Identifiability problems of defects with Robin condition

Carlo Domenico PAGANI – Dario PIEROTTI
Dipartimento di Matematica del Politecnico, Piazza L. da Vinci 32 - 20133 Milano (Italy)

Abstract

We consider the inverse problem of recovering the shape of an inclusion or of a crack contained in a connected domain Ω, and the problem of reconstructing part of the boundary $\partial\Omega$ itself, when a condition of the third kind (Robin condition) is prescribed on the defects. We prove a result of uniqueness by two measures: two different defects, with different coefficients of the Robin condition, cannot be compatible with same two pairs of Cauchy data on the (accessible) boundary. In case of cracks, we also prove that a single measure is sufficient if the coefficient of the Robin condition is known.

AMS Subject Classification: 35N10.

1 Introduction

The purpose of this work is to study some problems of identifiability arising in the field of non destructive evaluation. A specimen is given, which is marked by some imperfections due to various causes, which are located either in the interior of the specimen or on an inaccessible part of its boundary. The most common techniques used to detect these imperfections are the electrical impedance tomography (where static voltage and surface current measurements are used to determine the conductivity distribution at the interior) or the thermal imaging (where static temperature and heat flow are measured).

The body to be inspected is represented by a connected bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ and we assume that the voltage potential (or the static temperature) u is harmonic through the undamaged part of Ω. The defects are modeled either by a simply connected domain D (an inclusion) with boundary σ (a closed curve) contained in Ω or by a curve σ (a crack) or by a piece of the boundary $\partial\Omega$ itself (still denoted by σ).

These three typical situations (and other more complicated) have been largely considered in the literature, when the most common conditions assumed on σ are the Dirichlet (perfectly conducting defect) or Neumann (perfectly insulating defect) conditions. In this paper, we allow for a more general boundary condition, namely a condition of the third kind (or Robin condition)

$$\frac{\partial u}{\partial \nu} + \lambda u = 0, \qquad \text{on} \quad \sigma, \tag{1.1}$$

where the normal vector is pointing inward D in the case of the inclusion and outward Ω in the case of a surface defect (for the case of a crack see § 3 below). The coefficient $\lambda = \lambda(x)$ is strictly positive:

$$\lambda = \lambda(x) \geq \bar{\lambda} > 0, \qquad \text{on} \quad \sigma. \tag{1.2}$$

Condition (1.1) is called *Robin condition* and is sometime used to model damages due to corrosion (especially when σ is part of the boundary of Ω). For sure, corrosion is a complicated electrochemical phenomenon and the linear condition (1.1) can not be assumed to model it realistically; however, from a mathematical point of view, it shares some crucial features with more effective nonlinear conditions.

The coefficient λ, in the electrostatic context, represents the reciprocal of the surface impedance; in heat conduction problems, it is related to the surface conductivity. If we call Γ the accessible part of the external boundary $\partial\Omega$ (which is the whole of $\partial\Omega$ in the case of the inclusion and of the crack, only a subset of $\partial\Omega$ in the third case) the avaible data are represented by a fixed choice of the current density on Γ and the measurement of the corresponding boundary voltage on some arc of Γ (or, viceversa, we apply some voltage and measure the current); our main goal is to address the question of *uniqueness of σ: are two different defects compatible with the same pair of Cauchy data on Γ ?*
This problem was addressed in [1]-[5]. An answer comes in an unexpectedly simple way by the application of an integral identity found by Martin (in 1958) to positive solutions of the classical Steklov problem, i.e. to positive harmonic functions satisfying the boundary condition $\partial_\nu u = hu$, with h a given function on a boundary of a domain [6]. In [7] it is noticed that from Martin's result the unique identification of an unknown $\mathcal{C}^1$ boundary immediately derives. It should be remarked, however, that the same integral identity gives a solution also for the other problems and we will prove this under suitable regularity assumptions of the domain (see § 4). Moreover, we stress that such identity is a special case of a general one constructed in [6] which holds for any pair of smooth enough functions in a bounded domain with smooth boundary; in this paper, we will deduce from Martin's results another identity for the solutions of the Steklov problem, which applies to identification of cracks by a *single* measure (see § 5).
The paper is organized as follows: in § 2 we present the inverse problems for the inclusion and for the boundary and illustrate the relation of the uniqueness question to the Steklov problem. Section 3 is devoted to the study of the cracks: we define this kind of defect and study in some detail the direct problem, stressing the behaviour of the solution in the neighborhood of the end points of the crack. In § 4 we present the Martin's integral identity and apply it to the three problems: as an immediate consequence we get that two measures suffice to identify uniquely the defects together with the coefficient λ. By exploiting the above mentioned new integral identity for the solutions of the Steklov problem, we refine the result for the crack by proving that a single measure is sufficient for identifying σ.

