathbb{E}\big[\exp\big\{ -|\xi|^\alpha k M_\infty^{(\alpha)}(1 - i\eta \tan(\pi\alpha/2) \operatorname{sign} \xi)\big\}\big], \tag{45}$$

where the limit is pointwise in $\xi \in \mathbb{R}$. Since the limiting function is independent of the arbitrarily chosen sequence (n'), a classical argument shows that (45) is true with $n \to +\infty$ in place of $n'' \to +\infty$. In view of Proposition 2.1, the stated convergence follows.

By Lemma 3.1, the assertion about (non)-degeneracy of V_∞ follows immediately from the representation (21). To verify the claim about moments for $\gamma > \alpha$, observe that (21) implies that

$$\mathbb{E}[|V_\infty|^p] = \int_{\mathbb{R}} |x|^p dF_\infty(x) = \mathbb{E}\big[\big(M_\infty^{(\alpha)}\big)^p\big] \int_{\mathbb{R}} |u|^p dG_\alpha(u), \tag{46}$$

where G_α is the distribution function of the centered α-stable law with characteristic function $\hat{g}_\alpha$ defined in (15).

The p-th moment of $M_\infty^{(\alpha)}$ is finite at least for all $p < \gamma$ by Lemma 3.1. On the other hand, the p-th moment of G_α is finite if and only if $p < \alpha$. □

The following lemma replaces Lemma 4.4 in the case $\alpha = 2$.

Lemma 4.5. *Let the assumptions of Theorem 3.4 hold. Then for every divergent sequence (n') of integer numbers, there exists a divergent subsequence $(n'') \subset (n')$ and a set Ω_0 of probability one such that*

$$\lim_{n''\to+\infty} \mathbb{E}[e^{i\xi W_{n''}}|\mathcal{B}](\omega) = e^{-\xi^2 \frac{\sigma^2}{2} M_\infty^{(2)}(\omega)} \qquad (\xi \in \mathbb{R})$$

for every ω in Ω_0.

Proof. Let (n'') and Ω_0 have the properties stated in Lemma 4.3. The claim follows if for every ω in Ω_0,

$$\lim_{n''\to+\infty} \zeta_{n''}(x) = 0 \qquad (x \neq 0), \tag{47}$$

$$\lim_{\epsilon\to 0^+} \lim_{n''\to+\infty} \sigma^2_{n''}(\epsilon) = \sigma^2 M_\infty^{(2)}, \tag{48}$$

$$\lim_{n''\to+\infty} \eta_{n''} = 0 \tag{49}$$

are simultaneously satisfied.

First of all note that since

$$y^2(1 - F_0(y)) \leq \int_{(y,+\infty)} x^2 dF_0(x) \quad \text{and} \quad y^2 F_0(-y) \leq \int_{(-\infty,-y]} x^2 dF_0(x) \quad (y > 0)$$

and $\int_{\mathbb{R}} x^2 dF_0(x) < +\infty$, it follows that $\lim_{y\to+\infty} y^2(1 - F_0(y)) = \lim_{y\to-\infty} y^2(F_0(y)) = 0$. Hence, given $\epsilon > 0$, there exists a $Y = Y(\epsilon)$ such that $y^2(1 - F_0(y)) < \epsilon$ for every $y > Y$. Since

$$\zeta_{n''}(x) = \sum_{j=1}^{n''} (\beta^{-1}_{j,n''} x)^2 (1 - F_0(\beta^{-1}_{j,n''} x)) \beta^2_{j,n''}/x^2 \qquad (x > 0),$$

one gets

$$\zeta_{n''}(x) \leq \epsilon M^{(2)}_{n''} \frac{1}{x^2}$$

whenever $x > \beta_{(n'')} Y$. In view of property (33), the first relation (47) follows for $x > 0$. The argument for $x < 0$ is analogous.

We turn to the proof of (48). A simple computation reveals

$$s^2_{n''}(\epsilon) := \sum_{j=1}^{n''} \int_{(-\epsilon,\epsilon]} x^2 dQ_{j,n''}(x) = \sigma^2 M^{(2)}_{n''} - R_{n''},$$

with the remainder term

$$R_{n''} := \sum_{j=1}^{n''} \beta^2_{j,n''} \int_{|\beta_{j,n''} x| > \epsilon} x^2 dF_0(x) \leq M^{(2)}_{n''} \int_{|\beta_{(n'')} x| > \epsilon} x^2 dF_0(x).$$

Invoking property (33) again, it follows that $R_{n''} \to 0$ as $n'' \to \infty$; recall that F_0 has finite second moment by hypothesis. Consequently,

$$\lim_{n''\to+\infty} s^2_{n''}(\epsilon) = \sigma^2 M^{(2)}_\infty \tag{50}$$

for every ϵ. Since $\int_{\mathbb{R}} x\, dQ_{j,n}(x) = 0$,

$$\sum_{j=1}^{n''} \Big(\int_{(-\epsilon,\epsilon]} x\, dQ_{j,n}(x) \Big)^2 \leq \sum_{j=1}^{n''} \beta^2_{j,n''} \Big(\int_{|\beta_{j,n''} x| \geq \epsilon} x dF_0(x) \Big)^2 \leq \Big(\int_{|\beta_{(n'')} x| \geq \epsilon} |x| dF_0(x) \Big)^2 \sum_{j=1}^{n''} \beta^2_{j,n''},$$

which yields that

$$\lim_{n''\to+\infty} \sum_{j=1}^{n''} \Big(\int_{(-\epsilon,\epsilon]} x\, dQ_{j,n}(x) \Big)^2 = 0.$$

Combining this last fact with (50) gives (48).

Finally, in order to obtain (49), we use (47) and the dominated convergence theorem; the argument is the same as for (44) in the proof of Lemma 4.4. □

Proof of Theorem 3.4. Use Lemma 4.5 and repeat the proof of Theorem 3.2. A trivial adaptation is needed in the calculation of moments if $\gamma > 2$: consider (46) with $G_\alpha = G_2$, the distribution function of a Gaussian law, and note that it posses finite moments of every order. Hence $\mathbb{E}[|V_\infty|^p]$ is finite if and only if $\mathbb{E}[(M_\infty^{(2)})^p]$ is finite, which, by Lemma 3.1, is the case if and only if $\mathcal{S}(p) < 0$. □

4.3. **Proof of convergence for $\alpha = 1$ (Theorems 3.3 and 3.5).** Let us first prove Theorem 3.5. We shall apply the central limit theorem to the random variables

$$W_n^* = \sum_{j=1}^n (\beta_{j,n} X_j - q_{j,n})$$

with $q_{j,n}$ defined in (23). In what follows,

$$Q_{j,n}(x) := F_0\Big(\frac{x + q_{j,n}}{\beta_{j,n}}\Big).$$

The next Lemma is the analogue of Lemma 4.3 above.

