Positive solutions to critical growth biharmonic elliptic problems under Steklov boundary conditions

Filippo GAZZOLA – Dario PIEROTTI
Dipartimento di Matematica del Politecnico, Piazza L. da Vinci 32 - 20133 Milano (Italy)

Abstract

We study the fourth order nonlinear critical problem $\Delta^2 u = u^{2^*-1}$ in the unit ball of $\mathbb{R}^n$ $(n \geq 5)$, subject to the Steklov boundary conditions $u = \Delta u - du_\nu = 0$ on ∂B. We provide the exact range of the parameter d for which this problem admits a positive (radial) solution. We also show that the solution is unique in this range and in the class of radially symmetric functions. Finally, we study the behavior of the solution when d tends to the extremals of this range. These results complement previous results in [3].
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1 Introduction

Let $B \subset \mathbb{R}^n$ $(n \geq 5)$ be the unit ball, let $2^* = \frac{2n}{n-4}$ denote the critical Sobolev exponent for the embedding $H^2(B) \subset L^{2^*}(B)$, let $d \in \mathbb{R}$. We consider the following fourth order elliptic problem with purely critical growth and Steklov boundary conditions:

$$\begin{cases} \Delta^2 u = u^{2^*-1}, \quad u > 0 & \text{in } B \\ u = 0, \quad \Delta u - du_\nu = 0 & \text{on } \partial B. \end{cases} \tag{1}$$

Here u_ν denotes the outer normal derivative of u on ∂B.
The widely studied second order semilinear elliptic equation shows that nonlinearities at critical growth present highly interesting phenomena concerning the existence/nonexistence of positive solutions, see the seminal paper by Brezis-Nirenberg [4] and [5, Chapter III] for a survey. For fourth order equations the existence/nonexistence problem is even more challenging, since the available techniques strongly depend on the imposed boundary conditions. The present paper is motivated by the growing interest in recent years for the corresponding Dirichlet boundary value problem (corresponding to $d = -\infty$ in (1)) and Navier boundary value problem (corresponding to $d = 0$). We refer to the introduction in [3] for a survey of the known results.
By *solution* of (1) we mean here a function $u \in H^2 \cap H^1_0(B)$ such that $u > 0$ a.e. in B and

$$\int_B \Delta u \Delta \varphi - d \int_{\partial B} u_\nu \varphi_\nu = \int_B u^{2^*-1} \varphi \qquad \text{for all } \varphi \in H^2 \cap H^1_0(B)\ . \tag{2}$$

A solution in this sense is in fact a strong (classical) solution in $C^{4,\alpha}(\overline{B})$, see [1, Proposition 23] and also [6].
Preliminary results concerning (1) were obtained in [3] where it is shown that (1) admits no solution whenever $d \leq 4$ or $d \geq n$. It was conjectured in [3] that the existence range for (1) is $d \in (4, n)$ although the existence of solutions was shown only for $d \in (\sigma_n, n)$ for a suitable $\sigma_n > 4$. The purpose of the present paper is to fill the gap and to prove existence in the whole range $d \in (4, n)$. Moreover,

in such range we show that the solution is unique in the class of radially symmetric functions and we determine the asymptotic behavior of the solution in the limit cases where $d \to n$ and $d \to 4$:

Theorem 1. *If $d \leq 4$ or $d \geq n$, then (1) admits no solutions.*
If $4 < d < n$, then (1) admits a unique radially symmetric solution u_d.
Moreover:
(i) as $d \to n^-$ we have $u_d \to 0$ uniformly in $\overline{B}$;
(ii) as $d \to 4^+$ we have $u_d(0) \to +\infty$, $u_d(x) \to 0$ for all $x \in B \setminus \{0\}$ and $(u_d)_\nu \to 0$ on ∂B.

