Partially overdetermined elliptic problems

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Abstract

We consider semilinear elliptic Dirichlet problems in bounded domains, overdetermined with a Neumann condition on a proper part of the boundary. Under different kinds of assumptions, we show that these problems admit a solution only if the domain is a ball. When these assumptions are not fulfilled, we discuss possible counterexamples to symmetry. We also consider Neumann problems overdetermined with a Dirichlet condition on a proper part of the boundary, and the case of partially overdetermined problems on exterior domains.

Keywords: overdetermined boundary value problem.

2000MSC: 35J70, 35B50.

1 Introduction

In a celebrated paper [33], Serrin studied elliptic equations of the kind

$$-\Delta u = f(u) \qquad \text{in } \Omega \ , \tag{1}$$

overdetermined with both a Dirichlet and Neumann data

$$u = 0 \quad \text{and} \quad u_\nu = -c \qquad \text{on } \partial\Omega \ . \tag{2}$$

Here Ω is assumed to be an open bounded connected domain in $\mathbb{R}^n$ with smooth boundary, ν is the unit outer normal to $\partial\Omega$, $c > 0$ and f is a smooth function. Serrin proved that if a solution exists to (1)-(2) then necessarily the domain Ω is a ball and the solution u is radially symmetric, see Theorem 17 below (actually he got the same conclusion also for more general elliptic problems). His proof is based on what is nowadays called the "moving planes method", which is due to Alexandrov [1], and has been later used to derive further symmetry results for elliptic equations, see *e.g.* [4, 16]. Actually, a huge amount of literature originated from the pioneering work by Serrin, in both the directions of finding different proofs and as much general as possible extensions. So, in giving hereafter some related bibliographical items, we drop any attempt of being complete.
About alternative proofs, which hold just in the special case $f \equiv 1$, we refer to [8, 28, 38]. About the many existing generalizations, different investigation directions have been the case when the Laplacian is replaced by a possibly degenerate elliptic operator [6, 10, 11, 12, 13, 15], the case when the elliptic problem is stated on an exterior domain [14, 31, 34] or on a ring-shaped domain [2, 17], and also the case when the assumed regularity of $\partial\Omega$ is weaker than C^2 [30, 37].
The aim of this paper is to study not *totally* by just *partially* overdetermined boundary value problems. More precisely, we try to answer the following question, which has been raised by C. Pagani:

"Can we conclude that Ω is a ball if (1) *admits a solution u satisfying the Dirichlet and the Neumann conditions in* (2) *respectively on $\partial\Omega$ and on a proper subset of $\partial\Omega$?*

This question has several physical motivations, that we postpone to next section.
Let us also remark that, as long as no global regularity assumption is made on $\partial\Omega$, the above question may be seen a stronger version of the following one, which was set by H. Berestycki as reported in [30]: *"Does Serrin's result remain true when the Neumann condition holds everywhere on $\partial\Omega$ except that at a possible corner or cusp?"*
From a mathematical point of view, the problem can be stated as follows. Let Γ be a nonempty, proper and connected subset of $\partial\Omega$, relatively open in $\partial\Omega$. This set Γ is the overdetermined part of $\partial\Omega$, that is the region where both the Dirichlet and the Neumann conditions hold. Extending the Dirichlet condition also on $\partial\Omega \setminus \Gamma$, we obtain a Dirichlet problem partially overdetermined with a Neumann condition:

$$\begin{cases} -\Delta u = f(u) & & \text{in } \Omega \\ u = 0 \quad \text{and} \quad u_\nu = -c & & \text{on } \Gamma \\ u = 0 & & \text{on } \partial\Omega \setminus \Gamma \ . \end{cases} \tag{3}$$

In a dual way, we may also extend the Neumann condition to $\partial\Omega \setminus \Gamma$, instead of the Dirichlet one. Thus we obtain a Neumann problem partially overdetermined with a Dirichlet condition:

$$\begin{cases} -\Delta u = f(u) & & \text{in } \Omega \\ u = 0 \quad \text{and} \quad u_\nu = -c & & \text{on } \Gamma \\ |\nabla u| = c & & \text{on } \partial\Omega \setminus \Gamma \ . \end{cases} \tag{4}$$

Notice that, since *a priori* $\partial\Omega \setminus \Gamma$ is not known to be a level surface for u, the Neumann condition in (4) is stated under the form $|\nabla u| = c$ rather than $u_\nu = -c$.
Problems of the same kind as (3) or (4) can be also formulated on exterior domains, again supported by meaningful physical motivations, as described in next section. Denoting by ν the unit normal to $\partial\Omega$ pointing outside the open bounded domain Ω, we consider the problem

$$\begin{cases} -\Delta u = f(u) & & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 \quad \text{and} \quad u_\nu = -c & & \text{on } \Gamma \\ u = 1 & & \text{on } \partial\Omega \setminus \Gamma \\ u \to 0 & & \text{as } |x| \to \infty \end{cases} \tag{5}$$

and its dual version

$$\begin{cases} -\Delta u = f(u) & & \text{in } \mathbb{R}^n \setminus \overline{\Omega} \\ u = 1 \quad \text{and} \quad u_\nu = -c & & \text{on } \Gamma \\ |\nabla u| = c & & \text{on } \partial\Omega \setminus \Gamma \\ u \to 0 & & \text{as } |x| \to \infty \ . \end{cases} \tag{6}$$

For all the partially overdetermined problems (3), (4), (5) and (6), the common question is:

$$\text{“If } \Gamma \subsetneq \partial\Omega\text{, does the existence of a solution imply that } \Omega \text{ is a ball?”} \tag{7}$$

Clearly, without additional assumptions, the answer to question (7) is *no*: for instance one can easily check that, when $f \equiv 1$, problem (3) admits a (radial) solution if Ω is an annulus and Γ is a connected component of its boundary.
In this paper we show that the answer to question (7) becomes *yes* under different kinds of additional assumptions. Roughly speaking, we require that some information is available in one of the following aspects:

(I) regularity of Γ;

(II) maximal mean curvature of Γ;

(III) geometry of Γ.

In order to prove symmetry, our approach is completely different in each of these three situations. In case (I) we treat partially overdetermined problems as *initial value problems* in the spirit of Cauchy-Kowalewski Theorem; in cases (II) and (III) we take advantage respectively of the P-function and the moving planes methods already existing in literature, adapting them to our framework through suitable modifications. The only common feature between cases (I), (II), (III), is that each time our proof strategy consists in showing that the overdetermined condition must hold also on $\partial\Omega \setminus \Gamma$, so that we deal with a *totally* overdetermined problem, to which some known symmetry result applies. This proof line is adopted for both interior and exterior problems, though with a number of nontrivial points of difference.
It is natural to ask how far our assumptions are from being optimal, namely whether the answer to question (7) remains *yes* or becomes *no* under weaker requirements. In view of the above mentioned example of the annulus, one needs to impose at least that $\partial\Omega$ is a connected hypersurface. We believe that such condition is not enough to ensure that Ω is a ball. In fact, we set up counterexamples to symmetry, where the connectedness of the boundary is preserved. To that aim, we use an approach based on shape optimization and domain derivative. In order to make these counterexample complete, we would need some very delicate regularity results for the involved free boundaries; this will be investigated in a forthcoming work.
However, though the possible counterexamples we indicate may legitimate the assumptions under which we prove symmetry, at the same time they are not subtle enough to indicate that such assumptions are sharp. Therefore, in our opinion the problem of finding the minimal hypotheses which ensure symmetry deserves further investigation.
Finally, let us also mention that our results may be partially extended to more general kinds of problems, for instance involving degenerate elliptic operators, or different types of Neumann conditions. This goes beyond the purpose of this paper, which is mostly to attack the problem rather than to treat it in the highest generality.

The paper is organized as follows. Section 2 outlines some physical motivations for studying partially overdetermined problems. The main symmetry results are stated in Section 3, and proved in Section 5. Possible counterexamples to symmetry are discussed in Section 4. The Appendix is devoted to some known results, revisited and refined in the most convenient form for our purposes.

2 Physical motivations

We give here a short list of sample models which enlighten the relevance of question (7) in mathematical physics.