2 Inclusions and boundaries

Let us consider the boundary value problem

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega\backslash\overline{D} \\ u &= f \quad \text{on } \Gamma = \partial\Omega \\ \frac{\partial u}{\partial \nu} + \lambda u &= 0 \quad \text{on } \sigma = \partial D \end{aligned} \tag{2.1}$$

Here Ω is a bounded domain of $\mathbb{R}^n$ $(n = 2, 3)$ with Lipschitz boundary Γ and D (the inclusion) is a simply connected domain with Lipschitz boundary σ and such that $\overline{D} \subset \Omega$. Moreover, the coefficient λ satisfies (1.2) and the normal ν points inward D.
For $f \in H^{1/2}(\Gamma)$ problem (2.1) admits a unique variational solution, thanks to (1.2) that guarantees the coercivity of the associated bilinear form.
The inverse problem for the inclusion consists of recovering σ and λ when, in addition to the prescribed voltage f, the measured current

$$\frac{\partial u}{\partial \nu} = g$$

is known on some arc of Γ.

The uniqueness for the inverse problem is easily proved if $\lambda = 0$ (corresponding to a perfect insulating body D) and if $\lambda = +\infty$ (corresponding to a perfectly conducting body). Let us sketch the argument for $\lambda = 0$.

Let D and D' denote two inclusions bounded by simple curves σ, σ'; u and u' are two harmonic functions in $\Omega\backslash\overline{D}$ and $\Omega\backslash\overline{D}'$, satisfying condition (2.1) on σ, σ' respectively and assuming the same value f (not constant) on $\partial\Omega$. If the traces of the normal derivatives of u and u' on some arc of $\partial\Omega$ are also equal, then $u = u'$ on $\Omega\backslash(D \cup D')$ (by Holmgren's theorem); since $\frac{\partial u'}{\partial\nu} = 0$ on σ', then $\frac{\partial u}{\partial\nu} = 0$ on $\sigma' \cap \partial(D \cup D')$, thus $\frac{\partial u}{\partial\nu} = 0$ on the boundary of $D'\backslash(D \cap D')$; but this implies $u =$ constant in $D'\backslash(D \cap D')$ and then constant on $\overline{\Omega}\backslash D$, contradicting the assumption on f.

Clearly, if $\lambda > 0$ the assertion that $u = u'$ on $\Omega\backslash(D \cup D')$ remains true, but the subsequent argument cannot be applied. In this case, the real problem is the treatment of (partially) overlapping domains D, D'; if $D \cap D' = \emptyset$, uniqueness follows trivially. For, the function v which equals u in $\Omega\backslash D$ and u' in $\Omega\backslash D'$ is harmonic in the whole of Ω; if we take a voltage $f > 0$ on $\partial\Omega$, then $v > 0$ on $\overline{\Omega}$. In particular, $u'|_D$ is a positive harmonic function such that (by condition (2.1)) $\int_{\partial D} \frac{\partial u'}{\partial\nu} < 0$, a contradiction. The same argument applies to $u|_{D'}$.

On the other hand, on one connected component of $D'\backslash(\overline{D} \cap D')$, C say, we have

$$\frac{\partial u}{\partial\nu} + \lambda u = 0 \quad \text{on } \sigma \cap \partial C \tag{2.2}$$

$$\frac{\partial u}{\partial\nu} - \lambda u = 0 \quad \text{on } \sigma' \cap \partial C \tag{2.3}$$

where ν is pointing outward C. This boundary value problem (with a Robin condition on part of the boundary and a Steklov condition on the remaining part) is no more variational; if we take λ constant, we may speak of the Robin-Steklov (or generalized Steklov) eigenvalue problem. In principle such a problem (for a fixed domain) has infinitely many eigenvalues, each with finite multiplicity; but, since in our problem the domain is unknown, nothing in general can be said about the eigenfunctions u. Notice that a similar problem has to be satisfied by u' in any connected component of $D\backslash D'$. The harmonic functions u and u' extend to $\Omega\backslash D$ and $\Omega\backslash D'$ respectively and identically coincide on $\Omega\backslash(D \cup D')$. We have shown before that this is impossible if $D \cap D'$ is empty. If $D' \subset D$, next example illustrates many cases in which two different inclusions produce the same Cauchy data on Γ.

