Uniqueness for the determination of unknown boundary and impedance with homogeneous Robin condition

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Abstract

We consider the problem of determining the corroded portion of the boundary of a n-dimensional body (n=2, 3) and the impedance by two measures on the accessible portion of the boundary. On the unknown boundary part it is assumed the Robin homogeneous condition.

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1 Introduction

In this paper we deal with a classical inverse problem. Assume Ω be a bounded connected domain in $\mathbb{R}^n$, whose boundary $\partial \Omega$ belongs to $C^{2,\alpha}$ class, $0 < \alpha < 1$; suppose $\partial \Omega = \Gamma^a \cup \Gamma^i$, where $\Gamma^i$ and $\Gamma^a$ are two open connected disjoint portions of $\partial \Omega$. Assume that $\Gamma^i$ is unknown and inaccessible (perhaps $\Gamma^i$ is some interior connected component of $\partial \Omega$ or some inaccessible portion of the exterior component of $\partial \Omega$), while $\Gamma^a$ is known and accessible for input and output measurements. Let us consider the solution $u$ of the following mixed boundary value problem

\[
\begin{aligned}
\triangle u &= 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= g, & \text{on } \Gamma^a, \\
\frac{\partial u}{\partial \nu} + \gamma u &= 0, & \text{on } \Gamma^i,
\end{aligned}
\]

where $\nu$ is the exterior unit normal to $\partial \Omega$, $g$ is an assigned function, $\gamma \not\equiv 0$.

Suppose $g \in C^{1,\alpha}(\Gamma^a)$, $\text{supp } g \subset \Gamma^a$, $\gamma \in C^{1,\alpha}(\Gamma^i)$, $\gamma \geq 0$, $\gamma \not\equiv 0$, $\text{supp } \gamma \subset \Gamma^i$; it is known [10] that the direct problem (1) has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

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The inverse problem consists in determining $Γ^i$ and $γ$ provided $u|_Σ$, $Σ ⊂ Γ^a$, is known.

This problem arises from non-destructive testing in corrosion detection, where $Γ^i$ represents a corroded portion of $∂Ω$ and we will determine $Γ^i$ and the impedance $γ$ by suitable inspections and measurements on the accessible portion $Γ^a$ of the boundary of $Ω$. We consider such an inverse problem also where $Ω = ˜Ω \setminus D$, $Ω$ bounded connected, $D ⊂⊂ ˜Ω$, $Γ^i = ∂D$, $Γ^a = ∂ ˜Ω$: we are interested, by electrostatic measures or thermal imaging techniques, in identifying $D$ and the coefficient $γ$ by measurements on $Γ^a$, the external and accessible part of $∂Ω$.

Many authors have treated uniqueness and stability of $Γ^i$ in the case where on $Γ^i$ it is assumed a Neumann or a Dirichlet condition (see, e.g., [1], [3], [12], [13], [14], [16], [19]).

Concerning the Robin condition, we recall that in [11], assuming $Ω$ a thin rectangular plate, local uniqueness of $Γ^i$ is proved. In [18] two different algorithms are presented in order to reconstruct $Γ^i$. Regarding the impedance $γ$, in [8] it is introduced a numerical algorithm for recovering such a coefficient. Moreover we recall that different stability estimates for $γ$ have been proved: in [5] a monotone Lipschitz stability estimate, in [6] a local Lipschitz stability estimate, in [2] a log-type stability estimate.

In [4] it is proved, by counterexamples, that a single measurement $(g, u|_Σ)$ is not sufficient to determine simultaneously the shape $Γ^i$ and the impedance $γ$ and the same holds if, fixed $γ$ a known constant, the only aim is to determine $Γ^i$.

In the present paper we are able to show that two Cauchy data pairs, that is $(g, u|_Σ)$, $(˜g, ˜u|_Σ)$, guarantee simultaneously uniqueness of $Γ^i$ and $γ$, provided $g$, $˜g$ are linearly independent and one of them, say $g$, is positive.

2 The uniqueness theorem

Theorem 1 Let $Ω_j$, $j = 1, 2$, be a bounded connected domain in $\mathbb{R}^n$, whose boundary $∂Ω_j$ is of $C^{2,α}$ class, $0 < α < 1$. Let us assume that $∂Ω_j = Γ^a_j \cup ˜Γ_j$, $j = 1, 2$, where $Γ^a_j$, $Γ^o_j$ are two open connected disjoint sets. Suppose $Γ^a_j \subset C^{1,α}(Γ^o_j)$, $j = 1, 2$, $γ_j ≥ 0$, $γ_j \neq 0$, $supp γ_j \subset Γ^i_j$. Let be assigned two non trivial functions $g$, $˜g \in C^{1,α}(Γ^a_j)$, $supp g$, $supp ˜g \subset Γ^a_j$; suppose $g$, $˜g$ be linearly independent and $g ≥ 0$. Let $u_j$, $j = 1, 2$, be the solution to (1), where $Ω = Ω_j$, $γ = γ_j$ and the Neumann datum on $Γ^a_j$ is $g$. Let $u_j$, $j = 1, 2$, be the solution to (1), where $Ω = Ω_j$, $γ = γ_j$ and the Neumann datum on $Γ^a_j$ is $˜g$. Let be $Σ \subset Γ^a_j$, $Σ$ open in the relative topology of $∂Ω$.

