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de Branges spaces and characteristic operator function: the quaternionic case

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0.1 Abstract

This work inserts in the very fruitful study of quaternionic linear operators. This study is a generalization of the complex case, but the noncommutative setting of quaternions shows several interesting new features, see e.g. the so-called S-spectrum and S-resolvent operators. In this work, we study de Branges spaces, namely the quaternionic counterparts of spaces of analytic functions (in a suitable sense) with some specific reproducing kernels, in the unit ball of quaternions or in the half space of quaternions with positive real parts. The spaces under consideration will be Hilbert or Pontryagin or Krein spaces. These spaces are closely related to operator models that are also discussed. We also introduce a notion of the characteristic operator function of a bounded linear operator A with finite real part and we address several questions like the study of J-contractive functions, where J is self-adjoint and unitary, and we also treat the inverse problem namely to characterize which J-contractive functions are characteristic operator functions of an operator. In particular, we prove the counterpart of Potapov's factorization theorem in this framework. Besides other topics, we also consider canonical differential equations in the setting of slice hyperholomorphic functions. We define the lossless inverse scattering problem in the present setting. We also consider the inverse scattering problem associated to canonical differential equations. These equations provide a convenient unifying framework to discuss a number of questions pertaining, for example, to inverse scattering, non-linear partial differential equations and are studied in the last section of this paper.
AMS Classification: 46E22; 47S10; 30G35 ; 46C20
Key words: de Branges Rovnyak spaces, inverse scattering, quaternionic analysis, operator models

Contents

Chapter 1

Introduction

1.1 Foreword

The problem of determining the invariant subspaces of a linear closed operator is one of the crucial problems in operator theory. Working on Hilbert spaces the spectral theorem for normal operators is one of the most important achievements of the last century that characterizes the operator and give a complete reduction theory. Even though there has been a lot for works regarding the problem of extending the reduction theory to non-normal linear operators, still a lot of problems are unsolved nowadays.

In order to give an advanced spectral analysis to a number of non-selfadjoint operators one has to extend the reduction theory to non-normal operators in a Hilbert space and to operators in Banach spaces; this has been done with the theory of spectral operators, see [52]. An important contribution to the reduction theory of a non normal operators was the introduction of the characteristic operator function introduced by Livsic.

In the quaternionic setting things are much more complicated since the appropriate notion, namely the S-spectrum of a quaternionic linear operator, was introduced 70 years after the paper of Birkhoff and von Neumann on the logic of quantum mechanics that was published in 1936. Moreover, the spectral theorem for quaternionic normal linear operators (bounded or unbounded) was proved in 2015 and appeared in the literature in 2016. We recall that, if T is a bounded linear quaternionic operator then the S-spectrum is defined as

$$\sigma_S(T) = \{s \in \mathbb{H} \quad : \quad T^2 - 2\mathrm{Re}(s)T + |s|^2 I \quad \text{is not invertible}\},$$

while the S-resolvent set is

$$\rho_S(T) := \mathbb{H} \setminus \sigma_S(T).$$

Let us restrict to that case when T is a bounded normal quaternionic linear operator on a quaternionic Hilbert space $\mathcal{H}$. Then there exist three quaternionic

linear operators A, J, B such that $T = A + JB$, where A is self-adjoint and B is positive, J is an anti self-adjoint partial isometry (called imaginary operator). Moreover, A, B and J mutually commute.
Let us set $\mathbb{C}_j^+ = \{u + jv, (u,v) \in \mathbb{R} \times \mathbb{R}^+\}$, for $j \in \mathbb{S}$, where $\mathbb{S}$ is the unit sphere of purely imaginary quaternions. So the spectral theorem is as follows. There exists a unique spectral measure E_j on $\sigma_S(T) \cap \mathbb{C}_j^+$ so that for any slice continuous intrinsic function $f = f_0 + f_1 j$ and $x, y \in \mathcal{H}$

$$\langle f(T)x, y\rangle = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} f_0(q)\, d\langle E_j(q)x, y\rangle + \int_{\sigma_S(T)\cap\mathbb{C}_j^+} f_1(q)\, d\langle JE_j(q)x, y\rangle. \tag{1.1}$$

With the spectral theorem and the S-functional calculus, that is the quaternionic analogue of the Riesz-Dunford functional calculus it turned out to be clear that to replace complex spectral theory with quaternionic spectral theory we have to replace the classical spectrum with the S-spectrum.

The first direction of research of operator theory in the quaternionic setting, beyond the spectral theorem based on the S-spectrum, was done in the recent long paper [59], where the author studies quaternionic spectral operators. The present paper considers a different avenue, and begins the investigation of the quaternionic characteristic operator function. In classical operator theory the notion of resolvent operator plays a key role. More precisely, let T be a possibly unbounded linear operator acting on a Hilbert space $\mathcal{H}$. The resolvent operator $R(z) = (T - zI_{\mathcal{H}})^{-1}$ is an operator-valued function, analytic on the resolvent set $\rho(T)$ of T, assumed non-empty, and its properties and those of T are closely related. Characteristic operator functions are possibly simpler analytic functions, built on $R(z)$, and still allowing to deduce properties of the operator from properties of the functions. The characteristic operator function associated to a close-to-unitary operator originates with the work of Livsic; see [78]. Properties and applications of the characteristic operator function of an operator which is close-to-selfadjoint are discussed in particular in the book [45].

The purpose of this work is to study the characteristic operator function and related topics in the quaternionic setting; we focus on the case of a close to anti-selfadjoint operator. We first recall some definitions from the classical complex case.

1.2 The complex numbers setting

Let T be a bounded linear operator in a Hilbert space $\mathcal{H}$, with finite dimensional imaginary part, meaning that the operator $\frac{T-T^*}{2i}$ has a finite dimensional everywhere defined extension (say of rank n), which we write as

$$\frac{T - T^*}{2i} = KJK^*, \tag{1.2}$$

where $J \in \mathbb{C}^{n\times n}$ is both self-adjoint and unitary (i.e. is a signature matrix) and where K is a linear bounded operator from $\mathbb{C}^n$ into $\mathcal{H}$. The characteristic operator function of the operator T is then defined by

$$W(z) = I - 2iK^*(T - zI)^{-1}KJ, \tag{1.3}$$

see [45]. Note that the imaginary part of T may be infinite dimensional in [45], while the present work focuses on the finite dimensional case.

The function W is analytic in $\mathbb{C}\setminus\sigma(T)$, where we denoted by $\sigma(T)$ the spectrum of T; it is J-expansive in the open upper half-plane $\mathbb{C}_+$ and J-contractive in the open lower half-plane $\mathbb{C}_-$, namely

$$\begin{aligned} W(z)^*JW(z) &\geq J, \quad z \in \rho(T)\cap\mathbb{C}_+, \\ W(z)^*JW(z) &\leq J, \quad z \in \rho(T)\cap\mathbb{C}_-. \end{aligned} \tag{1.4}$$

Note that often, for a given J, one considers functions J-contractive rather than J-expansive in $\mathbb{C}_+$.

The study of the relationships between the properties of the function $W(z)$ and of the operator T leads to a number of important problems, of which we mention in particular:

- Relate the spectrum of T and the singularities of W.
- Relate factorizations of W and invariant subspaces of T.
- Inverse problem: when a J-contractive function is the characteristic operator of some operator?
- The indefinite setting case, where the Hilbert space is replaced by a Pontryagin space, or possibly by a Krein space.

Operator models are closely related to Hilbert (and Pontryagin) spaces of analytic functions of different kind, and were introduced in a series of work by de Branges and de Branges and Rovnyak, see e.g. [41, 42, 43, 44]. In particular spaces with reproducing kernel of one of the following forms (and their counterparts with denominator equal to $1 - z\overline{w}$) play an important role:

(a) $\mathcal{H}(A, B)$ spaces, with reproducing kernel

$$\frac{A(z)A(w)^* - B(z)B(w)^*}{z + \overline{w}}, \tag{1.5}$$

where A and B are $\mathbb{C}^{n\times n}$-valued and analytic in some open subset of the open right half-plane $\mathbb{C}_r$, with $\det A(z) \not\equiv 0$. When $S = A^{-1}B$ extends to an inner function (i.e. the boundary values are almost everywhere unitary), one has that $\mathcal{H}(A, B) = A\left(\mathbf{H}_2(\mathbb{C}_r) \ominus S\mathbf{H}_2(\mathbb{C}_r)\right)$, and the operator of multiplication by the variable is an Hermitian operator. The description of its self-adjoint extensions is a problem of interest and with, for instance, applications to interpolation problems (see [11] for the latter). It is sometimes easier for the arguments to rewrite (1.5) as

$$\frac{E_+(z)E_+(w)^* - E_-(z)E_-(w)^*}{z + \overline{w}}, \tag{1.6}$$

with

$$E_+(z) = \frac{A(z) + B(z)}{\sqrt{2}} \quad \text{and} \quad E_-(z) = \frac{A(z) - B(z)}{\sqrt{2}}.$$

(b) $\mathcal{H}(\Theta)$ spaces, with reproducing kernel

$$\frac{J - \Theta(z)J\Theta(w)^*}{z + \overline{w}}, \tag{1.7}$$

where Θ is $\mathbb{C}^{2n\times 2n}$-valued and analytic in some open subset of the open right half-plane. Let

$$J_1 = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \tag{1.8}$$

and set $J = J_1$. Multiplying the left side of (1.7) by $\begin{pmatrix} I_n & 0 \end{pmatrix}$ and by the transpose of this matrix on the left, we get a kernel of the form

$$-\frac{\Theta_{11}(z)\Theta_{12}(w)^* + \Theta_{12}(z)\Theta_{11}(w)^*}{z + \overline{w}}$$

which is of the form (1.5) with

$$A(z) = \frac{\Theta_{11}(z) - \Theta_{12}(z)}{\sqrt{2}} \quad \text{and} \quad B(z) = \frac{\Theta_{11}(z) + \Theta_{12}(z)}{\sqrt{2}}.$$

The same argument can be made by multiplying the left side of (1.7) by $\begin{pmatrix} 0 & I_n \end{pmatrix}$ and by the transpose of this matrix on the left. Also here we get a kernel of the form (1.5). In a number of cases there is a natural isometry between the two spaces, which is a generalization of the map sending orthogonal polynomials of the first kind to orthogonal polynomials of the second kind.

(c) $\mathcal{L}(\Phi)$ spaces, with reproducing kernel

$$\frac{\Phi(z) + \Phi(w)^*}{z + \overline{w}}, \tag{1.9}$$

where Φ is $\mathbb{C}^{n\times n}$-valued and analytic in some open subset of the open right half-plane (the associated reproducing kernel Hilbert spaces are models for self-adjoint operators and pairs of self-adjoint operators; see [43, 31]).

$\mathcal{L}(\Phi)$ spaces and $\mathcal{H}(\Theta)$ spaces are related by linear fractional transformations. More precisely, assume that in (1.7) we have $J = J_1$. A theorem of de Branges and Rovnyak, see [43], states that the map

$$F \mapsto \begin{pmatrix} \Phi & I_n \end{pmatrix} F$$

is a contraction from $\mathcal{H}(\Theta)$ into $\mathcal{L}(\Phi)$ if and only if one can write

$$\Phi = (\Theta_{22}\varphi - \Theta_{12})(\Theta_{11} - \Theta_{12}\varphi)^{-1},$$

where φ is analytic and has a real positive part in the left open half-plane (directions in which φ is identically equal to ∞ have to be suitably interpreted). To avoid this problem one can rewrite the above linear fractional transformation as

$$\Phi = (\Theta_{22}(I_n + s) - \Theta_{21}(I_n - s))(\Theta_{11}(I_n - s) - \Theta_{12}(I_n + s))^{-1} \tag{1.10}$$

where s is analytic and contractive in $\mathbb{C}_r$; see [43, p. 306].

Definition 1.2.1. *The problem of finding all the linear fractional expressions* (1.10) *associated to a given function Φ is called the lossless inverse scattering problem (LISP), and allows to put under a common setting a wide range of questions. It was studied in particular in [20, 21]. When Θ is entire, it is the inverse spectral problem studied, in particular, in the books [36, 58].*

The term *lossless* in the above definition refers to the fact that Θ is usually assumed J-inner and is the chain-scattering matrix-function of a lossless system. See [4] for more information.

Remarks 1.2.2. (1) It is not true that any $\mathcal{H}(A, B)$-space is the upper part of a $\mathcal{H}(\Theta)$ space. See [20, Theorem 3.1 p. 600] for such an embedding result for subspaces of $\mathbf{L}_2(d\mu)$, where $d\mu$ is a positive measure on the unit circle or on the real line, and see Theorem 3.4.3 below for its quaternionic counterpart.
(2) We also note that finite dimensional $\mathcal{H}(A, B)$ spaces of polynomials are related to the Gohberg-Heinig and Christoffel-Darboux formulas; see Section 2.2.
(3) Finally, we remark that all the kernels defined above are of the form

$$\frac{X(z)\Sigma X(w)^*}{\rho_w(z)} \tag{1.11}$$

where Σ is a signature matrix, X is a matrix-valued analytic function of appropriate size and $\rho_w(z)$ is equal to either $z + \overline{w}$ or $1 - z\overline{w}$. The general theory of such spaces (for more general denominators) was initiated in [20] and we refer to [25, 54, 57] for further information.

A related important notion is that of canonical differential expressions. These are ordinary differential equations of the form

$$\frac{\mathrm{d}G}{\mathrm{d}t}(t, z) = izJH(t)G(t, z), \tag{1.12}$$

where z is a complex parameter, H is a given $\mathbb{C}^{2n\times 2n}$-valued function, $J \in \mathbb{C}^{2n\times 2n}$ is a signature matrix and the unknown G is a $\mathbb{C}^{2n\times m}$-valued function for some $m \in \mathbb{N}$. A simpler family of canonical differential expressions are differential operators of the form

$$iJ\frac{\mathrm{d}F}{\mathrm{d}t}(t, z) = (zI_{2n} + V(t))\,F(t, z), \tag{1.13}$$

where

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \quad \text{and} \quad V(t) = \begin{pmatrix} 0 & v(t) \\ v(t)^* & 0 \end{pmatrix},$$

and where the $\mathbb{C}^{n\times n}$-valued function v is called the potential. To see the connection between (1.12) and (1.13) consider the solution T of the equation

$$iJ\frac{\mathrm{d}T}{\mathrm{d}t}(t) = V(t)T(t)$$

where V is as in (1.13). Let F be a solution of (1.13) and define G by $F = TG$. Then,

$$iJ(T'G + TG') = (zI_{2n} + V)TG$$

and so G satisfies (1.12) with H defined by

$$H = -JT^{-1}JT.$$

It is not true that, conversely, any equation (1.12) leads to an equation (1.13) in such a way.

Associated to these expressions there are a number of functions of z (among which we mention the scattering function, the Weyl function and the spectral function). Direct problems consist in computing these functions when H (or V in case (1.13)) are given, while inverse problems consist of recovering H (or V) from one of these functions. The study of these expressions form a convenient framework to study a wide range of problems, including non-linear partial differential equations. See [86].

Remark 1.2.3. The connection between the theory of canonical differential equations and the notion of characteristic function follows in particular from the fact that the solution to (1.13) subject to the initial condition $F(0,z) = I$ is J-expansive.

The multiplicative structure of matrix-valued functions J-contractive in the open unit disk was given by Potapov in [80]. More precisely, he proved that any function Θ which is J-contractive in the unit disk can be written in a unique way (up to multiplicative constant factors) as a product

$$\Theta(z) = \Theta_1(z)\Theta_2(z)\Theta_3(z) \tag{1.14}$$

where Θ_1 is a (possibly infinite) Blaschke product analytic in $\mathbb{D}$, Θ_2 is a (possibly infinite) Blaschke product analytic in $|z| > 1$ and Θ_3 is J-contractive and with no zeros nor poles in $\mathbb{D}$. Both Θ_1 and Θ_2 are J-unitary on the unit circle. More precisely, let $b_a(z) = \dfrac{z-a}{1-z\overline{a}}$, where $a \in \mathbb{C}$ of modulus different from 1. The function Θ_1 (resp. Θ_2) is a (possibly infinite) product of terms of the form

$$\theta_{a,u}(z) = I + \left(\frac{b_a(z)}{b_a(1)} - 1\right)\frac{uu^*J}{u^*Ju} \tag{1.15}$$

where $|a| < 1$ and $u \in \mathbb{C}^n$ is such that $u^*Ju > 0$ (resp. $|a| > 1$ and $u \in \mathbb{C}^n$ is such that $u^*Ju < 0$), and called (normalized) Blaschke-Potapov factors of the first and second kind, respectively. Factors of the second kind will appear if and only if J has negative eigenvalues. In case of a rational J-unitary Θ, the function Θ_3 is a finite product of Blaschke-Potapov factors of the third kind, or Brune sections. When normalized to be identity at the point $z = 1$ (and in particular not to have a pole there), these are functions of the form

$$\left(I + e\frac{z+a}{z-a}uu^*J\right)\left(I + e\frac{1+a}{1-a}uu^*J\right)^{-1} = I + \frac{2e}{1-a}\frac{1-z}{z-a}uu^*J, \tag{1.16}$$

where now $|a| = 1$, $e > 0$ and $u^*Ju = 0$. We will refer to Θ_3 as to the singular factor. The function $\Theta_3(z)$ is expressed in terms of a multiplicative integral in the form

$$\Theta_3(z) = \overset{\frown}{\int_0^{\ell}} \exp\left(\frac{z+e^{i\theta(t)}}{z-e^{i\theta(t)}}dE(t)\right) \tag{1.17}$$

where $\theta(t)$ is increasing and $E(t)J$ is Hermitian, increasing, and normalized by imposing $\mathrm{Tr}E(t)J = 1$.

We will refer to (1.14) as to the Potapov decomposition of the given function Θ.

We recall that the multiplicative integral $\overset{\frown}{\int_0^{\ell}} e^{f(t)dt}$ is defined as the limit of the products

$$\overset{\frown}{\prod_{j=0}^{k-1}} e^{f(s_j)(t_{j+1}-t_j)} = \overset{\frown}{\prod_{j=0}^{k-1}} (I + f(s_j)(t_{j+1}-t_j))$$

where $t_0 = 0 < t_1 < \cdots < t_k = \ell$ is a partition of $[0, \ell]$ and $s_j \in [t_j, t_{j+1}]$. The limit exists when f is continuous. We refer to the Appendix in Potapov's paper [80] for the basics on multiple integrals. Of particular importance is the following differential equation (see [80, Theorem p. 241]) satisfied by a multiple integral. Let $M(s) = \overset{\frown}{\int_0^{s}} e^{f(t)dt}$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}s}M(s) = M(s)f(s).$$

Potapov's decomposition leads to:

Theorem 1.2.4. *An entire function Θ which is J-inner in the open upper half-plane can be written as a multiplicative integral of the form*

$$\Theta(z) = \overset{\frown}{\int_0^{\ell}} e^{-izH(t)dt}\Theta(0) \tag{1.18}$$

where $H(t)$ is integrable and such that $H(t)J \geq 0$ on $[0, \ell]$.

Furthermore, de Branges proved that H is unique when $n = 2$. We refer to [45] and to the book of Arov and Dym [37] for uniqueness conditions when $n > 2$. We conclude this section with the following consequence of Theorem 1.2.4.

Corollary 1.2.5. *An entire function Θ J-inner in the open upper half-plane is of finite exponential type.*

1.3 The quaternionic setting

In [14] we defined the characteristic function for quaternionic linear operators. However, in [14] we considered A instead of A^* in (1.20) below, but in view of the connections with canonical differential systems (see Section 10.1) it is more convenient to use the present definition. The notations in use will be explained in the sequel. Note also the symmetry between A and A^* in (1.19) below. In the classical case, the operator and its adjoint have an anti-symmetric role rather than a symmetric role: replacing T by T^* in (1.2) changes J to $-J$.

Definition 1.3.1. *Let A be a continuous right linear operator in a right quaternionic space, with finite dimensional real part (say of rank n), and write*

$$A + A^* = -C^*JC, \tag{1.19}$$

where $J \in \mathbb{H}^{n\times n}$ is both self-adjoint and unitary, and C is linear bounded from the quaternionic Hilbert space $\mathcal{H}$ into $\mathbb{H}^n$. The function

$$S(p) = I_n - pC^* \star (I - pA^*)^{-\star}CJ \tag{1.20}$$

is called the characteristic operator function of A.

In this paper we wish to address the following questions in the setting of linear operators on quaternionic Hilbert spaces:

- What are the J-contractive functions.
- What is the analogue, if any, of (1.18).
- Which J-contractive functions are characteristic operator functions (inverse problem).
- How to associate to a given operator a canonical differential expression.
- What is the operator associated to a canonical differential expression (inverse problem).
- What is now the lossless inverse scattering problem (see Definition 1.2.1).