The nonexistence part and the asymptotic behavior as $d \to n^-$ were already proved in [3]. We also refer to [2] for results concerning *sign-changing* solutions to (1).
In order to prove the existence of solutions (Section 2), we use a refined compactness method. More precisely, we use a sequence of suitably modified Sobolev minimizers which tend to concentrate and show that their energy lies below the compactness threshold. The "optimal" modification is determined by solving a variational problem which leads to a quite simple Euler-Lagrange equation, see (15) below. Concerning uniqueness of radially symmetric solutions (Section 3), we consider the related ode, see (16). Then, with a suitable change of variables we transform it into an autonomous ode, see (19). By using the boundary conditions and the corresponding integral equation we can exclude the existence of two different solutions. Finally, in Section 4 we prove the asymptotic behavior of the solution u_d in the limit cases by refining some results in [3].

2 Proof of existence

We denote by $\|\cdot\|_p$ the L^p-norm (both on B and on $\mathbb{R}^n$) and we put

$$\|u\|_{\partial_\nu}^2 = \int_{\partial B} u_\nu^2 \qquad \text{for } u \in H^2 \cap H_0^1(B).$$

Set

$$S = \min_{u \in \mathcal{D}^{2,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\Delta u\|_2^2}{\|u\|_{2_*}^2},$$

and recall that the minimum is achieved by the radial entire functions

$$u_\varepsilon(x) := \frac{1}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}}$$

for any $\varepsilon > 0$. Moreover, from (7.3) and (7.4) in [3] we have

$$\int_{\mathbb{R}^n} |u_\varepsilon|^{2^*} = \frac{\omega_n}{2\varepsilon^n} \frac{[\Gamma(\frac{n}{2})]^2}{\Gamma(n)} =: \frac{K_2}{\varepsilon^n}$$

and

$$\int_{\mathbb{R}^n} |\Delta u_\varepsilon|^2 = S\frac{K_2^{2/2^*}}{\varepsilon^{n-4}} =: \frac{K_1}{\varepsilon^{n-4}}.$$

Here and in the sequel, ω_n denotes the surface measure of the unit ball:

$$\omega_n := |\partial B| = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}.$$

Let $\mathcal{H} = \{u \in H^2 \cap H^1_0(B);\, u = u(|x|)\}$ denote the closed subspace of radially symmetric functions and for all nontrivial $u \in \mathcal{H}$ consider the ratio

$$Q_d(u) := \frac{\|\Delta u\|_2^2 - d\|u\|_{\partial_\nu}^2}{\|u\|_{2^*}^2}\ . \tag{3}$$

Finally, we consider the minimization problem

$$\Sigma_d \ := \inf_{u \in \mathcal{H}\setminus\{0\}} \ Q_d(u) \tag{4}$$

and we recall [3, Proposition 13]:

Proposition 2. *Assume that $0 < d < n$. Then if $\Sigma_d < S$ the infimum in (4) is achieved. Moreover, up to a change of sign and up to a Lagrange multiplier, any minimizer is a positive radial solution of (1).*

In view of Proposition 2, the existence part of Theorem 1 is proved if we exhibit a nontrivial radial function $U_{\varepsilon,\delta} \in \mathcal{H}$ such that

$$Q_d(U_{\varepsilon,\delta}) < S. \tag{5}$$

Assume that $d > 4$, fix a real number

$$0 < \delta < \sqrt[n]{\frac{d-4}{n+d-4}} \tag{6}$$

and consider the following two-parameters family of functions

$$U_{\varepsilon,\delta}(x) = g_\delta(|x|)u_\varepsilon(x) \equiv \frac{g_\delta(|x|)}{(\varepsilon^2+|x|^2)^{\frac{n-4}{2}}},$$

where $g_\delta \in C^1[0,1] \cap W^{2,\infty}(0,1)$, $g_\delta(r) = 1$ for $0 \le r \le \delta$ and $g_\delta(1) = 0$. Then, $U_{\varepsilon,\delta} \in \mathcal{H}$ and

$$U_{\varepsilon,\delta}(x) = u_\varepsilon(x) = \frac{1}{(\varepsilon^2+|x|^2)^{\frac{n-4}{2}}} \qquad \text{in } B_\delta = \{x \in \mathbb{R}^n;\, |x| < \delta\}.$$