Example 1. (D and D' are concentric circles).

This example appears also in [4], but is presented here in a different and more complete version.

Consider the sequence of disjoint open intervals

$$I_n \equiv \Big\{\mu : \quad \frac{1}{2}(\sqrt{1+4n} - 1) < \mu < n\Big\}, \qquad n = 1, 2, ... \tag{2.4}$$

Assume $D = B_R$ (the ball of radius R centered at the origin) and $D' = B_r$ with $0 < r < R$. For a fixed constant $\lambda > 0$, we say that R is an admissible radius if $\lambda R \in I_n$ for some n. For every admissible R there exists precisely one associated r such that the pairs of functions u_n, v_n given below are eigenfunctions (corresponding to the same eigenvalue λ) of the Robin-Steklov problem in the annulus $B_R\backslash B_r$:

$$\begin{aligned} u_n(\rho, \theta) &= \Big(\frac{\rho^n}{R^n} + \frac{n - \lambda R}{n + \lambda R}\frac{R^n}{\rho^n}\Big)\cos(n\theta) \\ v_n(\rho, \theta) &= \Big(\frac{\rho^n}{R^n} + \frac{n - \lambda R}{n + \lambda R}\frac{R^n}{\rho^n}\Big)\sin(n\theta), \quad n = 1, 2, ... \end{aligned} \tag{2.5}$$

For, it is easy to check that: i) u_n, v_n are harmonic in $\mathbb{R}^2\backslash\{0\}$; ii) $w_\rho = \lambda w$ for $\rho = R$ (where w is either u_n or v_n); iii) $w_\rho = \lambda w$ for $\rho = r$ provided the ratio $t = r/R$ satisfies the equation

$$\frac{t^n - \Lambda t^{-n}}{t^n + \Lambda t^{-n}} = -\frac{\lambda R}{n} t, \qquad \text{where} \quad \Lambda = \frac{n - \lambda R}{n + \lambda R}. \tag{2.6}$$

Inspecting the function

$$\phi_n(t; \Lambda) : \quad [0,1] \ni t \mapsto (t^{2n} - \Lambda)(t^{2n} + \Lambda)^{-1}$$

one easily proves that, for a fixed n, ϕ_n intersects the straight line $t \mapsto \lambda R t/n$ exactly once in the *open* interval $(0,1)$ provided $\lambda R < n$ and $\sqrt{(\lambda R)^2 + \lambda R} > n$, i.e., if and only if R is an admissible value. Thus, for every positive integer n, we have a *continuum of admissible pairs* (R, r), so that $\sigma = \partial B_R$ and $\sigma' = \partial B_r$ are two different solutions of our inverse problem whenever we assign on Γ either the restriction of u_n or v_n.

The inverse problem for an *unknown boundary* has quite similar features. Let Γ be the accessible part of the boundary and σ the unknown part; we denote by Ω_σ the domain. Then the potential u is harmonic in Ω_σ and satisfies the Robin condition $\frac{\partial u}{\partial \nu} + \lambda u = 0$ on σ; we prescribe a voltage f on Γ and measure the resulting current g on some arc of Γ. If we have two domains Ω_σ and $\Omega_{\sigma'}$ corresponding to the same Cauchy pair (f, g), then we have two harmonic functions u and u' which identically coincide on $\Omega_\sigma \cap \Omega_{\sigma'}$. On one connected component of $\Omega_\sigma \backslash (\Omega_\sigma \cap \Omega_{\sigma'})$ u solves the Robin-Steklov problem.

Examples of non-uniqueness have been illustrated by [1]; here is another example of different nature.

Example 2

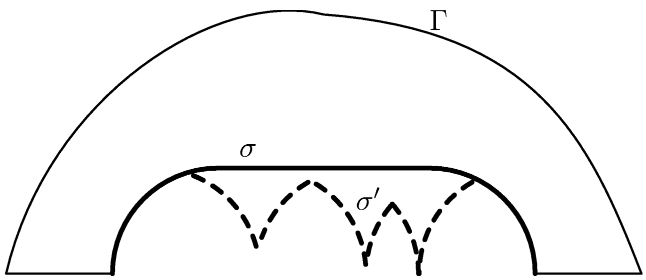

Fig. 1.