Note that usually we will take J with real entries; in particular we set

$$J_0 = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}, \tag{1.21}$$

where I_n (often denoted by I) denotes the identity matrix of order n.

Remark 1.3.2. The point of views and methods in the present work and in our book [18] are completely different. There, the emphasis was on the notion of realization, and a key role was played by a result of Shmul'yan [93] on extensions of linear relations in Pontyragin spaces. In the present work the emphasis is on spaces themselves and on their connections to underlying problems such as the lossless inverse scattering problem (see Definition 9.2.3 for the latter).

Remark 1.3.3. We note that there several substantial differences between the complex and the quaternionic case. First of all, linear spaces and linear operators can be considered on the left or on the right. The two cases are somewhat equivalent but different. Moreover, we replace the imaginary line by the real line and the (complex) open upper half-plane by the (quaternionic) right half-space. Furthermore analytic functions and rational functions are replaced by slice hyperholomorphic functions and rational slice hyperholomorphic functions, and the pointwise product of analytic functions is replaced by the so-called $\star$-product defined in the next section.

Remark 1.3.4. A recurring theme in the paper, and in some of our previous works, is the following: we are given a $\mathbb{H}^{n\times n}$-valued function $K(p,q)$, positive definite in an axially symmetric open subset of the open unit ball of the quaternion, and slice hyperholomorphic on the left in p and on the right in $\overline{q}$ there. We restrict to $\Omega\cap\mathbb{R}$ and consider $p=x$ and $q=y$ real, then we apply the map χ (see (2.2)) to reduce to the case of matrix-valued functions whose entries are complex. For the quaternionic kernel $K(p,q)$, the obtained kernel is of the form

$$A(x)\frac{V(x)+V(y)^*}{1-xy}A(y)^*, \tag{1.22}$$

where A is, say $\mathbb{C}^{2n\times 2n}$-valued, analytic and invertible, and V is *defined* on $\Omega\cap\mathbb{R}$. Assume that the kernel is positive definite in $\Omega\cap\mathbb{R}$. Loewner's theorem (see [51, Theorem 1, p. 95]) or arguments using function theory in the Hardy space (see [4]) will allow us to assert that V extends to a function analytic in the open unit disk, and with a positive real part there. This allows us to use methods of complex analysis to study the original kernel $K(p,q)$, and the case where (1.22) is replaced by

$$A(x)\frac{V(x)+V(y)^*}{x+y}A(y)^*.$$

A weaker fact holds when the above kernels are assumed to have a finite number of negative squares rather than being positive definite in $\Omega\cap\mathbb{R}$. The fact that the kernel has a finite number of negative squares will not imply analyticity. One then resort to a result of Krein and Langer (see [74, 19], and [16] for the quaternionic case) to obtain a slice hyperholomorphic extension from a given open set.

This work consists of ten sections, this introduction being the first. Sections 2-4 may be seen of a preliminary nature, although they also contain some new

material. In Section 2 we recall some facts on quaternions, quaternionic matrices and quaternionic functional analysis. The main aspects needed in this work on the theory of slice hyperholomorphic functions as well as on the S-resolvent operators and the S-spectrum are recalled in Section 3. The original part in the section consists in the study of slice hyperholomorphic weights, both in the case of the unit ball and of the half space. A key tool is a map, denoted by ω, which allows to rewrite the values of a quaternionic valued function in terms of 2×2 matrices with complex entries.
The theory of slice hyperholomorphic rational functions and their symmetries is considered in Section 4. Operator models in the sense of Rota are studied in Section 5. In Section 6 we consider quaternionic $\mathcal{H}(A, B)$ spaces and we provide the counterparts of various results in this framework, including the operator of multiplication in the half-space case and in the unit ball case and the study of the reproducing kernels. The case of J-contractive functions is presented in Section 7. The characteristic operator function is defined and studied in Section 8, where we also provide examples and we discuss inverse problems. Some classes of functions with a positive real part in the half-space or the unit ball are studied in Section 9. Finally, Section 10 is devoted to the canonical differential systems in the quaternionic setting, also those associated to an operator and, in particular, we study the matrizant.

Chapter 2

Quaternions, matrices, and functional analysis

This section contains some basic knowledge on quaternions, Toeplitz and Hankel matrices and we introduce some useful maps which allow to consider, instead of quaternionic matrices, complex matrices of double size. We also recall the notions of Pontryagin and Krein spaces which will be useful in the sequel.

2.1 Quaternions

The set of quaternions, denoted by $\mathbb{H}$, consists of the elements of the form $p = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, where the three imaginary units $\mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}.$$

The sum and the product of two quaternions $p = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $q = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$ are defined by

$$\begin{aligned} p + q &= (x_0 + y_0) + (x_1 + y_1)\mathbf{i} + (x_2 + y_2)\mathbf{j} + (x_3 + y_3)\mathbf{k}, \\ pq &= (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3) + (x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)\mathbf{i} + \\ &\quad + (x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1)\mathbf{j} + (x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)\mathbf{k}, \end{aligned}$$

and with these operations, $\mathbb{H}$ turns out to be a skew field. Given a quaternion p as above, its conjugate is defined to be $\bar{p} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$. The modulus (or norm) of a quaternion is given by the Euclidean norm, i.e.

$$|p| = \sqrt{p\bar{p}} = \sqrt{\bar{p}p} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

Given $p = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, its real (or scalar) part x_0 will be denoted also by $\mathrm{Re}(p)$ while $x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is the imaginary (or vector) part of p, denoted

also by $\mathrm{Im}(p)$.
Let

$$\mathbb{S} = \{p = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}$$

be the set of unit purely imaginary quaternions. It is a 2-dimensional sphere in $\mathbb{H}$ identified with $\mathbb{R}^4$. Any element $j \in \mathbb{S}$ satisfies $j^2 = -1$ and thus will be called imaginary unit and the set

$$\mathbb{C}_j = \{z = x + jy,\ x, y \in \mathbb{R}\}$$

is a complex plane.
Given any $p = x + iy \in \mathbb{H}$, we define the 2-sphere associated with it and denoted by $[p]$:

$$[p] = \{x + ky \ :\ k \in \mathbb{S}\},$$

and we note that if $p \in \mathbb{R}$, then $p = x$ and $[p]$ contains only the element p. For the sequel, it is useful to note that the 2-sphere is defined by the second degree equation

$$x^2 - 2\mathrm{Re}(p)x + |p|^2 = 0, \tag{2.1}$$

in fact x is a root of this polynomial if and only if $x \in [p]$.
To any nonreal quaternion $p = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, one associates the imaginary unit j_p defined by $j_p = \frac{\mathrm{Im}(p)}{|\mathrm{Im}(p)|}$ and thus $p \in \mathbb{C}_{j_p}$.
Assume to fix the imaginary units $i, j, k \in \mathbb{S}$ such that they form a new basis for $\mathbb{H}$. Then, a quaternion p can be written in the form $p = z_1 + z_2 j$ with

$$z_1 = x_0 + ix_1 \text{ and } z_2 = x_2 + ix_3 \ \in \mathbb{C},$$

where we identify $\mathbb{C}$ with the subset of $\mathbb{H}$ given by the elements of the form $x + iy$, $x, y \in \mathbb{R}$.
Let $\chi :\ \mathbb{H} \to \mathbb{C}^{2\times 2}$ be the map

$$\chi(p) = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}. \tag{2.2}$$

To be precise, the map χ should be denoted by χ_i as it has values in $\mathbb{C}^{2\times 2}$, but for the sake of simplicity we denote it χ. This map allows to translate problems from the quaternionic to the complex matricial setting, since it is an injective homomorphism of rings, i.e.

$$\chi(p + q) = \chi(p) + \chi(q), \qquad \chi(pq) = \chi(p)\chi(q).$$

The map χ can be extended to matrices in at least two ways. Let $M \in \mathbb{H}^{m\times n}$, $M = [m_{\ell k}]$, and write $M = A + Bj$ we can extend the map χ by

$$\chi(M) = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \quad \text{or} \quad \chi_1(M) = (\chi(m_{\ell k}))\,.$$

Both maps carry the same additive and multiplicative properties. This is well-know for χ. We now prove the result for χ_1

Lemma 2.1.1. *Let M and N be two matrices with quaternionic entries and of compatible sizes. Then*

$$\begin{aligned}\chi_1(M+N)&=\chi_1(M)+\chi_1(N)\\ \chi_1(MN)&=\chi_1(M)\chi_1(N)\\ \chi_1(M^*)&=\chi_1(M)^*.\end{aligned}\tag{2.3}$$

Proof. To simplify the notation we write the proof for square matrices $M, N \in \mathbb{H}^{u\times u}$. The general case is proved in the same way. By definition, $\chi(M) = (\chi(m_{jk})_{j,k=1,\ldots,u})$ and $\chi_1(N) = (\chi(n_{k\ell}))_{j,k=1,\ldots,u}$. Thus

$$\begin{aligned}(\chi_1(M+N))_{jk}&=\chi(m_{jk}+n_{jk})\\ &=\chi(m_{jk})+\chi(n_{jk})\\ &=(\chi_1(M))_{jk}+(\chi_1(N))_{jk},\quad j,k=1,\ldots u,\end{aligned}$$

and

$$\begin{aligned}(\chi_1(MN))_{j\ell}&=\chi((MN)_{j\ell})\\ &=\chi(\sum_{k=1}^{u}(m_{jk}n_{k\ell}))\\ &=\sum_{k=1}^{u}\chi(m_{jk})\chi(n_{k\ell})\\ &=(\chi_1(M)\chi_1(N))_{k\ell}\,.\end{aligned}$$

This proves the first and second equality in (2.3). The proof of last equality is omitted. $\square$

The previous result shows that χ_1 shares the same properties as the more classical map χ. The following two properties will be of key importance (see the lemma below and Sections 2.2 and 2.3), since they hold for χ_1 but they are not satisfied by χ. Recall that a matrix $T \in (\mathbb{H}^{u\times u})^{v\times v}$ is called a block Toeplitz matrix is a matrix constant on the block diagonals, while it is called a block Hankel matrix if it is constant on the block anti-diagonals. In other words,

$$T=(T_{j-k})_{j,k=1,\ldots,v}\quad\text{and}\quad H=(H_{j+k-1})_{j,k=1,\ldots,v}$$

where the matrices $T_{-v},\ldots,T_v$ and $H_1,\ldots,H_{2v-1}$ belong to $\mathbb{H}^{u\times u}$.

Lemma 2.1.2. *Assume that $M \in (\mathbb{H}^{u\times u})^{v\times v}$ is a block Toeplitz (resp. block Hankel) matrix. Then the $\mathbb{C}^{2uv\times 2uv}$ matrix $\chi_1(M)$ is block-Toeplitz (resp. block-Hankel). Let $Z_n \in \mathbb{H}^{2u\times 2u}$ be defined by*

$$Z_v=\begin{pmatrix}0 & I_u & 0 & \ldots & 0 & 0\\ 0 & 0 & I_u & 0 & \ldots & 0\\ & & & \ddots & & \\ 0 & 0 & 0 & \ldots & I_u & 0\\ 0 & 0 & 0 & \ldots & 0 & 0\end{pmatrix}.$$

Then

$$\chi_1(Z_v) = \begin{pmatrix} 0 & I_{2u} & 0 & \dots & 0 & 0 \\ 0 & 0 & I_{2u} & 0 & \dots & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & I_{2u} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} = Z_{2uv}. \tag{2.4}$$

Proof. The proof easily follows using standard arguments. □

Hermitian non-degenerate Toeplitz and Hankel matrices appear in the finite dimensional theory of $\mathcal{H}(A,B)$ spaces that will be treated in Section 6. For this reason, we introduce some basic facts about these matrices.

2.2 Toeplitz matrices

Let $T = (T_{j-k})_{j,k=0}^N$ be a Hermitian Toeplitz block matrix with blocks in $\mathbb{H}^{u\times u}$. It holds that

$$T - ZTZ^* = C^* J_0 C \tag{2.5}$$

where

$$J_0 = \begin{pmatrix} I_u & 0 \\ 0 & -I_u \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} I_u & 0 & \cdots & 0 \\ \frac{T_0}{2} & T_1 & \cdots & T_{N-1} \end{pmatrix}.$$

Note that (2.5) is a special case of the Stein equation (6.12), and that C^*JC expresses the displacement rank of T with respect to Z. See [39, 47, 68, 69, 89, 90] for more on displacement ranks and structured matrices, and [23, 24, 26, 27, 56] for a study of these topics using reproducing kernel methods.

The Gohberg-Heinig (see [64, 65]) formula to invert block Toeplitz matrices holds for general algebras and, in particular, in the present setting and since the map χ_1 keeps the underlying structure we can work in the complex setting. The first step is to consider the equations

$$\sum_{k=0}^N a_{j-k}x_k = \delta_{0j}I_u \tag{2.6}$$

$$\sum_{k=0}^N a_{k-j}z_{-k} = \delta_{0j}I_u \tag{2.7}$$

$$\sum_{k=0}^N w_k a_{j-k} = \delta_{0j}I_u \tag{2.8}$$

$$\sum_{k=0}^N y_{-k}a_{k-j} = \delta_{0j}I_u \tag{2.9}$$

where the unknowns are in $\mathbb{H}^{u\times u}$. Assuming that they are solvable, one has $x_0 = y_0$ and $z_0 = w_0$. Note that (x_k) is the first block column of A^{-1} while

(z_{-k}) is the first block column of $SA^{-1}S$. Similarly (w_k) is the first block row of A^{-1} while (y_{-k}) is the first block column of $SA^{-1}S$, where

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 & I_u \\ 0 & 0 & \cdots & I_u & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & I_u & \cdots & & \\ I_u & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.10}$$

AS we mentioned, the Gohberg-Heinig formula gives:

Theorem 2.2.1. *Assume that the equations* (2.6)-(2.9) *are solvable, and that x_0 or z_0 is invertible. Then the other is also invertible, T is invertible and one has the formula*

$$\begin{aligned} T^{-1} = & \begin{pmatrix} x_0 & 0 & \cdots & 0 \\ x_1 & x_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_N & x_{N-1} & \cdots & x_0 \end{pmatrix} x_0^{-1} \begin{pmatrix} y_0 & y_{-1} & \cdots & y_{-N} \\ 0 & y_0 & \cdots & y_{1-N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_0 \end{pmatrix} - \\ & - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ z_{-N} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ z_{-1} & \cdots & z_{-N} & 0 \end{pmatrix} z_0^{-1} \begin{pmatrix} 0 & w_N & \cdots & w_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_N \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned} \tag{2.11}$$

Theorem 2.2.2. *It holds that*

$$T^{-1} - Z_u^* T^{-1} Z_u = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix} x_0^{-1} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}^* - \begin{pmatrix} 0 \\ z_0 \\ \vdots \\ z_N \end{pmatrix} z_0^{-1} \begin{pmatrix} 0 \\ z_0 \\ \vdots \\ z_N \end{pmatrix}^* \tag{2.12}$$

and one has the Christoffel-Darboux formula

$$\sum_{a,b=0}^{N} x^a y^b (T^{-1})_{ab} = \frac{Q_N(x) x_0^{-1} Q_N(y)^* - xy P_N(x) z_0^{-1} P_N(y)^*}{1 - xy} \tag{2.13}$$

where

$$P_N(x) = \gamma_{0N}^{(N)} + z\gamma_{1N}^{(N)} + \cdots + z^N \gamma_{NN}^{(N)} \tag{2.14}$$

and

$$Q_N(x) = \gamma_{00}^{(N)} + z\gamma_{10}^{(N)} + \cdots + z^N \gamma_{N0}^{(N)},$$

where $(\gamma_{ab}^{(N)})_{a,b=0}^N$ is the block entry decomposition of T_N^{-1} into $\mathbb{H}^{u\times u}$ blocks.

Proof. The proof is a consequence of Theorem 2.2.1 and mimics the one in the complex case, see [28, Section 4]. □

Remark 2.2.3. Endow the space of matrix-valued polynomials of the form

$$p(z) = \sum_{a=0}^{N} p_a z^a, \quad p_0, \ldots, p_N \in \mathbb{H}^{u\times u}$$

with the possibly indefinite inner product

$$\langle p, q\rangle = \begin{pmatrix} q_0^* & q_1^* & \cdots q_N^* \end{pmatrix} T_N \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_N \end{pmatrix}, \quad q(z) = \sum_{a=0}^{N} q_a z^a, \quad q_0, \ldots, q_N \in \mathbb{H}^{u\times u}.$$

We see that

$$\langle P_N, q\rangle = 0$$

for every polynomial $q(z) = \sum_{a=0}^{N} q_a z^a$ for which $q_N = 0$ since the coefficients of P_N are the entries of the last (block) column of T_N^{-1}. This justifies the terminology *orthogonal polynomial* for P_N, even in the non positive case. In the positive case, the entries of T_N are moments of a positive measure, and one gets back the classical definition of orthogonal polynomials.

In the complex case, a theorem of Krein characterizes the distribution of the zeros of P_N with respect to the unit circle in terms of the signature of T_N; see [73]. It has been extended to the matrix-valued case in the papers [28, 67].

Krein's theorem is a particular case of the following result, when T is assumed to be a Toeplitz matrix. See [53, 55] for more general results. The quaternionic counterpart of this result appears in Section 6.5. See Theorem 6.5.1.

Theorem 2.2.4. *Let $T \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix, with $\nu \geq 0$ negative eigenvalues. Assume furthermore that*

$$\begin{pmatrix} 1 & z & \ldots & z^n \end{pmatrix} T^{-1} \begin{pmatrix} 1 & \overline{w} & \ldots & \overline{w}^n \end{pmatrix}^t = \frac{A(z)\overline{A(w)} - z\overline{w}B(z)\overline{B(w)}}{1 - z\overline{w}} \tag{2.15}$$

where A and B are polynomials of degree n. Then, A has ν zeros inside $\mathbb{D}$ and B has ν zeros outside $\mathbb{D}$. They have no zeros on the unit circle.

Proof. We split the proof in several steps.

STEP 1: *The number of negative squares of the kernel*

$$\begin{pmatrix} 1 & z & \ldots & z^n \end{pmatrix} T^{-1} \begin{pmatrix} 1 & \overline{w} & \ldots & \overline{w}^n \end{pmatrix}^t$$

is equal to the number of negative eigenvalues of T.

This follows from the spectral theorem for Hermitian matrices.

STEP 2: *A and B have no common zeros.*

Indeed, a common zero, say z_0, will be such that

$$\begin{pmatrix}1 & z_0 & \cdots & z_0^n\end{pmatrix} T^{-1} \begin{pmatrix}1 & \overline{w} & \cdots & \overline{w}^n\end{pmatrix}^t = 0, \quad \forall w \in \mathbb{C},$$

and hence we would have

$$\begin{pmatrix}1 & z_0 & \cdots & z_0^n\end{pmatrix} T^{-1} = \begin{pmatrix}0 & 0 & \cdots & 0\end{pmatrix},$$

contradicting the invertibility of T.

STEP 3: *The polynomials A and B have no zeros on the unit cirlce.*

Assume by contractiction that $A(z_0) = 0$, with $|z_0| = 1$ (the same argument would work for B). Rewritting formula (2.15) as

$$(1 - z\overline{w}) \begin{pmatrix}1 & z & \ldots & z^n\end{pmatrix} T^{-1} \begin{pmatrix}1 & \overline{w} & \ldots & \overline{w}^n\end{pmatrix}^t = A(z)\overline{A(w)} - z\overline{w}B(z)\overline{B(w)} \tag{2.16}$$

and setting $z = w = z_0$ implies that $B(z_0) = 0$, which cannot be by the previous step.

STEP 4: *The function $A^{-1}B$ is a generalized Schur function and can be written as*

$$A^{-1}B = S_1 S_2^{-1}$$

where S_1 is a Blaschke product of degree $n - \nu$ and S_2 is a Blaschke product of degree ν, without common zeros.

The fact that $A^{-1}B$ is a generalized Schur functions follows from the definition. The second part of the assertion follows from elementary facts on rational functions, but it follows also from the Krein-Langer's theorem.

