SOME STOCHASTIC DIFFERENTIAL EQUATIONS IN QUANTUM OPTICS AND MEASUREMENT THEORY: THE CASE OF COUNTING PROCESSES

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ABSTRACT

Stochastic differential equations of jump type are used in the theory of measurements continuous in time in quantum mechanics and have a concrete application in describing direct detection in quantum optics (counting of photons). In the paper the connections are explained among various types of stochastic equations: linear for Hilbert–space unnormalized vectors, non–linear for Hilbert space normalized vectors, linear for trace–class operators, non–linear for density matrices. These equations allow to construct "a posteriori" states and probabilities for the counting process describing the direct detection. Relations with master equations and "a priori" states are also explained. Two concrete applications related to a two–level atom are presented.

1. Counting Processes

Stochastic differential equations (SDE's) for trace–class operators or for vectors in Hilbert spaces appeared in the literature related to quantum mechanics in contexts such as measurements continuous in time, detection theory in quantum optics, dynamical reduction theories, simulations of master equations. Many authors contributed to this subject: a long and incomplete list of references is given in the article [1]. In this paper I want to discuss significant examples of SDE's of jump type related to measurements continuous in time and detection theory in quantum optics. All the statements given here will be based on heuristic arguments, while the mathematical theory of this class of SDE's is developed in the paper [2].

Photon detection theory, when both the emitting system and the electromagnetic field are included in the description, can be developed by using *quantum stochastic calculus* [3–5]. At this level of description standard quantum mechanics applies: evolutions are given by unitary operators and the observables representing the continuously measured quantities (such as "number of photons up to time t" in the case of direct detection) are associated to selfadjoint operators commuting, in the Heisenberg description, even for different times [6,7]; this fact implies that there is no need of the projection postulate nor of its generalizations in obtaining joint probabilities at different times.

By partial trace over the Hilbert space of the electromagnetic field, a description involving only the degrees of freedom of the emitting system is obtained [3]. It is at this lower level that classical SDE's appear [8–10], which are the subject of the present

paper, and unusual concepts, as *effects, operations* and *instruments*[11,12] and as *a priori* and *a posteriori* states[8], are introduced. SDE's of jump type arise in the case of direct detection (counting of photons).

Let us consider some quantum system emitting photons. The reduced dynamics of the system is given by a master equation of the type

$$\frac{\mathrm{d}\overline{\varrho_t}}{\mathrm{d}t} = \mathcal{L}[\,\overline{\varrho_t}\,]\,, \tag{1.1}$$

where $\mathcal{L}$ is the generic generator of a quantum dynamical semigroup[13,14], which we write in the following way:

$$\mathcal{L} = \widetilde{\mathcal{L}} + J\,, \qquad J[\varrho] = \sum_\alpha R_\alpha \varrho R_\alpha^\dagger\,, \qquad \widetilde{\mathcal{L}}[\varrho] = \mathcal{L}_0[\varrho] - \frac{1}{2}\sum_\alpha \big(R_\alpha^\dagger R_\alpha \varrho + \varrho R_\alpha^\dagger R_\alpha\big), \tag{1.2}$$

$$\mathcal{L}_0[\varrho] = -\mathrm{i}[H,\varrho] + \frac{1}{2}\sum_j \left(\big[L_j\varrho, L_j^\dagger\big] + \big[L_j, \varrho L_j^\dagger\big]\right), \tag{1.3}$$

where H, R_α and L_j are system operators and $H = H^\dagger$. To avoid mathematical difficulties we assume that our system is an n-level atom, so that the Hilbert space is finite–dimensional.

Now we want to discuss the direct detection (counting) of the emitted photons. In order to have a complete description of the counting process and of the dynamics of the atom, we have to construct the probability law $\widehat{P}$ for the counting process (distribution of the counts, of the arrival times, ...), which we shall refer to as the physical probability, and the state $\widehat{\varrho}_t$ of the system at time t given a certain sequence of counts (the statistical operator conditioned by the data up to time t), which can be called the *a posteriori* state. Let us denote by $\mathrm{E}_{\widehat{P}}[\cdot]$ the statistical mean with respect to the physical probability $\widehat{P}$ (the expectation with respect to the probability measure P). By the meaning of $\overline{\varrho_t}$, $\widehat{\varrho}_t$ and P we must have

$$\overline{\varrho_t} = \mathrm{E}_{\widehat{P}}[\widehat{\varrho}_t]\,, \tag{1.4}$$

i.e. the average of the *a posteriori* states with respect to all possible data must give the reduced state of the system.

The statistical operator $\overline{\varrho_t}$ is the state we attribute to the system when we do not know the results of the measurement; in this sense, we can call $\overline{\varrho_t}$ the *a priori* state. We can read Eq. (1.4) as: the *a posteriori* states are a demixture of the *a priori* state and the coefficients of this demixture are the physical probabilities.