We now estimate the ratio $Q_d(U_{\varepsilon,\delta})$. A lower bound for the denominator in (3) is readily obtained:

$$\int_B |U_{\varepsilon,\delta}(x)|^{2^*} = \int_{\mathbb{R}^n} |u_\varepsilon(x)|^{2^*} - \int_{\mathbb{R}^n\setminus B} |u_\varepsilon(x)|^{2^*} - \int_{B\setminus B_\delta} \frac{1-g_\delta(|x|)^{2^*}}{(\varepsilon^2+|x|^2)^n} \ge \frac{K_2}{\varepsilon^n} + O(1). \tag{7}$$

We now look for an upper bound of the numerator; in radial coordinates $r = |x|$, after some computations we find

$$\Delta U_{\varepsilon,\delta}(r) = U''_{\varepsilon,\delta}(r) + \frac{n-1}{r}U'_{\varepsilon,\delta}(r)$$

$$= \frac{g''_\delta(r)}{(\varepsilon^2+r^2)^{(n-4)/2}} + \frac{g'_\delta(r)}{r(\varepsilon^2+r^2)^{(n-2)/2}}\Big[(7-n)r^2 + (n-1)\varepsilon^2\Big] - (n-4)\frac{g_\delta(r)}{(\varepsilon^2+r^2)^{n/2}}(2r^2+n\varepsilon^2)\ .$$

Let us recall that $g'_\delta(r) = g''_\delta(r) = 0$ for $r < \delta$. Furthermore, as $\varepsilon \to 0$, we have

$$\Delta U_{\varepsilon,\delta}(r) = \frac{g''_\delta(r)}{r^{n-4}} + (7-n)\frac{g'_\delta(r)}{r^{n-3}} - 2(n-4)\frac{g_\delta(r)}{r^{n-2}} + o(1)$$

uniformly with respect to $r \in [\delta, 1]$. By squaring, we get

$$|\Delta U_{\varepsilon,\delta}(r)|^2 = \frac{g_\delta''(r)^2}{r^{2n-8}} + (7-n)^2 \frac{g_\delta'(r)^2}{r^{2n-6}} + 4(n-4)^2 \frac{g_\delta(r)^2}{r^{2n-4}} +$$

$$+2(7-n) \frac{g_\delta''(r) g_\delta'(r)}{r^{2n-7}} - 4(n-4) \frac{g_\delta''(r) g_\delta(r)}{r^{2n-6}} + 4(n-4)(n-7) \frac{g_\delta'(r) g_\delta(r)}{r^{2n-5}} + o(1).$$

We may now compute

$$\int_B |\Delta U_{\varepsilon,\delta}|^2 = \int_{\mathbb{R}^n} |\Delta u_\varepsilon|^2 - \int_{B \setminus B_\delta} |\Delta u_\varepsilon|^2 + \int_{B \setminus B_\delta} |\Delta U_{\varepsilon,\delta}|^2 - \int_{\mathbb{R}^n \setminus B} |\Delta u_\varepsilon|^2$$

$$= \frac{K_1}{\varepsilon^{n-4}} - 4(n-4)\omega_n + o(1) + \int_{B \setminus B_\delta} \left(|\Delta U_{\varepsilon,\delta}|^2 - |\Delta u_\varepsilon|^2 \right) \tag{8}$$

where we used formula (7.7) in [3]. We now rewrite in simplified radial form the terms contained in the last integral in (8). With some integrations by parts, and taking into account the behavior of $g_\delta(r)$ for $r \in \{1, \delta\}$, we obtain

$$\int_\delta^1 \frac{g_\delta''(r) g_\delta'(r)}{r^{n-6}} \, dr = \frac{n-6}{2} \int_\delta^1 \frac{g_\delta'(r)^2}{r^{n-5}} \, dr + \frac{g_\delta'(1)^2}{2} \ , \tag{9}$$