– *A model in fluid mechanics.*
When f is constant, equation (1) describes a viscous incompressible fluid which moves in straight parallel streamlines through a pipe with planar section $\Omega \subset \mathbb{R}^2$: the function u represents the flow velocity, the Dirichlet condition in (2) is the adherence condition to the wall, whereas u_ν represents the stress which is created per unit area on the pipe wall. In this framework, question (7) formulated for problem (3) reads: when the adherence condition holds on the entire pipe wall, and the stress on the pipe wall is the same on a proper portion of it, is it true that necessarily the pipe has a circular cross section?

– *A model in solid mechanics.*
When a solid bar of cross section $\Omega \subset \mathbb{R}^2$ is subject to torsion, the warping function u satisfies equation (1) with a constant source f, while the traction which occurs ar the surface of the bar is represented by the normal derivative u_ν. In this framework, question (7) formulated for problem (3) reads: when the traction occurring at the surface of the bar is constant on a proper portion of it, is it true that necessarily the bar has a circular cross section?

– *A model in thermodynamics.*
Let $\Omega \subset \mathbb{R}^3$ be a conductor heated by the application of a uniform electric current $I > 0$. If the body Ω is homogeneous with unitary thermal conductivity, the electric resistance R is a function of the temperature u, $R = R(u)$. Then, if the radiation is negligible, the resulting stationary equation for u, written in some dimensionless form, is exactly equation (1) with $f(u) = I^2 R(u)$. The homogeneous Dirichlet condition is satisfied as soon as the temperature is kept equal to 0 on the boundary, whereas the gradient ∇u represents the transfer of heat. In this framework, question (7) formulated for problem (3) reads: when the outgoing flux of heat is constant on a proper portion of the boundary, is it true that necessarily the conductor Ω has a spherical shape?

– *A model in electrostatics.*
Consider problem (5) or (6) when $\Gamma \equiv \partial\Omega$. In this case it is known that, whenever a solution exists, necessarily Ω is a ball (see [31, 34]). This symmetry result can be read as an electrostatic characterization of spheres, which answers positively the following conjecture by Gruber: if a source distribution ($= u_\nu$) is constant on the boundary of an open smooth domain $\Omega \subset \mathbb{R}^3$ and induces on it a constant single-layer potential ($=u$), then the domain must be a ball. In this framework, question (7) formulated for problem (5) can be read as a stronger version of Gruber's conjecture, where the source distribution is assumed to be constant only on a proper portion of the boundary.

3 Main results

Without further mention, throughout the paper we make the following assumptions:

- $\Omega \subset \mathbb{R}^n$ is an open bounded connected domain;
- $\Gamma \subset \partial\Omega$ is nonempty connected and relatively open in $\partial\Omega$, with $\Gamma \in C^1$;

- $c > 0$;
- $f \in C^1(\mathbb{R})$.

Additional assumptions will be specified in each statement.
We point out that we do not take care about possible compatibility conditions between f and c which ensure existence, since in all our statements a solution is always *assumed* to exist. In this respect, whenever we assume that (3) or (4) admit a solution u, we always mean that

$$u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \ ,$$

so that both the equation and the boundary conditions are satisfied in classical sense. In particular, the overdetermined condition will hold by continuity on $\overline{\Gamma}$. Similarly, when we assume that (5) or (6) admit a solution u, we always mean that $u \in C^2(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^n \setminus \Omega)$.
Under the above hypotheses, by Theorem 16 in the Appendix, we infer

$$\Gamma \in C^{2,\alpha} \qquad \text{and} \qquad u \text{ is } C^2 \text{ up to } \Gamma \ . \tag{8}$$

We are now ready to state our symmetry results. We present them in the three separate Subsections 3.1, 3.2, 3.3, which correspond respectively to the situations labelled as (I), (II), (III) in the Introduction. In each subsection, we consider both the interior problems (3)-(4) and their counterparts for the exterior problems (5)-(6). Each statement is complemented with several related comments.

3.1 The case when Γ is "analytically continuable"

We consider here the case where no assumption is made on $\partial\Omega \setminus \Gamma$, but Γ has a strong regularity property: not only it is analytic, but it is part of a globally analytic surface $\partial\widetilde{\Omega}$, a priori different from $\partial\Omega$. The precise statement for interior problems reads

Theorem 1 *Assume that $\partial\Omega$ is connected, that $\Gamma \subseteq \partial\widetilde{\Omega}$ for some open set $\widetilde{\Omega}$ with connected analytic boundary $\partial\widetilde{\Omega}$ and that f is an analytic function. If one of the following conditions holds:*

(a) there exists a solution u of (3),

(b) f is nonincreasing and there exists a solution u of (4),

then, $\Omega = \widetilde{\Omega}$, Ω is a ball and u is radially symmetric.

Theorem 1 establishes in particular that, if Γ is the portion of any (possibly unbounded) quadric surface different from a sphere, then problems (3)-(4) have no solution. This rules out the possibility of finding simple counterexamples to symmetry via explicit computations. On the other hand, if Γ is the portion of a sphere, then Theorem 1 implies that a solution exists if and only if Ω is a ball.

In case of exterior problems Theorem 1 has the following counterpart

Theorem 2 *Assume that $\partial\Omega$ is connected, that $\Gamma \subseteq \partial\widetilde{\Omega}$ for some open set $\widetilde{\Omega}$ with connected analytic boundary $\partial\widetilde{\Omega}$ and that f is an analytic nonincreasing function. If one of the following conditions holds:*

(a) there exists a solution u of (5) with $u < 1$ on $\mathbb{R}^n \setminus \overline{\Omega}$,

(b) there exists a solution u of (6) with $u < 1$ on $\mathbb{R}^n \setminus \overline{\Omega}$ and $\nabla u \to 0$ at ∞,

then $\Omega = \widetilde{\Omega}$, Ω is a ball and u is radially symmetric.

3.2 The case when Γ has a "large maximal mean curvature"

In this section we assume global regularity but no connectedness for $\partial\Omega$; the latter assumption is replaced by asking a suitable behaviour of u and a sufficiently large maximal mean curvature of Γ.

Theorem 3 *Assume that $\partial\Omega \in C^{2,\alpha}$, with mean curvature $H(x)$ satisfying $\sup_{x\in\Gamma} H(x) \geq f(0)/nc$. Assume also that f is nonincreasing. If one of the following conditions holds:*

(a) there exists a solution u of (3) such that the maximum of $|\nabla u|$ over $\partial\Omega$ is attained on Γ,

(b) $f > 0$ on $\mathbb{R}^-$ and there exists a solution u of (4) such that the maximum of u over $\partial\Omega$ is attained on Γ,

then $f(0) > 0$, Ω is a ball of radius $nc/f(0)$, and u is radially symmetric.

To prove Theorem 3, we make use of the maximum principle for a suitable P-function, which requires the hypothesis f nonincreasing. In such case, the maximum points of $|\nabla u|$ over $\overline{\Omega}$ lie on $\partial\Omega$ (*cf.* inequality (19)). We stress that these points are of relevant interest in the physical situations modeled by our boundary value problems. For instance, referring to the torsion problem described in Section 2, they are the so-called "fail points" or "points dangereux", which mark the onset of plasticity; in case of planar convex domains with special regularity and symmetry properties, their location has been studied in [24].

As far as we are aware, in case of exterior domains the P-function is fruitfully applied only to harmonic functions. Therefore, we assume $f \equiv 0$ and we prove

Theorem 4 *Let $n \geq 3$ and $f \equiv 0$. Assume that $\partial\Omega \in C^{2,\alpha}$, with mean curvature $H(x)$ satisfying $\inf_{x\in\Gamma} H(x) \leq c/(n-2)$. If one of the following conditions holds:*

(a) there exists a solution u of (5) such that the maximum of $|\nabla u|$ over $\partial\Omega$ is attained on Γ,

(b) there exists a solution u of (6) such that the minimum of u over $\partial\Omega$ is attained on Γ,

then Ω is a ball of radius $(n-2)/c$ and u is radially symmetric.