Lemma 4.6. *Suppose the assumptions of Theorem 3.5 are in force. Then, for every $\delta \in (0,1)$,*

$$|q_{j,n}| = \Big|\int \sin(\beta_{j,n}s)dF_0(s)\Big| \leq C_\delta \beta_{j,n}^{1-\delta} \tag{51}$$

with $C_\delta = \int_{\mathbb{R}} |x|^{1-\delta} dF_0(x) < +\infty$. Furthermore, for every divergent sequence (n') of integer numbers, there exists a divergent subsequence $(n'') \subset (n')$ and a set Ω_0 of probability one such that for every ω in Ω_0 and for every $\epsilon > 0$, the properties (33) and (34) are verified.

Proof. First of all note that $C_\delta < +\infty$ for every $\delta \in (0,1)$ because of hypothesis (16). Using further that $|\sin(x)| \leq |x|^{1-\delta}$ for $\delta \in (0,1)$, one immediately gets

$$|q_{j,n}| = \Big|\int_{\mathbb{R}} \sin(\beta_{j,n}s)dF_0(s)\Big| \leq \beta_{j,n}^{1-\delta} \int_{\mathbb{R}} |s|^{1-\delta} dF_0(s).$$

To prove (34) note that, as a consequence of (51),

$$\frac{\epsilon + q_{j,n}}{\beta_{j,n}} \geq \beta_{(n)}^{-1}(\epsilon - C_\delta \beta_{(n)}^{1-\delta}).$$

Clearly, the expression inside the bracket is positive for sufficiently small $\beta_{(n)}$. Defining (n'') and Ω_0 in accordance to Lemma 4.3, it thus follows

$$\max_{1 \leq j \leq n''} \Big(1 - Q_{j,n''}(\epsilon) + Q_{j,n''}(-\epsilon)\Big) \leq 1 - F_0(\bar{c}\beta_{(n'')}^{-1}) + F_0(-\bar{c}\beta_{(n'')}^{-1})$$

for a suitable constant $\bar{c}$ depending only on ϵ, δ and F_0. An application of (33) yields (34). □

Lemma 4.7. *Suppose the assumptions of Theorem 3.5 are in force, then for every divergent sequence (n') of integer numbers there exists a divergent subsequence $(n'') \subset (n')$ and a measurable set Ω_0 with $P(\Omega_0) = 1$ such that*

$$\lim_{n'' \to +\infty} \mathbb{E}[e^{i\xi W_{n''}^*}|\mathcal{B}](\omega) = \exp\{-|\xi| k_1 M_\infty^{(1)}(\omega)(1 + i2\eta \log|\xi| \operatorname{sign}\xi)\} \quad (\xi \in \mathbb{R}) \tag{52}$$

for every ω in Ω_0.

Proof. Define (n'') and Ω_0 according to Lemma 4.6, implying the convergencies (33), and the UAN condition (34). In the following, let $\omega \in \Omega_0$ be fixed. In view of Proposition 11 in Section 17.3 of Fristedt and Gray (1997) the claim (52) follows if (36), (37) and (38) are satisfied with $\alpha = 1$, and in addition

$$\lim_{n \to +\infty} \sum_{j=1}^n \int_{\mathbb{R}} \chi(t) dQ_{j,n''}(t) = M_\infty^{(1)}(c^+ - c^-) \int_0^\infty \frac{\chi(t) - \sin(t)}{t^2} dt \tag{53}$$

with $\chi(t) = -\mathbb{I}\{t \leq -1\} + t\mathbb{I}\{-1 < t < 1\} + \mathbb{I}\{t \geq 1\}$.

Let us verify (36) for an arbitrary $x > 0$. Given $\epsilon > 0$, there exists some $Y = Y(\epsilon)$ such that

$$c^+ - \epsilon \leq y(1 - F_0(y)) \leq c^+ + \epsilon \tag{54}$$

for all $y \geq Y$ because of hypothesis (16). Moreover, in view in Lemma 4.6,

$$\hat{y}_{j,n''} := \frac{x + q_{j,n''}}{\beta_{j,n''}} \geq \frac{x - C_{1/2}\beta_{(n'')}^{1/2}}{\beta_{(n'')}},$$

which clearly diverges to $+\infty$ as $n'' \to \infty$ because of (33); in particular, $\hat{y}_{j,n''} \geq Y$ for n'' large enough. It follows by (54) that for those n'',

$$\frac{c^+ - \epsilon}{x + q_{j,n''}}\beta_{j,n''} \leq 1 - F(\hat{y}_{j,n''}) \leq \frac{c^+ + \epsilon}{x + q_{j,n''}}\beta_{j,n''}. \tag{55}$$

Recalling that $\zeta_{n''}(x) = \sum_{j=1}^{n''}[1 - F(\hat{y}_{j,n''})]$, summation of (55) over $j = 1, \dots, n''$ gives

$$\frac{c^+ - \epsilon}{x + q_{j,n''}} M_{n''}^{(1)} \leq \zeta_{n''}(x) \leq \frac{c^+ + \epsilon}{x + q_{j,n''}} M_{n''}^{(1)}.$$

Finally, observe that $|q_{j,n''}| \leq C_{1/2}\beta_{(n'')}^{1/2} \to 0$ as $n'' \to \infty$, and that $M_{n''}^{(1)} \to M_\infty^{(1)}$ by (33). Since $\epsilon > 0$ has been arbitrary, the claim (36) follows. The proof of (37) for arbitrary $x < 0$ is completely analogous.

Concerning (38), it is obviously enough to prove that

$$\lim_{\epsilon \to 0} \limsup_{n'' \to +\infty} s_{n''}^2(\epsilon) = 0 \tag{56}$$

where $s_{n''}^2(\epsilon) := \sum_{j=1}^{n''} \int_{(-\epsilon,\epsilon]} x^2 dQ_{j,n''}(x)$. We split the domain of integration in the definition of $s_{n''}^2$ at $x = 0$, and integrate by parts to get

$$\begin{aligned} s_{n''}^2(\epsilon) =& - \epsilon^2 \sum_{j=1}^{n''} Q_{j,n''}(-\epsilon) - \sum_{j=1}^{n''} \int_{(-\epsilon,0]} Q_{j,n''}(u) 2u du \\ &- \epsilon^2 \sum_{j=1}^{n''}(1 - Q_{j,n''}(\epsilon)) + \sum_{j=1}^{n''} \int_{(0,\epsilon]} (1 - Q_{j,n''}(u)) 2u du \\ =:& A_{n''}(\epsilon) + B_{n''}(\epsilon) + C_{n''}(\epsilon) + D_{n''}(\epsilon). \end{aligned}$$

Having already proven (36) and (37), we conclude

$$\lim_{\epsilon \to 0^+} \lim_{n'' \to +\infty} \{|A_{n''}(\epsilon)| + |C_{n''}(\epsilon)|\} = 0. \tag{57}$$