$$\int_\delta^1 \frac{g_\delta''(r) g_\delta(r)}{r^{n-5}} \, dr = - \int_\delta^1 \frac{g_\delta'(r)^2}{r^{n-5}} \, dr + (n-5) \int_\delta^1 \frac{g_\delta'(r) g_\delta(r)}{r^{n-4}} \, dr \ , \tag{10}$$

$$\int_\delta^1 \frac{g_\delta'(r) g_\delta(r)}{r^{n-4}} \, dr = \frac{n-4}{2} \int_\delta^1 \frac{g_\delta(r)^2}{r^{n-3}} \, dr - \frac{1}{2\delta^{n-4}} \ . \tag{11}$$

Using (9), (10) and (11) we find

$$\int_{B \setminus B_\delta} \left(|\Delta U_{\varepsilon,\delta}|^2 - |\Delta u_\varepsilon|^2 \right) = \omega_n \int_\delta^1 \left(\frac{g_\delta''(r)^2}{r^{n-7}} + 3(n-3) \frac{g_\delta'(r)^2}{r^{n-5}} \right) dr + (7-n)\omega_n g_\delta'(1)^2 + 4(n-4)\omega_n. \tag{12}$$

Moreover, simple computations show that

$$\int_{\partial B} (U_{\varepsilon,\delta})_\nu^2 = \omega_n g_\delta'(1)^2 + o(1)$$

which, combined with (8) and (12), yields

$$\int_B |\Delta U_{\varepsilon,\delta}|^2 - d \int_{\partial B} (U_{\varepsilon,\delta})_\nu^2 = \frac{K_1}{\varepsilon^{n-2}} + \omega_n \int_\delta^1 \left(\frac{g_\delta''(r)^2}{r^{n-7}} + 3(n-3) \frac{g_\delta'(r)^2}{r^{n-5}} \right) dr + \omega_n (7-n-d) g_\delta'(1)^2 + o(1) \ . \tag{13}$$

Putting $f = g_\delta'$, we are so led to minimize the functional

$$J(f) := \int_\delta^1 \left(\frac{f'(r)^2}{r^{n-7}} + 3(n-3) \frac{f(r)^2}{r^{n-5}} \right) dr + (7-n-d) f(1)^2$$

among functions $f \in C^0[\delta, 1] \cap W^{1,\infty}(\delta, 1)$ such that

$$f(\delta) = 0 \qquad \text{and} \qquad \int_\delta^1 f(r) \, dr = -1. \tag{14}$$

The Euler-Lagrange equation relative to the integral part of the functional J reads

$$r^2 f''(r) + (7-n) r f'(r) - 3(n-3) f(r) = 0 \qquad \delta \le r \le 1, \tag{15}$$

whose solutions have the following general form $f(r) = ar^{n-3} + br^{-3}$ for any $a, b \in \mathbb{R}$. The first condition in (14) yields $b = -a\delta^n$. To determine the value of a, we use the second condition in (14) and obtain

$$a = -\frac{2(n-2)}{2 - n\delta^{n-2} + 2\delta^n} \ .$$

So, let us consider the function $f(r) = a(r^{n-3} - \delta^n r^{-3})$ and let us compute

$$\frac{J(f)}{a^2} = \int_\delta^1 \left[n(n-3) r^{n-1} + \frac{3n\delta^{2n}}{r^{n+1}} \right] dr + (7-n-d)(1-\delta^n)^2 = (1-\delta^n)\Big[(4-d)(1-\delta^n) + n\delta^n\Big] =: \gamma < 0$$

where the sign of γ follows from our initial choice of δ in (6). Summarizing, with the above choice of f and recalling that $g_\delta(r) = \int_r^1 f(s) ds$, from (13) we obtain

$$\int_B |\Delta U_{\varepsilon,\delta}|^2 - d \int_{\partial B} (U_{\varepsilon,\delta})_\nu^2 = \frac{K_1}{\varepsilon^{n-4}} + \omega_n a^2 \gamma + o(1) \ .$$