The construction of $\widehat{P}$ and $\widehat{\varrho}_t$ can be achieved via two auxiliary objects: a reference probability measure P and some unnormalized positive trace–class operators ϱ_t, such that

$$\widehat{\varrho}_t = \varrho_t \,/\, \mathrm{Tr}\{\varrho_t\}\,, \tag{1.5}$$

$$\mathrm{E}_P[\mathrm{Tr}\{\varrho_t\}X] = \mathrm{E}_{\widehat{P}}[X]\,, \tag{1.6}$$

for any random variable X adapted to the time interval $(0,t]$. In words Eq. (1.6) says that $\mathrm{Tr}\{\varrho_t\}$ is the (local) probability density of the physical probability $\widehat{P}$, with respect to the reference measure P. By Eq. (1.5), we can call ϱ_t the unnormalized *a posteriori* (UNAP) states. By combining Eqs. (1.4)–(1.6), we have in particular

$$\mathrm{E}_P[\varrho_t] = \mathrm{E}_{\widehat{P}}[\widehat{\varrho}_t] = \overline{\varrho_t}\,. \tag{1.7}$$

In the case of a counting process the reference measure P is the probability law of a Poisson process $N(t)$ of intensity $\mu > 0$ and the UNAP states are the solution of the Itô's type linear SDE

$$\mathrm{d}\varrho_t = \mathcal{L}[\varrho_{t-}]\mathrm{d}t + \left(\frac{1}{\mu}\,J[\varrho_{t-}] - \varrho_{t-}\right)\big(\mathrm{d}N(t) - \mu\mathrm{d}t\big), \tag{1.8}$$

with initial condition $\varrho_0 = \widehat{\varrho}_0 = \overline{\varrho_0} = \varrho$ (the initial state of the system).

In order to understand the meaning of equations like (1.8) we recall that, roughly speaking, the probability of one count in an interval of amplitude $\mathrm{d}t$ is of order $\mathrm{d}t$, while the probability of more than one count is of higher order. We can translate this into the rules

$$\mathrm{d}N(t) = \begin{cases} 1, & \text{if there is a count in the time interval } (t, t+\mathrm{d}t], \\ 0, & \text{otherwise,} \end{cases} \tag{1.9}$$

$$\big(\mathrm{d}N(t)\big)^2 = \mathrm{d}N(t)\,, \qquad \mathrm{d}N(t)\mathrm{d}t = 0\,. \tag{1.10}$$

In Eq. (1.8), the minus sign in ϱ_{t-} means "just before the jump, if any". By the fact that we have a process with independent increments and mean value

$$\mathrm{E}_P[N(t)] = \mu t \tag{1.11}$$

the P-expectation of Eq. (1.8) gives the master equation (1.1) and, thus, the validity of Eq. (1.7) is guaranteed.

In order to show that $\mathrm{Tr}\{\varrho_t\}$ is indeed the probability density of the physical probability $\widehat{P}$, we show that our description is equivalent to the Davies [15] and Srinivas–Davies [16] theory of quantum counting processes.

A typical trajectory of a Poisson process, or of any other (regular) counting process, is completely determined by giving the instants of the counts (the arrival times). Thus, we can specify a trajectory ω_t up to time t by $\omega_t = \{t_1, t_2, \ldots, t_r\}$, $0 < t_1 < t_2 < \cdots < t_r \leq t$, $r = 0, 1, \ldots$. By Eq. (1.9), the solution of Eq. (1.8), given the trajectory ω_t, is

$$\varrho_t \equiv \varrho(\omega_t) = \mu^{-r}\,\mathrm{e}^{\mu t}\,\mathrm{e}^{\widetilde{\mathcal{L}}(t-t_r)} \circ J \circ \cdots \circ J \circ \mathrm{e}^{\widetilde{\mathcal{L}}(t_2-t_1)} \circ J \circ \mathrm{e}^{\widetilde{\mathcal{L}}t_1}[\varrho]\,. \tag{1.12}$$

First let us consider the trajectories with no count up to time t. The probability of no count for the Poisson process is $\mathrm{e}^{-\mu t}$ and, for these trajectories, the density $\mathrm{Tr}\{\varrho_t\}$ is $\mathrm{e}^{\mu t}\ \mathrm{Tr}\left\{\mathrm{e}^{\widetilde{\mathcal{L}}t}[\varrho]\right\}$. By multiplying these two contributions we get the physical probability of no count up to time t

$$\widehat{P}[N(t)=0] = \mathrm{Tr}\left\{\mathrm{e}^{\widetilde{\mathcal{L}}t}[\varrho]\right\}. \tag{1.13}$$

Let us consider now trajectories with $r \geq 1$ counts up to time t. For a Poisson process the probability of a count in the time interval $(t_1, t_1 + \mathrm{d}t_1]$, a count in $(t_2, t_2 + \mathrm{d}t_2]$, ..., and no other count in between is $\mu^r\, \mathrm{e}^{-\mu t}\, \mathrm{d}t_1 \cdots \mathrm{d}t_r$ $(0 < t_1 < \cdots < t_r \leq t)$. By multiplying by the trace of (1.12) we get

$$\widehat{p}_t(t_1, \ldots, t_r)\, \mathrm{d}t_1 \cdots \mathrm{d}t_r =$$

$$= \mathrm{Tr}\left\{\mathrm{e}^{\widetilde{\mathcal{L}}(t-t_r)} \circ J \circ \cdots \circ J \circ \mathrm{e}^{\widetilde{\mathcal{L}}(t_2-t_1)} \circ J \circ \mathrm{e}^{\widetilde{\mathcal{L}}t_1}[\varrho]\right\} \mathrm{d}t_1 \cdots \mathrm{d}t_r\,. \tag{1.14}$$

The quantities $\widehat{p}_t(t_1, \ldots, t_r)$ are known as exclusive probability densities (EPD). The expressions (1.13) and (1.14) were essentially introduced in the paper[16] by Srinivas and Davies, where it is shown that they describe *consistently* the counting process.