Fix $\epsilon > 0$; assume that n'' is sufficiently large to have $|q_{j,n''}| < \epsilon/2$ for $j = 1, \dots, n''$. Then

$$\begin{aligned} |B_n(\epsilon)| &\leq \sum_{j=1}^{n''} 2 \int_0^\epsilon w F_0 \left(\frac{-w + q_{j,n''}}{\beta_{j,n''}}\right) dw \\ &\leq \sum_{j=1}^{n''} \left\{ \int_0^{2|q_{j,n''}|} 2w \, dw + 2 \int_{2|q_{j,n''}|}^\epsilon w F_0 \left(\frac{-w + q_{j,n''}}{\beta_{j,n''}}\right) dw \right\} \\ &\leq \sum_{j=1}^{n''} \left\{ 4|q_{j,n''}|^2 + 2 \int_{2|q_{j,n''}|}^\epsilon w \Big(\frac{K\beta_{j,n''}}{w - q_{j,n''}}\Big) dw \right\} \\ &\leq \sum_{j=1}^{n''} \left\{ 4C_{1/4}^2 \beta_{j,n''}^{3/2} + \beta_{j,n''} \int_0^\epsilon 4K \, dw \right\} \leq \big(4C_{1/4}^2 \beta_{(n'')}^{1/2} + 4K\epsilon\big) M_{n''}^{(1)}, \end{aligned}$$

with the constant K defined in (42). In view of (33), it follows

$$\lim_{\epsilon \to 0^+} \limsup_{n'' \to +\infty} |B_{n''}(\epsilon)| = 0 \tag{58}$$

as desired. A completely analogous reasoning applies to $D_{n''}$. At this stage we can conclude (56), and thus also (38).

In order to verify (53), let us first show that

$$\lim_{n''\to\infty}\sum_{j=1}^{n''}\int_{\mathbb{R}}\sin(x)\,dQ_{j,n''}(x)=0. \tag{59}$$

We find

$$\begin{aligned}\Big|\sum_{j=1}^{n''}\int_{\mathbb{R}}\sin(x)dQ_{j,n''}(x)\Big| &= \Big|\sum_{j=1}^{n''}\int_{\mathbb{R}}\sin(t\beta_{j,n''}-q_{j,n''})dF_0(t)\Big|\\ &\le\sum_{j=1}^{n''}\Big|\cos(q_{j,n''})\int_{\mathbb{R}}\sin(t\beta_{j,n''})dF_0(t)-\sin(q_{j,n''})\int_{\mathbb{R}}\cos(t\beta_{j,n''})dF_0(t)\Big|\\ &=\sum_{j=1}^{n''}\Big|(\cos(q_{j,n''})-1)q_{j,n''}+(q_{j,n''}-\sin(q_{j,n''}))+\sin(q_{j,n''})\int_{\mathbb{R}}(1-\cos(t\beta_{j,n''}))dF_0(t)\Big|\\ &\le\sum_{j=1}^{n''}\big(|I_1|+|I_2|+|I_3|\big).\end{aligned}$$

The elementary inequalities $|\cos(x)-1|\le x^2/2$ and $|x-\sin(x)|\le x^3/6$ provide the estimate

$$\sum_{j=1}^{n''}\big(|I_1|+|I_2|\big)\le\sum_{j=1}^{n''}|q_{j,n''}|^3\le C_{1/2}^3\beta_{(n'')}^{1/2}M_{n''}^{(1)}.$$

By (33), the last expression converges to zero as $n''\to\infty$. In order to estimate I_3, observe that, since $|1-\cos(x)|\le 2x^{3/4}$ for all $x\in\mathbb{R}$,

$$\int_{\mathbb{R}}(1-\cos(t\beta_{j,n''}))dF_0(t)\le 2\beta_{j,n''}^{3/4}\int_{\mathbb{R}}|t|^{3/4}dF_0(dt)=2C_{1/4}\beta_{j,n''}^{3/4}.$$

Consequently, applying Lemma 4.6 once again,

$$\sum_{j=1}^{n''}|I_3|\le\sum_{j=1}^{n''}|q_{j,n''}|\Big|\int_{\mathbb{R}}(1-\cos(t\beta_{j,n''}))dF_0(t)\Big|\le 2C_{1/4}^2\beta_{(n'')}^{1/2}M_{n''}^{(1)},$$

which converges to zero on grounds of (33).

Having proven (59), the condition (53) becomes equivalent to

$$\sum_{j=1}^{n''}\int_{\mathbb{R}}(\chi(t)-\sin(t))dQ_{j,n''}(t)\to M_\infty^{(1)}(c^+-c^-)\int_{\mathbb{R}^+}\frac{\chi(t)-\sin(t)}{t^2}dt. \tag{60}$$

The proof of this fact follows essentially the line of the proof of Theorem 12 of Fristedt and Gray (1997). Let us first prove that, if $-\infty<x<0<y<+\infty$,

$$\lim_{n''\to+\infty}\int_{(x,y]}d\nu_{n''}(t)=M_\infty^{(1)}(c^+y-c^-x), \tag{61}$$

where the sequence (ν_n) of measures on $\mathbb{R}$ is defined by

$$\nu_n[B]=\sum_{j=1}^{n}\int_B t^2dQ_{j,n}(t)$$

for Borel sets $B \subset \mathbb{R}$. For fixed $\epsilon \in (0, y)$, one uses (36) to conclude

$$\begin{aligned}\lim_{n''\to\infty}\int_{(\epsilon,y]} d\nu_{n''}(t) &= \lim_{n''\to\infty}\left(t^2\sum_{j=1}^{n''}\big(1-Q_{j,n''}(t)\big)\Big|_y^\epsilon + 2\int_{(\epsilon,y]} t\sum_{j=1}^{n''}\big(1-Q_{j,n''}(t)\big)dt\right)\\ &= \epsilon^2\frac{c^+M_\infty^{(1)}}{\epsilon} - y^2\frac{c^+M_\infty^{(1)}}{y} + 2\int_{(\epsilon,y]} t\frac{c^+M_\infty^{(1)}}{t}dt\\ &= (y-\epsilon)c^+M_\infty^{(1)}.\end{aligned}$$

Notice that we have used the dominated convergence theorem to pass to the limit under the integral; this is justified in view of the upper bound provided by (40). In a similar way, one shows for fixed $\epsilon \in (0, |x|)$ that

$$\lim_{n''\to\infty}\int_{(x,-\epsilon]} d\nu_{n''}(t) = (|x| \ -\epsilon)c^-M_\infty^{(1)}.$$

