Uniqueness for the determination of unknown boundary and impedance with homogeneous Robin condition

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Abstract

We consider the problem of determining the corroded portion of the boundary of a n-dimensional body (n=2, 3) and the impedance by two measures on the accessible portion of the boundary. On the unknown boundary part it is assumed the Robin homogeneous condition.

2000 *Mathematical Subject Classification.* 35R30, 35R25, 35R3510.

Key words. Inverse boundary value problems, corrosion, thermal imaging, unique continuation.

1 Introduction

In this paper we deal with a classical inverse problem. Assume Ω be a bounded connected domain in $\mathbb{R}^n$, whose boundary $\partial\Omega$ belongs to $C^{2,\alpha}$ class, $0 < \alpha < 1$; suppose $\partial\Omega = \overline{\Gamma^a} \cup \overline{\Gamma^i}$, where Γ^i and Γ^a are two open connected disjoint portions of $\partial\Omega$. Assume that Γ^i is unknown and inaccessible (perhaps Γ^i is some interior connected component of $\partial\Omega$ or some inaccessible portion of the exterior component of $\partial\Omega$), while Γ^a is known and accessible for input and output measurements. Let us consider the solution u of the following mixed boundary value problem

$$\begin{cases} \triangle u = 0, & \text{in } \Omega \ , \\ \dfrac{\partial u}{\partial \nu} = g, & \text{on } \Gamma^a, \\ \dfrac{\partial u}{\partial \nu} + \gamma u = 0, & \text{on } \Gamma^i, \end{cases} \tag{1}$$

where ν is the exterior unit normal to $\partial\Omega$, g is an assigned function, $\gamma \not\equiv 0$.

Suppose $g \in C^{1,\alpha}(\Gamma^a)$, $supp\, g \subset \Gamma^a$, $\gamma \in C^{1,\alpha}(\Gamma^i)$, $\gamma \geq 0$, $\gamma \not\equiv 0$, $supp\, \gamma \subset \Gamma^i$; it is known [10] that the direct problem (1) has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

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The inverse problem consists in determining Γ^i and γ provided $u\,|_\Sigma$, $\Sigma \subset \Gamma^a$, is known.

This problem arises from non-destructive testing in corrosion detection, where Γ^i represents a corroded portion of $\partial\Omega$ and we will determine Γ^i and the impedance γ by suitable inspections and measurements on the accessible portion Γ^a of the boundary of Ω. We consider such an inverse problem also where $\Omega = \widetilde{\Omega}\backslash D$, Ω bounded connected, $D \subset\subset \widetilde{\Omega}$, $\Gamma^i = \partial D$, $\Gamma^a = \partial\widetilde{\Omega}$: we are interested, by electrostatic measures or thermal imaging techniques, in identifying D and the coefficient γ by measurements on Γ^a, the external and accessible part of $\partial\Omega$.

Many authors have treated uniqueness and stability of Γ^i in the case where on Γ^i it is assumed a Neumann or a Dirichlet condition (see, e.g., [1], [3], [12], [13], [14], [16], [19]).

Concerning the Robin condition, we recall that in [11], assuming Ω a thin rectangular plate, local uniqueness of Γ^i is proved. In [18] two different algorithms are presented in order to reconstruct Γ^i. Regarding the impedance γ, in [8] it is introduced a numerical algorithm for recovering such a coefficient. Moreover we recall that different stability estimates for γ have been proved: in [5] a monotone Lipschitz stability estimate, in [6] a local Lipschitz stability estimate, in [2] a log-type stability estimate.

In [4] it is proved, by counterexamples, that a single measurement $(g, u\,|_\Sigma)$ is not sufficient to determine simultaneously the shape Γ^i and the impedance γ and the same holds if, fixed γ a known constant, the only aim is to determine Γ^i.

In the present paper we are able to show that two Cauchy data pairs, that is $(g, u\,|_\Sigma)$, $(\widetilde{g}, \widetilde{u}\,|_\Sigma)$, guarantee simultaneously uniqueness of Γ^i and γ, provided g, $\widetilde{g}$ are linearly independent and one of them, say g, is positive.

2 The uniqueness theorem

Theorem 1 *Let Ω_j, $j = 1, 2$, be a bounded connected domain in $\mathbb{R}^n$, whose boundary $\partial\Omega_j$ is of $C^{2,\alpha}$ class, $0 < \alpha < 1$. Let us assume that $\partial\Omega_j = \overline{\Gamma^a} \cup \overline{\Gamma^i_j}$, $j = 1, 2$, where Γ^a, Γ^i_j are two open connected disjoint sets. Suppose $\gamma_j \in C^{1,\alpha}(\Gamma^i_j)$, $j = 1, 2$, $\gamma_j \geq 0$, $\gamma_j \not\equiv 0$, supp $\gamma_j \subset \Gamma^i_j$. Let be assigned two non trivial functions g, $\widetilde{g} \in C^{1,\alpha}(\Gamma^a)$, supp g, supp $\widetilde{g} \subset \Gamma^a$; suppose g, $\widetilde{g}$ be linearly independent and $g \geq 0$. Let u_j, $j = 1, 2$, be the solution to (1), where $\Omega = \Omega_j$, $\gamma = \gamma_j$ and the Neumann datum on Γ^a is g. Let $\widetilde{u}_j$, $j = 1, 2$, be the solution to (1), where $\Omega = \Omega_j$, $\gamma = \gamma_j$ and the Neumann datum on Γ^a is $\widetilde{g}$. Let be $\Sigma \subset \Gamma^a$, Σ open in the relative topology of $\partial\Omega$.*