STEP 5: *We conclude the proof.*

We set

$$S_1(z) = \prod_{k=1}^{n-\nu} \frac{z - w_k}{1 - z\overline{w_k}} \quad \text{and} \quad S_2(z) = \prod_{j=1}^{\nu} \frac{z - v_j}{1 - z\overline{v_j}}.$$

Write $A(z)S_1(z) = B(z)S_2(z)$, i.e.

$$A(z) \prod_{k=1}^{n-\nu} \frac{z - w_k}{1 - z\overline{w_k}} = B(z) \prod_{j=1}^{\nu} \frac{z - v_j}{1 - z\overline{v_j}}$$

or

$$A(z) \prod_{j=1}^{\nu} (1 - z\overline{v_j}) \prod_{k=1}^{n-\nu} (z - w_k) = zB(z) \prod_{j=1}^{\nu} (z - v_j) \prod_{k=1}^{n-\nu} (1 - z\overline{w_k}),$$

and assume that $A(w) = 0$ with $w \in \mathbb{D}$. We note that the v_j's and the w_k's may be repeated. Since A and B have no common zeros and $A(0) \neq 0$, we have $S_2(w) = 0$, and A has ν zeros inside $\mathbb{D}$. The other $n-\nu$ zeros correspond to the poles of S_1. Suppose now that $S_1(\nu) = 0$. Since S_1 and S_2 have no common zeros we have that $B(\nu) = 0$. So B has $n-\nu$ zeros inside $\mathbb{D}$. The other ν zeros correspond to the poles of S_1. $\square$

2.3 Hankel matrices

Hermitian Hankel matrices are automatically real valued; so the case of interest is that of block Hankel matrices $H = (H_{ab})_{a,b=0}^N$, where each $H_{j+k} \in \mathbb{H}^{u\times u}$ is self-adjoint. Let S be as in (2.10) Then,

$$T = HS = \left(H_{N+|a-b|}\right)_{a,b=0}^N . \tag{2.17}$$

is a (non-Hermitian) Toeplitz matrix, to which the Gohberg-Heinig formula is applicable.

It follows from (2.17) that

$$\begin{aligned} H^{-1} = &\begin{pmatrix} x_n & x_{n-1} & \cdots & x_0 \\ x_{n-1} & \cdots & x_0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_0 & 0 & \cdots & 0 \end{pmatrix} x_0^{-1} \begin{pmatrix} y_0 & y_{-1} & \cdots & y_{-n} \\ 0 & y_0 & \cdots & y_{1-n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_0 \end{pmatrix} - \\ &- \begin{pmatrix} z_{-1} & \cdots & z_{-n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ z_{-n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} z_0^{-1} \begin{pmatrix} 0 & w_n & \cdots & w_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_n \\ 0 & 0 & \cdots & 0 \end{pmatrix} . \end{aligned} \tag{2.18}$$

Similarly to Theorem 2.2.4, one has the following result which holds in particular for Hankel matrices.

Theorem 2.3.1. *Let $T \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix, with $\nu \geq 0$ negative eigenvalues. Assume furthermore that*

$$\begin{pmatrix} 1 & z & \ldots & z^n \end{pmatrix} T^{-1} \begin{pmatrix} 1 & \overline{w} & \ldots & \overline{w}^n \end{pmatrix}^t = \frac{A(z)\overline{A(w)} - B(z)\overline{B(w)}}{z + \overline{w}} \tag{2.19}$$

where A and B are polynomials of degree n. Then, A has ν zeros inside $\mathbb{C}_r$ and B has ν zeros outside $\mathbb{C}_r$. They have no zeros on the imaginary axis.

2.4 Functional analysis

We will need the quaternionic counterpart of classical results from functional analysis; these can be found in the works [15, 17, 18]. Some of our results

are in the Pontryagin space setting, so we will recall the notion of quaternionic Pontryagin space and quaternionic reproducing kernel Pontryagin space. In this section we only briefly recall some of the definitions, to set the framework and the notation. Notions and results related to the spectrum of an operator in a quaternionic vector space involve the notion of slice hyperholomorphic functions, and are postponed to the next section.
Let $\mathcal{V}$ be a right vector space on $\mathbb{H}$ namely a linear space over $\mathbb{H}$ where the scalar are multiplied on the right. A map $T : \mathcal{V} \to \mathcal{V}$ is said to be a right linear operator if

$$T(u+v) = T(u) + T(v), \qquad T(up) = T(u)p, \quad \text{for all } p \in \mathbb{H},\ u, v \in \mathcal{V}.$$

Definition 2.4.1. *Let $\mathcal{V}$ be a right quaternionic vector space. The $\mathbb{H}$-valued map $(h, k) \mapsto [h, k]$ is called a Hermitian form if it satisfies the following conditions for all $u, v, w \in \mathcal{V}$ and $p, q \in \mathbb{H}$:*

$$\begin{aligned} [u, v + w] &= [u, v] + [u, w], && (2.20)\\ [u, v] &= \overline{[v, u]}, && (2.21)\\ [up, vq] &= \overline{q}[u, v]p. && (2.22) \end{aligned}$$

When one endows $\mathcal{V}$ with a two-sided quaternionic structure one requires moreover that

$$[pu, v] = [u, \overline{p}v]. \tag{2.23}$$

We will call such a form a (possibly degenerate and non-positive) inner product.

Definition 2.4.2. *Let $\mathcal{V}$ be a right quaternionic vector space and let $[\cdot, \cdot]$ be an associated Hermitian form. The pair $(\mathcal{V}, [\cdot, \cdot])$ is called a Krein space if it can be written as a direct and orthogonal sum*

$$\mathcal{V} = \mathcal{V}_+ \dot{+} \mathcal{V}_-,$$

where $(\mathcal{V}_+, [\cdot, \cdot])$ and $(\mathcal{V}_-, -[\cdot, \cdot])$ are quaternionic right Hilbert spaces. It is called a Pontryagin space if $\mathcal{V}_-$ is finite dimensional in one (and hence all) fundamental decompositions,

The quaternionic versions of the closed-graph theorem and the inverse mapping theorem (see e.g. the proof of Proposition 10.3.2) will be needed later; we refer the reader to [18]. Functional analysis in Pontryagin spaces can be seen as a "finite dimensional perturbation" of functional analysis in Hilbert space. For instance an important result, which does not extend to the Krein space is that the adjoint of a contraction between two quaternionic Pontryagin spaces of same index is still a contraction, see [18, Theorem 5.7.10].

For the sake of completeness, we recall the definition of a reproducing kernel Pontryagin space.

Definition 2.4.3. *The Pontryagin space* $(\mathcal{P}, [\cdot, \cdot])$ *of* $\mathbb{H}^n$*-valued functions defined on some set* Ω *is called a reproducing kernel Pontryagin space if there exists a* $\mathbb{H}^{n\times n}$*-valued function* $K(a, b)$*, called the reproducing kernel, and with the following properties:*
(1) *The function* $K_b : a \mapsto K(a, b)h$ *belongs to* $\mathcal{P}$ *for every choice of* $b \in \Omega$ *and* $h \in \mathbb{H}^n$*.*
(b) *With* b *and* h *as above, it holds that*

$$h^* f(b) = [f, K_b h] \tag{2.24}$$

for every $f \in \mathcal{P}$*.*

An example of a finite dimensional reproducing kernel Pontryagin space can be built from the kernel in (2.13), in fact that kernel has a finite number of negative squares.

We note that there is a one-to-one correspondence between quaternionic reproducing kernel Pontryagin spaces and Hermitian kernels which can be written as difference of two positive kernels, one being of finite rank. Definition 2.4.3 extends to Krein spaces. A necessary and sufficient condition for a function to be the reproducing kernel of a quaternionic reproducing kernel Krein space is that it can written as a difference of two positive definite functions. The associated reproducing kernel Krein space will not be unique in general; see [91, Section 13] and [3].

Chapter 3

Slice hyperholomorphic functions

Functions of a quaternionic variable with properties generalizing holomorphicity from the complex to the quaternionic setting can be defined in various different ways. The notion of hyperholomorphicity which looks more suitable for the applications to operator theory is the so-called slice hyperholomorphicity. In this section we recall some basic facts related to this function theory.

3.1 Slice hyperholomorphic functions

In the development of slice hyperholomorphic functions we follow our books [18] and [46] but for the scalar valued case see also [61]. There are different ways of defining slice hyperholomorphic functions. Here we follow the definition related with the Fueter mapping theorem (which is a construction giving Fueter regular functions starting from holomorphic functions), which is also a simplified version of the approach used in [63, 62] where the authors makes use of the so-called stem functions.

Definition 3.1.1. *Let $U \subseteq \mathbb{H}$. We say that U is* axially symmetric *if $[q] \subset U$ for any $q \in U$.*
We say that U is a slice domain (s-domain for short) if $U \cap \mathbb{R}$ is non empty and if $U \cap \mathbb{C}_j$ is a domain in $\mathbb{C}_j$ for all $j \in \mathbb{S}$.

Definition 3.1.2 (Slice functions)**.** *Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : u + jv \in U \text{ for some } j \in \mathbb{S}\}$. A function $f : U \to \mathbb{H}$ is called a left slice function, if it is of the form*

$$f(q) = f_0(u,v) + jf_1(u,v) \qquad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{H}$ that satisfy the compatibility conditions

$$f_0(u,-v) = f_0(u,v), \qquad f_1(u,-v) = -f_1(u,v). \tag{3.1}$$

A function $f : U \to \mathbb{H}$ is called a right slice function if it is of the form

$$f(q) = f_0(u, v) + f_1(u, v)j \qquad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{H}$ that satisfy (3.1).
If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called intrinsic.

Definition 3.1.3 (Slice hyperholomorphic functions)**.** *Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + jv \in U \text{ for some } j \in \mathbb{S}\}$. Let $f : U \to \mathbb{H}$ be a left slice function*

$$f(q) = f_0(u, v) + jf_1(u, v) \qquad \text{for } q = u + jv \in U.$$

If f_0 and f_1 satisfy the Cauchy-Riemann-equations

$$\frac{\partial}{\partial u} f_0(u, v) - \frac{\partial}{\partial v} f_1(u, v) = 0 \tag{3.2}$$

$$\frac{\partial}{\partial v} f_0(u, v) + \frac{\partial}{\partial u} f_1(u, v) = 0, \tag{3.3}$$

then f is called left slice hyperholomorphic. If f is a right slice function

$$f(q) = f_0(u, v) + f_1(u, v)j \qquad \text{for } q = u + jv \in U$$

and f_0 and f_1 satisfy the Cauchy-Riemann-equation (3.2)*, then f is called right slice hyperholomorphic.*
We denote the sets of left and right slice hyperholomorphic functions on U by $\mathcal{SH}_L(U)$ and $\mathcal{SH}_R(U)$, respectively. The set of intrinsic slice hyperholomorphic functions on U will be denoted by $\mathcal{N}(U)$.

A fundamental property of slice functions is the following structure formula.

Theorem 3.1.4 (The Structure Formula (or Representation Formula))**.** *Let $U \subset \mathbb{H}$ be axially symmetric and let $i \in \mathbb{S}$. A function $f : U \to \mathbb{H}$ is a left slice function on U if and only if for any $q = u + jv \in U$*

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}ji\Big[f(\overline{z}) - f(z)\Big] \tag{3.4}$$

with $z = u + iv$. A function $f : U \to \mathbb{H}$ is a right slice function on U if and only if for any $q = u + jv \in U$

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}\Big[f(\overline{z}) - f(z)\Big]ij \tag{3.5}$$

with $z = u + iv$.

The pointwise multiplication of two slice (hyperholomorphic) functions is, in general, a function of the same type. Thus we need a suitable notion of multiplication:

Definition 3.1.5. *Let $U \subset \mathbb{H}$ be an axially symmetric open set. If f, g are slice functions in U with $f(q) = f_0 + jf_1$ and $g = g_0 + jg_1$ for $q = u + jv \in U$, we define their left slice product as*

$$f \star_l g := f_0 g_0 - f_1 g_1 + j\left(f_0 g_1 + f_1 g_0\right). \tag{3.6}$$

In particular, the $\star_l$-product, in short $\star$-product, is defined for $f, g \in \mathcal{SH}_L(U)$ and $f \star_l g \in \mathcal{SH}_L(U)$.
If f, g are right slice functions with $f(q) = f_0(u,v) + f_1(u,v)j$ and $g(q) = g_0(u,v) + g_1(u,v)j$ for $q = u + jv \in U$, we define their right slice product as

$$f \star_r g := f_0 g_0 - f_1 g_1 + \left(f_0 g_1 + f_1 g_0\right) j. \tag{3.7}$$

In particular, the $\star_r$-product is defined for $f, g \in \mathcal{SH}_R(U)$ and $f \star_r g \in \mathcal{SH}_R(U)$.

Remark 3.1.6. We note that if $f \in \mathcal{N}(U)$ and $g \in \mathcal{SH}_L(U)$, then $f \star g = g \star f = fg \in \mathcal{SH}_L(U)$ and similarly when $g \in \mathcal{SH}_R(U)$.

We are also in need of the two definitions below:

Definition 3.1.7. *If $f \in \mathcal{SH}_L(U)$ with $f(q) = f_0 + jf_1$ we define*

- *the conjugate $f^c = \overline{f_0} + j\overline{f_1}$,*
- *the symmetrization $f^s = f \star_l f^c = f^c \star_l f$, which is given by*

$$f^s = |f_0|^2 - |f_1|^2 + 2jRe(f_1\overline{f_2}).$$

If $f \in \mathcal{SH}_R(U)$ with $f(q) = f_0 + f_1 j$ we define

- *the conjugate $f^c = \overline{f_0} + \overline{f_1}j$,*
- *the symmetrization $f^s = f \star_r f^c = f^c \star_r f$, which is given as above.*

Lemma 3.1.8. *The slice hyperholomorphic inverse is given in the following result.*

- *Let $f \in \mathcal{SH}_L(U)$ be non identically zero, then the left slice hyperholomorphic inverse $f^{-\star_l}$ given by*

  $$f^{-\star_l} = (f^s)^{-1} \star_l f^c = (f^s)^{-1} f^c$$

  is defined on $U \setminus \{s \in U \ : \ f(s) = 0\}$ and satisfies $f^{-\star_l} \star_l f = f \star_l f^{-\star_l} = 1$.

- *Let $f \in \mathcal{SH}_R(U)$ be non identically zero, then the right slice hyperholomorphic inverse $f^{-\star_r}$ given by*

  $$f^{-\star_r} = f^c \star_r (f^s)^{-1} = f^c (f^s)^{-1}$$

  is defined on $U \setminus \{s \in U \ : \ f(s) = 0\}$ and satisfies $f^{-\star_r} \star_r f = f \star_r f^{-\star_r} = 1$.

- *When $f \in \mathcal{N}(U)$ is non identically zero, then $f^{-\star_r} = f^{-\star_l} = f^{-1}$.*

As a consequence of the Structure formula and the Residue Theorem, one can prove the Cauchy formulas with slice hyperholomorphic Cauchy kernels. These kernels are defined outside a 2-sphere, see (2.1).

Definition 3.1.9 (Slice hyperholomorphic Cauchy kernels)**.** *Let $q, s \in \mathbb{H}$ with $q \notin [s]$.*

- *The left Cauchy kernel $S_L^{-1}(s,q)$ is defined as*

$$S_L^{-1}(s,q) := -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}).$$

- *The right Cauchy kernel $S_R^{-1}(s,q)$ is defined as*

$$S_R^{-1}(s,q) := -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}.$$

Theorem 3.1.10 (The Cauchy formulas)**.** *Let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$ and set $ds_j = ds(-j)$. If f is a (left) slice hyperholomorphic function on a set that contains $\overline{U}$ then*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s), \qquad \text{for any} \quad q \in U. \tag{3.8}$$

If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, then

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,q), \qquad \text{for any} \quad q \in U. \tag{3.9}$$

These integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

3.2 The S-resolvent operators and the S-spectrum

Slice hyperholomorphic functions can be defined also for vector-valued functions. As it happens for the class of holomorphic functions there is the concept of strong and weakly slice hyperholomorphicity. Here we just need a readaptation of the quaternionic valued case previously introduced.

Definition 3.2.1 (Slice hyperholomorphic functions vector-valued)**.** *Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let*

$$\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : u + jv \in U,\ j \in \mathbb{S}\}.$$

A function $f : U \to \mathcal{X}_L$ with values in a quaternionic left Banach space $\mathcal{X}_L$ is called a left slice function, if is of the form

$$f(q) = f_0(u,v) + jf_1(u,v) \qquad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathcal{X}_L$ that satisfy the compatibility condition (3.1)*. If in addition f_0 and f_1 satisfy the Cauchy-Riemann-equations* (3.2)*, then f is called* strongly left slice hyperholomorphic.
A function $f : U \to \mathcal{X}_R$ with values in a quaternionic right Banach space is called a right slice function if it is of the form

$$f(q) = f_0(u,v) + f_1(u,v)j \qquad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathcal{X}_R$ that satisfy the compatibility condition (3.1)*. If in addition f_0 and f_1 satisfy the Cauchy-Riemann-equations* (3.2)*, then f is called* strongly right slice hyperholomorphic.

Functions with values in a quaternionic Banach algebra can be multiplied, adapting the definition in the scalar valued case:

Definition 3.2.2. *Let $U \subset \mathbb{H}$ be an axially symmetric open set and let $\mathcal{X}$ be a two-sided quaternionic Banach algebra. For two functions $f, g \in \mathcal{SH}_L(U, \mathcal{X})$ with $f(q) = f_0 + jf_1$ and $g = g_0 + jg_1$ for $q = u + jv \in U$, we define their left slice hyperholomorphic product as*

$$f \star_l g := f_0 g_0 - f_1 g_1 + j\left(f_0 g_1 + f_1 g_0\right). \tag{3.10}$$

For two functions $f, g \in \mathcal{SH}_R(U, \mathcal{X})$ with $f(q) = f_0(u,v) + f_1(u,v)j$ and $g(q) = g_0(u,v) + g_1(u,v)j$ for $q = u + jv \in U$, we define their right slice hyperholomorphic product as

$$f \star_r g := f_0 g_0 - f_1 g_1 + \left(f_0 g_1 + f_1 g_0\right) j. \tag{3.11}$$

Remark 3.2.3. It is immediate that the $\star_l$-product of two left-slice hyperholomorphic functions is again left slice hyperholomorphic and that the $\star_r$-product of two right slice hyperholomorphic functions is again right slice hyperholomorphic. If moreover $U = B_r(0)$, then f, g admit power series expansions. If f and g are left slice hyperholomorphic with $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ with $a_n, b_n \in \mathcal{X}$, then

$$(f \star_l g)(q) := \sum_{n=0}^{+\infty} q^n \left(\sum_{\ell=0}^{n} a_\ell b_{n-\ell}\right).$$

Similarly, if f and g are right slice hyperholomorphic with $f(q) = \sum_{n=0}^{+\infty} a_n q^n$ and $g(q) = \sum_{n=0}^{+\infty} b_n q^n$ with $a_n, b_n \in \mathcal{X}$, then

$$(f \star_r g)(q) := \sum_{n=0}^{+\infty} \left(\sum_{\ell=0}^{n} a_\ell b_{n-\ell}\right) q^n.$$

In the sequel, we will be in need of the following definition:

Definition 3.2.4. *Let Ω be an axially symmetric s-domain in $\mathbb{H}$. We say that a function $f : \Omega \to \mathbb{H}$ is slice hypermeromorphic in Ω if f is slice hyperholomorphic in $\Omega' \subset \Omega$ such that $(\Omega \setminus \Omega') \cap \mathbb{C}_i$ has no accumulation point in $\Omega \cap \mathbb{C}_i$ for $i \in \mathbb{S}$, and every point in $\Omega \setminus \Omega'$ is a pole.*

In the case of functions with values in a quaternionic Banach space, we have

Definition 3.2.5. *Let $\mathcal{X}$ be a two-sided quaternionic Banach space. We say that a function $f : \Omega \to \mathcal{X}$ is (weakly) slice hypermeromorphic if for any $\Lambda \in \mathcal{X}^*$ the function $\Lambda f : \Omega \to \mathbb{H}$ is slice hypermeromorphic in Ω.*

The crucial objects in quaternionic operator theory are the notion of S-spectrum and of S-resolvent set which replace the classical concept of spectrum and resolvent set.

Definition 3.2.6. *Let $\mathcal{X}$ be a two-sided quaternionic Banach space. Let $T \in \mathcal{B}(\mathcal{X})$. For $s \in \mathbb{H}$, we set*

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2 I.$$

The S-resolvent set *$\rho_S(T)$ of T is*

$$\rho_S(T) := \{s \in \mathbb{H} : \mathcal{Q}_s(T) \text{ is invertible in } \mathcal{B}(\mathcal{X})\},$$

while the S-spectrum *$\sigma_S(T)$ of T is*

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1} \in \mathcal{B}(\mathcal{X})$ is called the pseudo-resolvent *of T at s.*

Observe that if $T \in \mathcal{B}(\mathcal{X})$ then the sets $\rho_S(T)$ and $\sigma_S(T)$ are axially symmetric, in fact we have, see [46]:

Theorem 3.2.7. *(Structure of the S-spectrum) Let T be a linear operator acting on a quaternionic linear space and let $p = p_0 + jp_1 \in \sigma_S(T)$. Then all the elements of the sphere $[p_0 + jp_1]$ belong to $\sigma_S(T)$.*

Moreover, we can give the definition of S-spectral radius, see [46]:

Definition 3.2.8. *Let $\mathcal{H}$ be a quaternionic Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. We call S-spectral radius of T the nonnegative real number*

$$r_S(T) := \sup\{|s| \ : \ s \in \sigma_S(T)\}.$$

We have:

Theorem 3.2.9 (The S-spectral radius of T)**.** *Let $\mathcal{H}$ be a quaternionic Hilbert space, $T \in \mathcal{B}(\mathcal{H})$, and let $r_S(T)$ be its S-spectral radius. Then*

$$r_S(T) = \lim_{n\to\infty} \|T^n\|^{1/n}. \tag{3.12}$$

As a consequence of the fact that for slice hyperholomorphic functions there are two different Cauchy kernels the functional calculus based on slice hyperholomorphicity has two resolvent operators.