3.3 The case when Γ "contains a hat"

In this section we modify further our assumptions by asking less a priori regularity on $\partial\Omega$ and a special geometric property of Γ in place of the hypothesis on its mean curvature. We need a preliminary

Definition 5 Assume $\partial\Omega \in C^1$. Consider a (hyper)plane $T \perp \partial\Omega$, namely such that T contains the unit outer normal $\nu(A)$ at some point $A \in \partial\Omega$. Take then a connected component of $\partial\Omega \setminus T$ such that its closure γ contains A. We say that γ is a *hat* of $\partial\Omega$ determined by T if the bounded open domain delimited by γ and T is entirely contained into Ω.

Remark 6 If Ω is convex, any plane T which intersects $\partial\Omega$ orthogonally determines two hats. On the other hand, if Ω is nonconvex then $\partial\Omega \setminus T$ may have more than two connected components, as it happens for instance in Figure 1 below. Therein, according to Definition 5, γ_1 is a hat, whereas γ_2 is not.

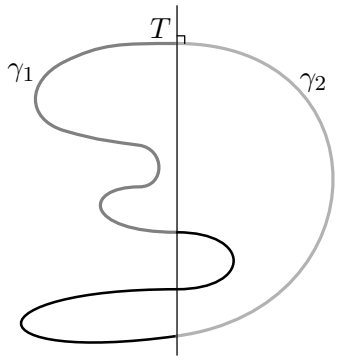

Figure 1: About Definition 5

For interior problems, we can now state the following result.

Theorem 7 *Assume that $\partial\Omega \in C^1$, and that Γ contains a hat of $\partial\Omega$ determined by some plane $T \perp \partial\Omega$. If one of the following conditions holds:*

(a) there exists a solution u of (3) such that the maximum of $|\nabla u|$ over $\partial\Omega$ is attained on Γ,

(b) f is nonincreasing and there exists a solution u of (4) such that the maximum of u over $\partial\Omega$ is attained on Γ,

then Ω is a ball and u is radially symmetric.

The geometric assumption that Γ contains a hat may be slightly weakened; this is explained in detail in Remark 15 after the proof of Theorem 7.

For exterior problems, we have:

Theorem 8 *Assume that $\partial\Omega \in C^1$, and that there exists a plane $T \perp \partial\Omega$ such that $\partial\Omega \setminus T$ has exactly two connected components and Γ contains a hat of $\partial\Omega$ determined by T. Assume that f is nonincreasing. If one of the following conditions holds:*

(a) there exists a solution u of (5), with $u < 1$ on $\mathbb{R}^n \setminus \overline{\Omega}$, such that the minimum of $|\nabla u|$ over $\partial\Omega$ is attained on Γ,

(b) there exists a solution u of (6), with $u < 1$ on $\mathbb{R}^n \setminus \overline{\Omega}$,

then Ω is a ball and u is radially symmetric.

We conclude with further remarks about the assumptions on u in the above statements.

Remark 9 Contrary to [33, Theorem 2], in Theorems 1 and 7 we do not assume that $u > 0$. This is because we take $c > 0$, see Theorem 17 in the Appendix. Similarly, contrary to [31, Theorem 1], in Theorems 2 and 8 we do not assume that $u \geq 0$, see Theorem 18 in the Appendix. Finally we point out that, in case (a) of Theorems 2 and 8, the assumption $u < 1$ on $\mathbb{R}^n \setminus \overline{\Omega}$ can be relaxed by arguing as in [34].

4 On the way towards counterexamples

4.1 An eigenvalue problem

Consider the minimization problem of the second Dirichlet eigenvalue $\lambda_2(\Omega)$ of $-\Delta$ among all planar convex domains of given area. By [19, Theorems 4,6,8], we know that

(i) there exists an optimal domain Ω^*;

(ii) $\partial\Omega^* \in C^1$;

(iii) $\partial\Omega^*$ contains no arc of circle;

(iv) if $\partial\Omega^* \in C^{1,1}$, then $\lambda_2(\Omega^*)$ is simple.

We observe that $\partial\Omega^*$ contains a strictly convex part which we call Γ^* and which is of class C^1. But this regularity is *not enough* to provide a full counterexample. We need to fill a regularity gap assuming that

$$\Gamma^* \in C^{1,1} \ . \tag{9}$$

If (9) is satisfied (and we have no proof of this fact!), then $\lambda_2(\Omega^*)$ is simple in view of Theorem 19 in the Appendix. We can so use Hadamard formula for simple eigenvalues [18, Theorem 2.5.1]; hence, arguing as in the proof of [19, Theorem 7], we obtain that $|\nabla e_2(x)| = c > 0$ for all $x \in \Gamma^*$, where e_2 denotes a corresponding (nontrivial) eigenfunction. Since $\partial\Omega^* \in C^1$ and we assumed (9), we know that $e_2 \in C^2(\Omega^*) \cap C^1(\Omega^* \cup \Gamma^*) \cap C^0(\overline{\Omega^*})$. Since $e_2 = 0$ and $|\nabla e_2(x)| = c > 0$ on Γ^*, by Theorem 16 we know that $\Gamma^* \in C^{2,\alpha}$ and that $e_2 \in C^2(\Omega^* \cup \Gamma^*) \cap C^0(\overline{\Omega^*})$.
We have so shown that, in absence of some specific assumptions, a counterexample to symmetry for problem (3) may be constructed provided one can fill the gap given by (9). Indeed, the obtained regularity of e_2 is appropriate because, as far as problem (3) is concerned, the assumption $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ can be relaxed to $u \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma) \cap C^0(\overline{\Omega})$. And, by (8), the latter condition implies $u \in C^2(\Omega \cup \Gamma) \cap C^0(\overline{\Omega})$.

4.2 A boundary-locked problem

Let γ be a smooth convex arc in the plane and take a constant $c > 0$. Consider the shape optimization problem

$$\min_{\Omega \in \mathcal{A}} \ J(\Omega) \ , \tag{10}$$

where the admissible domains Ω vary in the class

$$\mathcal{A} = \{\Omega \subset \mathbb{R}^2 : \ \Omega \text{ is bounded and convex, } \partial\Omega \supset \gamma\} \ ,$$

and the cost is given by the integral functional

$$J(\Omega) = \int_\Omega \left(\frac{|\nabla u_\Omega|^2}{2} - u_\Omega + c^2 \right) \ ,$$

being u_Ω the (unique) solution to the torsion problem in Ω:

$$\begin{cases} -\Delta u_\Omega = 1 & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega \ . \end{cases}$$

Firstly, we claim that there exists an optimal domain Ω^* for problem (10). This follows by the continuity of the map $\Omega \mapsto J(\Omega)$ with respect to the Hausdorff convergence of domains (which is a standard fact), and by compactness of the class $\mathcal{A}$ (which holds by Blaschke selection Theorem [32, Theorem 1.8.6] and the closedness of the constraint $\partial\Omega \supset \gamma$).
Next, playing on the initial choice of the constant c, one can show that there exists a part Γ^* of $\partial\Omega^* \setminus \gamma$ which is strictly convex. By the results in [7], it turns out that $\Gamma^* \in C^1$. Again, this regularity is *not enough* to provide a full counterexample and we need to fill a regularity gap assuming now that

$$\Gamma^* \in C^{1,\alpha} \ . \tag{11}$$

Next, we analyze the boundary behaviour of the state function $u_{\Omega^*} \in C^2(\Omega^*)$. Since Ω^* is convex, it has a Lipschitz boundary so that $u_{\Omega^*} \in C^0(\overline{\Omega^*})$. Moreover, if (11) holds, then $u_{\Omega^*} \in C^1(\Omega^* \cup \Gamma^*)$. By the strict convexity of Γ^*, we may use the results in [23] to infer that the stationarity of Ω^* implies the pointwise equality $|\nabla u_{\Omega^*}(x)| = c$ on Γ^*. Since by construction we also have $u_{\Omega^*} = 0$ on Γ^*, by Theorem 16 we conclude that $\Gamma^* \in C^{2,\alpha}$ and $u_{\Omega^*} \in C^2(\Omega^* \cup \Gamma^*)$. Up to filling the gap given by assumption (11), we have so obtained another counterexample to symmetry for problem (3), this time with a constant source. We also remark that Γ^* might cover a very small part of the boundary, and the latter might be globally not C^1 (differently from the case of the previous subsection).