By Eqs. (1.5), (1.8), (1.10) we get the non–linear SDE for *a posteriori* states

$$\mathrm{d}\widehat{\varrho}_t = \mathcal{L}[\widehat{\varrho}_{t-}]\mathrm{d}t + \left(\frac{J[\widehat{\varrho}_{t-}]}{I(t)} - \widehat{\varrho}_{t-}\right)\big(\mathrm{d}N(t) - I(t)\mathrm{d}t\big), \tag{1.15}$$

where

$$I(t) := \mathrm{Tr}\{J[\widehat{\varrho}_{t-}]\} \equiv \sum_\alpha \mathrm{Tr}\{R_\alpha^\dagger R_\alpha\, \widehat{\varrho}_{t-}\}\,. \tag{1.16}$$

Moreover, under the physical law $\widehat{P}$, $N(t)$ is a counting process of stochastic intensity $I(t)$, or

$$\mathrm{E}_{\widehat{P}}[\mathrm{d}N(t)|\omega_t] = I(t)\mathrm{d}t\,, \tag{1.17}$$

where $\mathrm{E}_{\widehat{P}}[\cdot|\omega_t]$ is the conditional expectation "given the data" up to time t. This equation guarantees, in particular, the fact that the $\widehat{P}$-expectation of Eq. (1.15) gives the master equation (1.1) and, therefore, the fact that Eq. (1.4) holds. By taking the expectation of both sides of Eq. (1.17), we have, by the properties of conditional expectations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\ \mathrm{E}_{\widehat{P}}[N(t)] = \mathrm{E}_{\widehat{P}}[I(t)]\,. \tag{1.18}$$

Note that no physical result depend on the intensity μ of the reference Poisson process.

In order to prove Eq. (1.17) and to compute moments of the counting process, we give here two useful formulae. First, by Eqs. (1.8), (1.10) and (1.2), we get

$$\varrho_{t+\mathrm{d}t}\,\mathrm{d}N(t) \equiv \big(\varrho_{t-} + \mathrm{d}\varrho_t\big)\mathrm{d}N(t) = \frac{1}{\mu} J[\varrho_{t-}]\mathrm{d}N(t)\,. \tag{1.19}$$

Second, by Eqs. (1.8), (1.11) and the fact that a Poisson process has independent increments, we have

$$\mathrm{E}_P[\varrho_t|\omega_s] = \mathrm{e}^{\mathcal{L}(t-s)}[\varrho_s]\,, \qquad s<t\,. \tag{1.20}$$

Let now X be a generic random variable adapted to $(0,t]$ (i.e. it depends on $N(s)$ only for times $s \leq t$). By using (1) properties of conditional expectations, (2) the fact that $X\mathrm{d}N(t)$ is adapted to $(0,t+\mathrm{d}t]$ and Eq. (1.6), (3) Eq. (1.19), (4) the independence of increments of the Poisson process and Eq. (1.11), (5) Eqs. (1.16) and (1.5), (6) the fact that a jump exactly at time t has zero probability, (7) Eq. (1.6), we can write

$$\mathrm{E}_{\widehat{P}}\left[X\,\mathrm{E}_{\widehat{P}}[\mathrm{d}N(t)|\omega_t]\right] = \mathrm{E}_{\widehat{P}}[X\mathrm{d}N(t)] = \mathrm{E}_P[X\,\mathrm{Tr}\{\varrho_{t+\mathrm{d}t}\}\mathrm{d}N(t)] =$$

$$= \frac{1}{\mu}\,\mathrm{E}_P\left[X\,\mathrm{Tr}\{J[\varrho_{t-}]\}\mathrm{d}N(t)\right] = \mathrm{E}_P\left[X\,\mathrm{Tr}\{J[\varrho_{t-}]\}\right]\mathrm{d}t =$$

$$= \mathrm{E}_P[XI(t)\,\mathrm{Tr}\{\varrho_{t-}\}]\mathrm{d}t = \mathrm{E}_P[XI(t)\,\mathrm{Tr}\{\varrho_t\}]\mathrm{d}t = \mathrm{E}_{\widehat{P}}[XI(t)]\mathrm{d}t\,.$$

This is statement (1.17).

Moments of the counting process can be computed by differentiation of the Fourier transform (characteristic functional) of the probability law $\widehat{P}$, as done in reference [3], or by arguments similar to those used just now. By using (a) point (1), (b) Eqs. (1.16) and (1.17), (c) point (6) and Eq. (1.4), we obtain

$$\mathrm{E}_{\widehat{P}}[\mathrm{d}N(t)] = \mathrm{E}_{\widehat{P}}\left[\,\mathrm{E}_{\widehat{P}}[\mathrm{d}N(t)|\omega_t]\right] = \mathrm{E}_{\widehat{P}}[\mathrm{Tr}\{J[\widehat{\varrho}_{t-}]\}]\mathrm{d}t = \mathrm{Tr}\{J[\overline{\varrho_t}]\}\mathrm{d}t\,,$$

which gives

$$\mathrm{E}_{\widehat{P}}[N(t_0+t)-N(t_0)] = \int_{t_0}^{t_0+t}\mathrm{d}s\;\mathrm{Tr}\{J[\overline{\varrho_s}]\} \equiv \int_0^t \mathrm{d}s\;\mathrm{Tr}\{J\circ\mathrm{e}^{\mathcal{L}s}[\,\overline{\varrho_{t_0}}\,]\}\,. \tag{1.21}$$