Combining this with (56), one concludes

$$\begin{aligned}&\limsup_{n''\to\infty}\int_{(x,y]} d\nu_{n''}(t)\\ &\quad\le \limsup_{\epsilon\to 0}\limsup_{n''\to\infty}\Big(\int_{(x,-\epsilon]} d\nu_{n''}(t) + \int_{(-\epsilon,\epsilon]} d\nu_{n''}(t) + \int_{(\epsilon,y]} d\nu_{n''}(t)\Big)\\ &\quad\le \lim_{\epsilon\to 0}(y-\epsilon)c^+M_\infty^{(1)} - \lim_{\epsilon\to 0}(x+\epsilon)c^-M_\infty^{(1)} + \lim_{\epsilon\to 0}\limsup_{n''\to\infty}\sum_{j=1}^{n''}\int_{(-\epsilon,\epsilon]} t^2 dQ_{j,n''}(t)\\ &\quad= (c^+y - c^-x)M_\infty^{(1)}.\end{aligned}$$

On the other hand, trivially,

$$\liminf_{n''\to\infty}\int_{(x,y]} d\nu_{n''}(t) \ge \liminf_{\epsilon\to 0}\liminf_{n''\to\infty}\Big(\int_{(x,-\epsilon]} d\nu_{n''}(t) + \int_{(\epsilon,y]} d\nu_{n''}(t)\Big) = (c^+y - c^-x)M_\infty^{(1)}.$$

This proves (61). Now fix $0 < R < +\infty$, and note that (61) yields that for every bounded and continuous function $f : [-R, R] \to \mathbb{R}$

$$\lim_{n''\to\infty}\int_{[-R,R]} f(t)d\nu_{n''}(t) = M_\infty^{(1)}c^-\int_{-R}^0 f(t)dt + M_\infty^{(1)}c^+\int_0^R f(t)dt \tag{62}$$

holds true. In particular, using $f(t) = (\chi(t) - \sin t)/t^2$, for every $0 < R < +\infty$, one gets

$$\begin{aligned}&\lim_{n''\to\infty}\sum_{j=1}^{n''}\int_{[-R,R]}(\chi(t)-\sin(t))dQ_{j,n''}(t)\\ &\qquad\qquad = M_\infty^{(1)}(c^+-c^-)\int_0^R\frac{(\chi(t)-\sin(t))}{t^2}dt.\end{aligned}\tag{63}$$

Moreover, since $|\chi(t) - \sin t| \le 2$,

$$|\sum_{j=1}^{n''}\int_{[-R,R]^c}(\chi(t)-\sin(t))dQ_{j,n''}(t)| \le 2[\zeta_{n''}(-R) + \zeta_{n''}(R)].$$

Applying (36) and (37) one obtains

$$\limsup_{n''\to+\infty}|\sum_{j=1}^{n''}\int_{[-R,R]^c}(\chi(t)-\sin(t))dQ_{j,n''}(t)| \le 2M_\infty^{(\alpha)}(c^++c^-)\frac{1}{R},$$

which gives

$$\limsup_{R\to+\infty}\limsup_{n''\to+\infty}|\sum_{j=1}^{n''}\int_{[-R,R]^c}(\chi(t)-\sin(t))dQ_{j,n''}(t)| = 0. \tag{64}$$

Combining (63) with (64) one gets (60). □

Proof of Theorem 3.5. Use Lemma 4.7 and repeat the proof of Theorem 3.2. □

Proof of Theorem 3.3. The theorem is a corollary of Theorem 3.5. Since $m_0 = \int_{\mathbb{R}} x\,dF_0(x) < \infty$ by hypothesis, it follows that $c^+ = c^- = 0$, and so V_t^* converges to 0 in probability. Now write

$$V_t = m_0 M_{\nu_t}^{(1)} + V_t^* - R_{\nu_t},$$

with the remainder

$$R_n := \sum_{j=1}^{n} (q_{j,n} - \beta_{j,n} m_0).$$

Thanks to Lemma 3.1, $m_0 M_{\nu_t}^{(1)}$ converges in distribution to $m_0 M_\infty^{(1)}$. It remains to prove that R_{ν_t} converges to 0 in probability. Since

$$\Big|\frac{\sin(x)}{x} - 1\Big| \le H(x) := 1/6\big[x^2 \mathbb{I}\{|x| < 1\} + \mathbb{I}\{|x| \ge 1\}\big] \le 1/6,$$

it follows that

$$\begin{aligned} |R_n| &\le \sum_{j=1}^{n} \beta_{j,n} \int_{\mathbb{R}} \Big|\frac{\sin(\beta_{j,n}x)}{\beta_{j,n}x} - 1\Big| |x| dF_0(x) \\ &\le \sum_{j=1}^{n} \beta_{j,n} \int_{\mathbb{R}} H(\beta_{j,n}x)|x|dF_0(x) \le M_n^{(1)} \int_{\mathbb{R}} H(\beta_{(n)}x)|x|dF_0(x). \end{aligned}$$

Recall that $M_n^{(1)}$ converges a.s. to $M_\infty^{(1)}$ and $\beta_{(n)}$ converges in probability to 0 by (33). By dominated convergence it follows that also $\int_{\mathbb{R}} H(\beta_{(n)}x)|x|dF_0(x)$ converges in probability to 0.

The (non-)degeneracy of V_∞ and the (in)finiteness of its moments is an immediate consequence of Lemma 3.1. □

4.4. **Estimates in Wasserstein metric (Proposition 3.6).**

Proof of Proposition 3.6. The proof uses the techniques employed in Lemma 3.1.

We shall assume that $\mathcal{W}_\gamma(X_0, V_\infty) < +\infty$, since otherwise the claim is trivial. Then, there exists an optimal pair (X^*, Y^*) realizing the infimum in the definition of the Wasserstein distance,

$$\Delta := \mathcal{W}_\gamma^{\max(\gamma,1)}(X_0, V_\infty) = \mathcal{W}_\gamma^{\max(\gamma,1)}(X^*, Y^*) = \mathbb{E}|X^* - Y^*|^\gamma. \tag{65}$$

Let $(X_j^*, Y_j^*)_{j\ge1}$ be a sequence of independent and identically distributed random variables with the same law of (X^*, Y^*), which are further independent of $B_n = (\beta_{1,1}, \beta_{1,2}, \dots, \beta_{n,n})$. Consequently, $\sum_{j=1}^n X_j^* \beta_{j,n}$ has the same law of W_n, and $\sum_{j=1}^n Y_j^* \beta_{j,n}$ has the same law of V_∞. By definition of $\mathcal{W}_\gamma$,

$$\begin{aligned} \mathcal{W}_\gamma^{\max(\gamma,1)}(W_n, V_\infty) &\le \mathbb{E}\Big[\Big|\sum_{j=1}^{n} X_j^* \beta_{j,n} - \sum_{j=1}^{n} Y_j^* \beta_{j,n}\Big|^\gamma\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big[\Big|\sum_{j=1}^{n} (X_j^* - Y_j^*)\beta_{j,n}\Big|^\gamma \Big| B_n\Big]\Big]. \end{aligned}$$