4.3 An exterior problem

Let Ω be the bounded planar region delimited by the curve represented in Figure 2 and defined parametrically as

$$\begin{cases} x(\vartheta) = 3\cos\vartheta + \cos 3\vartheta \\ y(\vartheta) = 3\sin\vartheta - \sin 3\vartheta \ . \end{cases} \tag{12}$$

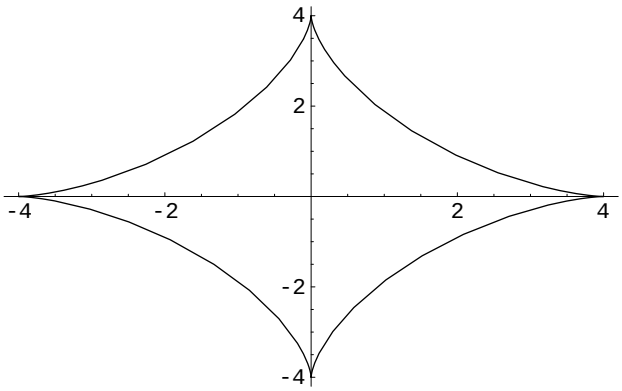

Figure 2: The astroid Ω

Following [20], we take $p = 2$, $n = 3$, $a = 3$ and $b = 1$ in formula (6.6) therein. So we set

$$\phi(z) := 3z + z^{-3} \ , \qquad f(z) := \frac{3i}{2}(z^2 + z^{-2}) \ ,$$

and

$$u(x, y) := 1 + \mathbb{R}\mathrm{e}\left[f(\phi^{-1}(x + iy))\right] \ .$$

It is readily verified (see [20]) that the function u satisfies

$$\Delta u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}\ , \qquad u = 1 \quad \text{on } \partial\Omega\ .$$

Moreover, if $\Gamma \subset \partial\Omega$ is the analytic curve defined parametrically by (12) with θ varying in $(0, \pi/2)$, (or in any of the intervals $(\pi/2, \pi)$, $(\pi, 3\pi/2)$, $(3\pi/2, 2\pi)$, there holds (see (2.2) and (1.11) in [20])

$$|\nabla u| = c \quad \text{on } \Gamma\ .$$

This example shows that a partially overdetermined exterior problem such as (5) may admit a solution on a domain which is not the complement of a ball, when the boundary lacks global regularity and the solution does not vanish at infinity.

5 Proofs

5.1 Proof of Theorem 1

Since by assumption the hypersurface $\partial\widetilde{\Omega}$ and the function f are analytic, by the Cauchy-Kowalewski Theorem (see for instance [29]) there exists a neighbourhood $\mathcal{U}$ of $\partial\widetilde{\Omega}$ and a unique function v analytic in $\mathcal{U}$ such that

$$\begin{cases} -\Delta v = f(v) & \text{in } \mathcal{U} \\ v = 0 & \text{on } \partial\widetilde{\Omega} \\ v_\nu = -c & \text{on } \partial\widetilde{\Omega}\ . \end{cases}$$

By assumption there exists a solution u to problem (3) (in case of assumption (a)) or to problem (4) (in case of assumption (b)). And an analytic elliptic problem with analytic Dirichlet data along an analytic portion of the boundary of its domain of definition can be extended analytically up to that portion of the boundary [25]. Therefore, if we set $\mathcal{U}_r(\Gamma) = \{x - t\nu(x) \ : \ x \in \Gamma\ ,\ 0 \le t < r\}$, for $r > 0$ sufficiently small the function u is a solution, analytic in $\mathcal{U}_r(\Gamma)$, to the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathcal{U}_r(\Gamma) \\ u = 0 & \text{on } \Gamma \\ u_\nu = -c & \text{on } \Gamma\ . \end{cases} \tag{13}$$

The geometric situation is illustrated in Figure 3.
Since by the Cauchy-Kowalewski Theorem the solution to (13) in the class of analytic functions over $\mathcal{U}_r(\Gamma)$ is unique, we deduce that

$$u \equiv v \quad \text{in} \quad \overline{\mathcal{U}_r(\Gamma) \cap \mathcal{U}}\ .$$

Then the function u, which is analytic in Ω because it solves (3) or (4), provides an analytic extension to Ω of the function v. By continuity, and since by assumption $\partial\Omega$ is connected, we infer that

$$v = 0 \qquad \text{on } \partial\Omega \tag{14}$$

in case of assumption (a), or that

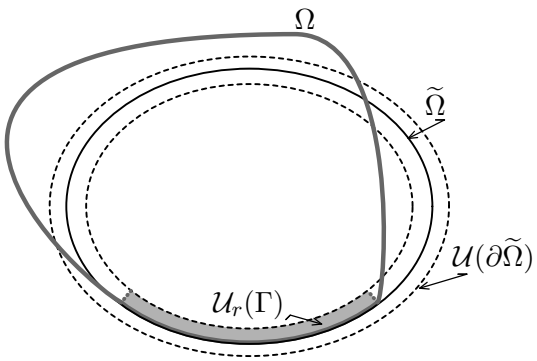

Figure 3: About the proof of Theorem 1

$$|\nabla v| = c \qquad \text{on } \partial\Omega \tag{15}$$

in case of assumption (b). We are now in a position to show that

$$\partial\Omega = \partial\widetilde{\Omega} \ . \tag{16}$$

To this end, notice first that the portion of $\partial\Omega$ where both the conditions $u = 0$ and $u_\nu = -c$ hold may be extended from Γ to $\partial\Omega \cap \partial\widetilde{\Omega}$ by analyticity of $\partial\widetilde{\Omega}$. Therefore, we may assume without loss of generality that

$$\Gamma = \partial\Omega \cap \partial\widetilde{\Omega} \ . \tag{17}$$

Then, since by assumption $\partial\widetilde{\Omega}$ is connected and (17) holds, to prove (16) it is enough to show that $\partial\Omega$ cannot "bifurcate" from $\partial\widetilde{\Omega}$, or more precisely that the *strict* inclusion

$$\Gamma = (\partial\Omega \cap \partial\widetilde{\Omega}) \subsetneqq \partial\Omega \tag{18}$$

cannot hold. To this end, we distinguish between assumptions (a) and (b).

– Under assumption (a), (18) cannot hold due to the implicit function Theorem for analytic functions, see for instance [36]. Indeed, the function v vanishes on both $\partial\widetilde{\Omega}$ and $\partial\Omega$ (respectively, by definition and by (14)), whereas ∇v does not vanish on $\partial\widetilde{\Omega}$ because the constant c is assumed to be positive.

– Under assumption (b), (18) cannot hold again by the implicit function Theorem. Indeed, the analytic function $|\nabla v|^2 - c^2$ vanishes on both $\partial\widetilde{\Omega}$ and $\partial\Omega$ (respectively, by definition and by (15)), whereas $\nabla\big(|\nabla v|^2 - c^2\big)$ does not vanish on $\partial\widetilde{\Omega}$. The last assertion is a consequence of Hopf's boundary Lemma after noticing that (since f nonincreasing) the following inequality holds

$$\Delta(|\nabla v|^2) = 2\Big(|\nabla^2 v|^2 - \nabla v \cdot \nabla(\Delta v)\Big) = 2\Big(|\nabla^2 v|^2 - f'(v)|\nabla v|^2\Big) \geq 0 \qquad \text{in } \Omega \ . \tag{19}$$

In any case, we have shown that (18) cannot hold. Then, both (14) and (15) are satisfied, and $u \equiv v$ in Ω; in particular, since the condition $|\nabla u| = c$ holds on the level surface $\partial\Omega = \{u = 0\}$ and the solution u is C^2 up to the boundary, there holds $u_\nu = -c$ on $\partial\Omega$.
Therefore, the statement follows from Theorem 17 in the Appendix. □

Remark 10 One may wonder whether our proof of Theorem 1 may be fruitfully applied also to fourth order problems such as

$$\begin{cases} \Delta^2 u = f(u) & \text{in } \Omega \\ u = u_\nu = 0 & \text{on } \partial\Omega \\ \Delta u = c & \text{on } \Gamma \ . \end{cases} \tag{20}$$

If $\Gamma \equiv \partial\Omega$, it is known that when $f \equiv 1$ (20) admits a solution only if Ω is a ball (see [5, 9], and also [26, Section 5] for an attempt to use a suitable P-function). If $\Gamma \subsetneqq \partial\Omega$, the method of introducing a Cauchy problem with initial data on Γ as done in the proof of Theorem 1 does not apply because, although the condition $\Delta u = c$ readily translates into $u_{\nu\nu} = c$, no condition on $u_{\nu\nu\nu}$ is available.