Let us now take $s<t$. By using (i) point (1), (ii) Eq. (1.17), (iii) Eq. (1.6) and point (6), (iv) point (1), (v) Eq. (1.20), (vi) Eq. (1.19), (vii) points (4) and (6) and Eq. (1.7), we obtain

$$\mathrm{E}_{\widehat{P}}[\mathrm{d}N(s)\mathrm{d}N(t)] = \mathrm{E}_{\widehat{P}}\left[\mathrm{d}N(s)\,\mathrm{E}_{\widehat{P}}[\mathrm{d}N(t)|\omega_t]\right] = \mathrm{E}_{\widehat{P}}[\mathrm{d}N(s)I(t)]\mathrm{d}t =$$

$$= \mathrm{E}_P\left[\mathrm{d}N(s)\,\mathrm{Tr}\{J[\varrho_t]\}\right]\mathrm{d}t = \mathrm{Tr}\left\{J\Big[\,\mathrm{E}_P\left[\mathrm{d}N(s)\,\mathrm{E}_P[\varrho_t|\omega_{s+\mathrm{d}s}]\right]\Big]\right\}\mathrm{d}t =$$

$$= \mathrm{Tr}\left\{J\circ \mathrm{e}^{\mathcal{L}(t-s)}\big[\,\mathrm{E}_P[\mathrm{d}N(s)\varrho_{s+\mathrm{d}s}]\big]\right\}\mathrm{d}t =$$

$$= \frac{1}{\mu}\,\mathrm{Tr}\left\{J\circ \mathrm{e}^{\mathcal{L}(t-s)}\circ J\big[\,\mathrm{E}_P[\mathrm{d}N(s)\varrho_{s-}]\big]\right\}\mathrm{d}t = \mathrm{Tr}\{J\circ \mathrm{e}^{\mathcal{L}(t-s)}\circ J[\,\overline{\varrho_s}\,]\}\mathrm{d}s\mathrm{d}t\,.$$

By writing

$$\big(N(t_0+t_1)-N(t_0)\big)\big(N(t_0+t_2)-N(t_0)\big) = \int_{t_0<t\le t_0+t_2}\int_{t_0<s\le t_0+t_1}\mathrm{d}N(s)\mathrm{d}N(t)$$

and decomposing the double integral into the regions $s<t$, $s=t$, $s>t$, the previous result and Eq. (1.10) give

$$\mathrm{E}_{\widehat{P}}\left[\big(N(t_0+t_1)-N(t_0)\big)\big(N(t_0+t_2)-N(t_0)\big)\right] = \int_0^{t_1\wedge t_2}\mathrm{d}t\;\mathrm{Tr}\{J\circ \mathrm{e}^{\mathcal{L}t}[\,\overline{\varrho_{t_0}}\,]\}+$$

$$+\left(\int_0^{t_2}\mathrm{d}t\int_0^{t\wedge t_1}\mathrm{d}s+\int_0^{t_1}\mathrm{d}t\int_0^{t\wedge t_2}\mathrm{d}s\right)\mathrm{Tr}\{J\circ \mathrm{e}^{\mathcal{L}(t-s)}\circ J\circ \mathrm{e}^{\mathcal{L}s}[\,\overline{\varrho_{t_0}}\,]\}\,, \qquad (1.22)$$

where $h\wedge k := \min\{h,k\}$.

From first and second moments one can obtain the so called Mandel Q-parameter

$$Q_{t_0}(t) := \frac{\mathrm{var}_{\widehat{P}}[N(t_0+t)-N(t_0)]-\mathrm{E}_{\widehat{P}}[N(t_0+t)-N(t_0)]}{\mathrm{E}_{\widehat{P}}[N(t_0+t)-N(t_0)]}\,, \qquad (1.23)$$

which measures the deviation from the Poisson variance; obviously, $\mathrm{var}[X] = \mathrm{E}[X^2]-\mathrm{E}[X]^2$.

A particularly interesting case is when the master equation (1.1) has a unique equilibrium solution ϱ_{eq} such that

$$\lim_{t\to+\infty}\mathrm{e}^{\mathcal{L}t}[\varrho] = \varrho_{\mathrm{eq}}\,, \qquad \mathcal{L}[\varrho_{\mathrm{eq}}] = 0\,, \qquad (1.24)$$

for any initial state ϱ. In this case one can introduce the equilibrium Q-parameter

$$Q(t) := \lim_{t_0\to+\infty} Q_{t_0}(t) = \frac{2\int_0^t \mathrm{d}s\,\big(1-\frac{s}{t}\big)\,\mathrm{Tr}\left\{J\circ \mathrm{e}^{\mathcal{L}s}\circ J[\varrho_{\mathrm{eq}}]\right\} - t\big(\,\mathrm{Tr}\{J[\varrho_{\mathrm{eq}}]\}\big)^2}{\mathrm{Tr}\{J[\varrho_{\mathrm{eq}}]\}}\,. \qquad (1.25)$$

Another interesting special case is when

$$J[\varrho] = \widetilde{\varrho}\;\mathrm{Tr}\{J[\varrho]\}\,, \qquad (1.26)$$

for any state ϱ; here $\widetilde{\varrho}$ is a fixed statistical operator, independent of ϱ. In this case the EPD's take the product structure

$$\widehat{p}_t(t_1,\dots,t_r) = \mathrm{Tr}\left\{\mathrm{e}^{\widetilde{\mathcal{L}}(t-t_r)}[\widetilde{\varrho}]\right\}\left(\prod_{l=1}^{r-1}\mathrm{Tr}\left\{J\circ\mathrm{e}^{\widetilde{\mathcal{L}}(t_{l+1}-t_l)}[\widetilde{\varrho}]\right\}\right)\mathrm{Tr}\left\{J\circ\mathrm{e}^{\widetilde{\mathcal{L}}t_1}[\varrho]\right\}. \tag{1.27}$$