Definition 3.2.10. *Let $T \in \mathcal{B}(\mathcal{X})$. For $s \in \rho_S(T)$, we define the* left S-resolvent operator *as*
$$S_L^{-1}(s,T) = -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\, I),$$
and the right S-resolvent operator *as*
$$S_R^{-1}(s,T) = -(T - \overline{s}I)\mathcal{Q}_s(T)^{-1}.$$

Theorem 3.2.11 (The S-resolvent equation)**.** *Let $T \in \mathcal{B}(\mathcal{X})$ and let $s, q \in \rho_S(T)$ with $q \notin [s]$. Then the equation*

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(p,T) = \big[& \left(S_R^{-1}(s,T) - S_L^{-1}(q,T)\right) q \\ & -\overline{s}\left(S_R^{-1}(s,T) - S_L^{-1}(q,T)\right)\big] (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} \end{aligned} \tag{3.13}$$

holds true.

The left S-resolvent $S_L^{-1}(s,T)$ is a $\mathcal{B}(\mathcal{X})$-valued right-slice hyperholomorphic function of the variable s on $\rho_S(T)$. The right S-resolvent $S_R^{-1}(s,T)$ is a $\mathcal{B}(\mathcal{X})$-valued left-slice hyperholomorphic function of the variable s on $\rho_S(T)$. So we can define the S-functional calculus.

Remark 3.2.12. We point out two main differences with respect to the classical operator theory:
(i) if A is a complex linear operator on a complex Banach Y space the resolvent set and the spectrum are associated with the invertibility of the operator $\lambda I - A$ and the operator $(\lambda I - A)^{-1} : \rho(A) \to B(Y)$ is a holomorphic function with values in the set of all bounded linear operators $B(Y)$. In the quaternionic setting the S-resolvent set and the S-spectrum are associated the the invertibility of $\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2 I$. The the pseudo-resolvent operator
$$(T^2 - 2\mathrm{Re}(s)T + |s|^2 I)^{-1} : \rho_S(T) \to \mathcal{B}(\mathcal{X})$$
is not slice hyperholomorphic. The slice hyperholomorphicity is associated with the S-resolvent operators.
(ii) Finally we observe that the resolvent equation (3.13) contains both the resolvent operators.

Definition 3.2.13. *Let $T \in \mathcal{B}(\mathcal{X})$. We denote by $\mathcal{SH}_L(\sigma_S(T))$, $\mathcal{SH}_R(\sigma_S(T))$ and $\mathcal{N}(\sigma_S(T))$ the set of all left, right and intrinsic slice hyperholomorphic functions f with $\sigma_S(T) \subset \mathcal{D}(f)$.*

Definition 3.2.14 (The S-functional calculus)**.** *Let $T \in \mathcal{B}(\mathcal{X})$. For any function $f \in \mathcal{SH}_L(\sigma_S(T))$, we define*
$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s), \tag{3.14}$$

where j is an arbitrary imaginary unit and U is an arbitrary slice Cauchy domain. For any $f \in \mathcal{SH}_R(\sigma_S(T))$, we define

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T), \tag{3.15}$$

where j is an arbitrary imaginary unit and U is an arbitrary slice Cauchy domain.

The S-functional calculus is well defined because the above integrals do not depend on U and $j \in \mathbb{S}$.

Theorem 3.2.15. *Let $T \in \mathcal{B}(\mathcal{X})$. For any $f \in \mathcal{SH}_L(\sigma_S(T))$, the integral in* (3.14) *that defines the operator $f(T)$ is independent of the choice of the slice Cauchy domain U and the imaginary unit $j \in \mathbb{S}$. Similarly, for any $f \in \mathcal{SH}_R(\sigma_S(T))$, the integral in* (3.15) *that defines the operator $f(T)$ is also independent of the choice of U and $j \in \mathbb{S}$.*

Remark 3.2.16. Thanks to the functional calculus we can define functions of an operator T. In particular, we can define $(I - pT)^{-\star_r}$ using the function $(1 - pq)^{-\star_r}$ (where the $\star_r$ is computed with respect to the variable p). Note that for $p \neq 0$ we have

$$(I - pT)^{-\star_r} = p^{-1} S_R(p,T),$$

moreover

$$(I - pT)^{-\star_r} = \sum_{n \geq 0} p^n T^n \qquad \text{for} \quad |p| \|T\| < 1.$$

For the sake of simplicity, in the sequel we will write $(I - pT)^{-\star}$. This function is left slice hyperholomorphic in p. It is also interesting to note that $S_L(p,T) = (pI - T)^{-\star_l}$ and $S_R(p,T) = (pI - T)^{-\star_r}$, where both the $\star$-inverses are computed with respect to the variable p.

Some of the results that we mention in this section are stated for two-sided quaternionic Banach spaces, even though later we will mainly work with Hilbert spaces. Moreover, sometimes the Banach spaces under consideration are not two-sided. In the following proposition we recall an extension result, see [14, Proposition 3.24], which is valid in a more general setting and will be useful in the sequel:

Proposition 3.2.17. *Let A be a bounded linear operator from a right-sided quaternionic Banach $\mathcal{P}$ space into itself, and let G be a bounded linear operator from $\mathcal{P}$ into $\mathcal{Q}$, where $\mathcal{Q}$ is a two sided quaternionic Banach space. The slice hyperholomorphic extension of $G(I - xA)^{-1}$, $1/x \in \sigma_S(A) \cap \mathbb{R}$, is*

$$(G - \overline{p} GA)(I - 2\mathrm{Re}(p)\, A + |p|^2 A^2)^{-1}.$$

Remark 3.2.18. We also note that the Identity Principle, see [18], implies that two slice hyperholomorphic functions defined on an s-domain and with values in a two sided quaternionic Banach space $\mathcal{X}$ coincide if their restrictions to the real axis coincide. More in general, any real analytic function $f : [a, b] \subseteq \mathbb{R} \to \mathcal{X}$ can be extended to a function $\mathrm{ext}(f)$ slice hyperholomorphic on an axially symmetric s-domain Ω containing $[a, b]$. The fact that the extension exists is assured by the fact that for any $x_0 \in [a, b]$ the function f can be written as $f(x) = \sum_{n\geq 0} x^n A_n$, $A_n \in \mathcal{X}$, and x such that $|x - x_0| < \varepsilon_{x_0}$ and thus $(\mathrm{ext} f)(p) = \sum_{n\geq 0} p^n A_n$ for $|p - x_0| < \varepsilon_{x_0}$. Thus the claim holds setting $B(x_0, \varepsilon_{x_0}) = \{p \in \mathbb{H} : |p - x_0| < \varepsilon_{x_0}\}$ and $\Omega = \cup_{x_0\in[a,b]} B(x_0, \varepsilon_{x_0})$.

Remark 3.2.19. The function

$$k(p, q) = (\bar{p} + \bar{q})(|p|^2 + 2\mathrm{Re}(p)\bar{q} + \bar{q}^2)^{-1} \tag{3.16}$$

is slice hyperholomorphic in p and $\bar{q}$ on the left and on the right, respectively in its domain of definition, i.e. for $p \notin [\bar{q}]$. It is positive definite in the open half-space $\mathbb{H}_+$. Its associated reproducing kernel Hilbert space is the Hardy space $\mathbf{H}_2(\mathbb{H}_+)$ of the right half space of quaternions with positive real part.

The function $k(p, q) = \sum_{a=0}^{\infty} p^a \overline{q}^a$ is positive definite in the quaternionic unit open ball $\mathbb{B}_1$, with associated reproducing kernel Hilbert space $\mathbf{H}_2(\mathbb{B}_1)$. The corresponding space $\mathbf{H}_2(\mathbb{B}_1, \mathcal{H})$ of $\mathcal{H}$-valued functions (where $\mathcal{H}$ is a quaternionic Hilbert space) plays a key role in interpolation theory and model theory; see Proposition 5.1.3 for the latter.

3.3 The map ω_i and applications

Let f be a slice hyperholomorphic function and let i, j be a pair of orthogonal imaginary units. For $p = z \in \mathbb{C}_i$, and selecting an imaginary unit $j \in \mathbb{S}$ such that j is orthogonal to i, we can write the restriction of f to $\mathbb{C}_i$ in the form

$$f(z) = F(z) + G(z)j, \tag{3.17}$$

where F and G are $\mathbb{C}_i$-valued and also analytic from $\mathbb{C}_i$ into itself, since f is slice hyperholomorphic. We define (see [10, p. 400])

$$\omega_i(f)(z) = \begin{pmatrix} F(z) & G(z) \\ -\overline{G(\overline{z})} & \overline{F(\overline{z})} \end{pmatrix}. \tag{3.18}$$

We recall that

$$\omega_i(f \star g) = \omega_i(f)\omega_i(g) \tag{3.19}$$

and that ω_i coincides with the map χ_i when f is constant. In the sequel, when no confusion arises, we will write ω instead of ω_i.

Let now $K(p, q)$ be $\mathbb{H}^{n\times n}$-valued, positive definite in the axially symmetric domain Ω. Assume moreover that f is left slice hyperholomorphic in p and right slice hyperholomorphic in $\overline{q}$, and let $\mathcal{H}(K)$ denote the associated reproducing kernel Hilbert space. It is separable in view of the slice hyperholomorphicity. Let $e_1, e_2 \ldots$ be an orthonormal basis of $\mathcal{H}(K)$. One can write the reproducing kernel as

$$K(p,q) = \sum_{u=1}^{\infty} e_u(p) e_u(q)^*,$$

and a function f belongs to $\mathcal{H}(K)$ if and only if it can be written as

$$f(p) = \sum_{u=1}^{\infty} e_u(p) c_u \tag{3.20}$$

where $c_1, c_2 \ldots \in \mathbb{H}^n$ and are such that

$$\sum_{u=0}^{\infty} c_u^* c_u < \infty.$$

Applying the map ω to (3.20) we obtain

$$\omega(f) = \sum_{u=1}^{\infty} \omega(e_u) \chi(c_u) \tag{3.21}$$

Theorem 3.3.1. *Let* $K(p,q) = \sum_{u=1}^{\infty} e_u(p) e_u(q)^*$.
(1) *The function*

$$\sum_{u=0}^{\infty} \left(\omega(e_u)(z)\right) \left(\omega(e_u)(w)\right)^* \tag{3.22}$$

is positive definite on $\mathbb{C}_i$.
(2) *The associated reproducing kernel Hilbert space of* $\mathbb{C}^{2n}$*-valued functions is the set of functions of the form*

$$F(z) = \sum_{u=0}^{\infty} \omega(e_u) d_u, \tag{3.23}$$

where $d_0, d_1 \ldots \in \mathbb{C}^{2n}$, *such that*

$$\sum_{u=0}^{\infty} d_u^* d_u < \infty, \tag{3.24}$$

with norm which is the infimum of (3.24) *over all representations.*
(3) *The associated reproducing kernel Hilbert* $\chi(\mathbb{C}^{n\times n})$*-module of* $\mathbb{C}^{2n\times 2n}$ *functions is the set of all functions of the form*

$$F(z) = \sum_{u=0}^{\infty} \omega(e_u) D_u, \tag{3.25}$$

where $D_0, D_1 \ldots \in \chi(\mathbb{C}^{n\times n})$, *such that*

$$\mathrm{Tr}\left(\sum_{u=0}^{\infty} D_u^* D_u\right) < \infty, \tag{3.26}$$

and associated $\mathbb{C}^{2n\times 2n}$*-valued form*

$$[F, G] = \sum_{u=0}^{\infty} G_u^* D_u, \quad \textit{where} \quad G(z) = \sum_{u=0}^{\infty} \omega(e_u) G_u. \tag{3.27}$$

Proof. The proof follows with standard arguments. $\square$

Remark 3.3.2. A similar theorem could be stated with the map χ instead of the map ω_i. The functions will not be analytic then.

3.4 Slice hyperholomorphic weights: half-plane case

As recalled in the introduction (see Remarks 1.2.2), it is of interest to find $\mathcal{H}(A, B)$ spaces isometrically included in a $\mathbf{L}_2(d\mu)$ space, where $d\mu$ is a positive measure on the unit circle or on the real line. We consider here this question in the quaternionic setting, and two questions pop up: What is the backward-shift operators now and what are the measures to be considered. Let us begin wih the backward-shift operators. For f slice hyperholomorphic in Ω and $\mathbb{H}^n$-valued and for $x, a \in \Omega \cap \mathbb{R}$, we define

$$(R_a f)(x) = \frac{f(x) - f(a)}{x - a}.$$

This function has slice hyperholomorphic extension to $\Omega \setminus \{a\}$ equal to

$$(R_a f)(p) = (p - a)^{-\star} \star (f(p) - f(a)) = (p - a)^{-1}(f(p) - f(a)). \tag{3.28}$$

Since

$$(R_a)f(x) - (R_b)f(x) = (a - b)(R_a R_b)f(x)$$

(the proof is as in the complex-valued case since x, a and b are chosen real) we have

$$(R_a f)(p) - (R_b f)(p) = (a - b)(R_a R_b f)(p), \quad p \in \Omega.$$

We now turn to the measures. These will be now on $i\mathbb{R}$, for some arbitrary but fixed $i \in \mathbb{S}$, or on a unit circle. Let dn be a $\mathbb{H}^{n\times n}$-valued positive measure and let $\mathcal{M}$ be a space of quaternionic slice hyperholomorphic functions isometrically included $\mathbf{L}_2(dn)$: For $f \in \mathcal{M}$ we have

$$\int_{\mathbb{R}} f(it)^* dn(t) f(it) < \infty. \tag{3.29}$$

We get

$$\operatorname{Tr} \int_{\mathbb{R}} \chi(f(it))^* \chi(dn(t)) \chi(f(it)) < \infty, \tag{3.30}$$

where we consider the boundary value of the function

$$\chi(f(z)) = \begin{pmatrix} F(z) & G(z) \\ -\overline{G(z)} & \overline{F(z)} \end{pmatrix},$$

where we wrote $f(z) = F(z) + G(z)j$ with F, G holomorphic, see (3.17). The problem is that the function $z \mapsto \chi(f(z))$ is not analytic in $\mathbb{C}_i$. One could consider the map ω and note that for $z, a \in \mathbb{C}_i$ and $f = F + Gj$ we have

$$R_a f = R_a F + R_a G j$$

and so, for *real* a,

$$\omega(R_a f) = R_a(\omega(f)).$$

But there seems to be no direct connection between a natural norm for $\omega(f)$ and (3.29). We thus proceed along a different line and this subsection contains a key result, which allows to make the connection, as in the complex setting, between the quaternionic counterparts of the $\mathcal{H}(A, B)$ spaces and $\mathcal{H}(\Theta)$ spaces. We fix a pair (i, j) of orthogonal squareroots of -1 as above. Let W_+ be a $\mathbb{H}^{n\times n}$-valued function slice hyperholomorphic in an axially symmetric Ω which contains $i\mathbb{R}$, and such that $\det \omega(W_+) \not\equiv 0$. In view of (3.19), this condition implies that

$$W_+ \star f \equiv 0 \quad \Longleftrightarrow \quad f = 0 \tag{3.31}$$

where f is slice hyperholomorphic in Ω. Thus we can define an inner product on the space of functions slice hyperholomorphic in Ω by

$$\langle f, g \rangle = \int_{\mathbb{R}} ((W_+ \star g)(p))^*_{|p=it} ((W_+ \star f)(p))_{|p=it} dt \tag{3.32}$$

which are such that

$$\int_{\mathbb{R}} \|((W_+ \star f)(p))_{|p=it}\|^2 dt < \infty. \tag{3.33}$$

We note that in the above formulas, we first compute the $\star$-product between two functions and then we restrict to $i\mathbb{R}$.
So we can now give the following:

Definition 3.4.1. *We denote by* $\mathbf{L}_2(W_+^* W_+, dt)$ *the closure of the space of slice hyperholomorphic functions satisfying* (3.33).

As the complex case already illustrates with $W_+ = 1$, this set contains functions which are not, in general, slice hyperholomorphic.

Remark 3.4.2. We note that already in the complex setting case, analytic weights are of importance, and are related to spectral factorizations (see [77] and later the part on the Wiener algebra). The notion of spectral factorization also intervenes in Subsection 10.2.

We note that for real a and b

$$W_+\star R_af=\frac{1}{p-a}(W_+\star(f-f(a))\quad\text{and}\quad W_+\star R_bg=\frac{1}{p-b}(W_+\star(g-g(b))$$

and, for $p=it$,

$$\frac{1}{p-a}+\frac{1}{\overline{p}-b}=-\frac{a+b}{(p-a)(\overline{p}-b)}.$$

Hence we can write

$$\begin{aligned}
&(W_+\star g)^*(W_+\star R_af)+(W_+\star R_bg)^*(W_+\star f)+(a+b)(W_+\star R_bg)^*(W_+\star R_af)\\
&\ =(W_+\star g)^*\frac{1}{p-a}(W_+\star(f-f(a))+(W_+\star(g-g(b)))^*\frac{1}{\overline{p}-b}-\\
&\quad -(W_+\star(g-g(b)))^*\left(\frac{1}{\overline{p}-b}+\frac{1}{p-a}\right)(W_+\star(f-f(a))\\
&\ =(W_+g(b))^*(W_+\star R_af)+(W_+\star R_bg)^*(W_+f(a)),
\end{aligned}$$

and so

$$\langle R_af,g\rangle+\langle f,R_bg\rangle+(a+b)\langle R_af,R_bg\rangle=G(b)^*JF(a)\tag{3.34}$$

with

$$F(a)=\begin{pmatrix}f(a)\\ f_-(a)\end{pmatrix}\quad\text{and}\quad J_1=\begin{pmatrix}0&I_n\\ I_n&0\end{pmatrix},\tag{3.35}$$

the function $f_-(a)$ being defined by

$$f_-(a)=\langle R_af,1\rangle\tag{3.36}$$

in the scalar case and by

$$c^*f_-(a)=\langle R_af,c\rangle$$

in the matrix-valued case.
We define a new space consisting of pairs F,G as in (3.35) equipped with inner product

$$\langle F,G\rangle=\langle f,g\rangle.\tag{3.37}$$

Formula (3.34) and (3.37) give

$$\langle R_aF,G\rangle+\langle F,R_bG\rangle+(a+b)\langle R_aF,R_bG\rangle=G(b)^*JF(a).\tag{3.38}$$

One recognizes f_- as the function introduced in [20] and (3.38) as the structural identity of de Branges (see [40, 42]) characterizing a certain family of reproducing kernel Hilbert spaces.

Theorem 3.4.3. *Let $\mathcal{M}\subset\mathbf{L}_2(W_+^*W_+,dt)$ be a Hilbert space of functions slice hyperholomorphic in the open axially symmetric domain Ω, and assume $\Omega\cap\mathbb{R}\neq\emptyset$. Assume moreover that $R_a\mathcal{M}\subset\mathcal{M}$ for $a\in\Omega\cap\mathbb{R}$. Then there exists a J_1-inner function Θ such that the reproducing kernel of $\mathcal{M}$ is equal to*

$$-\frac{\Theta_{11}(z)\Theta_{12}(w)^*+\Theta_{12}(z)\Theta_{11}(w)^*}{z+\overline{w}}.$$

Proof. We proceed in a number of steps.

STEP 1: *Let* $a, b \in \mathbb{R}$. *We first check that*

$$\begin{aligned}\frac{(p-a)^{-1}(f(p)-f(a))-(p-b)^{-1}(f(p)-f(b))}{a-b} &= \\ = (p-a)^{-1}\left((p-b)^{-1}(f(p)-f(b))-(a-b)^{-1}(f(a)-f(b))\right)\end{aligned} \tag{3.39}$$

It suffices to compare the coefficients of $f(p)$, $f(a)$ and $f(b)$ on both sides. For $f(p)$ we have

$$\frac{1}{a-b}\left((p-a)^{-1}-(p-b)^{-1}\right)$$

for the left side, and this is equal to $(p-a)^{-1} \star (p-b)^{-1} = (p-a)^{-1}(p-b)^{-1}$, which is the coefficient of $f(p)$ on the right side. The coefficients of $f(a)$ and $f(b)$ are treated in the same way.