5.2 Proof of Theorem 2

One can follow line by line the proof of Theorem 1. The only differences are:

– the neighbourhood $\mathcal{U}_r(\Gamma)$ must be defined as $\{x + t\nu(x) \ : \ x \in \Gamma \ , \ 0 < t < r\}$;

– in order to exclude (18) under assumption (b), one needs the hypothesis that $\nabla u \to 0$ at ∞ (it ensures that the maximum over $\mathbb{R}^n \setminus \Omega$ of the subharmonic function $(|\nabla v|^2 - c^2)$ is attained on $\partial\Omega$, so that Hopf's lemma can be applied);

– to conclude, one has to invoke Theorem 18 in the Appendix (this is why it is needed $u < 1$ on $\mathbb{R}^n \setminus \overline{\Omega}$ and f nonincreasing also under assumption (a)). □

5.3 Proof of Theorem 3

Assume for contradiction that $f(0) \leq 0$, so that $f \leq 0$ on $\mathbb{R}_+$. Let $\mathcal{U} \subset \Omega$ denote the largest open neighbourhood of Γ where $u > 0$; possibly $\mathcal{U} = \Omega$, but certainly $\mathcal{U} \neq \emptyset$ since $c > 0$. Then $-\Delta u = f(u) \leq 0$ in $\mathcal{U}$ and $u = 0$ on $\partial\mathcal{U}$, so that $u \leq 0$ by the maximum principle, contradiction.

We now follow the approach in [27]. We put

$$F(s) := \int_0^s f(t)\, dt \ ,$$

and consider the P-function defined by

$$P(x) := |\nabla u(x)|^2 + \frac{2}{n} F[u(x)] \ , \qquad x \in \overline{\Omega} \ . \tag{21}$$

Then, we have

Lemma 11 *Under the assumptions of Theorem 3 the P-function defined by* (21) *either is constant in $\overline{\Omega}$ or satisfies $P_\nu > 0$ on $\overline{\Gamma}$.*

Proof. Since by assumption f is nonincreasing, P satisfies the elliptic inequality (2.39) in [27] over Ω and thus takes its maximum on the boundary [27, Theorem 4]. We claim that the maximum of P over $\partial\Omega$ is attained on Γ (and hence by continuity on $\overline{\Gamma}$), that is

$$\max_{x \in \partial\Omega} P(x) = c^2 \ . \tag{22}$$

Indeed:
– In case of assumption (a), we have

$$P(x) = |\nabla u(x)|^2 \qquad \text{on } \partial\Omega$$

and (22) follows from the assumption that the maximum of $|\nabla u|$ over $\partial\Omega$ is attained on Γ.
– In case of assumption (b), we have

$$P(x) = c^2 + \frac{2}{n} F[u(x)] \qquad \text{on } \partial\Omega$$

and (22) follows from the assumptions that the maximum of u over $\partial\Omega$ is attained on Γ and that $f > 0$ on $\mathbb{R}_-$.
Since $\partial\Omega \in C^{2,\alpha}$, elliptic regularity gives $u \in C^2(\overline{\Omega})$, so that $P \in C^1(\overline{\Omega})$. Then, according to Hopf's boundary Lemma, either P is constant on $\overline{\Omega}$, or $P_\nu > 0$ on $\overline{\Gamma}$. □

According to Lemma 11, two cases may occur. Let us rule out the second case arguing by contradiction. Since $P \in C^1(\overline{\Omega})$, we may compute P_ν on $\overline{\Gamma}$ as

$$P_\nu = -2c \left(u_{\nu\nu} + \frac{f(0)}{n} \right) \qquad \text{on } \overline{\Gamma}\ , \tag{23}$$

and we may also write pointwise the equation on $\overline{\Gamma}$ as

$$u_{\nu\nu} - (n-1)cH(x) = -f(0) \qquad \text{on } \overline{\Gamma}\ . \tag{24}$$

Now, if $P_\nu > 0$ on Γ, by combining (23) and (24), and taking into account that $u_\nu = -c < 0$ on $\overline{\Gamma}$, we readily obtain $H(x) < f(0)/nc$ for all $x \in \overline{\Gamma}$, against the assumption $\sup_{x\in\Gamma} H(x) \geq f(0)/(nc)$. This contradiction shows that the first alternative of Lemma 11 occurs. Hence, since $P(x) = c^2$ on $\overline{\Gamma}$ and P is constant on $\overline{\Omega}$, we infer that

$$P(x) = c^2 \qquad \text{on } \partial\Omega\ . \tag{25}$$

– In case of assumption (a), (25) implies that $|\nabla u(x)| = c$ for all $x \in \partial\Omega$. Since $\partial\Omega$ is the level surface $\{u = 0\}$ and since $u \in C^2(\overline{\Omega})$, the equality $|\nabla u| = c$ implies $u_\nu = -c$ on $\partial\Omega$.
– In case of assumption (b), (25) implies that $F(u) = 0$ for all $x \in \partial\Omega$; in turn, this implies that $u(x) = 0$ for all $x \in \partial\Omega$ because we assumed $\max_{\partial\Omega} u = 0$ and $f > 0$ on $\mathbb{R}^-$.
In both cases, the problem is totally overdetermined. Hence (23) and (24) hold on $\partial\Omega$. Then the condition $P_\nu = 0$ on $\partial\Omega$ may be reformulated as $H(x) = f(0)/nc$ on $\partial\Omega$. By Alexandrov Theorem [1], this implies that Ω is a ball of radius $nc/f(0)$. □

5.4 Proof of Theorem 4

We follow the approach in [35]. We consider the P-function defined by

$$P(x) := \frac{|\nabla u(x)|^2}{u(x)^{\frac{2(n-1)}{n-2}}}\ , \qquad x \in \mathbb{R}^n \setminus \Omega\ . \tag{26}$$

Then, we have

Lemma 12 *Under the assumptions of Theorem 4, the P-function defined by* (26) *either is constant in $\mathbb{R}^n \setminus \Omega$ or satisfies $P_\nu < 0$ on Γ.*

Proof. One first verifies that P is subharmonic over $\mathbb{R}^n \setminus \overline{\Omega}$ and thus takes its maximum either on $\partial\Omega$ or at infinity, see [14, Theorem 2.2]. One then compares the values of P on $\partial\Omega$ and at infinity, and using [14, Theorem 3.1] one obtains that P attains its maximum over $\partial\Omega$. We claim that such maximum is attained on $\overline{\Gamma}$, that is

$$\max_{x\in\partial\Omega} P(x) = c^2 \ . \tag{27}$$

Indeed:
– In case of assumption (a), we have

$$P(x) = |\nabla u(x)|^2 \qquad \text{on } \partial\Omega$$

and (27) follows from the assumption that the maximum of $|\nabla u|$ over $\partial\Omega$ is attained on Γ.
– In case of assumption (b), we have

$$P(x) = \frac{c^2}{u(x)^{\frac{2(n-1)}{n-2}}} \qquad \text{on } \partial\Omega$$

and (27) follows from the assumption that the minimum of u over $\partial\Omega$ is attained on Γ.
Since $P \in C^1(\mathbb{R}^n \setminus \Omega)$, the statement follows from Hopf's boundary Lemma. □