If also Eq. (1.24) holds, the equilibrium Q-parameter (1.25) takes the form

$$Q(t) = 2\int_0^t \mathrm{d}s\left(1-\frac{s}{t}\right)\mathrm{Tr}\left\{J\left[\mathrm{e}^{\mathcal{L}s}[\widetilde{\varrho}]-\varrho_{\mathrm{eq}}\right]\right\}. \tag{1.28}$$

2. A Two–level Atom

As an example we consider a two–level atom stimulated by a monocromatic laser in resonance with the atomic transition. By eliminating the explicit time dependence due to the laser wave by means of a unitary transformation, we obtain a Liouvillian $\mathcal{L}$ with

$$R_\alpha \to R = \sqrt{\Gamma}|0\rangle\langle 1|\,,\qquad L_j = 0\,,\qquad H = \frac{\Omega}{2}\big(|0\rangle\langle 1| + |1\rangle\langle 0|\big), \tag{2.1}$$

where $|0\rangle$ is the ground state, $|1\rangle$ is the excited one, the parameter $\Omega > 0$ is proportional to the laser intensity and it is called the Rabi frequency, the parameter $\Gamma > 0$ is the electromagnetic transition rate. Moreover, Eqs. (1.2), (1.3) and (1.14) give

$$\widetilde{\mathcal{L}}[\varrho] = -\mathrm{i}\widetilde{H}\varrho + \mathrm{i}\varrho\widetilde{H}^\dagger\,,\qquad \widetilde{H} := H - \frac{\mathrm{i}}{2}|1\rangle\Gamma\langle 1|\,, \tag{2.2}$$

$$J[\varrho] = \Gamma|0\rangle\langle 1|\varrho|1\rangle\langle 0|\,,\qquad I(t) = \Gamma\langle 1|\widehat{\varrho}_{t-}|1\rangle\,. \tag{2.3}$$

Now Eqs. (1.8) and (1.15) for UNAP and *a posteriori* states become

$$\mathrm{d}\varrho_t = \left(\frac{\Gamma}{\mu}\,|0\rangle\langle 1|\varrho_{t-}|1\rangle\langle 0| - \varrho_{t-}\right)\mathrm{d}N(t) + \widetilde{\mathcal{L}}[\varrho_{t-}]\mathrm{d}t + \mu\varrho_{t-}\mathrm{d}t\,, \tag{2.4}$$

$$\mathrm{d}\widehat{\varrho}_t = (|0\rangle\langle 0| - \widehat{\varrho}_{t-})\,\mathrm{d}N(t) + \widetilde{\mathcal{L}}[\,\widehat{\varrho}_{t-}\,]\mathrm{d}t + I(t)\widehat{\varrho}_{t-}\mathrm{d}t\,. \tag{2.5}$$

Essentially, these equations say that in between the jumps the system evolution is governed by a non–selfadjoint Hamiltonian $\widetilde{H}$ and at any jump the atom goes into the ground state. Note that we have not considered the time–delay from emission to detection, the efficiency of the counter and any other characteristic of the detector; thus, more than the detection process we are describing the emission process. The passage from emission process to detection process could be treated in principle by classical filtering theory.

Now Eq. (1.26) holds with $\widetilde{\varrho}$ given by the ground state; moreover, we choose the origin of the time axis coinciding with an emission, so that the initial state is again the ground state. Summarizing, we have

$$\widetilde{\varrho} = |0\rangle\langle 0|\,, \qquad \varrho = |0\rangle\langle 0|\,. \tag{2.6}$$

The probabilities (1.13) and (1.27) become

$$\widehat{P}[N(t)=0] = \sum_{i=0}^{1} \left|\langle i|\mathrm{e}^{-\mathrm{i}\widetilde{H}t}|0\rangle\right|^2\,, \tag{2.7}$$

$$\widehat{p}_t(t_1,\dots,t_r) = \widehat{P}[N(t-t_r)=0]w(t_r-t_{r-1})\cdots w(t_2-t_1)w(t_1)\,, \tag{2.8}$$

$$w(t) := \Gamma\left|\langle 1|\mathrm{e}^{-\mathrm{i}\widetilde{H}t}|0\rangle\right|^2\,. \tag{2.9}$$

The quantity $w(t)$ turns out to be the probability density of the waiting time between two photon emissions. Indeed, let us denote by T such a waiting time; for any $t>0$, we have $\widehat{P}[T\le t] = 1-\widehat{P}[N(t)=0]$, which, together with Eqs. (2.7) and (2.9), gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\widehat{P}[T\le t] = -\,\frac{\mathrm{d}}{\mathrm{d}t}\,\widehat{P}[N(t)=0] = w(t)\,. \tag{2.10}$$

In principle $w(t)$ determines completely the whole statistics of the counting process.