Finally, by combining this estimate with (7) and recalling the definition in (3), we find

$$Q_d(U_{\varepsilon,\delta}) \le \frac{\frac{K_1}{\varepsilon^{n-4}} + \omega_n a^2 \gamma + o(1)}{[\frac{K_2}{\varepsilon^n} + O(1)]^{2/2^*}} = S + \frac{\omega_n a^2 \gamma}{K_2^{2/2^*}} \varepsilon^{n-4} + o(\varepsilon^{n-4}) \qquad \text{as } \varepsilon \to 0$$

so that (5) holds for sufficiently small ε. This completes the proof of the existence part in Theorem 1.

3 Proof of uniqueness

If we consider radially symmetric solutions and put $r = |x|$, then the equation in (1) reads

$$u^{iv}(r) + \frac{2(n-1)}{r} u'''(r) + \frac{(n-1)(n-3)}{r^2} u''(r) - \frac{(n-1)(n-3)}{r^3} u'(r) = u^{\frac{n+4}{n-4}}(r) \qquad r \in [0,1) \ , \tag{16}$$

while the boundary conditions become

$$u(1) = 0 \ , \qquad u''(1) + (n-1-d) u'(1) = 0 \ . \tag{17}$$

In turn, with the change of variables

$$u(r) \ = \ r^{-\frac{n-4}{2}} \, v(\log r) \quad (0 < r \le 1) \ , \qquad v(t) \ = \ e^{\frac{n-4}{2} t} \, u(e^t) \quad (t \le 0), \tag{18}$$

equation (16) may be rewritten as

$$v^{iv}(t) - \mathcal{K}_2 v''(t) + \mathcal{K}_1 v(t) = v^{\frac{n+4}{n-4}}(t) \qquad t \in (-\infty, 0) \ , \tag{19}$$

where

$$\mathcal{K}_1 \ = \ \left(\frac{n(n-4)}{4} \right)^2 \ , \qquad \mathcal{K}_2 \ = \ \frac{n^2 - 4n + 8}{2} > 0. \tag{20}$$

Moreover, by [3, Lemma 18], we know that

$$v(0) = 0, \quad v'(0) < 0, \quad v''(0) = (d-2) v'(0), \quad v'''(0) = \frac{n^2 - 4n + 2d^2 - 8d + 16}{4} v'(0). \tag{21}$$

Assume that there exist two solutions u_1 and u_2 of (16) and let v_1 and v_2 be the corresponding functions obtained through the change of variables (18). Then, both v_1 and v_2 satisfy (19) and (21). Put

$$w_i(t) := \frac{v_i(t)}{|v_i'(0)|} \qquad (i=1,2)$$

so that

$$w_i^{iv}(t) - \mathcal{K}_2 w_i''(t) + \mathcal{K}_1 w_i(t) = \lambda_i w_i^{(n+4)/(n-4)}(t), \qquad t \in (-\infty, 0), \tag{22}$$

with

$$w_i(0)=0, \quad w_i'(0)=-1, \quad w_i''(0)=2-d, \quad w_i'''(0) = -\frac{n^2-4n+2d^2-8d+16}{4},$$
$$\lambda_i = |v_i'(0)|^{8/(n-4)} > 0.$$

With no loss of generality we may assume that $\lambda_1 \geq \lambda_2$. Let $w := w_1 - w_2$ so that w satisfies

$$w^{iv}(t) - \mathcal{K}_2 w''(t) + \mathcal{K}_1 w(t) = \lambda_1 w_1^{(n+4)/(n-4)}(t) - \lambda_2 w_2^{(n+4)/(n-4)}(t)$$

with homogeneous initial conditions at $t=0$. This equation may be rewritten as

$$w^{iv}(t) - \mathcal{K}_2 w''(t) + \mathcal{K}_1 w(t) = (\lambda_1 - \lambda_2) w_1^{(n+4)/(n-4)}(t) + f(t) w(t), \tag{23}$$

where, by Lagrange Theorem,

$$f(t) = \frac{n+4}{n-4} \lambda_2 \int_0^1 [s w_1(t) + (1-s) w_2(t)]^{8/(n-4)}\, ds \geq 0.$$