For further estimates, we distinguish two cases. In the first case, $0 < \alpha < \gamma \le 1$, we apply the elementary inequality

$$\Big|\sum_{j=1}^{n} z_j\Big|^\gamma \le \sum_{j=1}^{n} |z_j|^\gamma$$

for real numbers $z_1, \dots, z_n$ to obtain

$$\mathcal{W}_\gamma(W_n, V_\infty) \le \mathbb{E}\Big[\mathbb{E}\Big[\sum_{j=1}^{n} \beta_{j,n}^\gamma |X_j^* - Y_j^*|^\gamma \Big| B_n\Big]\Big] = \mathbb{E}\Big[\sum_{j=1}^{n} \beta_{j,n}^\gamma\Big]\Delta;$$

where Δ is defined in (65). In the second case, $1 \leq \alpha < \gamma \leq 2$, we can apply the Bahr-Esseen inequality (31) since $E(X_j^* - Y_j^*) = E(X_1) - E(V_\infty) = 0$ and $E|X_j^* - Y_j^*|^\gamma = W_\gamma^\gamma(X_0, V_\infty) < +\infty$. Thus,

$$\mathcal{W}_\gamma^\gamma(W_n, V_\infty) \leq \mathbb{E}\Big[\Big[2\sum_{j=1}^n \beta_{j,n}^\gamma |X_j^* - Y_j^*|^\gamma \Big| F_n\Big] = 2\mathbb{E}\big[\sum_{j=1}^n \beta_{j,n}^\gamma\big]\Delta.$$

By convexity of the Wasserstein metric,

$$\mathcal{W}_\gamma^{\max(\gamma,1)}(V_t, V_\infty) \leq \sum_{n\geq 1} e^{-t}(1-e^{-t})^{n-1}\mathcal{W}_\gamma^{\max(\gamma,1)}(W_n, V_\infty).$$

Combining the previous estimates with Proposition 4.1, we obtain

$$\begin{aligned}\mathcal{W}_\gamma^{\max(\gamma,1)}(V_t, V_\infty) &\leq a\Delta \sum_{n\geq 1} e^{-t}(1-e^{-t})^{n-1}\mathbb{E}\Big[\sum_{j=1}^n \beta_{j,n}^\gamma\Big] \\ &= a\Delta e^{t\mathcal{S}(\gamma)},\end{aligned}$$

with $a = 1$ if $0 < \alpha < \gamma \leq 1$ and $a = 2$ if $1 \leq \alpha < \gamma \leq 2$. □

Lemma 3.7 is a corollary of the following.

Lemma 4.8. *Let two random variables X_1 and X_2 be given, and assume that their distribution functions F_1 and F_2 both satisfy the conditions* (28) *and* (29) *with the same constants $\alpha > 0$, $0 < \epsilon < 1$, K and $c^+, c^- \geq 0$. Then $\mathcal{W}_\gamma(X_1, X_2) < \infty$ for all γ that satisfy $\alpha < \gamma < \frac{\alpha}{1-\epsilon}$.*

Proof. Define the auxiliary functions H, H_+ and H_- on $\mathbb{R}\setminus\{0\}$ by

$$H(x) = \mathbb{I}\{x > 0\}(1 - c^+x^{-\alpha}) + \mathbb{I}\{x < 0\}c^-|x|^{-\alpha}, \quad H_\pm(x) = H(x) \pm K|x|^{-(\alpha+\epsilon)},$$

so that $H_- \leq F_i \leq H_+$ for $i = 1, 2$ by hypothesis. It is immediately seen that $H(x)$, $H_+(x)$ and $H_-(x)$ all tend to one (to zero, respectively) when x goes to $+\infty$ ($-\infty$, respectively). Moreover, evaluating the functions' derivatives, one verifies that H and H_- are strictly increasing on $\mathbb{R}_+$, and that H_+ is strictly increasing on some interval $(R_+, +\infty)$. Let $\check{R} > 0$ be such that $H(\check{R}) > H_+(R_+)$; then, for every $x > \check{R}$, the equation

$$H_-(\hat{x}) = H(x) = H_+(\check{x}) \tag{66}$$

possesses precisely one solution pair $(\check{x}, \hat{x})$ satisfying $R_+ < \check{x} < x < \hat{x}$. Likewise, H and H_+ are strictly increasing on $\mathbb{R}_-$, and H_- is strictly increasing on $(-\infty, -R_-)$. Choosing $\hat{R} > 0$ such that $H(-\hat{R}) < H_-(-R_-)$, equation (66) has exactly one solution $(\check{x}, \hat{x})$ with $\check{x} < x < \hat{x} < -R_-$ for every $x < -\hat{R}$.

A well-known representation of the Wasserstein distance of measures on $\mathbb{R}$ reads

$$\mathcal{W}_\gamma^{\max(\gamma,1)}(X_1, X_2) = \int_0^1 |F_1^{-1}(y) - F_2^{-1}(y)|^\gamma\, dy,$$

where $F_i^{-1} : (0,1) \to \mathbb{R}$ denotes the pseudo-inverse function of F_i. We split the domain of integration $(0,1)$ into the three intervals $(0, H(-\hat{R}))$, $[H(-\hat{R}), H(\check{R})]$ and $(H(\check{R}), 1)$, obtaining:

$$\begin{aligned}\mathcal{W}_\gamma(X_1, X_2)^{\max(\gamma,1)} &= \int_{-\infty}^{-\hat{R}} |F_1^{-1}(H(x)) - F_2^{-1}(H(x))|^\gamma H'(x)\, dx \\ &+ \int_{H(-\hat{R})}^{H(\check{R})} |F_1^{-1}(y) - F_2^{-1}(y)|^\gamma\, dy \\ &+ \int_{\check{R}}^{\infty} |F_1^{-1}(H(x)) - F_2^{-1}(H(x))|^\gamma H'(x)\, dx.\end{aligned}$$

The middle integral is obviously finite. To prove finiteness of the first and the last integral, we show that

$$\int_{\check{R}}^{\infty} |F_1^{-1}(H(x)) - x|^\gamma H'(x)\, dx < \infty;$$

the estimates for the remaining contributions are similar. Let some $x \geq \check{R}$ be given, and let $\hat{x} > \check{x} > \check{R}$ satisfy (66). From $H_- < F_1 < H_+$, it follows that

$$F_1(\check{x}) < H(x) < F_1(\hat{x}),$$

which implies further that

$$\check{x} - x < F_1^{-1}(H(x)) - x < \hat{x} - x. \tag{67}$$

From the definition of H, it follows that

$$x = \hat{x}(1 + \kappa\hat{x}^{-\epsilon})^{-1/\alpha},$$

with $\kappa = K/c^+$. Combining this with a Taylor expansion, and recalling that $\hat{x} > x > \check{R} > 0$, one obtains