Let Ω_σ and $\Omega_{\sigma'}$ be the domains represented in fig. 1; in one case the unknown boundary σ consists of two arcs of circles (in the half plane $y > 0$) of equation

$$(x - c)^2 + y^2 = \frac{1}{\lambda^2}, \tag{2.7}$$

with a suitable c, joined by a segment on the line $y = \frac{1}{\lambda}$; in the other case σ' is made of various (arbitrarily chosen) arcs of the same family of circles with different c's. If we choose $f = y$ on Γ, then the harmonic function $u(x, y) = y$ satisfies the conditions of the problem on both σ and σ'. Notice that here we have infinitely many possible boundaries σ', which could be taken arbitrarily close to σ, but only one is of class $\mathcal{C}^1$.

3 Cracks

What is a crack? Roughly, in two dimensions is the limit of a thin imperfection

$$D_\epsilon \equiv \{x \in \Omega : \quad x = x_\sigma + t d(x_\sigma), \ x_\sigma \in \sigma, \ t \in (-\epsilon, \epsilon)\} \tag{3.1}$$

when $\epsilon \to 0$; here σ is a simple curve, ν a unit normal vector field to σ and d represents the thickness variation along σ. When $\epsilon \to 0$, D_ϵ tends to σ, but it is necessary to distinguish limits on the two sides of σ. A crack is to be considered as an abstract simple closed curve obtained from two copies of σ and glueing two by two the corresponding end points. Having chosen the normal direction ν, the direct problem for the crack is to find a harmonic function u in $\Omega\backslash\sigma$ satisfying a Dirichlet or Neumann condition on $\partial\Omega$ and the following two conditions on the two sides of σ:

$$\left.\frac{\partial u}{\partial\nu}\right|_{\pm} \pm \lambda u|_{\pm} = 0, \tag{3.2}$$

where $|_{\pm}$ denotes the trace on the two sides of σ. Conditions of this type has also been considered on open surfaces in three dimensions by Eller [5].

It can be readily shown that the direct problem has a unique weak solution in the Sobolev space $H^1(\Omega\backslash\sigma)$; here we state the result together with some additional remarks concerning regularity of the solution.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with $\mathcal{C}^1$ boundary $\partial\Omega$ and let $\sigma \subset \Omega$ be a non self-intersecting $\mathcal{C}^1$ curve with end points $P_1 \neq P_2$ in Ω. Then, for every $f \in H^{1/2}(\partial\Omega)$ there is a unique $u \in H^1(\Omega\backslash\sigma)$ satisfying*

$$\Delta u = 0, \quad \text{in} \quad \Omega\backslash\sigma \tag{3.3}$$

$$u = f, \quad \text{on} \quad \partial\Omega \tag{3.4}$$

and such that (3.2) *holds on σ. Moreover, if h is regular enough (e.g. $h \in H^{3/2}(\partial\Omega)$) u is continuous up to the boundary $\partial\Omega \cup \sigma$; in particular, the traces $u_{\pm}$ satisfy $u_+(P_i) = u_-(P_i)$, $i = 1, 2$.*

Remark 3.2. *The mapping $u \mapsto \left.\frac{\partial u}{\partial n}\right|_{\pm}$ is a continuous operator from the closed subspace of harmonic functions in $H^1(\Omega\backslash\sigma)$ to the dual space of $\tilde{H}^{1/2}(\sigma)$, a suitable trace space on (either side of) σ see [8] § 1.5 and 1.7. The boundary conditions* (3.2) *in the statement of theorem 3.1 are at first understood in this dual space.*

Proof. Consider the weak form of the problem: find $u \in H^1(\Omega\backslash\sigma)$ satisfying (5) and such that

$$\int_{\Omega\backslash\sigma} \nabla u \cdot \nabla v \, dx + \int_{\sigma} \lambda\{u_+v_+ + u_-v_-\} d\sigma = 0, \tag{3.5}$$

for every $v \in H^1(\Omega\backslash\sigma)$ with $v = 0$ on $\partial\Omega$; here $dx = dx_1dx_2$ and $d\sigma$ is the usual surface measure on σ. It is readily verified that the bilinear form at the right hand side is continuous and coercive on the subspace of the $H^1(\Omega\backslash\sigma)$ functions with vanishing trace on $\partial\Omega$; hence, by the surjectivity of the trace mapping $u \mapsto u|_{\partial\Omega}$ from $H^1(\Omega\backslash\sigma)$ into $H^{1/2}(\partial\Omega)$, unique solvability of (3.5) follows in the standard way. Then, u is (weakly) harmonic in $\Omega\backslash\sigma$ and (3.2) holds; in particular, this means that $\left.\frac{\partial u}{\partial n}\right|_{\pm} \in H^{1/2}(\sigma)$. Now, if we also assume $f \in H^{3/2}(\partial\Omega)$, we can apply known regularity results for weak solutions of elliptic problems in domains with cuts; in order to avoid compatibility conditions (typical for traces of H^m functions on polygonal boundaries) we look for regularity in the space W^2_p with $1 < p < 2$.