According to Lemma 12, two cases may occur. Arguing by contradiction, let us rule out the second case. Since P is C^1 up to $\partial\Omega$, we may compute the expression of P_ν on $\overline{\Gamma}$ as

$$P_\nu = 2c\left(\frac{n-1}{n-2}c^2 - u_{\nu\nu}\right) \qquad \text{on } \overline{\Gamma} \tag{28}$$

and we may rewrite the equation pointwise on $\overline{\Gamma}$ as

$$u_{\nu\nu} - (n-1)cH = 0 \qquad \text{on } \overline{\Gamma} \ . \tag{29}$$

If $P_\nu < 0$ on $\overline{\Gamma}$, by combining (28) and (29), we obtain that the inequality $H(x) > c/(n-2)$ holds on $\overline{\Gamma}$, against the assumption $\inf_{x\in\Gamma} H(x) \le c/(n-2)$. This contradiction shows that the first alternative of Lemma 12 occurs. Hence, since $P(x) = c^2$ on $\overline{\Gamma}$ and P is constant in $\mathbb{R}^n \setminus \Omega$, we infer that

$$P(x) = c^2 \qquad \text{on } \partial\Omega \ . \tag{30}$$

– In case of assumption (a), (30) implies that $|\nabla u(x)| = -u_\nu = c$ for all $x \in \partial\Omega$.
– In case of assumption (b), (30) implies that $u(x) = 1$ on $\partial\Omega$.
In both cases, the problem is totally overdetermined. Hence (28) and (29) are satisfied on $\partial\Omega$. Then the condition $P_\nu = 0$ on $\partial\Omega$ may be reformulated as $H(x) = c/(n-2)$ for all $x \in \partial\Omega$. By Alexandrov Theorem [1], this implies that Ω is a ball of radius $(n-2)/c$. □

5.5 Proof of Theorem 7

Firstly, we recall a restricted version of the maximum principle:

Lemma 13 *Let $\omega \subset \mathbb{R}^n$ be an open bounded domain, let $g \in C^1(\mathbb{R})$ and assume that $u, v \in C^2(\omega)$ satisfy*

$$-\Delta u = g(u) \qquad \text{and} \qquad -\Delta v = g(v) \qquad \text{in } \omega \ .$$

If $u \geq v$ in ω then either $u > v$ or $u \equiv v$ in ω.

Proof. See [21, p.149-150] or the reprinted version in [22, p.3-14]. □

Next, we recall a boundary point Lemma at a corner:

Lemma 14 *Let $D^* \subset \mathbb{R}^n$ be a domain with C^2 boundary and let T be a plane containing the normal to ∂D^* at some point Q. Let D denote the portion of D^* lying on some particular side of T. Assume that $a \in C^0(\overline{D})$ and $w \in C^2(\overline{D})$ satisfy*

$$w(Q) = 0\,, \qquad w > 0 \qquad \text{and} \qquad -\Delta w + a(x)w \geq 0 \qquad \text{in } D \ .$$

Then, for any vector $\vec{s}$ which exits nontangentially from D at Q we have

$$\text{either} \qquad \frac{\partial w}{\partial s}(Q) < 0 \qquad \text{or} \qquad \frac{\partial^2 w}{\partial s^2}(Q) < 0 \ .$$

Proof. With no loss of generality we may assume that Q coincides with the origin and that the outer normal to ∂D^* at Q coincides with $\vec{e}_1 = (1, 0, ..., 0)$. With the usual Hopf's trick we set $z(x) = e^{-bx_1} w(x)$ for $b > 0$ and we find that z satisfies

$$z > 0 \qquad \text{and} \qquad -\Delta z - 2b\frac{\partial z}{\partial x_1} \geq [b^2 - a(x)]z \geq 0 \qquad \text{in } D$$

provided b is large enough. Then we apply [33, Lemma 2] to obtain that

$$\text{either} \qquad \frac{\partial z}{\partial s}(Q) < 0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial s^2}(Q) < 0 \ . \tag{31}$$

As $\vec{s}$ forms an angle smaller than the right angle with $\vec{e}_1$, we have $\vec{e}_1 \cdot \vec{s} = c_s > 0$. Moreover, since $z(Q) = 0$, we have

$$\frac{\partial w}{\partial s}(Q) = \frac{\partial z}{\partial s}(Q) \qquad , \qquad \frac{\partial^2 w}{\partial s^2}(Q) = 2bc_s \frac{\partial z}{\partial s}(Q) + \frac{\partial^2 z}{\partial s^2}(Q) \ .$$

The statement then follows by (31), taking into account that $\frac{\partial z}{\partial s}(Q) \leq 0$. □

To prove Theorem 7, we use the moving planes method with some modifications with respect to [33]. In fact, the basic differences are: we are going to move just *one* plane (which is not arbitrary, but must be carefully chosen), and in addition such plane in its initial position is not necessarily exterior to Ω. For the sake of clearness we divide the proof into four steps.

Step 1: definition of the critical plane T_ℓ.

The plane T we move is the one which determines the hat γ contained into Γ. First, we move it to the limit parallel position T_0 such that γ is tangent to T_0 and entirely contained into the closed strip between T_0 and T. Up to a rotation and a translation, we may assume that $T = T_m := \{x_1 = m\}$ for some $m > 0$, that $T_0 = \{x_1 = 0\}$ and that the tangency point between T_0 and γ is the origin O. Then, we start moving T_0 towards T into the planes $T_\lambda := \{x_1 = \lambda\}$ for $\lambda \geq 0$: starting from $\lambda = 0$, we increase slightly λ so that T_λ intersects Ω close to O (see Figure 4). For any sufficiently small λ, we denote by C_λ the open cap bounded by T_λ and by $\gamma \cap \{x_1 < \lambda\}$. We point out that, for sufficiently small λ, $C_\lambda \subset \Omega$ since γ is a hat. Notice also that there may be subsets of Ω, delimited by T_λ and a portion of $\partial\Omega \setminus \gamma$, which lie into the half space $\{x_1 < \lambda\}$, but by construction they are not parts of the cap C_λ (see Figure 4).
For any C_λ, denote by C'_λ its (open) reflection with respect to T_λ. Clearly, $C'_\lambda \subset \Omega$ for sufficiently small λ; more precisely, we will have $C'_\lambda \subset \Omega$ until one of the following two facts depicted in Figure 5 occurs [3, Lemma A.1]:

(i) C'_λ becomes internally tangent to $\partial\Omega$ (which is C^1) at some point $P \notin T_\lambda$.

(ii) T_λ reaches a position where it is orthogonal to $\partial\Omega$ at some point Q; it may happen that $\lambda < m$, but certainly Q belongs to the closure of the strip between T_0 and T, so that $Q \in \Gamma$.

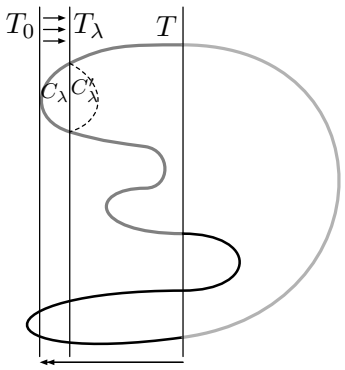

Figure 4: The moving plane T_λ and the reflected cap C'_λ

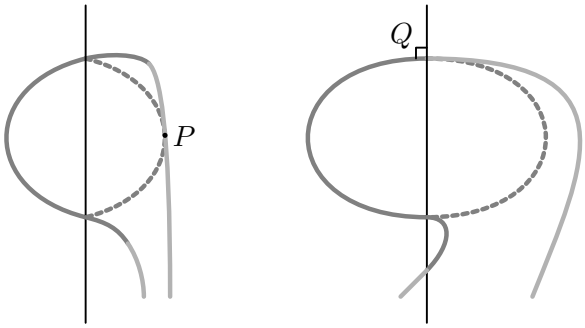

Figure 5: The two possible limiting positions

Let $\ell \in (0, m]$ be the smallest value of λ for which either (i) or (ii) occurs.