STEP 2: *We note that*

$$(R_b f_-)(a) = ((R_b f)_-)(a). \tag{3.40}$$

We need to check that

$$\int_{\mathbb{R}} W_+(it)^* \left[W_+(p) \star (p-a)^{-1}(R_b f(p) - R_b f(a))\right]_{p=it} dt = \frac{f_-(a)-f_-(b)}{a-b}.$$

Since

$$\begin{aligned}\frac{f_-(a)-f_-(b)}{a-b} = \frac{1}{a-b}\Bigg(\int_{\mathbb{R}} W_+(it))^* \left[W_+(p) \star (p-a)^{-1}(f(p)-f(a))\right]_{p=it} dt - \\ - \int_{\mathbb{R}} W_+(it))^* \left[W_+(p) \star (p-b)^{-1}((f(p)-f(b))\right]_{p=it} dt\Bigg)\end{aligned}$$

we have (3.39), after removing the integrals and the multiplications by W_+ and W_+^*.

STEP 3: *The set* $\mathcal{M}^{\square}$ *of functions* F *of the form given in* (3.35) *endowed with the inner product* (3.37) *is a Hilbert space of slice hyperholomorphic functions which is* R_b*-invariant for real* $b \in \Omega$ *and satisfies* (3.38).

This follows directly from Step 2. As in [20] we have

$$R_b F = R_b \begin{pmatrix} f \\ f_- \end{pmatrix} = \begin{pmatrix} R_b f \\ R_b(f_-) \end{pmatrix} = \begin{pmatrix} R_b f \\ (R_b f)_- \end{pmatrix}.$$

The proof is concluded by using the counterpart of [20, Theorem 3.1 p. 600], which is proved in [33]. $\square$

Definition 3.4.4. *We will call $\mathcal{M}^{\square}$ the extension of the space $\mathcal{M}$ associated to the weight $W_+^*W_+$.*

Remark 3.4.5. More generally, one can consider inner products of the form

$$\langle f, g\rangle = \int_{\mathbb{R}} ((W_+ \star g)(p))^*_{|p=it}((W_+ \star f)(p))_{|p=it} d\mu(t) \tag{3.41}$$

where $d\mu$ is a scalar positive measure on the real line.

3.5 Slice hyperholomorphic weights: the quaternionic unit ball case

We now discuss the quaternionic unit ball case. An analytic weight will now be a $\mathbb{H}^{n\times n}$-valued function invertible slice hyperholomorphic in a neighborhood of the close quaternionic unit ball. We associate with such a weight the set of slice hyperholomorphic functions f such that

$$\int_0^{2\pi} \|((W_+ \star f)(p))_{|p=e^{it}}\|^2 dt < \infty$$

$$f_-(a) = \int_0^{2\pi} W_+(e^{it})^* \left((2p \star f(p) - pf(a) - af(a)) \star (p-a)^{-1} \star W_+(p)\right)_{|p=e^{it}} dt.$$

We note that on $(-1,1)$ we have

$$f_-(x) = R_a(xf(x)) + xR_af \tag{3.42}$$

Proposition 3.5.1. *Let f, f_- be as above and $b \in \mathbb{R}$, then we have:*

$$(R_bf)_- = R_bf_-.$$

Proof. In view of (3.42) it is enough to prove that

$$R_b(R_a(xf(x)) + xR_af) = R_a(R_b(xf(x)) + xR_bf),$$

i.e. since R_a and R_b commute

$$R_a(xR_bf) = R_b(xR_af)$$

i.e.

$$\frac{x\dfrac{f(x)-f(b)}{x-b} - a\dfrac{f(a)-f(b)}{a-b}}{x-a} = \frac{x\dfrac{f(x)-f(a)}{x-a} - b\dfrac{f(b)-f(a)}{b-a}}{x-b}.$$

This latter equality is easy to check. $\square$

Chapter 4

Rational functions

The notion of a rational slice hyperholomorphic function was introduced and studied in [18]. These functions play an important role in various aspects of quaternionic operator theory. It is always of interest to study the finite dimensional version of general results, and rational functions intervene in such cases; see e.g. Corollaries 6.1.11 and 6.1.12.

4.1 Rational slice hyperholomorphic functions

Various equivalent definitions can be used to introduce the notion of rational function. The simplest seems to be the following:

Definition 4.1.1. *The $\mathbb{H}^{u\times v}$-valued function slice hyperholomorphic in an axially symmetric Ω is rational if the function*

$$x \mapsto \chi(R(x))$$

is a rational function of $x \in \Omega \cap \mathbb{R}$.

Following the realization theorem for complex-valued rational functions, one then has:

Theorem 4.1.2. *Let Ω and R be as above, and assume that $0 \in \Omega$. Then, R is rational if and only if it can be written as*

$$R(p) = D + pC \star (pI_m - A)^{-\star} B \tag{4.1}$$

where $D = R(0)$ and $(A, B, C) \in \mathbb{H}^{m\times m} \times \mathbb{H}^{m\times v} \times \mathbb{H}^{u\times m}$.

Expression (4.1) is called a realization of R; it is called a minimal realization if m in (4.1) is minimal.

When $(B, C) \in \mathbb{H}^{m\times v} \times \mathbb{H}^{u\times m}$ and $u = v$ and D is invertible we note the formula

$$R^{-\star}(p) = D^{-1} - pD^{-1}C \star (pI_m - (A - BD^{-1}C)^{-\star} BD^{-1}. \tag{4.2}$$

Definition 4.1.3. *Let $J \in \mathbb{R}^{u\times u}$ be a signature matrix, i.e. $J = J^* = J^{-1}$, and let Θ be a $\mathbb{H}^{u\times u}$ slice hyperholomorphic rational function. It is called J-unitary if*

$$\Theta(x)J\Theta(-x)^* = J \tag{4.3}$$

at all real points where it is defined.

The following theorem is taken from [18, Theorem 9.3.1 p. 245] and is the counterpart of a result proved in [29] in the complex setting.

Theorem 4.1.4. *Let Ω be an axially symmetric with $0 \in \Omega$. The function Θ is a slice hyperholomorphic rational function in Ω, with minimal realization*

$$\Theta(p) = D + C \star (pI - A)^{-\star}B. \tag{4.4}$$

if and only if D is J unitary, meaning $DJD^* = J$ and there exists a uniquely determined Hermitian matrix H such that

$$\begin{aligned} C &= JB^*H, & (4.5)\\ HA + A^*H &= C^*JC. & (4.6) \end{aligned}$$

When $J = I_n$, rational matrix-valued unitary functions slice hyperholomorphic in the right half-space are called finite Blaschke products. A (say $\mathbb{H}^{n\times n}$-valued) unitary rational function S, (that is $S(x)S(-x)^* = I_n$ at those real points x where the expression makes sense) can be written as

$$S(p) = B_1(p) \star B_2(p)^{-\star} \tag{4.7}$$

where B_1 and B_2 are $\mathbb{H}^{n\times n}$-valued finite Blaschke products, see [15]. This is the quaternionic version of a special case of a general result of Krein and Langer; see [74] for the latter and [28] for its rational complex-valued version.

More generally we will need the following definition:

Definition 4.1.5. The function B slice hypermeromorphic in $\mathbb{B}_1$ is called a Blaschke-Potapov product of the first kind (resp. second kind, resp. a singular factor) if $\chi(B)(x)$, $x \in (-1, 1)$, is the restriction to $(-1, 1)$ of a Blaschke-Potapov product of the first kind (resp. second kind, resp. singular factor) in the open unit ball.

We now turn to the notion of Wiener-Hopf factorization.

Definition 4.1.6. *The $\mathbb{H}^{n\times n}$-valued rational function R slice hyperholomorphic in a neighborhood of infinity with $R(\infty)$ invertible is said to admit a left Wiener-Hopf factorization if it can be written as $R = R_1 \star R_2$ where R_1 and R_2 are also $\mathbb{H}^{n\times n}$-valued rational functions slice hyperholomorphic and invertible in a neighborhood of infinity, and such that R_1 (resp. R_2) and its slice hyperholomorphic inverse have no poles in $\mathrm{Re}\, p \geq 0$ (resp. in $\mathrm{Re}\, p \leq 0$.*

Theorem 4.1.7. *Let R be a $\mathbb{H}^{n\times n}$-valued rational function such that $R(p) > 0$ for p in $\partial\mathbb{B}$, the boundary of the unit ball $\mathbb{B}$. Then R admits a Wiener-Hopf factorization of the form*

$$R(p) = R_+(p) \star R_+^c(p).$$

The above result follows from [13] for the scalar case and from [92] for the matrix-valued case since a rational function invertible on $\partial\mathbb{B}$ belongs to the Wiener algebra.
We will see an example of such factorization in Theorem 10.2.3.

Remark 4.1.8. In the case of Fueter variables, rational functions are defined in a similar way and appear first in the work [75]. They have been studied in [34] and [35].

4.2 Symmetries

The present section is in the complex setting, and will be used to prove Potapov's factorization theorem (Theorem 7.1.1 below) in the quaternionic setting.
The factorization of J-inner functions originates with the work of Potapov; see also [60], and [29] for the case of J-unitary rational functions. The corresponding problem here is to consider minimal factorization of J-inner (and J-unitary) functions into factors which are J-inner (or J-unitary) and satisfy moreover the symmetry (4.8) below; see [7].
We will set

$$E = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

and we will consider a $\mathbb{C}^{2n\times 2n}$-valued rational function X such that

$$X(z) = E\overline{X(\overline{z})}E^{-1},\ z \in \mathbb{C}. \tag{4.8}$$

Then we define the symmetry

$$\alpha(X)(x) = E\overline{X(x)}E^{-1},\ x \in \mathbb{R}. \tag{4.9}$$

Lemma 4.2.1.
(1) *The symmetry α is multiplicative i.e.*

$$\alpha(XY) = \alpha(X)\alpha(Y),$$

for arbitrary $\mathbb{C}^{2n\times 2n}$-valued rational function X and Y, and satisfies

$$\alpha(X^*) = (\alpha(X))^*. \tag{4.10}$$

(2) *Let J be a signature matrix which commutes with E. If R is a product of Blasckhe-Potapov factors of the first kind (resp. second kind) (resp. third kind) so is $\alpha(R)$.*

Proof.
(1) Let X, Y be $\in \mathbb{C}^{2n\times 2n}$-valued rational function. We have (where we remove the dependance on the variable to lighten the notation)

$$\begin{aligned}\alpha(XY) &= E\overline{XY}E^{-1}\\ &= E\overline{X}\cdot\overline{Y}E^{-1}\\ &= E\overline{X}E^{-1}E\overline{Y}E^{-1}\\ &= \alpha(X)\alpha(Y).\end{aligned}$$

Furthermore

$$\alpha(X^*) = E\overline{\overline{X^t}}E^{-1} = EX^tE^{-1}$$

while

$$(\alpha(X))^* = E^{-*}\overline{E}^*E^* = EX^tE^{-1}$$

since $E^* = -E^{-1}$.

(2) This claim follows from

$$\begin{aligned}\alpha\left(\frac{J - R(x)JR(y)^*}{1-xy}\right) &= \frac{\alpha(J) - \alpha(R(x))\alpha(J)\alpha(R(y)^*)}{1-xy}\\ &= \frac{\alpha(J) - \alpha(R(x))\alpha(J)(\alpha(R(y)))^*}{1-xy}\\ &= \frac{J - \alpha(R(x))J(\alpha(R(y)))^*}{1-xy},\end{aligned}$$

since $\alpha(J) = EJE^{-1} = JEE^{-1} = J$. □

We will be interested in slice hyperholomorphic rational functions R such that $X = \chi(R)$ satisfies the symmetry

$$X(x) = \alpha(X)(x), \quad x \in \mathbb{R}, \tag{4.11}$$

i.e, for complex z, and in order to preserve analyticity,

$$X(z) = E\overline{X(\overline{z})}E^{-1}, \quad z \in \mathbb{C}. \tag{4.12}$$

The following result is a special case of results in [6].

Theorem 4.2.2. *Let $X(z)$ be a $\mathbb{C}^{2n\times 2n}$-valued rational function, analytic at infinity, and with minimal realization*

$$X(z) = D + C(zI_u - A)^{-1}B.$$

Then, X satisfies (4.12) if and only if there exists a uniquely defined matrix $S \in \mathbb{C}^{u\times u}$ such that

$$\overline{S}S = -I_u \tag{4.13}$$

and

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \begin{pmatrix} S & 0\\ 0 & E\end{pmatrix}\begin{pmatrix} \overline{A} & \overline{B}\\ \overline{C} & \overline{D}\end{pmatrix}\begin{pmatrix} S^{-1} & 0\\ 0 & E^{-1}\end{pmatrix}. \tag{4.14}$$

Proof. The function $E\overline{X(\overline{z})}E^{-1}$ is rational, analytic at infinity, and with minimal realization

$$E\overline{X(\overline{z})}E^{-1} = E\overline{D}E^{-1} + E\overline{C}(zI_u - \overline{A})^{-1}\overline{B}E^{-1}.$$

By uniqueness up to a similarity matrix of the minimal realization of a rational function analytic at infinity (see e.g. [38]) there exists a uniquely defined matrix S such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & E^{-1} \end{pmatrix}. \tag{4.15}$$

Taking the conjugate of this equality and the fact that $E^{-1} = -E$ we see that it is also satisfied by $-\overline{S}^{-1}$, and hence the result. □

We note that (4.15) can be rewritten as

$$\begin{aligned} A &= S\overline{A}S^{-1} && (4.16)\\ B &= S\overline{B}E^{-1} && (4.17)\\ C &= E\overline{C}S^{-1} && (4.18)\\ D &= E\overline{D}E^{-1}. && (4.19) \end{aligned}$$

Corollary 4.2.3. *An elementary factor satisfying the symmetry* (4.8)*, normalized to be identity at $z = 1$ and with singularities on the unit circle is of the form*

$$X(z) = D \begin{pmatrix} \varphi(z)I & 0 \\ 0 & \overline{\varphi(z)}I \end{pmatrix} D^{-1} \tag{4.20}$$

where D satisfies (4.19) *and*

$$\varphi(z) = \frac{(z + e^{i\alpha})(1 - e^{-i\alpha})}{(z - e^{i\alpha})(1 + e^{i\alpha})},$$

i.e., with

$$D = \begin{pmatrix} D_1 & D_2 \\ -\overline{D_2} & \overline{D_1} \end{pmatrix}$$

$$X(z) = \begin{pmatrix} D_1\overline{D_1}\varphi(z) + D_2\overline{D_2}\overline{\varphi(z)} & -D_1D_2(\varphi(z) + \overline{\varphi(z)}) \\ \overline{D_1}\overline{D_2}(\varphi(z) + \overline{\varphi(z)}) & \overline{D_1}D_1\overline{\varphi(z)} + \overline{D_2}D_2\varphi(z) \end{pmatrix}. \tag{4.21}$$

Proof. By hypothesis we can take $A = \mathrm{diag}(e^{i\alpha}, e^{-i\alpha})$, where $\alpha \in \mathbb{R}$. Condition (4.16) together with (4.13) implies that S is of the form

$$S = \begin{pmatrix} 0 & S_2 \\ -\overline{S_2}^{-1} & 0 \end{pmatrix}.$$

□

Chapter 5

Operator models

In [82], Rota studied models for linear operators in a Hilbert space: he proved that every linear operator T with spectral radius less than 1 is similar to the restriction of the adjoint of the unilateral shift S to a suitable invariant subspace. Thus, the unilateral shift is a "universal model" for such operators T. In this short chapter we begin the study of operator models in the quaternionic framework.

5.1 Rota's model in the quaternionic setting

In this section we discuss Rota's model for linear operators in quaternionic Hilbert spaces. Although the results and arguments are essentially the same, they rely on tools which have been developed only in recent years, and are specific to the quaternionic case. This allows to pinpoint important differences between the complex and quaternionic cases.
We begin by proving:

Theorem 5.1.1. *Let $\mathcal{H}$ be a right quaternionic Hilbert space.Then:*
(1) *Every linear operator in $\mathcal{H}$ with S-spectrum in $\mathbb{B}_1$ is similar to a contraction of norm strictly less than* 1*.*
(2) *Every compact operator with S-spectrum in $\overline{\mathbb{B}}_1$ is similar to a contraction.*
(3) *Every quasi-nilpotent operator is similar to operators of arbitrary small norm.*

Proof. We first prove the existence of a universal model for contractions in $\mathcal{H}$. Following [82], consider the space $\ell_2(\mathbb{N}, \mathcal{H})$. As in the complex case, the S-spectrum is closed and the S-spectral radius $r_S(T)$ is the value of the spectrum of largest modulus, see Theorem 3.2.9. In view of (3.12) the power series $\sum_{k=0}^{\infty} \|T^n x\|^2$ converges for every $x \in \mathcal{H}$. Furthermore, the space

$$\mathcal{M}(T) = \left\{(x, Tx, T^2x, \ldots),\, x \in \mathcal{H}\right\}$$

is closed in $\ell_2(\mathbb{N}, \mathcal{H})$. By the closed-graph theorem the map $\tau_T(x) = (x, Tx, \ldots)$ is bounded invertible. Let F denote the forward shift in $\ell_2(\mathbb{N}, \mathcal{H})$. We then have

$$T = \tau_T^{-1} \mathsf{F} \tau_T. \tag{5.1}$$

Since the Hilbert spaces $\mathcal{H}$ and $\ell_2(\mathbb{N}, \mathcal{H})$ have the same cardinality, one can find a unitary operator U from $\mathcal{H}$ onto $\ell_2(\mathbb{N}, \mathcal{H})$. Thus

$$T = \tau_T^{-1} U^{-1} U \mathsf{F} U^{-1} U \tau_T. \tag{5.2}$$

The above argument shows that every linear operator T with S-spectrum inside the open unit ball is similar to a contraction. This contraction has possibly norm 1. With this result at hand we turn to the proof of (1). Replacing T by $(1+\epsilon)T$ dilates the spectrum by a factor $(1+\epsilon)$, and so will stay inside $\mathbb{B}_1$ for $\epsilon > 0$ small enough.

We now turn to the last item. By hypothesis, for every $\epsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$\|N^n\|^{1/n} \leq \epsilon, \quad n \geq m.$$

Hence, $\|\frac{N^n}{\epsilon^n}\| \leq 1$ for such n, and there exists $M > 0$ such that

$$\left\| \frac{N^n}{\epsilon^n} \right\| \leq M, \quad n \in \mathbb{N}.$$

We can apply the previous result to N/ϵ, and therefore N/ϵ is similar to a contraction S, which implies that N is similar to ϵS, which is of arbitrarily small norm. □

Remark 5.1.2. Let F be the forward shift operator. The operator $U\mathsf{F}U^{-1}$ is a universal operator in the sense of Rota.

In the sequel we will be interested in the operator M_p of $\star$-multiplication by the quaternionic variable, and we note the following:

Proposition 5.1.3. *The operator* F *is unitarily equivalent to* M_p *from* $\mathbf{H}_2(\mathbb{B}_1, \mathcal{H})$ *into itself, and its adjoint is given by the backward-shift operator* R_0.

Proof. Both the assertions follows as in the complex case. □

5.2 Operator models

Consider a bounded linear operator T acting on a right quaternionic Pontryagin space $(\mathcal{P}, [\cdot, \cdot])$; we assume that its spectrum intersects the real line on an open set O. We note that for any $x \in \mathbb{R}$ the S-resolvent operator coincides with $(T - xI)^{-1}$. Assume that there is a $u \in \mathcal{P}$ such that the linear span of the vectors $(T^* - xI)^{-1}u$ is dense in $\mathcal{P}$ when x runs in O. The map $\mathscr{I}$ which to $f \in \mathcal{P}$ associates the function

$$F(x) = [f, (T^* - xI)^{-1}u] = [(T - xI)^{-1}f, u] \tag{5.1}$$

is one-to-one, and defines a Pontryagin space structure on the sets of such F. Let now $x_0 \in O$. Using the resolvent identity we have:

$$\begin{aligned}[(T-x_0I)^{-1}f,(T^*-xI)^{-1}u] &= [(T-xI)^{-1}(T-x_0I)^{-1}f,u]\\ &= \frac{[(T-xI)^{-1}f,u]-[(T-x_0I)^{-1}f,u]}{x-x_0}\end{aligned}$$

and so

$$\mathscr{I}((T-x_0I)^{-1}f)=R_{x_0}(\mathscr{I}(f)),$$

i.e. the resolvent operator $(T-x_0I)^{-1}$ is unitarily equivalent to the backward-shift operator R_{x_0}. We note also that by Stone theorem $R_{x_0}=(T-x_0)^{-1}$ when $\ker R_{x_0}=\{0\}$. This argument cannot be extended in a direct way when x, x_0 are not real.