We first recall the Sobolev imbeddings $H^{1/2}(\sigma) \subset W_p^{1-\frac{1}{p}}(\sigma)$ and $H^{3/2}(\partial\Omega) \subset W_p^{2-\frac{1}{p}}(\partial\Omega)$; this implies that u is in W^2_p *outside any neighborhood of the end points P_1, P_2 of σ* (where the boundary forms two "corners" of angle 2π). Thus, again by Sobolev imbeddings, u is continuous in $\Omega\backslash\sigma$ (actually, Hölder continuous with exponent $\alpha = 2 - 2/p$) up to the boundary *except possibly at the end points of σ*. Let us now fix a ball B centered in P_1 (or P_2) and assume, for the sake of simplicity, that $\sigma \cap B$ is a segment; define polar coordinates (r, θ) with origin in the center of B and such that $\theta = 0$ on the upper side of σ and $\theta = 2\pi$ on the lower side. Then, by the results of [8] (theorem 5.1.3.5 and remark 5.1.3.7) we have $u(r, \theta) - c\, r^{1/2} \cos(\theta/2) \in W^2_p(B\backslash\sigma)$ for some constant c, so that continuity holds up to the whole boundary $\partial\Omega \cup \sigma$. □

Remark 3.3. *By recalling that u is smooth inside $\Omega\backslash\sigma$, it turns out that the continuity of u up to σ can be proved, by a slight modification of the above arguments, without further regularity assumptions on the datum f and with a Lipschitz boundary $\partial\Omega$. Also, the assumption that σ is a straight line in a neighborhood of the end points can be removed by using the results of [8] § 5.2 about curvilinear polygons.*

There is also an analogue of theorem 3.1 in the case of Neumann boundary conditions on $\partial\Omega$:

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain with $\mathcal{C}^1$ boundary $\partial\Omega$ and let $\sigma \subset \Omega$ be a non self-intersecting $\mathcal{C}^1$ curve with end points $P_1 \neq P_2$ in Ω. Then, for every $g \in L^2(\partial\Omega)$ there is a unique $u \in H^1(\Omega\backslash\sigma)$ satisfying*

$$\begin{aligned} \Delta u &= 0, \qquad \text{in} \quad \Omega\backslash\sigma \qquad (3.6)\\ \partial_\nu u &= g, \qquad \text{on} \quad \partial\Omega \qquad (3.7) \end{aligned}$$

and such that (3.2) *holds on σ. Moreover, if g is regular enough (e.g. $g \in H^{1/2}(\partial\Omega)$) u is continuous up to the boundary $\partial\Omega \cup \sigma$; in particular, the traces $u_\pm$ satisfy $u_+(P_i) = u_-(P_i)$, $i = 1, 2$.*

The proof follows by obvious modifications of the proof of theorem 3.1.

Remark 3.5. *In general, both u and $\partial_\nu u$ are discontinuous through σ. Either u or $\partial_\nu u$ could be continuous, but if both are continuous through σ at some point, they must be zero; if this happens along an arc of σ, u would identically vanish in Ω*

Let us now consider the *inverse problem for cracks.* Hence, we are given two cracks σ and σ', and ask if they are compatible with the same Cauchy data on $\Gamma = \partial\Omega$. Thanks to the previous remark, the uniqueness for the inverse problem is trivial if σ and σ' do not disconnect the domain Ω (e.g., they are separated, or intersect at one point or partially overlap). In that case the solutions of the direct problem u and u', corresponding to σ and λ and to σ' and λ' respectively, coincide (by Holmgren's theorem) on $\Omega\backslash(\sigma\cup\sigma')$. Then, if $|\sigma\cup\sigma'| = 0$, it follows that $u = u' = 0$ (a contradiction); furthermore, if σ and σ' overlap, we have no contradiction only if $\sigma = \sigma'$ (and then immediately follows that also $\lambda = \lambda'$).
Conversely, on one connected component D of $\Omega\backslash(\sigma\cup\sigma')$ we have

$$\begin{aligned} \text{on} \quad \sigma\cap\partial D: &\quad \partial_\nu u + \lambda u = 0 \quad \text{and} \quad \partial_\nu u' - \lambda' u' = 0 \qquad (3.8)\\ \text{on} \quad \sigma'\cap\partial D: &\quad \partial_\nu u - \lambda u = 0 \quad \text{and} \quad \partial_\nu u' + \lambda' u' = 0 \qquad (3.9) \end{aligned}$$

i.e., u and u' satisfy a generalized Steklov problem in D, with coefficients $-\lambda$ on σ, λ on σ' and λ' on σ, $-\lambda'$ on σ', respectively.