By construction, there holds

$$\big(\partial C_\ell \cap \partial\Omega\big) \subset \Gamma \ . \tag{32}$$

Step 2: definition and one sign property of the function w.
For any $\lambda \in (0, \ell]$ we introduce a new function v^λ defined in C'_λ by the formula $v^\lambda(x) = u(x^\lambda)$ where x^λ is the reflected value of x across T_λ. Since $u_\nu = -c < 0$ on Γ it is clear that for λ sufficiently small we have

$$u > v^\lambda \qquad \text{at interior points of } C'_\lambda \ . \tag{33}$$

Arguing by contradiction as in [33, p.311], we infer that (33) holds for any $\lambda < \ell$. Otherwise, we could find some $\lambda \in (0, \ell)$ such that either

$$u \geq v^\lambda \quad \text{in } C'_\lambda \qquad \text{and} \qquad u = v^\lambda \quad \text{at some interior point of } C'_\lambda \tag{34}$$

or

$$u > v^\lambda \quad \text{at interior points of } C'_\lambda \qquad \text{and} \qquad u_\nu = v^\lambda_\nu \quad \text{at some interior point of } T_\lambda \cap \Omega \ . \tag{35}$$

But (34) is ruled out by Lemma 13 whereas (35) is ruled out by Hopf boundary Lemma. Thus (33) holds for all $\lambda < \ell$ and, by continuity, we have $u \geq v^\ell$ in C'_ℓ. Put $w := u - v^\ell$ in C'_ℓ. If we invoke again Lemma 13, we obtain

$$\text{either} \qquad w > 0 \text{ in } C'_\ell \qquad \text{or} \qquad w \equiv 0 \text{ in } C'_\ell \ . \tag{36}$$

Step 3: vanishing of the function w.
Let us exclude the first alternative in (36). Assume for contradiction that $w > 0$ in C'_ℓ. We study separately the two possible situations (i)-(ii) for the limit cap C'_ℓ.

Case (i). This is the only part of the proof where we need to distinguish between assumptions (a) and (b). In any case, we are going to show that

$$w_\nu(P) \geq 0 \quad \text{and} \quad w(P) = 0. \tag{37}$$

Indeed, about u we know that
– under assumption (a), we have that $u_\nu(P) = -|\nabla u(P)| \geq -c$ (recall that $\max_{\partial\Omega} |\nabla u| = c$) and $u(P) = 0$ (because the Dirichlet condition holds on all $\partial\Omega$).
– under assumption (b), we have that $u_\nu(P) \geq -|\nabla u(P)| = -c$ (because the Neumann condition holds on all $\partial\Omega$) and $u(P) \leq 0$ (recall that $\max_{\partial\Omega} u = 0$).
Concerning v^ℓ, since P is the reflected of a point in Γ, we have $v^\ell(P) = 0$ and $v^\ell_\nu(P) = -c$. Summarizing, we have shown that in any case $w_\nu(P) \geq 0$ and $w(P) \leq 0$; but since $w > 0$ in C'_ℓ we readily infer that $w(P) = 0$. This completes the proof of (37).

Next, we note that w satisfies in C'_ℓ the (linear) equation $-\Delta w + a(x)w = 0$, where $a(x)$ is a bounded function given by the mean value Theorem, namely $a(x) = -f'(\tau(x))$ for a suitable $\tau(x) \in (v^\ell(x), u(x))$. Therefore, setting $z(x) = e^{-bx_1} w(x)$ for $b > 0$, we find that z satisfies

$$z > 0 \quad \text{and} \quad -\Delta z - 2b\frac{\partial z}{\partial x_1} = [b^2 - a(x)]z \geq 0 \qquad \text{in } C'_\ell$$

provided b is large enough. Hence, by Hopf's boundary Lemma (recall that $z(P) = 0$) we infer that $z_\nu(P) < 0$. Returning to w, we then obtain $w_\nu(P) < 0$, contradicting (37).

Case (ii). Denote by $Q \in \partial\Omega \cap T_\ell$ a point where $\partial\Omega$ and T_ℓ intersect orthogonally. By (32), we know that $Q \in \Gamma$. Moreover, by (8), we know that $u_\Omega \in C^2(\Omega \cup \Gamma)$. Therefore, the very same arguments used in [33, p.307] show that w vanishes of second order at Q. This contradicts Lemma 14.

In both cases (i)-(ii) we reached a contradiction. Hence the second alternative in (36) occurs.

Step 4: conclusion.
Since $u \equiv v^\ell$ in C'_ℓ, recalling (32), we infer that u solves the following *totally* overdetermined problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } C_\ell \cup C'_\ell \\ u = 0 \quad \text{and} \quad u_\nu = -c & \text{on } \partial\big(C_\ell \cup C'_\ell\big) \ . \end{cases}$$

By the implicit function Theorem as done in the last part of the proof of Theorem 1, we get

$$\partial\big(C_\ell \cup C'_\ell\big) = \partial\Omega \ ;$$

here in case of assumption (a) we use the condition $c > 0$, and in case of assumption (b) we use the condition f nonincreasing which gives the analogous of inequality (19).
By (32) and since Γ is open, there exists $\varepsilon > 0$ such that

$$\partial\big(C_\ell \cup C'_\ell\big) \cap \{x_1 < \ell + \varepsilon\} \subset \Gamma \ ;$$

then, by the regularity of Γ in (8), there holds $\partial\Omega \in C^{2,\alpha}$. By Theorem 17, Ω is a ball and u is radially symmetric. □

Remark 15 By inspection of the proof, one sees that the geometric assumptions made on Γ may be relaxed as follows. Let T and γ be as in Definition 5, and let ω be the bounded open domain delimited by γ and T: instead of asking that $\omega \subset \Omega$, it is enough to ask that $\omega \cap T_\lambda \neq \emptyset$ for $\lambda > 0$ sufficiently small.

5.6 Proof of Theorem 8

We use the moving planes method with some modifications with respect to [31]. Similarly as for Theorem 7, we divide the proof into four steps.

Step 1: definition of the critical plane T_ℓ.
We move the plane T to the limit parallel position T_0 where the hat $\gamma \subset \Gamma$ determined by T is tangent to T_0 and entirely contained into the closed strip between T_0 and T.
We may assume that $T_0 = \{x_1 = 0\}$, that $\Omega \subset \{x_1 > 0\}$ and that the tangency point is the origin O. Then, we start moving T_0 towards T into the planes $T_\lambda = \{x_1 = \lambda\}$ for $\lambda \geq 0$: starting from $\lambda = 0$, we increase slightly λ so that T_λ intersects Ω close to O. For any sufficiently small λ, we denote by C_λ the open cap bounded by T_λ and by $\gamma \cap \{x_1 < \lambda\}$. We also denote by Ω^λ the reflection of Ω with respect to T_λ and we put $\Sigma_\lambda = \{x_1 < \lambda\} \setminus \Omega^\lambda$. Clearly, $C_\lambda \subset \Omega^\lambda$ for sufficiently small λ; more precisely, we will have $C_\lambda \subset \Omega^\lambda$ until one of the following two facts occurs:
(i) C_λ becomes internally tangent to $\partial\Omega^\lambda$ (which is C^1) at some point $P \notin T_\lambda$.
(ii) T_λ reaches a position where it is orthogonal to $\partial\Omega$ (and to $\partial\Omega^\lambda$) at some point Q, which necessarily belongs to the closure of the strip between T_0 and T.
Let $\ell > 0$ be the smallest value of λ for which either (i) or (ii) occurs.