If $f_0\in\mathcal{M}$, $x,x_0\in\mathbb{R}$ and $f_0(x_0)\neq 0$ we can write

$$\frac{f(x)-f_0(x)\dfrac{f(x_0)}{f_0(x_0)}}{x-x_0}\in\mathcal{M}.$$

This can be written as

$$\frac{f(x)-f_0(x)\dfrac{f(x_0)}{f_0(x_0)}}{x-x_0}=f_0(x)\left(R_{x_0}f_0^{-1}f\right)(x)$$

We consider a (possibly) unbounded closed operator T such that there exists $u\in\mathcal{P}$ such that the elements

$$(xI-T)^{-*}u,\quad x\in\Omega\cap\mathbb{R},$$

are dense in $\mathcal{P}$. When T is bounded, this condition is equivalent to ask that u is cyclic for T^*, that is the span of the vectors $u, T^*u, T^{2*}u,\ldots$ is dense in $\mathcal{P}$. We denote by G the linear operator $Gf=[f,u]_{\mathcal{P}}$. Consider the function

$$L(x)=[(I-xT)^{-1}f,u]=G(I-xT)^{-1}f.$$

Assuming T bounded and using (3.2.17), the (unique) slice left-hyperholomorphic extension of L to a neighborhood of the origin is given by

$$L(p)=(G-\overline{p}GT)(I-2(\mathrm{Re}\,p)T+|p|^2T^2)^{-1}f. \tag{5.2}$$

We define

$$F(p)=\frac{1}{p}L(1/p)=\frac{1}{p}(G-\frac{p}{|p|^2}GT)(I-2(\frac{\mathrm{Re}\,p)}{|p|^2}T+\frac{1}{|p|^2}T^2)^{-1}f,$$

that is,

$$F(p)=\frac{1}{p}(|p|^2G-pGT)(|p^2|I-2(\mathrm{Re}\,p)T+T^2)^{-1}f. \tag{5.3}$$

Note that $F(x) = G(xI - T)^{-1}$ for $x \in \Omega \cap \mathbb{R}$. In view of the hypothesis on u, the function $F(p)$ is identically equal to 0 if and only if $f = 0$.

Assume we replace in (5.3) f by $S_L^{-1}(\alpha, T)f = (\alpha I - T)^{-1}f$. Then, the resolvent equation (3.2.11) leads to

$$\begin{aligned}
&[S_R^{-1}(x,T)S_L^{-1}(\alpha,T)f, u]_{\mathcal{P}} = \\
&= [\left(S_R^{-1}(x,T) - S_L^{-1}(\alpha,T)\right)(x-\alpha)^{-1}f, u]_{\mathcal{P}} \\
&= G\left(S_R^{-1}(x,T) - S_L^{-1}(\alpha,T)\right)(x-\alpha)^{-1}f.
\end{aligned}$$

The left slice hyperholomorphic extension of this expression is

$$G \star \left(S_R^{-1}(p,T) - S_L^{-1}(T,\alpha)\right) \star (p-\alpha)^{-\star} f, \tag{5.4}$$

which is the analogue of R_α for non real α.

Remark 5.2.1. The expression (3.28) is the counterpart, in the quaternionic setting, of the backward-shift operator R_α. For real p and α it reduces to the operator R_p applied to the function $s \mapsto S_R^{-1}(s,T)f$.

The proof of the next result follows easily from the previous arguments and will be omitted.

Theorem 5.2.2. *The space of functions F as in* (5.3) *endowed with the Hermitian form*

$$[F, F_1] = [f, f_1]_{\mathcal{P}} \tag{5.5}$$

is a reproducing kernel Pontryagin space with reproducing kernel $K(p,q)$ defined by

$$K(p,q) = G \star (pI - T)^{-\star}((qI - T)^{-\star})^* G^*. \tag{5.6}$$

Chapter 6

Structure theorems for $\mathcal{H}(A, B)$ spaces

In the complex setting, $\mathcal{H}(A, B)$ spaces (that is, reproducing kernel spaces which reproducing kernel of the form (1.5) (or their analogues with denominator replaced by $1 - z\overline{w}$) play an important role in the theory of operator models and related topics. In this section we focus on $\mathcal{H}(A, B)$ spaces in the present context.

6.1 $\mathcal{H}(A, B)$ spaces

Definition 6.1.1. *$\mathcal{H}(A, B)$ spaces are reproducing kernel spaces of quaternionic-valued (or more generally, $\mathbb{H}^n$-valued) functions, slice hyperholomorphic in an axially symmetric s-domain Ω, and with a reproducing kernel whose restriction on $\Omega \cap \mathbb{R}$ is of the form*

$$\frac{A(x)A(y)^* - B(x)B(y)^*}{x + y} \quad \text{or} \quad \frac{A(x)A(y)^* - B(x)B(y)^*}{1 - xy}, \tag{6.1}$$

where A and B are $\mathbb{H}^{n \times n}$-valued functions slice hyperholomorphic in Ω.

We first consider the uniqueness of the decomposition (6.1).

Definition 6.1.2. *The pair of $\mathbb{H}^{n \times n}$-valued functions (A, B) slice hyperholomorphic in Ω is full rank if the right linear span of the vectors*

$$\begin{pmatrix} A(x)^*c \\ B(x)^*c \end{pmatrix}, \quad x \in \Omega \cap \mathbb{R}, \quad c \in \mathbb{H}^n,$$

spans all of $\mathbb{H}^{2n}$. We also recall that J_0 denotes the signature matrix (1.21)*:*

$$J_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Proposition 6.1.3. *Assume the pair (A,B) in (6.1) to be full rank. Then, the pair (A,B) is unique up to a J_0-unitary constant factor.*

Proof. We consider the first kernel, and assume that

$$\frac{A(x)A(y)^*-B(x)B(y)^*}{x+y}=\frac{A_1(x)A_1(y)^*-B_1(x)B_1(y)^*}{x+y},\quad x,y\in\Omega\cap\mathbb{R}.$$

Then,

$$A(x)A(y)^*-B(x)B(y)^*=A(x)_1A_1(y)^*-B_1(x)B_1(y)^*,\quad x,y\in\Omega\cap\mathbb{R}. \tag{6.2}$$

Let $x_1,\ldots,x_{2n}\in\Omega\cap\mathbb{R}$ and $c_1,\ldots,c_{2n}\in\mathbb{H}^n$ be such that the vectors

$$d_j=\begin{pmatrix}a(x_j)^*c_j\\ B(x_j)^*c_j\end{pmatrix},\quad j=1,\ldots,2n,$$

span $\mathbb{H}^{2n}$, and let $D=\begin{pmatrix}d_1 & d_2 & \cdots & d_{2n}\end{pmatrix}$. We also define D_1 to be the corresponding matrix built from the pair (A_1,B_1). We have from (6.2)

$$D^*J_0D=D_1^*J_0D_1,$$

and the result follows since D is invertible. $\square$

Theorem 6.1.4. *Let Ω be an axially symmetric s-domain. Assuming in (1.5) that $\det\chi(A)\not\equiv 0$, we can rewrite*

$$\frac{A(x)A(y)^*-B(x)B(y)^*}{x+y}=A(x)\frac{I_n-S(x)S(y)^*}{x+y}A(y)^*. \tag{6.3}$$

with $S=A^{-1}B$. It follows that S is the restriction to $\Omega\cap\mathbb{R}$ of a Schur multiplier. If it is moreover inner, in the sense that the operator of multiplication by S is an isometry from $\mathbf{H}_2(\mathbb{H}_+)^n$ into itself. Then

$$\mathcal{H}(A,B)=A\star\left(\mathbf{H}_2(\mathbb{H}_+)^n\ominus S\star\mathbf{H}_2(\mathbb{H}_+)^n\right), \tag{6.4}$$

and the following condition holds in $\mathcal{H}(A,B)$: For $p_0\in\Omega\cap\mathbb{R}$, if $A(p_0)$ is invertible and $F\in\mathcal{H}(A,B)$ vanishes at $p_0\in\Omega\cap\mathbb{R}$, then the function f_{p_0}

$$p\mapsto\frac{p+p_0}{p-p_0}\star f(p)$$

belongs to $\mathcal{H}(A,B)$ and has same norm as f:

$$\|f\|=\|f_{p_0}\| \tag{6.5}$$

Proof. Assume S inner; then, $\mathbf{H}_2(\mathbb{H}_+)^n \ominus S\star\mathbf{H}_2(\mathbb{H}_+)^n$ is isometrically included in the Hardy space $\mathbf{H}_2(\mathbb{H}_+)^n$, and the restriction of its reproducing kernel to $(\Omega\cap\mathbb{R})^2$ is the function $\frac{I_n-S(x)S(y)^*}{x+y}$. Equality (6.4) follows then from [18]. Let now $F=A\star f\in\mathcal{H}(A,B)$ and such that $(A\star f)(p_0)=0$ for some real $p_0\in\Omega\cap\mathbb{R}$. Since p_0 is real we have

$$(A\star f)(p_0)=A(p_0)f(p_0)$$

and so $f(p_0)=0$ since $A(p_0)$ is invertible. Thus

$$(p+p_0)(p-p_0)^{-\star}\star(f(p)=(1+2p_0(p-p_0))^{-\star}\star(f(p)-f(p_0))\in\mathbf{H}_2(\mathbb{H}_+)^n\ominus S\mathbf{H}_2(\mathbb{H}_+)^n$$

since the space $\mathbf{H}_2(\mathbb{H}_+)^n\ominus S\mathbf{H}_2(\mathbb{H}_+)^n$ is R_{p_0}-invariant (see[18]). □

Theorem 6.1.5. *Let Ω be an axially symmetric s-domain. Assuming in* (1.5) *that* $\det\chi(A)\not\equiv 0$ *we can rewrite*

$$\frac{A(x)A(y)^*-B(x)B(y)^*}{1-xy}=A(x)\frac{I_n-S(x)S(y)^*}{1-xy}A(y)^*. \tag{6.6}$$

with $S=A^{-1}B$. It follows that S is the restriction to $\Omega\cap\mathbb{R}$ of a Schur multiplier. If it is moreover inner, in the sense that the operator of multiplication by S is an isometry from $\mathbf{H}_2(\mathbb{B}_1)^n$ into itself. Then

$$\mathcal{H}(A,B)=A\star\left(\mathbf{H}_2(\mathbb{B}_1)^n\ominus S\star\mathbf{H}_2(\mathbb{B}_1)^n\right), \tag{6.7}$$

and the following condition holds in $\mathcal{H}(A,B)$: If $f\in\mathcal{H}(A,B)$ vanishes at $p_0\in\Omega\cap\mathbb{R}$, then the function f_{q_0}

$$p\mapsto\frac{1-pp_0}{p-p_0}\star f(p)$$

belongs to $\mathcal{H}(A,B)$ and has same norm as f:

$$\|f\|=\|f_{q_0}\| \tag{6.8}$$

The proof is similar to the one of Theorem 6.1.4, and therefore omitted.

Theorem 6.1.6. *(Half-space case) The operator $f\mapsto p\star f$ is anti-hermitian from $\mathcal{H}(A,B)$ into itself*

Proof. We have for f,g in the domain of M_p

$$\begin{aligned}\langle M_pf,g\rangle&=\int_{\mathbb{R}}(g(it)^*(it)f(it)dt\\&=\int_{\mathbb{R}}(g(it)(-it))^*f(it)dt\\&=\langle f,-M_pg\rangle.\end{aligned}$$

□

Note that in the above statement, M_p need not be densely defined, let alone bounded. We prove:

Theorem 6.1.7. *In the notation of the previous theorem, the domain of M_p is given by the functions of the form $A \star f$, where $f \in \mathbf{H}_2(\mathbb{H}_+) \ominus S \star \mathbf{H}_2(\mathbb{B}_1)^n$ is such that $pf(p) \in \mathbf{H}_2(\mathbb{H}_+)$ and satisfies*

$$\langle f, S \rangle = 0, \tag{6.9}$$

in the sense that

$$\langle (p+1)f, (p+1)^{-1}S \rangle = 0. \tag{6.10}$$

Proof. In view of the characterization (6.7), $pA \star f$ is in the domain of M_p (with $f \in \mathbf{H}_2(\mathbb{H}_+)^n \ominus S \star \mathbf{H}_2(\mathbb{H}_+)^n$ if and only if $pf \in \mathbf{H}_2(\mathbb{H}_+)$ and f is orthogonal to $S \star \mathbf{H}_2(\mathbb{H}_+)^n$. The latter condition is equivalent to

$$\langle pf(p), (p+p_0)^{-\star} S(p) \rangle = 0, \quad \forall p_0 > 0.$$

Rewriting

$$p(p+p_0)^{-1} = 1 - p_0(p+p_0)^{-1}$$

we have

$$\langle pf(p), (p+p_0)^{-\star} S(p) \rangle = \langle f(p), S(p) - p_0(p+p_0)^{-\star} S(p) \rangle = 0, \quad \forall p_0 > 0,$$

and so the result since the expression $\langle f(p), p_0(p+p_0)^{-\star} S(p) \rangle$ is well defined for all $p_0 > 0$. $\square$

A similar result holds when $\mathbb{H}_+$ is replaced by $\mathbb{B}_1$.

Theorem 6.1.8. *(Quaternionic unit ball case) The operator $f \mapsto pf$ is a partial isometry from $\mathcal{H}(A,B)$ into itself.*

In the next two sections we study the converse statements: do the conditions imposed in the above two theorems force the form of the reproducing kernel? Before that we prove the following result set in the scalar case. The case of $\mathbb{H}^n$-valued function is similar, but f_0 is now a matrix made of elements of $\mathcal{M}$ and a full rank hypothesis has to be added.

Proposition 6.1.9. *Let $\mathcal{M}$ be a space of $\mathbb{H}^n$-valued slice hyperholomorphic functions with the property that if $f \in \mathcal{M}$ and $f(a) = 0$ for $a \in \Omega \cap \mathbb{R}$, then the function*

$$p \mapsto (p-a)^{-1} f(p)$$

also belongs to $\mathcal{M}$. Assume the following full rank hypothesis: There exists $f_1, \ldots, f_n \in \mathcal{M}$ such that $\det \chi \begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix} \not\equiv 0$. Let $F = \begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix}$, The space $F^{-1}\mathcal{M}$ is R_b-invariant for $b \in \Omega \cap \mathbb{R}$ for which $F(b)$ is invertible.

Proof. Indeed, let $g(x) = F(x)^{-1}f(x)$ (with extension $F(p)^{-\star} \star f(p)$). Then

$$\begin{aligned} R_b g(x) &= \frac{F(x)^{-1}f(x) - F(b)^{-1}f(b)}{x-b} \\ &= F(x)^{-1} \cdot \frac{f(x) - F(x)F(b)^{-1}f(b)}{x-b}. \end{aligned}$$

To conclude we note that the function $x \mapsto f(x) - F(x)F(b)^{-1}f(b)$ belongs to $\mathcal{M}$, vanishes at $x = b$ and extends uniquely to a slice hyperholomorphic function belonging to $\mathcal{M}$. □

Corollary 6.1.10. *Assuming that $\mathcal{M}$ in the previous proposition is finite dimensional, of dimension N, and assume regularity at the origin. The space $\mathcal{M}$ is the linear span of the columns of a matrix of the form*

$$F(p) \star C \star (I - pA)^{-\star}$$

where the pair $(C, A) \in \mathbb{H}^{n\times N} \times \mathbb{H}^{N\times N}$ is observable, i.e.

$$\cap_{u=0}^{\infty} \ker CA^u = \{0\}\,. \tag{6.11}$$

Proof. This follows from the structure of finite dimensional R_0-invariant spaces; see [18]. □

In the next two corollaries we endow the space $\mathcal{M}$ with a (possibly indefinite) inner product defined by an Hermitian invertible matrix $P \in \mathbb{H}^{N\times N}$

Corollary 6.1.11. *Assume that P is an invertible Hermitian matrix which satisfies*

$$P - A^*PA = C^*JC, \tag{6.12}$$

where the matrix $I - A$ is assumed invertible. Then the reproducing kernel of the space $\mathcal{M}$ is of the form

$$F(x)\frac{J - \Theta(x)J\Theta(y)^*}{1 - xy}F(y)^*$$

Proof. As in [18] (see e.g. [5, Exercise 7.7.17, p. 368] for the case of complex functions) we define

$$\Theta(x) = I - (1-x)C(I - xA)^{-1}P^{-1}(I - A)^{-*}C^*J.$$

Then,

$$C(I - xA)^{-1}P^{-1}(I - yA)^{-*}C^* = \frac{J - \Theta(x)J\Theta(y)^*}{1 - xy}$$

□

Corollary 6.1.12. *Assume that P is an invertible Hermitian matrix which satisfies*

$$AP + PA^* + C^*JC = 0.$$

Then the reproducing kernel of $\mathcal{M}$ is of the form

$$F(x)\frac{J - \Theta(x)J\Theta(y)^*}{x+y}F(y)^*$$

Proof. One now defines

$$\Theta(x) = I - C(xI - A)^{-1}P^{-1}C^*J.$$

Then,

$$C(xI - A)^{-1}P^{-1}(yI - A)^{-*}C^* = \frac{J - \Theta(x)J\Theta(y)^*}{x+y}$$

□

If $J = I$ in the previous results we get $\mathcal{H}(A,B)$ spaces.

6.2 The structure theorem: Half-space case

Extending the theory of de Branges to the non positive case is an involved topic. We mention in particular the papers [70, 71, 72]. In this section and in the next one it will be easier to consider the quaternionic counterpart of (1.6) rather than the form (1.5) itself. The arguments still work for Krein spaces, and this is the setting we chose in this section and the following one. One should keep in mind that there is no one-to-one correspondence between reproducing kernel Krein spaces and difference of positive definite functions (see the discussion at the end of Section 2.4).

Lemma 6.2.1. *Let f be slice hyperhomolorphic in a neighborhood of the point $q_0 \in \mathbb{H}$ and assume that $f(q_0) = 0$. Then, $[q_0]$ is a removable singularity of the function*

$$g_{q_0}(p) = (p + \overline{q_0}) \star (p - q_0)^{-\star} \star f(p)$$

and g_{q_0} vanishes at the point $-\overline{q_0}$.

Proof. Indeed, we have

$$(p - q_0)^{-\star} = (p^2 - 2(\mathrm{Re}\, q_0)p + |q_0|^2)^{-1}(p - \overline{q_0}), \quad p \notin [q_0].$$

Furthermore, $f(p) = (p - q_0) \star h(p)$ where h is slice hyperholomophic in a neighborhood of q_0. Hence, for $p \notin [q_0]$ we have

$$\begin{aligned}
g_{q_0}(p) &= (p + \overline{q_0}) \star (p - q_0)^{-\star} \star f(p) \\
&= (p + \overline{q_0}) \star (p^2 - 2(\mathrm{Re}\, q_0)p + |q_0|^2)^{-1}(p - \overline{q_0}) \star (p - q_0) \star h(p) \\
&= (p + \overline{q_0}) \star h(p)
\end{aligned}$$

since $(p-\overline{q_0})\star(p-q_0) = p^2-2(\operatorname{Re} q_0)p+|q_0|^2$. Thus g_{q_0} is slice hyperholomorphic in a neighborhood of q_0.

Furthermore, with $h(p) = \sum_{n=0}^{\infty} p^n h_n$ we have

$$(p+\overline{q_0}) \star h(p) = (p+\overline{q_0}) \star \left(\sum_{n=0}^{\infty} p^n h_n\right) = \sum_{n=0}^{\infty} p^n (p+\overline{q_0}) h_n,$$

and so $g_{q_0}(-\overline{q_0}) = 0$. $\square$

Conditions (6.15) and (6.24) below are restrictive. On the other hand, they are automatically satisfied in the positive case.

Theorem 6.2.2. *Let $\mathcal{K}$ be a reproducing kernel Krein space of $\mathbb{H}$-valued functions slice hyperholomorphic in an axially symmetric s-domain Ω of the quaternions, and assume that the following condition holds: If $f \in \mathcal{K}$ vanishes at the point $q_0 \in \Omega \setminus \widetilde{\mathbb{H}}$), then the function f_{q_0} defined by*

$$p \mapsto (p+\overline{q_0}) \star (p-q_0)^{-\star} \star f(p) \tag{6.13}$$

belongs to $\mathcal{K}$ and it holds that

$$[f, f] = [f_{q_0}, f_{q_0}]. \tag{6.14}$$

Assume furthermore that there exists a real $p_0 \neq 0 \in \Omega$ such that

$$K(p_0, p_0)K(-p_0, -p_0) > 0. \tag{6.15}$$

Then, the reproducing kernel of $\mathcal{K}$ is of the form

$$K(p,q) = E_+(p) \star k(p,q) \star_r \overline{E_+(q)} - E_-(p) \star k(p,q) \star_r \overline{E_-(q)}, \tag{6.16}$$

where the functions E_+ and E_- are slice hyperholomorphic in Ω, and where $k(p,q)$ is given by (3.16)*,*

$$k(p,q) = (\bar{p}+\bar{q})(|p|^2 + 2\mathrm{Re}(p)\bar{q} + \bar{q}^2)^{-1}.$$

Before proving the theorem we make some remarks.