Then, if

$$u_1|_Σ = u_2|_Σ, \quad ˜u_1|_Σ = ˜u_2|_Σ,$$

we have

$$Γ^i_1 = Γ^i_2, \quad γ_1 = γ_2.$$
Proof. The regularity assumptions on $\Omega_j$, $\gamma_j$, $j = 1, 2$, $g$, $\tilde{g}$ guarantee [10] that $u_j - \tilde{u}_j \in C^{2,\alpha}(\overline{\Omega}_j)$, $j = 1, 2$. We observe moreover that $u_j$ is positive on $\overline{\Omega}_j$, $j = 1, 2$; on the contrary, if there exists a point $P$ in $\overline{\Omega}_j$ such that $u_j(P) \leq 0$, by the maximum principle [17], denoting $Q \in \partial \Omega_j$ the minimum point of $u_j$ in $\overline{\Omega}_j$, also $u_j(Q) \leq 0$. The point $Q$ cannot belong to $\Gamma^a$, since this contradicts the Hopf maximum principle [17], being $g = \frac{\partial u_j}{\partial n}(Q) \geq 0$; the point $Q$ cannot belong to $\Gamma^b_j$, since the condition on $\Gamma^b_j$ implies $\frac{\partial u_j}{\partial n}(Q) \geq 0$ and that contradicts again the Hopf maximum principle.

We prove first that $\Gamma^b_1 = \Gamma^b_2$. By contradiction assume for instance that $\Omega_1 \backslash \Omega_2 \neq \emptyset$. Denote by $G$ the connected component of $\Omega_1 \cap \Omega_2$ such that $\Sigma \subset \overline{G}$. Since $u_1 | \Sigma = u_2 | \Sigma$, $\tilde{u}_1 | \Sigma = \tilde{u}_2 | \Sigma$, (4) and

$$
\frac{\partial u_1}{\partial \nu}|_\Sigma = \frac{\partial u_2}{\partial \nu}|_\Sigma, \quad \frac{\partial \tilde{u}_1}{\partial \nu}|_\Sigma = \frac{\partial \tilde{u}_2}{\partial \nu}|_\Sigma,
$$

Holmgren’s theorem implies $u_1 \equiv u_2$, $\tilde{u}_1 \equiv \tilde{u}_2$ in a small ball and then, by unique continuation property, we get that $u_1 \equiv u_2$ in $G$ and $\tilde{u}_1 \equiv \tilde{u}_2$ in $G$. Let us consider $\Omega_1 \backslash G$ and denote with $N$ the exterior unit normal to $\partial(\Omega_1 \backslash G)$. Then $u_1$ satisfies the problem

\[
\begin{align*}
\Delta u_1 &= 0, & \text{in } \Omega_1 \backslash G, \\
\frac{\partial u_1}{\partial \nu} + \gamma_1 u_1 &= 0, & \text{on } \partial(\Omega_1 \backslash G) \cap \Gamma^b_1, \\
-\frac{\partial u_1}{\partial \nu} + \gamma_2 u_1 &= 0, & \text{on } \partial(\Omega_1 \backslash G) \cap \Gamma^b_2,
\end{align*}
\]

that is $u_1$ satisfies on $\partial(\Omega_1 \backslash G) \cap \Gamma^b_1$ a Robin condition with coefficient $\gamma_1$, while on $\partial(\Omega_1 \backslash G) \cap \Gamma^b_2$ a Steklov condition with coefficient $\gamma_2$. The same is true for $\tilde{u}_1$. As $u_1 > 0$, the function $\lambda = \frac{\tilde{u}_1}{u_1}$ is regular in $\Omega_1 \backslash G$.

Since $\partial(\Omega_1 \backslash G) \subset \partial \Omega_1 \cup \partial \Omega_2$, we have $\mathcal{H}^{n-1}(\partial(\Omega_1 \backslash G)) < +\infty$ ($\mathcal{H}^{n-1}$ denotes the $n-1$ Hausdorff measure), so we get that $[7]$ $\Omega_1 \backslash G$ is a set of finite perimeter. Therefore, also by the regularity properties of $u_1$, $\tilde{u}_1$, we are able to apply in $\Omega_1 \backslash G$ the Gauss-Green formula (see for instance [9], [7]); more precisely we make use of the following equality (see [15]), that is an easy consequence of the Gauss-Green formula

$$
\int_{\Omega_1 \backslash G} \lambda (u_1 \Delta \tilde{u}_1 - \tilde{u}_1 \Delta u_1) + \int_{\Omega_1 \backslash G} u_1^2 |\nabla \lambda|^2
$$