The probability density $w(t)$ can be easily computed by solving a Schrödinger equation with the non–selfadjoint Hamiltonian $\widetilde{H}$. We consider only the case

$$\Omega > \Gamma/2\,. \tag{2.11}$$

Then, we obtain

$$w(t) = \frac{\Gamma}{2}\left(1+\frac{\Gamma^2}{4\widetilde{\Omega}^2}\right)\mathrm{e}^{-\Gamma t/2}\left(1-\cos\widetilde{\Omega}t\right)\,, \qquad \widetilde{\Omega} := \Omega\sqrt{1-\frac{\Gamma^2}{4\Omega^2}}\,. \tag{2.12}$$

The mean time between emissions is given by

$$\mathrm{E}_{\widehat{P}}[T] \equiv \int_0^{+\infty} t w(t)\,\mathrm{d}t = \frac{2}{\Gamma}+\frac{\Gamma}{\Omega^2}\,, \tag{2.13}$$

which, under condition (2.11), is bounded by $2/\Gamma < \mathrm{E}_{\widehat{P}}[T] < 6/\Gamma$.

The stochastic intensity (1.16), (2.3) takes the expression

$$\begin{aligned} I(t) &= \Gamma\,\frac{\langle 1|\varrho_{t-}|1\rangle}{\mathrm{Tr}\{\varrho_{t-}\}} = \frac{w(s)}{\widehat{P}[N(s)=0]} = \\ &= \frac{\Gamma}{2} - \Gamma\widetilde{\Omega}\,\frac{2\widetilde{\Omega}\cos\widetilde{\Omega}s+\Gamma\sin\widetilde{\Omega}s}{4\Omega^2-\Gamma^2\cos\widetilde{\Omega}s+2\widetilde{\Omega}\Gamma\sin\widetilde{\Omega}s}\,, \end{aligned} \tag{2.14}$$

where $s := t - t_r$ is the time elapsed from the last emission before time t (recall that $I(t)$ depends on the trajectory). Note how Rabi oscillations appear in the waiting time density (2.12) and in the stochastic intensity (2.14).

In our case, the master equation (1.1) has a unique equilibrium solution $\varrho_{\rm eq}$, which can be easily computed from the second of Eqs. (1.24). In particular, we obtain

$$\langle 1|\varrho_{\rm eq}|1\rangle = \frac{\Omega^2}{2\Omega^2 + \Gamma^2}\,. \tag{2.15}$$

Let us stress that Eq. (2.13) can be written also as

$$\mathrm{E}_{\widehat{P}}[T] = \big(\Gamma\langle 1|\varrho_{\rm eq}|1\rangle\big)^{-1}\,. \tag{2.16}$$

Equation (1.21) gives now

$$\lim_{t_0\to+\infty} \mathrm{E}_{\widehat{P}}[N(t_0+t) - N(t_0)] = t\Gamma\langle 1|\varrho_{\rm eq}|1\rangle\,, \tag{2.17}$$

while Eq. (1.28) becomes

$$Q(t) = 2\Gamma\int_0^t \mathrm{d}s\,\left(1 - \frac{s}{t}\right)\langle 1|\mathrm{e}^{\mathcal{L}s}[|0\rangle\langle 0|] - \varrho_{\rm eq}|1\rangle\,. \tag{2.18}$$

Note that $Q(0) = 0$ and $\dot{Q}(0) = -\Gamma\langle 1|\varrho_{\rm eq}|1\rangle < 0$, so that we have surely sub–Poissonian statistics ($Q(t) < 0$, at least for small times). In order to compute $Q(t)$ for all times we have to solve the master equation (1.1) with initial condition $\varrho = |0\rangle\langle 0|$. An explicit calculation gives

$$Q(t) = \alpha\left(1 + \frac{1 - \mathrm{e}^{\lambda t}}{t\lambda}\right) + \text{c.c.}\,, \tag{2.19}$$

$$\begin{aligned} \lambda &:= -\frac{3}{4}\,\Gamma + \mathrm{i}\sqrt{\Omega^2 - \frac{\Gamma^2}{16}}\,, \\ \alpha &:= \frac{2\Gamma\Omega^2}{2\Omega^2 + \Gamma^2}\,\frac{\lambda^*}{\lambda(\lambda^* - \lambda)}\,. \end{aligned} \tag{2.20}$$

Note that

$$\lim_{t\to+\infty} Q(t) = -\,\frac{6\Omega^2\Gamma^2}{\big(2\Omega^2 + \Gamma^2\big)^2} < 0\,. \tag{2.21}$$

All these results show that the emission process is a renewal counting process with a sub–Poissonian statistics. Only in the limit $\Omega \to +\infty$, a Poisson process of intensity $\Gamma/2$ is obtained.

Let us compare the previous results with the case of a thermally excited two–level atom. Equations (2.3), (2.4), (2.5), (2.6), (2.8), (2.10), (2.16), (2.17), (2.18)

hold true also in the new model. In particular, the operator J is the same as before, while $\widetilde{\mathcal{L}}$ is given by

$$\begin{aligned}\widetilde{\mathcal{L}}[\varrho] = \frac{\gamma}{2}\,(\overline{n}+1)\big(2|0\rangle\langle 1|\varrho|1\rangle\langle 0| - |1\rangle\langle 1|\varrho - \varrho|1\rangle\langle 1|\big) + \\ + \frac{\gamma}{2}\,\overline{n}\big(2|1\rangle\langle 0|\varrho|0\rangle\langle 1| - |0\rangle\langle 0|\varrho - \varrho|0\rangle\langle 0|\big) - \frac{\Gamma}{2}\big(|1\rangle\langle 1|\varrho + \varrho|1\rangle\langle 1|\big),\end{aligned} \tag{2.22}$$

where $\gamma > 0$, $\overline{n} = \big(\mathrm{e}^{\beta\omega} - 1\big)^{-1}$. This corresponds to set

$$\begin{aligned} R_\alpha \to R = \sqrt{\Gamma}|0\rangle\langle 1|\,, \qquad H = 0\,, \\ L_1 = \gamma(\overline{n}+1)|0\rangle\langle 1|\,, \qquad L_2 = \gamma\overline{n}|1\rangle\langle 0| \end{aligned} \tag{2.23}$$

in Eqs. (1.2) and (1.3).