Remarks 6.2.3.
(a) The conditions in the theorem is about zeros which are not in $\widetilde{\mathbb{H}}$, but elements in $\mathcal{K}$ may have zeros which are in $\widetilde{\mathbb{H}}$.
(b) The function (6.13) has a removable singularity at the sphere $[p_0]$. This has been explained in Lemma 6.2.1.
(c) In the positive case, condition (6.15) is automatically satisfied and, for entire complex functions, this theorem was first proved by L. de Branges; see [48, Theorem 23, p. 57]. We will give in Section 6.4 examples where (6.15) is in force.

Proof of Theorem 6.2.2. We divide the proof in a number of steps.

STEP 1: *Let $p_0 \in \Omega$ such that $K(p_0,p_0)$ is not zero. The function*

$$(p+\overline{p_0})\star(p-p_0)^{-\star}\star\left(K(p,q)-\frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}\right)$$

has a removable singularity at the sphere $[p_0]$, belongs to the space and vanishes at the point $-\overline{p_0}$.

This is a consequence of Lemma 6.2.1 and of the hypothesis of the theorem since the function

$$p \mapsto \left(K(p,q)-\frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}\right) \in \mathcal{K}. \tag{6.17}$$

STEP 2: *Assume that there exists a non zero $p_0 \in \Omega$ such that $K(p_0,p_0)$ and $K(-\overline{p_0},-\overline{p_0})$ are both non zeros. Then for real s it holds that*

$$\begin{aligned}(p+\overline{p_0})\star(p-p_0)^{-\star}\star&\left(K(p,s)-\frac{K(p,p_0)K(p_0,s)}{K(p_0,p_0)}\right)\\ =&\left(K(p,s)-\frac{K(p,-\overline{p_0})K(-\overline{p_0},s)}{K(-\overline{p_0},-\overline{p_0})}\right)(s-\overline{p_0})(s+p_0)^{-1}.\end{aligned} \tag{6.18}$$

The argument is that of de Branges, adapted to the fact that we are in the quaternionic setting. The idea is to compute in two different ways the inner product

$$[f(p),(p+\overline{p_0})\star(p-p_0)^{-\star}\star\left(K(p,q)-\frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}\right)]$$

where f vanishes at $p=-\overline{p_0}$. Note that the function

$$(p+\overline{p_0})\star(p-p_0)^{-\star}\star\left(K(p,q)-\frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}\right)$$

vanishes also at the point $-\overline{p_0}$.

In the first way, we take $q=s$ to be real and make use of the isometry hypothesis (with the point $q_0=-\overline{p_0}$) to write for real s

$$\begin{aligned}[f(p),(p+\overline{p_0})\star(p-p_0)^{-\star}\star&\left(K(p,s)-\frac{K(p,p_0)K(p_0,s)}{K(p_0,p_0)}\right)]=\\ &=[(p-p_0)\star(p+\overline{p_0})^{-\star}\star f(p),K(p,s)-\frac{K(p,p_0)K(p_0,s)}{K(p_0,p_0)}]\\ &=(s-p_0)(s+\overline{p_0})^{-1}f(s)\end{aligned}$$

since the function $(p-p_0)\star(p+\overline{p_0})^{-\star}\star f(p)$ vanishes at $p=p_0$ and hence

$$0=[(p-p_0)\star(p+p_0)^{-\star}\star f(p),\frac{K(p,p_0)K(p_0,s)}{K(p_0,p_0)}].$$

In the second way we rewrite $(s-p_0)(s+\overline{p_0})^{-1}f(s)$ as

$$(s-p_0)(s+\overline{p_0})^{-1}f(s) = [f(p), \left(K(p,s) - \frac{K(p,-\overline{p_0})K(-\overline{p_0},s)}{K(-\overline{p_0},-\overline{p_0})}\right)(s-\overline{p_0})(s+p_0)^{-1}]. \tag{6.19}$$

Above we used the fact that $f(-\overline{p_0}) = 0$ and hence

$$0 = [f(p), \frac{K(p,-\overline{p_0})K(-\overline{p_0},s)}{K(-\overline{p_0},-\overline{p_0})}(s-\overline{p_0})(s+p_0)^{-1}].$$

We thus have for every function f vanishing at $-\overline{p_0}$

$$\begin{aligned}&[f(p), (p+\overline{p_0}) \star (p-p_0)^{-\star} \star \left(K(p,s) - \frac{K(p,p_0)K(p_0,s)}{K(p_0,p_0)}\right)] = \\ &= [f(p), \left(K(p,s) - \frac{K(p,-\overline{p_0})K(-\overline{p_0},s)}{K(-\overline{p_0},-\overline{p_0})}\right)(s-\overline{p_0})(s+p_0)^{-1}].\end{aligned}$$

STEP 3: *The reproducing kernel is of the form*

$$K(p,q) = \frac{1}{u}E(p) \star k(p,q) \star_r \overline{E(q)} - \frac{1}{v}F(p) \star k(p,q) \star_r \overline{F(q)} \tag{6.20}$$

with $k(p,q)$ *as in* (3.16), *and*

$$E(p) = (p+\overline{p_0}) \star K(p,p_0), \quad F(p) = (p-p_0) \star K(p,-\overline{p_0}),$$

and

$$u = \frac{2(\operatorname{Re} p_0)}{K(p_0,p_0)} \quad \text{and} \quad v = \frac{2(\operatorname{Re} p_0)}{K(-\overline{p_0},-\overline{p_0})}. \tag{6.21}$$

For real $p = t$, and multiplying equality (6.18) on the left by $(t-p_0)$ and on the right by $s+p_0$ we obtain:

$$\begin{aligned}&(t+\overline{p_0})K(t,s)(s+p_0) - (t-p_0)K(t,s)(s-\overline{p_0}) = \\ &= \frac{(t+\overline{p_0})K(t,p_0)K(p_0,s)(s+p_0)}{K(p_0,p_0)} - \frac{(t-p_0)K(t,-\overline{p_0})K(-\overline{p_0},s)(s-\overline{p_0})}{K(-\overline{p_0},-\overline{p_0})}.\end{aligned} \tag{6.22}$$

Note that $K(t,s)$ commutes with $\overline{p_0}$ since it commutes with p_0. Thus we can rewrite (6.22) as

$$\begin{aligned}2(\operatorname{Re} p_0)K(t,s)(t+s) = &\frac{(t+\overline{p_0})K(t,p_0)K(p_0,s)(s+p_0)}{K(p_0,p_0)} \\ &- \frac{(t-p_0)K(t,-\overline{p_0})K(-\overline{p_0},s)(s-\overline{p_0})}{K(-\overline{p_0},-\overline{p_0})}\end{aligned}$$

and hence

$$K(t,s)=\frac{E(t)u\overline{E(s)}-F(t)v\overline{F(s)}}{t+s},\quad t,s\in\Omega\cap\mathbb{R}.$$

The result follows by slice hyperholomorphic extension.

STEP 4: *The reproducing kernel is of the form* (6.16).

By hypothesis, $uv>0$. It suffices to take

$$E_+(p)=\sqrt{u}E(p)\quad\text{and}\quad E_-(p)=\sqrt{v}F(p)$$

if $u>0$, and

$$E_+(p)=\sqrt{-v}F(p)\quad\text{and}\quad E_-(p)=\sqrt{-u}E(p)$$

if $u<0$. □

Remark 6.2.4. Assume that $\mathcal{K}$ is a Pontryagin space. Then $uv>0$ in the preceding proof. Indeed, assume $uv<0$ If $u<0$ and $v>0$, the kernel $K(t,s)$ has an infinite number of negative squares on $\Omega\cap\mathbb{R}$, contradicting the hypothesis that $\mathcal{K}$ is a Pontryagin space.

If $u>0$ and $v<0$, the kernel $K(p,q)$ (equal to (6.20)) is positive definite, and in particular both $K(p_0,p_0)$ and $K(-\overline{p_0},-\overline{p_0})$ are positive. This forces

$$uv=\frac{(\operatorname{Re}p_0))^2}{K(p_0,p_0)K(-\overline{p_0},-\overline{p_0})}>0$$

which cannot be for $u>0$ and $v<0$.

Remark 6.2.5. Assume that $K(p,-\overline{p_0})\equiv 0$. Then the reproducing kernel is of the form (6.16) with $E_-(p)\equiv 0$.

As an example of a space where conditions (6.13) and (6.14) holds for real p_0, consider a slice hyperholomorphic function A. Then for any slice hyperholomorphic function f we have

$$A\star\frac{p+p_0}{p-p_0}\star f=\frac{p+p_0}{p-p_0}A\star f$$

and hence for $p=it$ we have

$$|(A\star\frac{p+p_0}{p-p_0}\star f)(p)|=|A\star f|(p)$$

and

$$\int_{\mathbb{R}}\left|(A\star\frac{p+p_0}{p-p_0}\star f)(p)|_{p=it}\right|^2dt=\int_{\mathbb{R}}|(A\star f)(p)|_{p=it}|^2\,dt.$$

6.3 The unit ball case

Theorem 6.3.1. *Let $\mathcal{K}$ be a reproducing kernel Krein of $\mathbb{H}$-valued functions slice hyperhomorphic in an axially symmetric s-domain Ω of the quaternions, symmetric with respect to the unit sphere $\mathbb{H}_1$ of the quaternions (i.e. if $\beta \in \Omega \setminus \{0\}$ then $(\overline{\beta})^{-1} \in \Omega$) and assume that the following condition holds: If $f \in \mathcal{K}$ vanishes at the point $q_0 \in \Omega \setminus \mathbb{H}_1$ then the function f_{q_0} defined by*

$$p \mapsto (1 - p\overline{q_0}) \star (p - q_0)^{-\star} \star f(p) \tag{6.23}$$

belongs to $\mathcal{K}$ and it holds that

$$[f, f] = [f_{q_0}, f_{q_0}]. \tag{6.24}$$

Assume furthermore that there exists a point $p_0 \neq 0 \in \Omega$ such that (6.24) holds, as well as

$$K(p_0, p_0)K(\overline{p_0}^{-1}, \overline{p_0}^{-1}) > 0, \tag{6.25}$$

and

$$[F\frac{\overline{p_0}}{p_0}, F\frac{\overline{p_0}}{p_0}] = [F, F]. \tag{6.26}$$

Then, the reproducing kernel of $\mathcal{K}$ is of the form

$$K(p, q) = E_+(p) \star (1 - p\overline{q})^{-\star} \star_r \overline{E_+(q)} - E_-(p) \star (1 - p\overline{q})^{-\star} \star_r \overline{E_-(q)}, \tag{6.27}$$

where the functions E_+ and E_- are slice hyperholomorphic in Ω.

Remarks 6.3.2.
(a) The complex-valued version of the above theorem can be found in the Hilbert space case in [9, Theorem 6.1, p. 173]. In the case of spaces of polynomials it was proved earlier in [76, Theorem 1, p. 231].
(b) Equations (6.24) and (6.26) are satisfied in particular if p_0 can be chosen real. This will hold in particular in the Hilbert space case.
(c) As for the half-plane case, condition (6.27) will hold in the case of a Pontryagin space.

Proof of Theorem 6.3.1. We follow the proof as in [9], suitably modified to take into account the non commutativity of the quaternions, as in Theorem 6.2.2 above.

Let p_0 be as in the statement of the theorem. The function (6.17) belongs to $\mathcal{K}$ and so does the function

$$(1 - p\overline{p_0}) \star (p - p_0)^{-\star} \star \left(K(p, q) - \frac{K(p, p_0)K(p_0, q)}{K(p_0, p_0)} \right). \tag{6.28}$$

This last function vanishes at the point $(\overline{p_0})^{-1}$. So, with

$$\Delta(p) = (1 - pp_0^{-1}) \star (p - (\overline{p_0})^{-1})^{-\star} \star (1 - p\overline{p_0}) \star (p - p_0)^{-\star} = \frac{\overline{p_0}}{p_0}$$

we can write for every $F \in \mathcal{K}$ vanishing at $(\overline{p_0})^{-1}$

$$
\begin{aligned}
&[F(p), (1-p\overline{p_0})\star(p-p_0)^{-\star}\star\left(K(p,q)-\frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}\right)] = \\
&\qquad = [(p-(\overline{p_0})^{-1})^{-\star}\star(1-pp_0^{-1})\star F(p), \Delta(p)\star\left(K(p,q)-\frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}\right) \\
&\qquad = [(p-p_0)\star(1-p\overline{p_0})^{-\star}\star F(p)\frac{\overline{p_0}}{p_0}, \left(K(p,q)-\frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}\right)\frac{\overline{p_0}}{p_0}] \\
&\qquad = [(p-p_0)\star(1-p\overline{p_0})^{-\star}\star F(p), \left(K(p,q)-\frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}\right)] \\
&\qquad = [(p-p_0)\star(1-p\overline{p_0})^{-\star}\star F(p), K(p,q)] - \\
&\qquad\qquad - [(p-p_0)\star(1-p\overline{p_0})^{-\star}\star F(p), \frac{K(p,p_0)K(p_0,q)}{K(p_0,p_0)}]
\end{aligned}
$$

where we have used (6.26). Taking $q=s$ real and taking into account that

$$(p-p_0)\star(1-p\overline{p_0})^{-\star}\star F(p)$$

vanishes at p_0 we thus have

$$[F(p), (p-p_0)\star(1-p\overline{p_0})^{-\star}\star\left(K(p,s)-\frac{K(p,p_0)K(p_0,s)}{K(p_0,p_0)}\right)] = (s-p_0)(1-s\overline{p_0})^{-1}F(s).$$

On the other hand, since $F((\overline{p_0})^{-1})=0$, the reproducing kernel property gives

$$(s-p_0)(1-s\overline{p_0})^{-1}F(s) = [F, \left(K(p,s)-\frac{K(p,(\overline{p_0})^{-1})K((\overline{p_0})^{-1},s)}{K((\overline{p_0})^{-1},(\overline{p_0})^{-1})}\right)(s-\overline{p_0})(1-sp_0)^{-1}].$$

Hence we get

$$
\begin{aligned}
&(1-p\overline{p_0})\star(p-p_0)^{-\star}\star\left(K(p,s)-\frac{K(p,p_0)K(p_0,s)}{K(p_0,p_0)}\right) = \\
&\qquad = \left(K(p,s)-\frac{K(p,(\overline{p_0})^{-1})K((\overline{p_0})^{-1},s)}{K((\overline{p_0})^{-1},(\overline{p_0})^{-1})}\right)(s-\overline{p_0})(1-sp_0)^{-1}.
\end{aligned}
$$

Setting $p=t$ real, multiplying on the left by $(1-p\overline{p_0})$ and on the right by $(1-sp_0)$ we get:

$$
\begin{aligned}
&(1-t\overline{p_0})\left(K(t,s)-\frac{K(t,p_0)K(p_0,s)}{K(p_0,p_0)}\right)(1-sp_0) = \\
&\qquad = (t-p_0)\left(K(t,s)-\frac{K(t,(\overline{p_0})^{-1})K((\overline{p_0})^{-1},s)}{K((\overline{p_0})^{-1},(\overline{p_0})^{-1})}\right)(s-\overline{p_0}).
\end{aligned}
$$

Since (6.24) holds, we get the result as in the proof of Theorem 6.2.2. $\square$

6.4 The conditions (6.15) and (6.25)

Assume that $K(p,q)$ is positive definite, and not identically vanishing. The Cauchy-Schwarz inequality will imply that (6.15) or (6.25) hold. A similar conclusion holds if we only know that $K(0,0) \neq 0$.

Proposition 6.4.1. *Assume $K(p,q) \not\equiv 0$. There exist $p \in \Omega$ such that $K(p,p) \neq 0$.*

Proof. If the kernel is positive definite the claim follows trivially from the Cauchy-Schwarz inequality. In the general case, we adapt an argument from [25, Proof of Theorem 4.2, p. 50]. Let $x_0 \in \Omega \cap \mathbb{R}$, and let

$$K(p,q) = \sum_{n,m=0}^{\infty} (p-x_0)^n k_{n,m} (\overline{q} - x_0)^n$$

be the power series of $K(p,q)$ near the point (x_0, x_0), with quaternionic coefficients $k_{n,m}$. We write $k_{n,m} = a_{n,m} + b_{n,m} j$, where $a_{n,m}$ and $b_{n,m}$ are complex numbers in $\mathbb{C}_i$. To show that these numbers are equal to 0 we take various choices of p.

Case 1. $p = x_0 + re^{it}$ with $r > 0$ and $t \in \mathbb{R}$: Since

$$(a_{n,m} + b_{n,m} j) e^{-imt} = e^{-imt} a_{n,m} + e^{imt} b_{n,m} j,$$

the condition $K(p,p) \equiv 0$ can be rewritten as

$$\left(\sum_{n,m=0}^{\infty} r^{n+m} e^{i(n-m)t} a_{n,m} \right) + \left(\sum_{n,m=0}^{\infty} \rho^{n+m} e^{i(n+m)t} b_{n,m} \right) j \equiv 0,$$

and it follows that

$$\sum_{n,m=0}^{\infty} r^{n+m} e^{i(n-m)t} a_{n,m} \equiv 0,$$

The complex case argument gives then $a_{n,m} = 0$ for all $n, m \in \mathbb{N}_0$. Note that the vanishing of the factor of j will not lead to the similar conclusion for the $b_{n,m}$.

Case 2. $p = x_0 + re^{jt}$: In view of the first case, $K(p,p)$ can be rewritten as

$$\begin{aligned} \sum_{n,m=0}^{\infty} p^n b_{n,m} j \overline{p}^m &= \left(\sum_{n,m=0}^{\infty} r^{n+m} (\operatorname{Re} b_{n,m}) e^{j(n-m)t} \right) j + \left(\sum_{n,m=0}^{\infty} r^{n+m} e^{jnt} i (\operatorname{Im} b_{n,m}) e^{-jmt} \right) j \\ &= \left(\sum_{n,m=0}^{\infty} r^{n+m} (\operatorname{Re} b_{n,m}) e^{j(n-m)t} \right) j + \left(\sum_{n,m=0}^{\infty} r^{n+m} e^{j(n+m)t} (\operatorname{Im} b_{n,m}) \right) ij. \end{aligned}$$

Thus $K(p,p) \equiv 0$ leads to

$$\left(\sum_{n,m=0}^{\infty} r^{n+m} (\operatorname{Re} b_{n,m}) e^{j(n-m)t} \right) \equiv 0,$$

and we have that $\operatorname{Re} b_{n,m} = 0$ for all $n, m \in \mathbb{N}_0$.

Case 3. $p = x_0 + re^{kt}$ (with $k = ij$): Then

$$\sum_{n,m=0}^{\infty} p^n ij(\operatorname{Im} b_{n,m})\overline{p}^m = k \cdot \sum_{n,m=0}^{\infty} \rho^{n+m} e^{k(n-m)t}(\operatorname{Im} b_{n,m})$$

and so $\operatorname{Im} b_{n,m} \equiv 0$.

□

6.5 A theorem on the zeros of a polynomial

We now turn to the counterpart of Theorem 2.2.4.

Theorem 6.5.1. *Let $T \in \mathbb{H}^{n\times n}$ be an invertible Hermitian matrix, with $\nu \geq 0$ negative eigenvalues. Assume furthermore that*

$$\begin{pmatrix} 1 & x & \dots & x^n \end{pmatrix} T^{-1} \begin{pmatrix} 1 & y & \dots & y^n \end{pmatrix}^t = \frac{A(x)\overline{A(y)} - x\overline{y}B(z)\overline{B(w)}}{1 - xy} \tag{6.29}$$

where A and B are polynomials of degree n. Then, A has ν zeros inside $\mathbb{B}_1$ and B has ν zeros outside $\mathbb{B}_1$. They have no zeros on the boundary $\partial\mathbb{B}_1$.

We remark that (6.29) can be rewritten as

$$\begin{aligned} \begin{pmatrix} 1 & p & \dots & p^n \end{pmatrix} T^{-1} \begin{pmatrix} 1 & q & \dots & q^n \end{pmatrix}^* &= (A(p)\overline{A(q)} - pB(p)\overline{B(q)}\overline{q}) \star (1 - p\overline{q})^{-\star} \\ &= \sum_{u=0}^{\infty} p^u \left(A(p)\overline{A(q)} - pB(p)\overline{B(q)}\overline{q} \right) \overline{q}^u \end{aligned} \tag{6.30}$$

by slice hyperholomorphic extension.

Proof of Theorem 6.5.1. We follow the steps of the proof of Theorem 6.5.1. The first step still holds since the spectral theorem holds for quaternionic Hermitian matrices. The second step works also in the same way. To consider the third step in the quaternionic setting, we first rewrite (6.30) as

$$\begin{aligned} &A(p)\overline{A(q)} - pB(p)\overline{B(q)}\overline{q} = \\ &= \begin{pmatrix} 1 & p & \dots & p^n \end{pmatrix} T^{-1} \begin{pmatrix} 1 & q & \dots & q^n \end{pmatrix}^* - p \begin{pmatrix} 1 & p & \dots & p^n \end{pmatrix} T^{-1} \begin{pmatrix} 1 & q & \dots & q^n \end{pmatrix}^* \overline{q}. \end{aligned} \tag{6.31}$$

Let q_0 be a zero of (say A) on $\partial\mathbb{B}_1$. Remark that

$$\begin{pmatrix} 1 & q_0 & \dots & q_0^n \end{pmatrix} T^{-1} \begin{pmatrix} 1 & q_0 & \dots & {q_0}^n \end{pmatrix}^* \in \mathbb{R} \tag{6.32}$$

and so setting $p = q = q_0$ in (6.31) leads to $B(q_0) = 0$, contradicting the previous step. The next two steps are also the same since the Krein-Langer factorization theorem holds in the slice hyperholomorphic setting.