$$\hat{x} - x = x\big[(1 + \kappa\hat{x}^{-\epsilon})^{1/\alpha} - 1\big] < x\big[(1 + \kappa x^{-\epsilon})^{1/\alpha} - 1\big] < \hat{C}x^{1-\epsilon}, \tag{68}$$

where $\hat{C}$ is defined in terms of α, κ and $\check{R}$. In an analogous manner, one concludes from

$$x = \check{x}(1 - \kappa\check{x}^{-\epsilon})^{-1/\alpha},$$

in combination with $0 < R_+ < \check{x} < x$ and $0 < \epsilon < 1$ that

$$\check{x} - x = \check{x}\big[1 - (1 - \kappa\check{x}^{-\epsilon})^{-1/\alpha}\big] \geq \check{x}\big[1 - (1 + \check{C}\check{x}^{-\epsilon})\big] = -\check{C}\check{x}^{1-\epsilon} > -\check{C}x^{1-\epsilon}, \tag{69}$$

where $\check{C}$ only depends on α, κ and R_+. Substitution of (68) and (69) into (67) yields

$$\int_{\check{R}}^{\infty} |F_1^{-1}(H(x)) - x|^\gamma H'(x)\,dx < \max(\hat{C}, \check{C})^\gamma \int_{\check{R}}^{\infty} x^{\gamma(1-\epsilon)-\alpha-1}\,dx,$$

which is finite provided that $0 < \gamma < \alpha/(1-\epsilon)$. □

Proof of Lemma 3.7. In view of Lemma 4.8, it suffices to show that the distribution function F_∞ of V_∞ satisfies (28) and (29) with the same constants c^+ and c^- as the initial condition F_0 (possibly after diminishing ϵ and enlarging K).

The proof is based on the representation of F_∞ as a mixture of stable laws. More precisely, let G_α be the distribution function whose characteristic function is $\hat{g}_\alpha$ as in (15), then

$$F_\infty(x) = \mathbb{E}\Big[G_\alpha\Big(\big(M_\infty^{(\alpha)}\big)^{-1/\alpha}x\Big)\Big],$$

see (21). Since $\alpha < \gamma < 2\alpha$, then there exists a finite constant $K > 0$ such that

$$|1 - c_+x^{-\alpha} - G_\alpha(x)| \leq Kx^{-\gamma}$$

for $x > 0$, and similarly for $x < 0$; see, e.g. Sections 2.4 and 2.5 of Zolotarev (1986)). Using that $\mathbb{E}[M_\infty^{(\alpha)}] = 1$ and $C := \mathbb{E}[(M_\infty^{(\alpha)})^{\gamma/\alpha}] < \infty$ (since $\mathcal{S}(\gamma) < 0$) it follows further that

$$\begin{aligned}
\big|1 - c^+x^{-\alpha} - F_\infty(x)\big| &= \big|1 - c^+\mathbb{E}\big[M_\infty^{(\alpha)}\big]x^{-\alpha} - \mathbb{E}\big[G_\alpha((M_\infty^{(\alpha)})^{-1/\alpha}x)\big]\big| \\
&\leq \mathbb{E}\big[\big|1 - c^+\big((M_\infty^{(\alpha)})^{-1/\alpha}x\big)^{-\alpha} - G_\alpha((M_\infty^{(\alpha)})^{-1/\alpha}x)\big|\big] \\
&\leq \mathbb{E}\big[K(M_\infty^{(\alpha)})^{\gamma/\alpha}x^{-\gamma}\big] = CKx^{-\gamma}.
\end{aligned}$$

This proves (28) for F_∞, with $\epsilon = \gamma - \alpha$ and $K' = CK$. A similar argument proves (29). □

4.5. **Proofs of strong convergence (Theorem 3.8).** We shall use the Wild sum representation of the solution to the Boltzmann equation, see (10) and (11). The idea is to prove that certain ξ-pointwise a priori bounds on the characteristic functions $\hat{q}_n$ are preserved by the collisional operator, and hence are propagated from the initial condition to any later time.

A first intermediate result is

Lemma 4.9. *Under the hypotheses of Theorem 3.8, there exists a constant $\theta > 0$ and a radius $\rho > 0$, both independent of $n \geq 0$, such that $|\hat{q}_n(\xi)| \leq (1 + \theta|\xi|^\alpha)^{-1/r}$ for all $|\xi| \leq \rho$.*

Proof. By the explicit representation (21) or (22), respectively, we conclude that

$$|\phi_\infty(\xi)| \leq \Phi(\xi) := \mathbb{E}[\exp(-|\xi|^\alpha k M_\infty^{(\alpha)})],$$

with the parameter k from (17), or $k = \sigma^2/2$ if $\alpha = 2$. Notice further that, by (20), Φ satisfies

$$\Phi(\xi) = \mathbb{E}[\Phi(L\xi)\Phi(R\xi)]. \tag{70}$$

Moreover, since $M_\infty^{(\alpha)} \neq 0$, $\mathbb{E}[M_\infty^{(\alpha)}] = 1$ and $\mathbb{E}[(M_\infty^{(\alpha)})^{\gamma/\alpha}] < +\infty$, the function Φ is positive and strictly convex in $|\xi|^\alpha$, with $\Phi(\xi) = 1 - k|\xi|^\alpha + o(|\xi|^\alpha)$. It follows that for each $\kappa > 0$ with $\kappa < k$, there exists exactly one point $\Xi_\kappa > 0$ with $\Phi(\Xi_\kappa) + \kappa|\Xi_\kappa|^\alpha = 1$, and Ξ_κ decreases monotonically from $+\infty$ to zero as κ increases from zero to k.

Since $\hat{q}_0 = \phi_0$ is the characteristic function of the initial datum, satisfying the condition (16), it follows by Theorem 2.6.5 of Ibragimov and Linnik (1971) that

$$\hat{q}_0(\xi) = 1 - k|\xi|^\alpha(1 - i\eta\tan(\pi\alpha/2)\operatorname{sign}\xi) + o(|\xi|^\alpha),$$

with the same k as before, and η determined by (17). For $\alpha = 2$, clearly $\hat{q}_0(\xi) = 1 - \sigma^2\xi^2/2 + o(\xi^2)$. By the aforementioned properties of Φ, there exists a $\kappa \in (0, k)$ such that

$$|\hat{q}_0(\xi)| \leq \Phi(\xi) + \kappa|\xi|^\alpha \tag{71}$$

for all $\xi \in \mathbb{R}$. This is evident, since for small ξ,

$$|\hat{q}_0(\xi)| = |\phi_0(\xi)| = 1 - k|\xi|^\alpha + o(|\xi|^\alpha),$$

while inequality (71) is trivially satisfied for $|\xi| \geq \Xi_k$, since $|\phi_0| \leq 1$.