$$
= \int_{\partial^*(\Omega_1 \backslash G)} \lambda (u_1 \frac{\partial \tilde{u}_1}{\partial N} - \tilde{u}_1 \frac{\partial u_1}{\partial N}),
$$

where $\partial^*(\Omega_1 \backslash G)$ is the reducing boundary in the De Giorgi sense. Since on $\partial(\Omega_1 \backslash G) \cap \Gamma^b_1$ we have $u_1 \frac{\partial \tilde{u}_1}{\partial N} - \tilde{u}_1 \frac{\partial u_1}{\partial N} = u_1 (-\gamma_1 \tilde{u}_1) - \tilde{u}_1 (-\gamma_1 u_1) = 0$, while on $\partial(\Omega_1 \backslash G) \cap \Gamma^b_2$, we have $u_1 \frac{\partial \tilde{u}_1}{\partial N} - \tilde{u}_1 \frac{\partial u_1}{\partial N} = u_1 (\gamma_2 \tilde{u}_1) - \tilde{u}_1 (\gamma_2 u_1) = 0$, than by (6) we get $\lambda = const$, so that there exist $\alpha, \beta \in \mathbb{R}, (\alpha, \beta) \neq (0, 0)$, such that $\alpha u_1 + \beta \tilde{u}_1 \equiv 0$ in $\Omega_1 \backslash G$. Again by unique continuation property we have $\alpha g + \beta \tilde{g} \equiv 0$, that contradicts the assumption that $g$, $\tilde{g}$ are linearly independent.
Now we prove that $\gamma_1 = \gamma_2$. Since $u_1 \equiv u_2$ in $\Omega_1$, we get on $\Gamma^i$

$$\frac{\partial u_1}{\partial \nu} + \gamma_1 u_1 = 0, \quad \frac{\partial u_1}{\partial \nu} + \gamma_2 u_1 = 0.$$ 

Subtracting one to the other, we obtain $(\gamma_1 - \gamma_2)u_1 = 0$ on $\Gamma^i$; if, by contradiction, there exists $P \in \Gamma^i$ such that $(\gamma_1 - \gamma_2)(P) \not= 0$, we get, as $\gamma_1 - \gamma_2 \in C^{1,\alpha}(\Gamma^i)$, $(\gamma_1 - \gamma_2) \not= 0$ in $U(P) \cap \Gamma^i$, that implies $u_1 = 0$ in $U(P) \cap \Gamma^i$ and, at the same time, $\frac{\partial u_1}{\partial \nu} = 0$ in $U(P) \cap \Gamma^i$. This contradicts the assumption $g \not\equiv 0$.

Remark 2 We will remark that the uniqueness result of theorem 1 holds, without any change, also in the case, already presented in the introduction, where $\Omega$ is a bounded connected domain such that $\Omega = \Omega \backslash D$, $D \subset \subset \tilde{\Omega}$, and $\Gamma^i = \partial D$, $\Gamma^a = \partial \tilde{\Omega}$.

Remark 3 Instead of problem (1), we can consider the following

$$\begin{cases}
\triangle u = 0, & \text{in } \Omega, \\
u = f & \text{on } \Gamma^a, \\
\frac{\partial u}{\partial \nu} + \gamma u = 0, & \text{on } \Gamma^i,
\end{cases}$$

(7)

with $f$ assigned, $f \in C^{2,\alpha}(\Gamma^a)$, $\gamma \in C^{1,\alpha}(\Gamma^i)$, $\gamma \geq 0, \gamma \not= 0$. In such a case the inverse problem consists in determining $\Gamma^a$ and $\gamma$ by the knowledge of $\frac{\partial u}{\partial \nu} |_{\Sigma}$, $\Sigma \subset \Gamma^a$, being $u$ the solution to (7). Also in this case it is possible to state a theorem analogous to the previous one, that is one can determine $\Gamma^a$ and $\gamma$ with two pairs of measurements $(f, \frac{\partial u}{\partial \nu} |_{\Sigma})$, $(\tilde{f}, \frac{\partial \tilde{u}}{\partial \nu} |_{\Sigma})$, provided $f$, $\tilde{f}$ are linearly independent and one of them, say $f$, is positive.

Remark 4 The result of theorem 1 can be easily extended to the case in which we consider, instead of problem (1), the following

$$\begin{cases}
\text{div}(A \nabla u) = 0, & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = g, & \text{on } \Gamma^a, \\
A \nabla u \cdot \nu + \gamma u = 0, & \text{on } \Gamma^i,
\end{cases}$$

(8)

where $A = \{a_{ij}\}$, $i,j = 1,\ldots,n$, is a symmetric matrix satisfying the uniform ellipticity condition, with $a_{ij} \in C^{1,\alpha}(\overline{\Omega})$, $g$ is a non trivial assigned function and $\gamma \geq 0, \gamma \not= 0$. 

4
References