4 Uniqueness theorems

It is now clear that informations about solutions of the (homogeneous) generalized Steklov problem would help us to state the uniqueness for the inverse problems considered above. Before stating a first result in such direction, we recall the definition of the Sobolev space $H^1(\overline{C})$ (C a connected domain of $\mathbb{R}^n$): it is the space of all $u \in H^1(C)$ which are restriction to C of elements of $H^1(\mathbb{R}^n)$; the crucial fact is that the space of all functions which are restriction to C of smooth functions with compact support in $\mathbb{R}^n$ is dense in $H^1(\overline{C})$ *without any assumption on C.* Then, we have the following proposition :

Proposition 4.1. *Let $C \subset \mathbb{R}^n$ ($n = 2, 3$) be a bounded connected domain with piecewise $\mathcal{C}^1$ boundary and let $u_1 \in H^1(\overline{C}) \cap \mathcal{C}^0(\overline{C})$ be a* positive solution *of the Steklov problem*

$$\begin{aligned} \Delta u &= 0 \quad \text{in } C \\ \partial_\nu u &= \mu u \quad \text{on } \partial C \end{aligned} \tag{4.1}$$

where $\mu = \mu(x) \in L^\infty(\partial C)$ (and may change sign on ∂C). Then, any other finite energy solution u_2 continuous on $\overline{C}$ is linearly dependent on u_1.

Proof. The proposition follows at once from the integral identity

$$\int_{\partial C} \frac{u_2}{u_1}(u_1\partial_\nu u_2 - u_2\partial_\nu u_1) = \int_C \left[u_1^2\left|\nabla\frac{u_2}{u_1}\right|^2 + \frac{u_2}{u_1}(u_1\Delta u_2 - u_2\Delta u_1)\right] \tag{4.2}$$

which is readily verified for smooth $u_1 > 0$ and u_2 by using the classical Green's formula (see [6]). The identity is subsequently extended by continuity to any pair of functions u_1, u_2 in $H^1(\overline{C}) \cap \mathcal{C}^0(\overline{C})$, with $u_1 > 0$ and Δu_1, Δu_2 in $L^2(C)$. Now, if u_1, u_2 satisfy (4.1), then it follows that $\nabla(u_2/u_1) = 0$. □

Remark 4.2. *When applying* (4.2)*, we assume a priori that the harmonic functions u_1, u_2 exist, continuous in $\overline{C}$ and of finite energy in C and that are the resctrictions of elements of $H^1(\mathbb{R}^n)$. Notice that C (whose boundary in our applications consists partly of a piece of σ and partly of a piece of σ') may even not have a continuous boundary in a neighborhood of the contact points between σ and σ'.*

To apply Proposition 4.1 to our problems we need now to provide a positive solution for anyone of the three direct problems: for the inclusion, for the boundary, for the crack; moreover, such solution (restricted to C) should satisfy the regularity assumption of the Proposition.
The boundary condition (1.1) implies that, on σ, the harmonic function u cannot have a positive maximum as well as a negative minimum (for the crack, the assertion holds on both sides of σ); then, if we take $f > 0$ on Γ, it follows that $u \geq 0$ on $\overline{\Omega}$; but, if u vanishes at some point of σ, this point is a minimum and, again by condition (1.1), the normal derivative is zero. This contradicts the Hopf principle, provided this principle is applicable; for that, we need to assume some regularity for the points of σ, namely that they have the sphere property, or a uniform cone property with angle larger than $\pi/2$ [9] . For the sake of simplicity, we will assume σ of class $\mathcal{C}^1$.
If we are given the current g on Γ (and measure the potential f) we still may provide a positive solution; for, take $g > 0$ on Γ; then the minimum of u cannot be attained on Γ, but only on σ, where it must be greater than zero.
Finally, some regularity of the *whole boundary* $\Gamma \cup \sigma$ is also required in order to assure that the solution satisfies the conditions of Proposition 4.1 (see remark 4.2 above); the assumption that $\Gamma \cup \sigma$ is the boundary of a *curvilinear polygon* of class $\mathcal{C}^1$ [8] suffices to obtain the required regularity and extension properties in all the cases. Notice that we do *not* require $\mathcal{C}^1$ regularity of $\Gamma \cup \sigma$ as in [7] (such assumption implies much more a priori information on the unknown σ).
Then, we can state the following theorems of uniqueness.