Starting from (71), we shall now prove inductively that

$$|\hat{q}_\ell(\xi)| \leq \Phi(\xi) + \kappa|\xi|^\alpha. \tag{72}$$

Fix $n \geq 0$, and assume (72) holds for all $\ell \leq n$. Choose $j \leq n$. Using the invariance property (70) of Φ, as well as the uniform bound of characteristic functions by one, it easily follows that

$$\begin{aligned} |\widehat{Q}^+[\hat{q}_j, \hat{q}_{n-j}](\xi)| - \Phi(\xi) &\leq \mathbb{E}\big[|\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)| - \Phi(L\xi)\Phi(R\xi)\big] \\ &\leq \mathbb{E}\big[\big(|\hat{q}_j(L\xi)| - \Phi(L\xi)\big)|\hat{q}_{n-j}(R\xi)|\big] + \mathbb{E}\big[\Phi(L\xi)\big(|\hat{q}_{n-j}(R\xi)| - \Phi(R\xi)\big)\big] \\ &\leq \mathbb{E}\big[\kappa(L|\xi|)^\alpha\big] + \mathbb{E}\big[\kappa(R|\xi|)^\alpha\big] = \kappa|\xi|^\alpha. \end{aligned}$$

The final equality is a consequence of $\mathbb{E}[L^\alpha + R^\alpha] = 1$. By (11), it is immediate to conclude (72) with $\ell = n + 1$.

The proof is finished by noting that, since $\kappa < k$,

$$\big(1 + \theta|\xi|^\alpha\big)^{-1/r} \geq \Phi(\xi) + \kappa|\xi|^\alpha$$

holds for $|\xi| \leq \rho$, provided that $\rho > 0$ and $\theta > 0$ are sufficiently small. □

Lemma 4.10. *Under the hypotheses of Theorem 3.8, let $\rho > 0$ be the radius introduced in Lemma 4.9 above. Then, there exists a constant $\lambda > 0$, independent of $\ell \geq 0$,*

$$|\hat{q}_\ell(\xi)| \leq (1 + \lambda|\xi|^r)^{-1/r} \quad \textit{for all } |\xi| \geq \rho. \tag{73}$$

Proof. Since the density f_0 has finite Linnik–Fisher information by hypothesis (H2), it follows that

$$|\phi_0(\xi)| \leq \Big(\int_{\mathbb{R}} |\zeta|^2 |\widehat{h}(\zeta)|^2 \, d\zeta\Big)|\xi|^{-1}$$

for all $\xi \in \mathbb{R}$, where $h = \sqrt{f_0}$ and $\widehat{h}$ is its Fourier transform. See Lemma 2.3 in Carlen et al. (1999). For any sufficiently small $\lambda > 0$, one concludes

$$|\phi_0(\xi)| \leq (1 + \lambda|\xi|^r)^{-1/r} \tag{74}$$

for sufficiently large $|\xi|$.

Next, recall that the modulus of the characteristic function of a probability density is continuous and bounded away from one, locally uniformly in ξ on $\mathbb{R} \setminus \{0\}$. Diminishing the $\lambda > 0$ in (74) if necessary, this estimate actually holds for $|\xi| \geq \rho$.

Thus, the claim (73) is proven for $\ell = 0$. To proceed by induction, fix $n \geq 0$ and assume that (73) holds for all $\ell \leq n$. In the following, we shall conclude (73) for $\ell = n+1$.

Recall that $r < \alpha$ in hypothesis (H1); see Remark 1. Hence, defining

$$\rho_\lambda = (\lambda/\theta)^{1/(\alpha-r)}, \tag{75}$$

it follows that

$$(1+\theta|\xi|^\alpha)^{-1/r} \leq (1+\lambda|\xi|^r)^{-1/r}$$

if $|\xi| \geq \rho_\lambda$. Taking into account Lemma 4.9, estimate (74) for $\ell \leq n$ extends to all $|\xi| \geq \rho_\lambda$. We assume $\rho_\lambda < \rho$ from now on, which is equivalent to saying that $0 < \lambda < \lambda_0 := \theta\rho^{\alpha-r}$.

With these notations at hand, introduce the following "good" set:

$$M^G_{\lambda,\delta} := \big\{\omega : L^r(\omega) + R^r(\omega) \geq 1 + \delta^r \text{ and } \min(L(\omega), R(\omega))\rho \geq \rho_\lambda\big\}\ ,$$

depending on λ and a parameter $\delta > 0$. We are going to show that if $\delta > 0$ and $\lambda > 0$ are sufficiently small, then $M^G_{\lambda,\delta}$ has positive probability. First observe that the law of (L,R) cannot be concentrated in the two point set $\{(0,1),(1,0)\}$ because $\mathcal{S}(\gamma) < 0$ by the hypotheses of Theorem 3.8. Hence we can assume $P\{L^r + R^r > 1\} > 0$, possibly after diminishing $r > 0$ (recall that if (H1) holds for some $r > 0$, then it also holds for all smaller $r' > 0$ as well). Moreover, notice that $L^r + R^r > 1$ and $L = 0$ or $R = 0$ implies $L^\alpha + R^\alpha > 1$. But since $\mathbb{E}[L^\alpha + R^\alpha] = 1$, it follows that $P\{L > 0, R > 0, L^r + R^r > 1\} > 0$. In conclusion, the countable union of sets

$$\bigcup_{k=1}^\infty M^G_{\lambda_0/k,1/k} = \big\{\omega : L^r(\omega) + R^r(\omega) > 1,\, L(\omega) > 0,\, R(\omega) > 0\big\}$$

has positive probability, and so has one of the components $M^G_{\lambda_0/k,1/k}$.

Also, we introduce a "bad" set, that depends on λ and ξ,

$$M^B_{\lambda,\xi} := \big\{\omega : \min(L(\omega), R(\omega))|\xi| < \rho_\lambda\big\}\ .$$

Notice that $M^G_{\lambda,\delta}$ and $M^B_{\lambda,\xi}$ are disjoint provided $|\xi| \geq \rho$.

We are now ready to carry out the induction proof, for a given λ small enough. Fix $j \leq n$ and some $|\xi| \geq \rho$. We prove that

$$\widehat{Q}^+[\hat{q}_j, \hat{q}_{n-j}](\xi) \leq \mathbb{E}[|\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)|] \leq (1+\lambda|\xi|^r)^{-1/r}. \tag{76}$$