The waiting time density has now the expression

$$w(t) = \Gamma\langle 1|\mathrm{e}^{\widetilde{\mathcal{L}}t}[|0\rangle\langle 0|]|1\rangle = \frac{\overline{n}\gamma\Gamma}{\delta}\left(\mathrm{e}^{-\lambda_- t} - \mathrm{e}^{-\lambda_+ t}\right), \tag{2.24}$$

where

$$\delta := \sqrt{(\Gamma+\gamma)^2 + 4\overline{n}(\overline{n}+1)\gamma^2}\,, \quad \lambda_\pm := \frac{\lambda \pm \delta}{2}\,, \quad \lambda := (2\overline{n}+1)\gamma + \Gamma\,. \tag{2.25}$$

Then, the mean of the waiting time between emissions becomes

$$\mathrm{E}_{\widehat{P}}[T] = \frac{(2\overline{n}+1)\gamma + \Gamma}{\overline{n}\gamma\Gamma}\,; \tag{2.26}$$

now, we have

$$\langle 1|\varrho_{\mathrm{eq}}|1\rangle = \frac{\overline{n}\gamma}{(2\overline{n}+1)\gamma + \Gamma}\,. \tag{2.27}$$

Again with $s := t - t_r$, the stochastic intensity becomes

$$I(t) = \frac{w(s)}{\widehat{P}[N(s)=0]} = \frac{\overline{n}\gamma\Gamma\left(1 - \mathrm{e}^{-\delta s}\right)}{\lambda_+ - \lambda_-\mathrm{e}^{-\delta s}} \tag{2.28}$$

$\big($cf. Eq. (2.14)$\big)$.

Finally, by solving the master equation, we obtain from Eq. (1.28)

$$Q(t) = -\,\frac{2\overline{n}\gamma\Gamma}{\lambda^2}\left(1 - \frac{1 - \mathrm{e}^{-\lambda t}}{t\lambda}\right). \tag{2.29}$$

We have again sub–Poissonian statistics, which becomes Poissonian with intensity $\Gamma/2$ at infinite temperature or Poissonian with intensity $\overline{n}\gamma$ in the limit $\Gamma \to +\infty$;

note, however, that we are describing a *single* atom at inverse temperature β, *not a gas*.

3. Stochastic Differential Equations for Vectors

A qualitative difference between the two models of Sect. 2 is that Eqs. (2.4), (2.5) for the first model preserve pure states, while this does not hold in the case of the second model. In general Eqs. (1.8) and (1.15) preserve pure states only if $\mathcal{L}_0$ is of purely Hamiltonian form and J preserve pure states. In spite of this difference, Eqs. (1.8) and (1.15) have always a pure state version.

Let us consider the linear SDE for unnormalized vectors in the Hilbert space, with initial condition $\psi_0 = \psi$, $\|\psi\| = 1$:

$$\mathrm{d}\psi_t = \sum_\alpha \left(\frac{1}{\sqrt{\mu_\alpha}} R_\alpha - \mathbf{1} \right) \psi_{t-} \mathrm{d}N_\alpha(t) + \sum_j \left(\frac{1}{\sqrt{\nu_j}} L_j - \mathbf{1} \right) \psi_{t-} \mathrm{d}N_j(t) +$$

$$+ \left[\frac{1}{2} (\mu + \nu) \mathbf{1} - \mathrm{i}\widetilde{H} \right] \psi_{t-} \mathrm{d}t \,, \tag{3.1}$$

where

$$\mu = \sum_\alpha \mu_\alpha \,, \qquad \nu = \sum_j \nu_j \,, \tag{3.2}$$

$$\widetilde{H} = H - \frac{\mathrm{i}}{2} \sum_\alpha R_\alpha^\dagger R_\alpha - \frac{\mathrm{i}}{2} \sum_j L_j^\dagger L_j \,, \tag{3.3}$$

$N_\alpha(t)$ is a Poisson process of intensity μ_α, $N_j(t)$ is a Poisson process of intensity ν_j, all these Poisson processes are independent. Here and in the following all the undefined quantities are as in Sect. 1.

It can be proved [2] that $\|\psi_t\|^2$ is a (local) probability density with respect to the probability measure of the Poisson processes. Under this new probability law the $N_\alpha(t)$ and $N_j(t)$ become counting processes with intensities

$$r_\alpha(t) = \langle \widehat{\psi}_{t-} | R_\alpha^\dagger R_\alpha \, \widehat{\psi}_{t-} \rangle \,, \qquad l_j(t) = \langle \widehat{\psi}_{t-} | L_j^\dagger L_j \, \widehat{\psi}_{t-} \rangle \,, \tag{3.4}$$

respectively, with

$$\widehat{\psi}_t = \psi_t / \|\psi_t\| . \tag{3.5}$$

By normalization (3.5) and Itô's stochastic calculus, we obtain from Eq. (3.1) the non–linear SDE for normalized vectors