Theorem 4.3. *(inclusions) Let D and D' be two inclusions (simply connected domains with $\mathcal{C}^1$ boundaries σ and σ' respectively) in Ω with $D \cap D'$ not empty; let $\partial\Omega = \Gamma$ of class and $\mathcal{C}^1$ and let $H^{1/2}(\Gamma) \ni f_1 > 0$, u_1 and u_1' be the solutions of a problem 2.1 with $f = f_1$ and D, D', σ, σ', λ, λ' respectively. Let now $f_2 \in H^{1/2}(\Gamma)$ be a voltage independent of f_1 and u_2, u_2' be the solutions of the same problem with $f = f_2$ and the rest as before. If the measured currents are equal*

$$\frac{\partial u_i}{\partial \nu} = \frac{\partial u_i'}{\partial \nu} \qquad i = 1, 2, \tag{4.3}$$

on some arc of Γ*, then*

$$\sigma = \sigma' \qquad \text{and} \qquad \lambda = \lambda'. \tag{4.4}$$

Proof. Assume $\sigma \neq \sigma'$ and let C a connected component of $D'\backslash(\overline{D}\cap D')$. We have already established that, in the stated hypotheses, u_1 is a continuous positive solution of the generalized Steklov problem with coefficient $\mu = -\lambda$ on $\sigma \cap \partial C$ and $\mu = \lambda$ on $\sigma' \cap \partial C$ (see (2.2)); if u_2 is another solution of the same problem, then it is linearly dependent on u_1 by proposition (4.2). But the harmonic functions u_1, u_2, both extend outside C up to the boundary Γ where they are respectively equal to f_1 and f_2, contradicting the assumption that f_1 and f_2 are independent. Then, $D = D'$; as a consequence, we also have $u_1' = u_1$ in $\Omega\backslash D$ and this implies $\lambda' = \lambda$. □

The assertion in case of the boundaries is similar to that for the inclusions and the proof is the same (see also [7]).

Theorem 4.4. *(boundaries) Let* Γ*,* σ *and* σ' *be regular curves of class* $\mathcal{C}^1$*,* Ω_σ *and* $\Omega_{\sigma'}$ *two bounded connected domains (curvilinear polygons of class* $\mathcal{C}^1$*) whose boundaries are* $\Gamma\cup\sigma$ *and* $\Gamma\cup\sigma'$ *respectively. let* $H^{1/2}(\Gamma) \ni f_1 > 0$*,* u_1 *and* u_1' *be the solutions of the direct problem for a boundary with* $f = f_1$ *and* σ*,* σ'*,* λ*,* λ' *respectively. Let now* $f_2 \in H^{1/2}(\Gamma)$ *be a voltage independent of* f_1 *and* u_2*,* u_2' *be the solutions of the same problem with* $f = f_2$ *and the rest as before. If the measured currents are equal*

$$\frac{\partial u_i}{\partial \nu} = \frac{\partial u_i'}{\partial \nu} \qquad i = 1, 2, \tag{4.5}$$

on some arc of Γ*, then*

$$\sigma = \sigma' \qquad \text{and} \qquad \lambda = \lambda'. \tag{4.6}$$

A similar theorem holds for cracks; but in this case a refined result can be proved for the problem of identification of a crack with *known* coefficient λ: a single measure is sufficient. This comes from the following integral identity:

Lemma 4.5. *Let* u_1*,* u_2 *be two positive harmonic functions belonging to* $H^1(\overline{C}) \cap \mathcal{C}^0(\overline{C})$*, where* C *is a domain as in proposition 4.1. Then*

$$\int_{\partial C} \left(\sqrt{\frac{u_2}{u_1}} \frac{\partial u_1}{\partial \nu} + \sqrt{\frac{u_1}{u_2}} \frac{\partial u_2}{\partial \nu} \right) = -\frac{1}{2} \int_C \frac{\left| u_1 \nabla u_2 - u_2 \nabla u_1 \right|^2}{(u_1 u_2)^{3/2}} \tag{4.7}$$