We distinguish several cases. If ω does *not* belong to the bad set $M^B_{\lambda,\xi}$, then $L|\xi| \geq \rho_\lambda$ and $R|\xi| \geq \rho_\lambda$ so that by induction hypothesis

$$\begin{aligned}|\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)| &\leq \big((1+\lambda L^r|\xi|^r)(q+\lambda R^r|\xi|^r)\big)^{-1/r}\\ &\leq \big(1+\lambda(L^r+R^r)|\xi|^r\big)^{-1/r} \leq (1+\lambda|\xi|^r)^{-1/r};\end{aligned}$$

indeed, recall that $L^r + R^r \geq 1$ because of (H1). In particular, if ω belongs to the good set $M^G_{\lambda,\delta}$, then the previous estimate improves as follows,

$$|\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)| \leq \big(1+\lambda(1+\delta^r)|\xi|^r\big)^{-1/r} \leq \Big(\frac{1+\lambda\rho^r}{1+\lambda(1+\delta^r)\rho^r}\Big)^{1/r}(1+\lambda|\xi|^r)^{-1/r},$$

where we have used that $|\xi| \geq \rho$. Notice further that there exists some $c > 0$ — depending on δ, θ, λ_0, ρ and r, but not on λ — such that for all sufficiently small $\lambda > 0$,

$$\Big(\frac{1+\lambda\rho^r}{1+\lambda(1+\delta^r)\rho^r}\Big)^{1/r} \leq 1 - c\lambda.$$

Finally, suppose that ω is a point in the bad set $M^B_{\lambda,\xi}$, and assume without loss of generality that $L \geq R$. Then $L^r|\xi|^r \geq (1-R^r)|\xi|^r \geq |\xi|^r - \rho^r_\lambda$, and so, for sufficiently small λ and for any $\xi \geq \rho$,

$$|\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)| \leq (1+\lambda L^r|\xi|^r)^{-1/r} \leq (1+\lambda|\xi|^r - \lambda\rho^r_\lambda)^{-1/r} \leq (1+\lambda\rho^r_\lambda)^{1/r}(1+\lambda|\xi|^r)^{-1/r}.$$

Again, there exists a λ-independent constant C such that, for all sufficiently small $\lambda > 0$,

$$(1+\lambda\rho^r_\lambda)^{1/r} \leq 1 + C\lambda\rho^r_\lambda.$$

Putting the estimates obtained in the three cases together, one obtains

$$\begin{aligned}&\mathbb{E}[|\hat{q}_j(L\xi)||\hat{q}_{n-j}(R\xi)|]\\&\quad\le(1+\lambda|\xi|^r)^{-1/r}\big[\big(1-P(M^G_{\lambda,\delta})-P(M^B_{\lambda,\xi})\big)+P(M^G_{\lambda,\delta})(1-c\lambda)+P(M^B_{\lambda,\xi})(1+C\lambda\rho_\lambda^r)\big]\\&\quad\le(1+\lambda|\xi|^r)^{-1/r}\big[1+\lambda(C\rho_\lambda^r-cP\big(M^G_{\lambda,\delta})\big)\big].\end{aligned}$$

Notice that we have used the trivial estimate $P(M^B_{\lambda,\xi})\le 1$ in the last step, which eliminates any dependence of the term in the square brackets on ξ. To conclude (76), it sufficies to observe that as λ decreases to zero, ρ_λ tends to zero monotonically by (75), while the measure $P(M^G_{\lambda,\delta})$ is obviously non-decreasing and we have already proved that $P(M^G_{\lambda^*,\delta})>0$ for λ^* and δ suitably chosen. Hence $C\rho_\lambda^r\le cP(M^G_{\lambda,\delta})$ when $\lambda>0$ is small enough. From (76), it is immediate to conclude (73), recalling the recursive definition of $\hat{q}_{n+1}$ in (11).

Thus, the induction is complete, and so is the proof of the lemma. □

Proof of Theorem 3.8. The key step is to prove convergence of the characteristic functions $\phi(t)\to\phi_\infty$ in $L^2(\mathbb{R})$. To this end, observe that the uniform bound on $\hat{q}_n$ obtained in Lemma 4.10 above directly carries over to the Wild sum,

$$|\phi(t;\xi)|\le e^{-t}\sum_{n=0}^{\infty}(1-e^{-t})^n|\hat{q}_n(\xi)|\le(1+\lambda|\xi|^r)^{-1/r}\qquad(|\xi|\ge\rho).$$

Moreover, since $\lim_{t\to+\infty}\phi(t;\xi)=\phi_\infty(\xi)$ for every $\xi\in\mathbb{R}$, also

$$|\phi_\infty(\xi)|\le(1+\lambda|\xi|^r)^{-1/r}\qquad(|\xi|\ge\rho).$$

Let $\epsilon>0$ be given. Then there exists a $\Xi\ge\rho$ such that

$$\begin{aligned}\int_{|\xi|\ge\Xi}|\phi(t;\xi)-\phi_\infty(\xi)|^2\,d\xi&\le 2\int_{|\xi|\ge\Xi}\big(|\phi(t;\xi)|^2+|\phi_\infty(\xi)|^2\big)\,d\xi\\&\le 4\int_{\Xi}^{\infty}(1+\lambda|\xi|^r)^{-2/r}\,d\xi\le\frac{\epsilon}{2}.\end{aligned}$$

On the other hand, by weak convergence of V_t to V_∞, $\phi(t;\cdot)$ converges to ϕ_∞ uniformly on every compact set of $\mathbb{R}$ as $t\to+\infty$, hence there exists a time $T>0$ such that

$$|\phi(t;\xi)-\phi_\infty(\xi)|^2\le\frac{\epsilon}{4\Xi}$$

for every $|\xi|\le\Xi$ and $t\ge T$. In combination, it follows that

$$\|\phi(t)-\phi_\infty\|_{L^2}^2\le\epsilon$$

for all $t\ge T$. Since $\epsilon>0$ has been arbitrary, convergence of $\phi(t)$ to ϕ_∞ in $L^2(\mathbb{R})$ follows. By Plancherel's identity, this immediately implies strong convergence of the densities $f(t)$ of V_t to the density f_∞ of V_∞ in L^2.

Convergence in $L^1(\mathbb{R})$ is obtained by interpolation between weak and $L^2(\mathbb{R})$ convergence. Let $\epsilon>0$ be given, and choose $M>0$ such that

$$\int_{|x|\ge M}f_\infty(x)\,dx<\frac{\epsilon}{4}.$$

By weak convergence of V_t to V_∞ there exists a $T>0$ such that

$$\int_{|x|\ge M}f(t;x)\,dx<\frac{\epsilon}{2}$$

for all $t \geq T$. Now Hölder's inequality implies

$$\begin{aligned}\int_{\mathbb{R}} |f(t;x) - f_\infty(x)|\,dx &\leq (2M)^{1/2}\Big(\int_{|x|\leq M} |f(t;x) - f_\infty(x)|^2\,dx\Big)^{1/2} \\ &\quad + \int_{|x|>M} \big(|f(t;x)| + |f_\infty(x)|\big)\,dx \\ &< (2M)^{1/2}\|f(t) - f_\infty\|_{L^2} + \frac{3\epsilon}{4}\end{aligned}$$

Increasing T sufficiently, the last sum is less than ϵ for $t \geq T$.

Finally, convergence in $L^p(\mathbb{R})$ with $1 < p < 2$ follows by interpolation between convergence in $L^1(\mathbb{R})$ and in $L^2(\mathbb{R})$. □

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