$$\mathrm{d}\widehat{\psi}_t = \sum_\alpha \left(\frac{1}{\sqrt{r_\alpha(t)}} R_\alpha - \mathbf{1} \right) \widehat{\psi}_{t-} \mathrm{d}N_\alpha(t) + \sum_j \left(\frac{1}{\sqrt{l_j(t)}} L_j - \mathbf{1} \right) \widehat{\psi}_{t-} \mathrm{d}N_j(t) +$$

$$+\left[\frac{1}{2}\big(I(t)+K(t)\big)\mathbf{1}-\mathrm{i}\widetilde{H}\right]\widehat{\psi}_{t-}\mathrm{d}t\,, \tag{3.6}$$

where

$$I(t)=\sum_{\alpha}r_{\alpha}(t)\,,\qquad K(t)=\sum_{j}l_{j}(t)\,. \tag{3.7}$$

In Eq. (3.6) the processes $N_{\alpha}(t)$, $N_{j}(t)$ have to be considered under the new probability law, so that they have the intensities (3.4). In going from Eq. (3.1) to Eq. (3.6) one has to use the rules

$$\mathrm{d}N_{a}(t)\mathrm{d}N_{b}(t)=\delta_{ab}\mathrm{d}N_{a}(t)\,,\qquad \mathrm{d}N_{a}(t)\mathrm{d}t=0\,, \tag{3.8}$$

which generalize Eq. (1.10).

Let us consider now the trace–class operator

$$\sigma_{t}:=|\psi_{t}\rangle\langle\psi_{t}|\,. \tag{3.9}$$

By Eq. (3.8), we obtain from Eq. (3.1) the linear SDE for trace–class operators

$$\mathrm{d}\sigma_{t}=\mathcal{L}[\sigma_{t-}]\mathrm{d}t+\sum_{\alpha}\left(\frac{1}{\mu_{\alpha}}\,R_{\alpha}\sigma_{t-}R_{\alpha}^{\dagger}-\sigma_{t-}\right)\big(\mathrm{d}N_{\alpha}(t)-\mu_{\alpha}\mathrm{d}t\big)+$$

$$+\sum_{j}\left(\frac{1}{\nu_{j}}\,L_{j}\sigma_{t-}L_{j}^{\dagger}-\sigma_{t-}\right)\big(\mathrm{d}N_{j}(t)-\nu_{j}\mathrm{d}t\big)\,. \tag{3.10}$$

On the other side, starting from Eq. (3.6) or from Eq. (3.10), one obtains for

$$\widehat{\sigma}_{t}:=|\widehat{\psi}_{t}\rangle\langle\widehat{\psi}_{t}|\equiv\frac{\sigma_{t}}{\mathrm{Tr}\{\sigma_{t}\}} \tag{3.11}$$

the non–linear SDE

$$\mathrm{d}\widehat{\sigma}_{t}=\mathcal{L}[\widehat{\sigma}_{t-}]\mathrm{d}t+\sum_{\alpha}\left(\frac{1}{r_{\alpha}(t)}\,R_{\alpha}\widehat{\sigma}_{t-}R_{\alpha}^{\dagger}-\widehat{\sigma}_{t-}\right)\big(\mathrm{d}N_{\alpha}(t)-r_{\alpha}(t)\mathrm{d}t\big)+$$

$$+\sum_{j}\left(\frac{1}{l_{j}(t)}\,L_{j}\widehat{\sigma}_{t-}L_{j}^{\dagger}-\widehat{\sigma}_{t-}\right)\big(\mathrm{d}N_{j}(t)-l_{j}(t)\mathrm{d}t\big)\,. \tag{3.12}$$

In this equation the $N_{\alpha}(t)$, $N_{j}(t)$ are again counting processes with intensities (3.4), as in Eq. (3.6).

Let us consider now the case when only the process

$$N(t):=\sum_{\alpha}N_{\alpha}(t) \tag{3.13}$$

is observed, so that the processes $N_j(t)$ are completely unobserved and the processes $N_\alpha(t)$ are only partially observed through their sum. Under the original probability measure, $N(t)$ is a Poisson process of intensity μ and under the new one a counting process of intensity $I(t)$. The mathematical object which "takes the mean over all what is not observed" is the conditional expectation with respect to the σ-algebra generated by the process N. By taking this conditional expectation with reference to the original probability measure, one obtains immediately Eq. (1.8) from Eq. (3.10): the practical rules are

$$\sigma_t \to \varrho_t\,, \qquad \mathrm{d}N_j(t) \to \nu_j \mathrm{d}t\,, \qquad \mathrm{d}N_\alpha(t) \to \frac{\mu_\alpha}{\mu}\,\mathrm{d}N(t)\,. \tag{3.14}$$

By working in the same way with conditional expectations related to the new probability measure, one obtains Eq. (1.15) from Eq. (3.12): the rules are

$$\widehat{\sigma}_t \to \widehat{\varrho}_t\,, \qquad \mathrm{d}N_j(t) \to l_j(t)\mathrm{d}t\,, \qquad \mathrm{d}N_\alpha(t) \to \frac{r_\alpha(t)}{I(t)}\,\mathrm{d}N(t)\,. \tag{3.15}$$

Equations (3.10) and (3.12) always preserve pure states (by construction); this property is shared by Eqs. (1.8) and (1.15) only if the the conditional expectation just discussed is trivial. This is the case of the first model of Sect. 2, but not of the second one. We can say that one can reduce the second model to the pure–state preserving situation only if one adds unphysical processes which count the quanta exchanged with the thermal bath.

On Eqs. (3.1) and (3.6) they are based powerful numerical methods for solving master equations, known as quantum jump or Montecarlo wave simulation techniques [17,18].

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