By construction, there holds

$$\big(\partial C_\ell \cap \partial\Omega\big) \subset \Gamma \ . \tag{38}$$

Step 2: definition and one sign property of the function w.
We introduce a new function v defined in Σ_ℓ by the formula $v(x) = u(x^\ell)$, where x^ℓ is the reflected of x across T_ℓ. Then we set

$$w(x) := v(x) - u(x) \ , \qquad x \in \Sigma_\ell \ .$$

We point out that w is well-defined in Σ_ℓ. Indeed, since $\partial\Omega \setminus T$ has only two connected components, the inclusion $C_\lambda \subset \Omega^\lambda$ holding for $\lambda \leq \ell$ implies that $\{x_1 < \ell\} \cap \Omega \subseteq \Omega^\ell$. Therefore any point x of Σ_ℓ is outside Ω.
Next we observe that the function w satisfies in Σ_ℓ the equation

$$-\Delta w + a(x) w = 0 \ , \tag{39}$$

being $a(x) = -f'(\tau(x)) \leq 0$, with $\tau(x)$ between $u(x)$ and $v(x)$. Moreover, there holds $w \geq 0$ on $\partial\Sigma_\ell$: indeed $w = 0$ on T_ℓ, and $w \geq 0$ on $\partial\Omega^\ell \cap \{x_1 < \ell\}$ because of the assumption $u < 1$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Since we also have $w \to 0$ at infinity, we deduce from the maximum principle that $w \geq 0$ in Σ_ℓ.
By Lemma 13, we obtain

$$\text{either} \qquad w > 0 \text{ in } \Sigma_\ell \qquad \text{or} \qquad w \equiv 0 \text{ in } \Sigma_\ell \ . \tag{40}$$

Step 3: vanishing of the function w.
Let us exclude the first alternative in (40). Assume for contradiction that $w > 0$ in Σ_ℓ. We study separately the two possible situations (i)-(ii).

Case (i). We claim that

$$w_\nu(P) \leq 0 \quad \text{and} \quad w(P) = 1. \tag{41}$$

Indeed, about u we know that, since $P \in \Gamma$, there holds $u(P) = 1$ and $u_\nu(P) = -c$. About v we know that
– under assumption (a), we have that $v(P) = 1$ (because the Dirichlet condition holds on all $\partial\Omega$) and $v_\nu(P) = -|\nabla u(P^\ell)| \leq -c$ (recall that $\min_{\partial\Omega} |\nabla u| = c$).
– under assumption (b), we have that $v(P) = u(P^\ell) \leq 1$ (by continuity, since we assumed $u < 1$ in $\mathbb{R}^n \setminus \overline{\Omega}$), and that $v_\nu(P) \geq -|\nabla u(P^\ell)| = -c$ (because the Neumann condition holds on all $\partial\Omega$).
Now recall that w satisfies in Σ_ℓ the equation (39). Therefore, setting $z(x) = e^{-bx_1} w(x)$ for $b > 0$, we find that z satisfies

$$z > 0 \quad \text{and} \quad -\Delta z - 2b\frac{\partial z}{\partial x_1} = [b^2 - a(x)]z \geq 0 \qquad \text{in } \Sigma_\ell$$

provided b is large enough. Hence, by Hopf's boundary Lemma (recall that $z(P) = 0$) we infer that $z_\nu(P) > 0$. Returning to w, we then obtain $w_\nu(P) > 0$, contradicting (41).

Case (ii). Denote by $Q \in \partial\Omega \cap T_\ell$ a point where $\partial\Omega$ and T_ℓ intersect orthogonally. By (38), we know that $Q \in \Gamma$. Moreover, by (8), we know that u_Ω is C^2 up to Γ. Therefore, the very same arguments used in [31, p.389] show that w vanishes of second order at Q. This contradicts Lemma 14.

In both cases (i)-(ii) we reached a contradiction. Hence the second alternative in (40) occurs.

Step 4: conclusion.

Since we have $u \equiv v$ in Σ_ℓ, u is symmetric with respect to T_ℓ. This implies first that $u = 1$ on $\partial\Omega^\ell \cap \{x_1 < \ell\}$, which, by the assumption $u < 1$ in $\mathbb{R}^n \setminus \overline{\Omega}$, yields

$$\Big(\partial\Omega^\ell \cap \{x_1 < \ell\}\Big) \subseteq \Gamma \ .$$

Second, we get that both the conditions $u = 1$ and $|\nabla u| = c$ hold on $\partial\Omega \cap \{x_1 > \ell\}$. Hence u solves a totally overdetermined exterior problem. By the regularity of Γ in (8), we deduce that $\partial\Omega \in C^{2,\alpha}$. Hence Theorem 18 applies, so Ω is a ball, and u is radially symmetric. □

6 Appendix: Slight refinements of some known results

In this section we collect a number of known results slightly refined according to our needs.

We first recall that on the overdetermined part of the boundary there is a gain of regularity (in both the cases of interior and exterior problems):

Theorem 16 (Vogel [37])
Let $\mathcal{U} \subset \mathbb{R}^n$ be an open domain and let $\Gamma \in C^1$ be a nonempty connected and relatively open subset of $\partial\mathcal{U}$. Let $f \in C^1(\mathbb{R})$. Assume that there exists a solution $u \in C^2(\mathcal{U}) \cap C^1(\mathcal{U} \cup \Gamma)$ of (1) satisfying $u = 0$ and $|\nabla u| = c > 0$ on Γ. Then, $\Gamma \in C^{2,\alpha}$ and $u \in C^2(\mathcal{U} \cup \Gamma)$.

Proof. In view of the assumed regularity of u, we see that [37, (1.7)] is satisfied as $x \to \Gamma$. Since $c > 0$, we know that u is of one sign in a neighbourhood of Γ inside $\mathcal{U}$. Hence, all the assumptions of [37, Theorem 1] are satisfied with $\partial\mathcal{U}$ replaced by Γ, and the $C^{2,\alpha}$ regularity of Γ follows by observing that the proof in [37] is local. Finally, we deduce that $u \in C^2(\mathcal{U} \cup \Gamma)$ by standard elliptic regularity. □

Then, we restate Serrin's and Reichel's symmetry results in the following generalized versions:

Theorem 17 (Serrin [33])
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary and let $f \in C^1(\mathbb{R})$. Assume that there exists a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ to problem (1) − (2) with $c > 0$. Then, Ω is a ball and u is radially symmetric.

Proof. We only emphasize the two differences with respect to [33, Theorem 2]. First, thanks to Theorem 16, we may assume that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ in place of $u \in C^2(\overline{\Omega})$. Second, we replace the assumption that $u > 0$ in Ω with the assumption that $c > 0$: the latter condition enables to prove (16) in [33] and to start the moving plane procedure. The remaining of the proof works as in [33].□

Theorem 18 (Reichel [31])
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary and let $f \in C^1(\mathbb{R})$ be a nonincreasing function. Assume that there exists a solution $u \in C^2(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^n \setminus \Omega)$ to problem (5) with $\Gamma = \partial\Omega$ and $u < 1$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Then, Ω is a ball and u is radially symmetric.

Proof. We only emphasize the differences with respect to [31, Theorem 1]. First, similarly as above, Theorem 16 allows us to assume that $u \in C^2(\mathbb{R}^n \setminus \overline{\Omega}) \cap C^1(\mathbb{R}^n \setminus \Omega)$ in place of $u \in C^2(\mathbb{R}^n \setminus \Omega)$. Further, we assume neither that $\nabla u \to 0$ at infinity, nor that $u \geq 0$. Indeed, according to the remark following [31, Theorem 1], under our assumptions the vanishing condition for ∇u at infinity

is not needed, and the the proof can be reduced to steps I, II, VI, where one can check that the assumption $u \geq 0$ is not used. □

Finally, we give a refined statement concerning the simplicity of Dirichlet eigenvalues:

Theorem 19 (Henrot-Oudet [19])
Let $\Omega \subset \mathbb{R}^n$ be a convex domain minimizing the second Dirichlet eigenvalue $\lambda_2(\Omega)$ of $-\Delta$ among convex domains of given measure. Assume that $\partial\Omega$ contains a part Γ which is nonempty, relatively open in $\partial\Omega$, connected, strictly convex and of class $C^{1,1}$. Then $\lambda_2(\Omega)$ is simple.

Proof. Assume for contradiction that $\lambda_2(\Omega)$ is not simple. Then, under our assumptions, [19, Lemma 1] remains true: it is enough to repeat the same proof, by taking care that the vector field V constructed therein is chosen with support contained into Γ. Then such lemma, combined with the simplicity of the first Dirichlet eigenvalue, shows that there exists a convex domain $\widetilde{\Omega}$ with the same area as Ω and with $\lambda_2(\widetilde{\Omega}) < \lambda_2(\Omega)$, contradiction. □

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