We now prove a technical result:

Lemma 3. *Let $h \in C^0(-\infty, 0]$, then the unique solution $w \in C^4(-\infty,0]$ of the Cauchy problem*

$$\begin{cases} w^{iv}(t) - \mathcal{K}_2 w''(t) + \mathcal{K}_1 w(t) = h(t), & t \in (-\infty, 0) \\ w(0) = w'(0) = w''(0) = w'''(0) = 0 \end{cases}$$

is given by

$$w(t) = \frac{4}{n(n-4)} \int_t^0 \sinh\left[\frac{n}{2}(\tau - t)\right] \int_\tau^0 \sinh\left[\frac{n-4}{2}(s-\tau)\right] h(s)\, ds\, d\tau \;= \int_t^0 G(s-t)\, h(s)\, ds, \tag{24}$$

where

$$G(\sigma) = \frac{1}{n(n-2)} \sinh[n\sigma/2] - \frac{1}{(n-2)(n-4)} \sinh[(n-4)\sigma/2]$$

is positive for $\sigma > 0$.

Proof. It follows by combining three simple facts.
First, the unique solution w of the problem

$$\begin{cases} w''(t) - \frac{n^2}{4} w(t) = z(t), & t \in (-\infty, 0) \\ w(0) = w'(0) = 0 \end{cases}$$

is given by

$$w(t) = \frac{2}{n} \int_t^0 \sinh\left[\frac{n}{2}(\tau - t)\right] z(\tau)\, d\tau\ .$$

Second, the unique solution z of the problem

$$\begin{cases} z''(t) - \frac{(n-4)^2}{4} z(t) = h(t), & t \in (-\infty, 0) \\ z(0) = z'(0) = 0 \end{cases}$$

is given by

$$z(t) = \frac{2}{n-4} \int_t^0 \sinh\left[\frac{n-4}{2}(\tau - t)\right] h(\tau)\, d\tau \ .$$

Third, by (20) the left hand side of (23) may be written as

$$\left[\frac{d^2}{dt^2} - \frac{(n-4)^2}{4}\right]\left[\frac{d^2}{dt^2} - \frac{n^2}{4}\right] w.$$

Finally, by changing the order of integration in the second term of (24) we get

$$G(s-t) = \frac{4}{n(n-4)} \int_t^s \sinh\left[\frac{n-4}{2}(s-\tau)\right] \sinh\left[\frac{n}{2}(\tau - t)\right] d\tau, \quad t < s < 0,$$

and the explicit form of G follows by elementary calculations. □

By Lemma 3, the homogeneous Cauchy problem for (23) is equivalent to the following integral equation:

$$w(t) = (\lambda_1 - \lambda_2) \int_t^0 G(s-t)\, w_1^{(n+4)/(n-4)}(s)\, ds + \int_t^0 G(s-t)\, f(s) w(s)\, ds\,.$$

In turn, by putting

$$W(t) := (\lambda_1 - \lambda_2) \int_t^0 G(s-t)\, w_1^{(n+4)/(n-4)}(s)\, ds \ \geq 0\ ,$$

the above integral equation reads

$$w(t) = W(t) + \int_t^0 G(s-t)\, f(s) w(s)\, ds\,.$$

The solution to this problem is obtained by iteration; by recalling that $W \geq 0$, $f \geq 0$ on $(-\infty, 0]$ and that $G(s-t) \geq 0$ for $s > t$, we readily obtain

$$w(t) \geq 0 \qquad \text{for all } t \leq 0.$$

Finally, if we multiply (23) by $e^{nt/2}$ we may rewrite it as

$$\frac{d}{dt}\left[e^{nt/2}\left(w'''(t) - \frac{n}{2}w''(t) - \frac{(n-4)^2}{4}w'(t) + \frac{n(n-4)^2}{8}w(t)\right)\right] =$$
$$= e^{nt/2}\Big[(\lambda_1 - \lambda_2) w_1^{(n+4)/(n-4)}(t) + f(t) w(t)\Big]\ . \tag{25}$$