Then, if

$$u_1\,|_\Sigma = u_2\,|_\Sigma\ , \quad \widetilde{u}_1\,|_\Sigma = \widetilde{u}_2\,|_\Sigma\ , \tag{2}$$

we have

$$\Gamma^i_1 = \Gamma^i_2, \quad \gamma_1 = \gamma_2. \tag{3}$$

Proof. The regularity assumptions on Ω_j, γ_j, $j = 1, 2$, g , $\widetilde{g}$ garantee [10] that u_j, $\widetilde{u_j} \in C^{2,\alpha}(\overline{\Omega_j})$, $j = 1, 2$. We observe moreover that u_j is positive on $\overline{\Omega_j}$, $j = 1, 2$; on the contrary, if there exists a point P in $\overline{\Omega_j}$ such that $u_j(P) \leq 0$, by the maximum principle [17], denoting $Q \in \partial\Omega_j$ the minimum point of u_j in $\overline{\Omega_j}$, also $u_j(Q) \leq 0$. The point Q cannot belong to Γ^a, since this contradicts the Hopf maximum principle [17], being $g = \frac{\partial u_j}{\partial \nu}(Q) \geq 0$; the point Q cannot belong to Γ^i_j, since the condition on Γ^i_j implies $\frac{\partial u_j}{\partial \nu}(Q) \geq 0$ and that contradicts again the Hopf maximum principle.

We prove first that $\Gamma^i_1 = \Gamma^i_2$. By contradiction assume for istance that $\Omega_1 \backslash \Omega_2 \neq \emptyset$. Denote by G the connected component of $\Omega_1 \cap \Omega_2$ such that $\Sigma \subset \overline{G}$. Since

$$u_1 \,|_\Sigma = u_2 \,|_\Sigma \;, \quad \widetilde{u}_1 \,|_\Sigma = \widetilde{u}_2 \,|_\Sigma \;, \tag{4}$$

and

$$\frac{\partial u_1}{\partial \nu} \,|_\Sigma = \frac{\partial u_2}{\partial \nu} \,|_\Sigma \;, \quad \frac{\partial \widetilde{u}_1}{\partial \nu} \,|_\Sigma = \frac{\partial \widetilde{u}_2}{\partial \nu} \,|_\Sigma \;, \tag{5}$$

Holmgren's theorem implies $u_1 \equiv u_2$, $\widetilde{u_1} \equiv \widetilde{u_2}$ in a small ball and then, by unique continuation property, we get that $u_1 \equiv u_2$ in G and $\widetilde{u_1} \equiv \widetilde{u_2}$ in G. Let us consider $\Omega_1 \backslash G$ and denote with N the exterior unit normal to $\partial(\Omega_1 \backslash G)$. Then u_1 satisfies the problem

$$\begin{cases} \triangle u_1 = 0, & \text{in } \Omega_1 \backslash G, \\ \dfrac{\partial u_1}{\partial N} + \gamma_1 u_1 = 0, & \text{on } \partial(\Omega_1 \backslash G) \cap \Gamma^i_1, \\ -\dfrac{\partial u_1}{\partial N} + \gamma_2 u_1 = 0, & \text{on } \partial(\Omega_1 \backslash G) \cap \Gamma^i_2, \end{cases}$$

that is u_1 satisfies on $\partial(\Omega_1 \backslash G) \cap \Gamma^i_1$ a Robin condition with coefficient γ_1, while on $\partial(\Omega_1 \backslash G) \cap \Gamma^i_2$ a Steklov condition with coefficient γ_2. The same is true for $\widetilde{u_1}$. As $u_1 > 0$, the function $\lambda = \frac{\widetilde{u_1}}{u_1}$ is regular in $\overline{\Omega_1 \backslash G}$.

Since $\partial(\Omega_1 \backslash G) \subset \partial\Omega_1 \cup \partial\Omega_2$, we have $\mathcal{H}^{n-1}(\partial(\Omega_1 \backslash G)) < +\infty$ ($\mathcal{H}^{n-1}$denotes the $n-1$ Hausdorff measure), so we get that [7] $\Omega_1 \backslash G$ is a set of finite perimeter. Therefore, also by the regularity properties of u_1, $\widetilde{u_1}$, we are able to apply in $\Omega_1 \backslash G$ the Gauss-Green formula (see for istance [9], [7]); more precisely we make use of the following equality (see [15]), that is an easy consequence of the Gauss-Green formula