Proof. For the sake of brevity, we prove the result in the case $n = 2$; we start from the identity [6]

$$\int_{\partial C} \alpha_i \frac{\partial u_i}{\partial \nu} = \int_C \left[\frac{\partial \alpha_i}{\partial u_j} \left(\frac{\partial u_i}{\partial x}\frac{\partial u_j}{\partial x} + \frac{\partial u_i}{\partial y}\frac{\partial u_j}{\partial y} \right) + \alpha_i \Delta u_i \right] \tag{4.8}$$

where u_i, $i = 1, 2$ are smooth functions, $\alpha_i = \alpha_i(u_1, u_2)$ and the summation convention for repetead indices is understood. For positive u_1, u_2, choose $\alpha_1 = \sqrt{u_2/u_1}$, $\alpha_2 = \sqrt{u_1/u_2}$; then, if u_1, u_2 are harmonic, identity (4.7) follows. The more general case of u_1, u_2 in $H^1(\overline{C}) \cap \mathcal{C}^0(\overline{C})$ is proved as in proposition 4.1. □

Remark 4.6. *It is worthwhile to stress that the former identity* (4.2) *of proposition 4.1 follows from* (4.8) *by choosing* $\alpha_1 = -u_2^2/u_1$ *and* $\alpha_2 = u_2$*.*

Theorem 4.7. *(cracks) Let* σ*,* σ' *be two cracks of class* $\mathcal{C}^1$ *contained in* Ω*; for a given* $H^{1/2}(\partial\Omega) \ni f > 0$*, let* $u \in H^{1/2}(\Omega\backslash\sigma)$ *and* $u' \in H^{1/2}(\Omega\backslash\sigma')$ *be the solutions of problem (3.2)-(3.4) respectively with* σ*,* σ'*, but with the same* λ*. If the measured currents are equal*

$$\frac{\partial u}{\partial \nu} = \frac{\partial u'}{\partial \nu} \tag{4.9}$$

on some arc of Γ*, then*

$$\sigma = \sigma'.$$

Proof. We have already noticed that uniqueness follows at once when σ and σ' do not disconnect the domain Ω; on a connected component C of $\Omega \backslash (\sigma \cup \sigma')$ (not containing $\partial\Omega$) u and u' solve two Steklov problems corresponding respectively to a coefficient μ ($\mu = -\lambda$ on σ and $\mu = \lambda$ on σ') for u and $-\mu$ for u'.
Considering now the integral identity (4.7) with $u_1 = u$, $u_2 = u'$ and taking account of the boundary conditions satisfied by u and u', we see that the left member of (4.7) vanishes, so that

$$u\nabla u' = u'\nabla u \qquad \text{on } C$$

and therefore

$$u\frac{\partial u'}{\partial \nu} = u'\frac{\partial u}{\partial \nu} \qquad \text{on } \partial C.$$

Then, on $\sigma \cap \partial D$ we have

$$\lambda u u' = -\lambda u u',$$

which implies u (or u') $= 0$, a contradiction. □

We stress that in [5] this same conclusion was reached by assuming the knowledge of the entire Dirichlet-to-Neumann map.

References

[1] F. Cakoni and R. Kress, *Integral equations for inverse problems in corrosion detection from partial Cauchy data*, Inverse Problems and Imaging 1, 2007, 229-245

[2] D.Fasino and G.Inglese, *An inverse Robin problem for Laplace's equation: theoretical fresults and numerical methods*, Inverse Problems 15, 1999, 41-48

[3] G.Inglese and F.Mariani, *Corrosion detection in conducting boundaries*, Inverse Problems 20, 2004, 1207-1215

[4] W. Rundell *Recovering an obstacle and its impedance from Cauchy data*, Inverse Problems 24, 2008, 1-22

[5] M. Eller *Identification of cracks in three-dimensional bodies by many boundary measurements*, Inverse Problems 12, 1996, 395-408

[6] M.H.Martin *Linear and nonlinear boundary problems for Harmonic functions*, Proc. AMS 12, 1958, 258-266

[7] V.Bacchelli *uniqueness for the determination of unknown boundary and impedance with homogeneous Robin condition* preprint (2008)

[8] P.Grisvard *Elliptic problems in nonsmooth domains*, Inverse Problems Pitman Adv. Publ. Program, London, 1985.

[9] J.Serrin *A symmetry problem in potential theory*, Arch.Ration.Mech. 43, 1971, 304-318