By integrating (25) over $(-\infty, 0)$ and using the homogeneous boundary conditions we obtain

$$\int_{-\infty}^0 e^{nt/2}\Big[(\lambda_1 - \lambda_2) w_1^{(n+4)/(n-4)}(t) + f(t) w(t)\Big]\, dt = 0\ .$$

In view of the sign conditions

$$\lambda_1 - \lambda_2 \geq 0, \quad w_1 > 0, \quad f, w \geq 0,$$

this implies that $\lambda_1 = \lambda_2$, $w \equiv 0$ and $u_1 = u_2$. Uniqueness is so proved.

4 Asymptotic behavior in the limit cases

Statement (i) when $d \to n^-$ is proved in [3, Theorem 1].
When $d \to 4^+$, from [3, Theorem 7] we recall that

$$u_d(r) \to 0 \qquad \text{for all } r \in (0,1] \tag{26}$$

which is the second statement of (ii).
Next, [3, Theorem 7] also states that for all $d \in (4,n)$ the following identity holds:

$$\frac{d(d-n)}{2} u_d'(1) = \int_0^1 r^{n-1} u_d^{2^*-1}(r)\, dr. \tag{27}$$

By combining [3, Lemma 17] with the change of variables (18) we infer that for all $d \in (4,n)$ we have

$$u_d(r) \le \left(\frac{(n-4)n^3}{16}\right)^{(n-4)/8} r^{-(n-4)/2} \qquad \text{for all } r > 0.$$

In particular, this shows that $r^{n-1}u_d^{2^*-1}(r) \le C_n r^{(n-6)/2} \in L^1(0,1)$ since $n > 4$. This, combined with (26), enables us to apply Lebesgue Theorem and to obtain that

$$\lim_{d\to 4^+} \int_0^1 r^{n-1} u_d^{2^*-1}(r)\, dr = 0.$$

From (27) we then infer

$$\lim_{d\to 4^+} \; u_d'(1) \; = \; 0 \tag{28}$$

which is the third statement of (ii).
In order to prove the first statement of (ii) we recall that, due to the variational characterization in Proposition 2, we have

$$\frac{\|\Delta u_d\|_2^2 - d\|u_d\|_{\partial_\nu}^2}{\|u_d\|_{2^*}^2} = \Sigma_d < S \tag{29}$$

for all $d \in (4,n)$. Moreover, by taking $\varphi = u_d$ in (2) we obtain

$$\|\Delta u_d\|_2^2 - d\|u_d\|_{\partial_\nu}^2 = \|u_d\|_{2^*}^{2^*}$$

These two equalities yield

$$\|u_d\|_{2^*} = \Sigma_d^{(n-4)/8}. \tag{30}$$

By combining the Sobolev inequality in $H^2 \cap H_0^1(B)$ (see [6]) with (28) and (29) we obtain

$$S\|u_d\|_{2^*}^2 + o(1) \le \|\Delta u_d\|_2^2 - d\|u_d\|_{\partial_\nu}^2 = \Sigma_d \|u_d\|_{2^*}^2 < S\|u_d\|_{2^*}^2$$

which shows that $\Sigma_d \to S$ as $d \to 4^+$. Therefore, (30) entails $\|u_d\|_{2^*} \to S^{(n-4)/8} > 0$ proving that $u_d(0)$ cannot remain bounded since otherwise by (26) and Lebesgue Theorem we would get a contradiction.

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