$$\begin{aligned} &\int_{\Omega_1 \backslash G} \lambda \,(u_1 \triangle \widetilde{u_1} \; - \widetilde{u_1} \triangle u_1) + \int_{\Omega_1 \backslash G} u_1^2 \;\, |\nabla \lambda|^2 \\ &= \int_{\partial^*(\Omega_1 \backslash G)} \lambda (u_1 \tfrac{\partial \widetilde{u}_1}{\partial N} - \widetilde{u_1} \tfrac{\partial u_1}{\partial N}), \end{aligned} \tag{6}$$

where $\partial^*(\Omega_1 \backslash G)$ is the reducing boundary in the De Giorgi sense. Since on $\partial(\Omega_1 \backslash G) \cap \Gamma^i_1$ we have $u_1 \frac{\partial \widetilde{u}_1}{\partial N} - \widetilde{u_1} \frac{\partial u_1}{\partial N} = u_1(-\gamma_1 \widetilde{u_1}) - \widetilde{u_1}(-\gamma_1 u_1) = 0$, while on $\partial(\Omega_1 \backslash G) \cap \Gamma^i_2$, we have $u_1 \frac{\partial \widetilde{u}_1}{\partial N} - \widetilde{u_1} \frac{\partial u_1}{\partial N} = u_1(\gamma_2 \widetilde{u_1}) - \widetilde{u_1}(\gamma_2 u_1) = 0$, than by (6) we get $\lambda = const$, so that there exist $\alpha, \beta \in \mathbb{R}, (\alpha, \beta) \neq (0, 0)$, such that $\alpha u_1 + \beta \widetilde{u_1} \equiv 0$ in $\Omega_1 \backslash G$. Again by unique continuation property we have $\alpha g + \beta \widetilde{g} \equiv 0$, that contradicts the assumption that $g, \widetilde{g}$ are linearly independent.

Now we prove that $\gamma_1 = \gamma_2$. Since $u_1 \equiv u_2$ in Ω_1, we get on Γ^i

$$\frac{\partial u_1}{\partial \nu} + \gamma_1 u_1 = 0, \ \frac{\partial u_1}{\partial \nu} + \gamma_2 u_1 = 0.$$

Subtracting one to the other, we obtain $(\gamma_1 - \gamma_2)u_1 = 0$ on Γ^i; if, by contradiction, there exists $P \in \Gamma^i$ such that $(\gamma_1 - \gamma_2)(P) \neq 0$, we get, as $\gamma_1 - \gamma_2 \in C^{1,\alpha}(\Gamma^i)$, $(\gamma_1 - \gamma_2) \neq 0$ in $U(P) \cap \Gamma^i$, that implies $u_1 = 0$ in $U(P) \cap \Gamma^i$ and, at the same time, $\frac{\partial u_1}{\partial \nu} = 0$ in $U(P) \cap \Gamma^i$. This contradicts the assumption $g \not\equiv 0$. ∎

Remark 2 *We will remark that the uniqueness result of theorem 1 holds, without any change, also in the case, already presented in the introduction, where* Ω *is a bounded connected domain such that* $\Omega = \widetilde{\Omega} \backslash D$, $D \subset\subset \widetilde{\Omega}$, *and* $\Gamma^i = \partial D$, $\Gamma^a = \partial \widetilde{\Omega}$.

Remark 3 *Instead of problem (1), we can consider the following*

$$\begin{cases} \triangle u = 0, & \text{in } \Omega, \\ u = f & \text{on } \Gamma^a, \\ \dfrac{\partial u}{\partial \nu} + \gamma u = 0, & \text{on } \Gamma^i, \end{cases} \tag{7}$$

with f *assigned,* $f \in C^{2,\alpha}(\Gamma^a)$, $\gamma \in C^{1,\alpha}(\Gamma^i)$, $\gamma \geq 0$, $\gamma \not\equiv 0$. *In such a case the inverse problem consists in determining* Γ^i *and* γ *by the knowledge of* $\frac{\partial u}{\partial \nu}|_\Sigma$, $\Sigma \subset \Gamma^a$, *being* u *the solution to (7). Also in this case it is possible to state a theorem analogous to the previous one, that is one can determine* Γ^i *and* γ *with two pairs of measurements* $(f, \frac{\partial u}{\partial \nu}|_\Sigma), (\widetilde{f}, \frac{\partial \widetilde{u}}{\partial \nu}|_\Sigma)$, *provided* f, $\widetilde{f}$ *are linearly independent and one of them, say* f, *is positive.*

Remark 4 *The result of theorem 1 can be easily extended to the case in which we consider, instead of problem (1), the following*

$$\begin{cases} div(A \ \nabla u) = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = g, & \text{on } \Gamma^a, \\ A \nabla u \cdot \nu + \gamma u = 0, & \text{on } \Gamma^i \ , \end{cases} \tag{8}$$

where $A = \{a_{ij}\}$, $i, j = 1, ..., n$, *is a symmetric matrix satisfying the uniform ellipticity condition, with* $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$, g *is a non trivial assigned function and* $\gamma \geq 0$, $\gamma \not\equiv 0$.

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