□

Remark 6.5.2. The argument on the lack of zeros on the boundary $\partial\mathbb{B}_1$ will not hold in the matrix-valued case; then, (6.32) is not real but is a quaternionic Hermitian matrix which will not commute with q_0 in general.

Chapter 7

J-contractive functions

As explained in the introduction there are (at least) three important families of reproducing kernel Hilbert spaces introduced by de Branges and Rovnyak, and used in operator models and related topics. The previous section with the quaternionic counterpart of $\mathcal{H}(A,B)$ spaces. In this section and the next one we study $\mathcal{H}(\Theta)$ spaces in the quaternionic setting.

7.1 J-contractive functions in the quaternionic unit ball

REcalling the definition of hypermeromorphic functions, we can prove our next result:

Theorem 7.1.1. *Let Θ be a $\mathbb{H}^{n\times n}$-valued function slice hypermeromorphic in $\mathbb{B}_1$ and J-contractive there, and slice-hyperholomorphic in a neighborhood of $p=1$. Then it can be written as*

$$\Theta(p)=\Theta_1(p)\star\Theta_2(p)\star\Theta_3(p), \tag{7.1}$$

where Θ_1 is a Blaschke-Potapov product with all its zeros inside the open unit ball, Θ_2 is a Blaschke-Potapov product with all its zeros outside the closed unit ball, and Θ_3 is a singular factor in the following sense: $\chi(\Theta_3)$ is a singular $\chi(J)$-contractive function.

Proof. The idea of the proof is to go to the complex setting using the map χ, use an analytic extension theorem, and apply Potapov's theorem and apply then χ^{-1}. We proceed in a number of steps.

STEP 1: *The function $\chi(\Theta)$ has a meromorphic extension which is $\chi(J)$-contractive in the open unit disk $\mathbb{D}$.*

Indeed, restricting to $p, q \in (-1,1)$ and applying the map χ we have that the function

$$\frac{\chi(J) - \chi(\Theta)(x)\chi(J)(\chi(\Theta)(y))^*}{1-xy} \tag{7.2}$$

is positive for x, y in $(-1,1)$ where $\chi(\Theta)$ is defined. Let now

$$P = \frac{I+\chi(J)}{2} \quad \text{and} \quad Q = \frac{I+\chi(J)}{2}.$$

Since $\det(P + \chi(\Theta)Q) \not\equiv 0$ (to check this, reduce the situation to the case where $\chi(J)$ is replaced by a matrix of the form $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$), where I denotes the identity, one can define the Potapov-Ginzburg transform of $\chi(\Theta)$

$$\Sigma = (P + \chi(\Theta)Q)^{-1}(Q - \chi(\Theta)P).$$

It is such that:

$$\chi(J) - \chi(\Theta)(x)\chi(J)(\chi(\Theta)(y))^* = (P + \chi(\Theta)Q)\,(I - \Sigma(x)\Sigma(y)^*)\,(P + \chi(\Theta)Q)^*.$$

The kernel

$$\frac{I - \Sigma(x)\Sigma(y)^*}{1-xy}$$

is positive definite in $(-1,1)$. It follows that Σ is the restriction to $(-1,1)$ of an analytic contractive function; see [4, Théorème 2.6.3 p. 44]. Going back to Θ we get that $\chi(\Theta)$ is meromorphic and J-contractive in $\mathbb{D}$.
Following [80] another possibility is to consider the function

$$V(x) = (\chi(\Theta)(x) + I)^{-1}(\chi(\Theta)(x) - I)J.$$

Then

$$\frac{V(x) + V(y)^*}{2(1-xy)} = (\chi(\Theta)(x) + I)^{-1}\frac{J - \chi(\Theta)(x)J(\chi(\Theta)(y))^*}{1-xy}(\chi(\Theta)(y) + I)^{-*} \tag{7.3}$$

is positive definite on $(-1,1)$. It follows from Loewner's theorem (see [51, Theorem 1, p. 95] in the open half-plane case setting) that V is the restriction to $(-1,1)$ of a function analytic in $\mathbb{D}$, and hence the required conclusion for $\chi(\Theta)$.

In view of Step 1, we can introduce the Potapov's decomposition of $\chi(\Theta)$,

$$\chi(\Theta)(x) = P_1(x)P_2(x)P_3(x), \tag{7.4}$$

where each of the P_u is normalized by $P_u(1) = I$, and $\chi(\Theta)(x)$ is the restriction to $(-1,1)$ of an analytic function in the unit disc $\mathbb{D}$.

STEP 2: *There exist quaternionic Blasckhe-Potapov products of the first (resp. second) kind such that* $P_1(x) = \chi(\Theta_1(x))$ *and* $P_2(x) = \chi(\Theta_2(x))$.

The function $\chi(\Theta)$ satisfies the symmetry (4.8). The factors in the Potapov decomposition (1.14) of $\chi(\Theta)$ are invariant under the symmetry (4.8).

$$\chi(\Theta)(x) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \overline{\chi(\Theta)(x)} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{7.5}$$

The symmetry (4.8) is multiplicative (see Lemma 4.2.1), and the uniqueness of the factorization implies that each of the factors in the factorization (7.4) satisfies (7.5). In the case of P_1 and P_2 we see that non real poles or zeros will appear in pair. The same holds for P_3 when it is rational and J-unitary.

STEP 3: *We conclude by studying the structure of the factor P_3:*

We now study the factor $P_3(x)$, and slightly modify Potapov's original proof. In [80, pp. 216-220] Potapov proves that one can approximate uniformly on compact subsets of the open unit disk the term P_3 by rational functions. He then shows that the singularities of these rational functions converge to the unit circle $\mathbb{T}$. In fact one can directly construct such sequence of rational functions with singularities on $\mathbb{T}$. To that purpose, consider (as in (7.3) above) the function

$$V(x) = (P_3(x) + I)^{-1} (I - P_3(x)) J.$$

We have

$$\frac{V(x) + V(y)^*}{2(1 - xy)} = (P_3(x) + I))^{-1} \frac{J - P_3(x) J (P_3(y))^*}{1 - xy} (P_3(y) + I))^{-*}$$

for $x, y \in (-1, 1)$. By the already mentioned theorem of Loewner (see [51, Theorem 1, p. 95]), V has a unique analytic extension to the open unit disk, which as a real positive part there. By Herglotz's representation formula one can write in a unique way

$$V(x) = ia + \int_0^{2\pi} \frac{x + e^{it}}{x - e^{it}} dM(t) \tag{7.6}$$

and both a has the symmetry (7.5) and dM has the symmetry

$$\overline{dM(t)} = E\, dM(-t)\, E^{-1}$$

since the representation(7.6) is unique. Approximating dM by finite measures and so V by rational functions satisfying (7.5) we approximate $\chi(P_3)$ by finite products which satisfy (7.5) and with singularities on the unit circle, i.e. not of the form (1.16) as in [80], but of the form (4.20). Since these products converge, the limit satisfies (7.5), and we can write

$$P_3(x) = \chi(\Theta_3(x)).$$

The function $\chi(\Theta_3(x))$ is $\chi(J)$-contractive, as a limit of rational $\chi(J)$-inner functions. To finish the proof it is sufficient to take slice hyperholomorphic extension. $\square$

7.2 J-contractive functions in the right half-space

Definition 7.2.1. *Let J be a real signature operator. The $\mathbb{H}^{n\times n}$-valued function Θ slice hyperholomorphic in an open symmetric domain is called J-contractive if the (unique) solution of the equation*

$$J - \Theta(p)J\Theta(q)^* = pK_\Theta(p,q) + K_\Theta(p,q)\overline{q} \tag{7.7}$$

is positive definite in Ω.

Proposition 7.2.2. *The $\mathbb{H}^{n\times n}$-valued function Θ is J contractive if and only if the kernel*

$$Jk(p,q) - \Theta(p)J \star k(p,q) \star_r \Theta(q)^* \tag{7.8}$$

where

$$k(p,q) = (\overline{p}+\overline{q})(|p|^2 + 2(\mathrm{Re}(p))\overline{q} + \overline{q}^2)^{-1},$$

is positive definite in Ω.

Proof. For $x, y \in \Omega \cap (0,\infty)$ we have

$$K_\Theta(x,y) = \frac{J - \Theta(x)J\Theta(y)^*}{x+y}.$$

Then k is left slice hyperholomorphic in p, and right slice hyperholomorphic in $\overline{q}$. Moreover $k(x,y) = \frac{1}{x+y}$ for $x, y \in \Omega \cap (0,\infty)$. By taking the slice hyperholomorphic extension (left) in p and (right) in $\overline{q}$ we have

$$K_\Theta(p,q) = Jk(p,q) - \Theta(p)J \star k(p,q) \star_r \Theta(q)^*. \tag{7.9}$$

□

Corollary 7.2.3. *Assume that the $\mathbb{H}^{n\times n}$-valued function Θ is J-contractive. Then $\Theta(1/p)$ is J-contractive.*

Proof. Let

$$E = \{x \in \mathbb{R} \setminus \{0\}\,, \text{ such that } 1/x \in \Omega\}\,.$$

We have:

$$K_\Theta(1/x, 1/y) = \frac{J - \Theta(1/x)J\Theta(1/y)^*}{1/x + 1/y} = x\frac{J - \Theta(1/x)J\Theta(1/y)^*}{x+y}y, \quad x, y \in E.$$

Hence the kernel $\frac{J-\Theta(1/x)J\Theta(1/y)^*}{x+y}$ is positive definite on E. The claim follows then by slice hyperholomorphic extension. □

Proposition 7.2.4. *Assume that the $\mathbb{H}^{n\times n}$-valued function Θ is slice hyperholomorphic at ∞ and that $\Theta(\infty) = I_n$. Then:*

$$\lim_{x\to\infty} F(x) = 0 \tag{7.10}$$

and the limit $\lim_{x\to\infty} xF(x)$ *exists for all* $F \in \mathcal{H}(\Theta)$*, and defines a continuous linear operator from* $\mathcal{H}(\Theta)$ *into* $\mathbb{H}^n$*, with Riesz representation*

$$d^*\left(\lim_{x\to\infty} xF(x)\right) = \langle F, g_d\rangle \tag{7.11}$$

with

$$g_d(y) = Jd - \Theta(y)Jd$$

(and in particular the function $g_d \in \mathcal{H}(\Theta)$ *for every* $d \in \mathbb{H}^n$*).*

Proof. Let $F \in \mathcal{H}(\Theta)$ and $d \in \mathbb{H}^n$. For $x \in \mathbb{R}\cap\Omega$ we have by Cauchy-Schwarz inequality

$$\begin{aligned} |d^*F(x)| &= |\langle F(\cdot), K_\Theta(\cdot,x)d\rangle_{\mathcal{H}(\Theta)}| \\ &\le \|F\|\cdot\sqrt{d^*\frac{J-\Theta(x)J\Theta(x)^*}{2x}d} \\ &\to 0 \quad \text{as } x\to\infty \end{aligned} \tag{7.12}$$

To prove the second claim we note that the norm of the operator $x \mapsto xF(x)$ is $x^2\frac{\|J-\Theta(x)J\Theta(x)^*\|}{2x}$. In view of the analyticity at ∞, we can write

$$\Theta(x) = I_n + \frac{M}{x} + \frac{o(1/x)}{x}$$

where $\lim_{x\to\infty} o(1/x) = 0$. So

$$x\left(J-\Theta(x)J\Theta(x)^*\right) = -(MJ+JM^*) + \text{terms which go to 0 as } x\to\infty.$$

Thus

$$\sup_{x\in\mathbb{R}\cap\Omega} x^2\frac{\|J-\Theta(x)J\Theta(x)^*\|}{2x} < \infty.$$

It follows that for every $d \in \mathbb{H}^n$ the family of functions $xK(\cdot,x)d$ has a weak limit in $\mathcal{H}(\Theta)$. Let g_d be this limit. In a reproducing kernel Hilbert space, weak convergence implies pointwise convergence and so

$$g_d(y) = \lim_{x\to\infty} xK(y,x) = Jd - \Theta(y)Jd. \tag{7.13}$$

□

Definition 7.2.5. *Assume that* $\mathbb{H}^{n\times n}$*-valued function* Θ *is slice hyperholomorphic at* ∞ *and that* $\Theta(\infty) = I_n$*. We denote by* K *the operator from* $\mathbb{H}^n$ *into* $\mathcal{H}(\Theta)$ *defined by*

$$(Kd)(x) = Jd - \Theta(x)Jd \tag{7.14}$$

Lemma 7.2.6. *Assume that* $\mathbb{H}^{n\times n}$*-valued function* Θ *is slice hyperholomorphic at* ∞ *and that* $\Theta(\infty) = I_n$*. It holds that*

$$K^*f = \lim_{x\to\infty} xf(x) \tag{7.15}$$

Proof. Let $f \in \mathcal{H}(\Theta)$ and $d \in \mathbb{H}^n$. By definition of the weak limit, and using (7.13), we have:

$$\begin{aligned}\langle K^* f, d\rangle_{\mathbb{H}^n} &= \langle f, Jd - \Theta(\cdot)Jd\rangle \\ &= \lim_{x\to\infty} x\langle F, K(\cdot, x)d\rangle \\ &= \lim_{x\to\infty} x\langle f(x), d\rangle_{\mathbb{H}^n}.\end{aligned}$$

□

Theorem 7.2.7. *let J be a $\mathbb{R}^{n\times n}$-valued signature matrix. Then, the $\star$-product of J-contractive functions defined on a common axially symmetric set Ω is J-contractive.*

Proof. Let $K_1(p,q)$ and $K_2(p,q)$ be $\mathbb{H}^{n\times n}$-valued positive definite functions such that

$$J - \Theta_u(p)J\Theta_u(q)^* = pK_u(p,q) + K_u(p,q)\overline{q}, \quad u = 1,2.$$

Then,

$$\begin{aligned}&J - (\Theta_1(p)\star\Theta_2(p))J(\Theta_1(q)\star\Theta_2(q))^* = \\ &\quad = J - (\Theta_1(p)\star\Theta_2(p))J(\Theta_1(q)^*\star_r\Theta_2(q)^*) \\ &\quad = J - \Theta_1(p)J\Theta_1(q)^* + \Theta_1(p)\star(J - \Theta_2(p)J\Theta_2(q)^*)\star_r\Theta_1(q)^* \\ &\quad = pK(p,q) + K(p,q)\overline{q}\end{aligned}$$

with

$$K(p,q) = K_1(p,q) + \Theta_1(p)\star K_2(p,q)\star_r\Theta(q)^*.$$

This ends the proof since $K(p,q)$ is positive definite in Ω. □

Theorem 7.2.8. *Let $J \in \mathbb{R}^{n\times n}$ be a signature matrix and let $A_1,\ldots,A_N \in \mathbb{H}^{n\times n}$ be such that $A_jJ \geq 0$, $j = 1,\ldots,N$. Then*

$$\Theta(p) = e^{-\star pA_1}\star\cdots\star e^{-\star pA_N} \tag{7.16}$$

is J-inner.

Proof. We first prove the claim for $N = 1$. The claim for general N follows then from Theorem 7.2.7. Set $A_1 = A$ and let $x, y \in \mathbb{R}$. Taking into account that $AJ = JA^*$, integration by part leads to:

$$\begin{aligned}x\int_0^1 e^{-txA}AJe^{-tyA^*}dt &= \left[-e^{-txA}Je^{-tyA^*}\right]_{t=0}^{t=1} + y\int_0^1 e^{-txA}(-JA^*)e^{-tyA^*}dt \\ &= J - e^{-xA}Je^{-yA^*} - y\int_0^1 e^{-txA}AJe^{-tyA^*}dt,\end{aligned}$$

so that

$$\frac{J - e^{-xA}Je^{-yA^*}}{x+y} = \int_0^1 e^{-txA}AJe^{-tyA^*}dt, \tag{7.17}$$

and the claim follows from slice hyperholomorphic extension since $AJ \geq 0$. □

Remark 7.2.9. The preceding example is adapted from [79]. In that paper, and for complex matrices and $n = 2$ the decomposition (7.16) is shown to be unique, exhibiting a special case of the key result of de Branges [42] recalled in Theorem 1.2.4 above.

7.3 The case of entire functions

The following result is partial, but hints at new directions to be explored.

Theorem 7.3.1. *Let Θ be a $\mathbb{H}^{n\times n}$-valued slice hyperholomorphic entire J-inner functions, where $J \in \mathbb{R}^{n\times n}$ is a real signature matrix. Then $\chi(\Theta)$ extends to a $\chi(J)$-inner entire function. If it satisfies the uniqueness representation condition, Θ can then be written as a multiplicative integral of the form*

$$\Theta(x) = \overset{\curvearrowright}{\int_0^\ell} e^{xH(t)dt},$$

where H is a $\mathbb{H}^{n\times n}$-valued integrable function such that $H(t)J \geq 0$ on $[0, \ell]$.

Proof. By hypothesis, the kernel (7.9) is positive definite in the right half-space. Restricting to $p = x$ and $q = y$ in $(0, \infty)$, and applying the map χ defined by (2.2), we see that the kernel

$$\frac{\chi(J) - (\chi(\Theta)(x))\,\chi(J)\,(\chi(\Theta)(y))^*}{x + y} \tag{7.18}$$

is positive definite in $(0, \infty)$. The rest of the proof is divided into a number of steps.

STEP1: *The power series defining $\chi(\Theta)(x)$ extends to all $\mathbb{C}$ and the corresponding kernel*

$$\frac{\chi(J) - (\chi(\Theta)(z))\,\chi(J)\,(\chi(\Theta)(w))^*}{z + \overline{w}} \tag{7.19}$$

is positive definite in the open right half-plane.

This follows from Loewner's theorem; see Remark 1.3.4.

STEP 2: *There exist $\ell > 0$ and a $\mathbb{C}^{n\times n}$-valued function G defined on $[0, \ell]$ such that $G(t)\chi(J) \geq 0$ and*

$$\chi(\Theta)(z) = \overset{\curvearrowright}{\int_0^\ell} e^{zG(t)dt},$$

where we assume $\chi(\Theta)(0) = I_n$.

Indeed, since the function $\chi(\Theta)(x)$ is real analytic in the whole real line, the function $\chi(\Theta)(z)$ is entire, and we can apply Potapov's theorem (Theorem 1.2.4 above, but for the open right half-plane rather than the open upper half-plane) to write $\chi(\Theta)$ as a multiplicative integral (1.18).

STEP 3: *It holds that*

$$\chi(\Theta)(x) = \overset{\curvearrowright}{\int_0^\ell} e^{zG_1(t)dt}, \tag{7.20}$$

with

$$G_1(s) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \overline{G(s)} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

This follows from the limit of Θ as a limit of products of the form

$$\overset{\curvearrowright}{\prod_{k=0}^{n-1}} (I + G(s_j)(t_{j+1} - t_j))$$

from of the symmetry (7.5), and from the assumed uniqueness of the multiplicative integral representation of $\chi(\Theta)$.

STEP 4: *The function G may be chosen satisfying* (7.5).
Let $M(s,x) = \overset{\curvearrowright}{\int_0^s} e^{xG(t)dt}$. By definition of the multiplicative integral

$$\frac{\mathrm{d}}{\mathrm{ds}} M(s,x) = M(s,x)xG(s)$$

In view of (7.5) we can write

$$\frac{\mathrm{d}}{\mathrm{ds}} M(s,x) = M(s,x)xH(s)$$

with

$$H(s) = \frac{G(s) - EG(s)E}{2}.$$

So we can choose the matrix function in (1.18) to satisfy

$$H(s) = -EH(s)E,$$

and so $G = \chi(M)$ and the result follows.

$\square$

Chapter 8

The characteristic operator function

In this section we first briefly discuss the case of close to self-adjoint operators. We then focus on the close to anti self-adjoint operators, which is the case of interest in quaternionic analysis.

8.1 The problem with close to self-adjoint operators in the quaternionic case

Let A be a (say bounded) right linear operator from the right quaternionic Hilbert space $\mathcal{H}$ into itself, and assume that $A - A^*$ has finite rank equal to n. Then we can write

$$A - A^* = C^* E C, \tag{8.1}$$

where $E \in \mathbb{H}^{n\times n}$ satisfies

$$E^* = E^{-1} = -E^{-1}, \tag{8.2}$$

and following (1.2), set

$$\Theta(p) = I_n - C^* \star (A - pI)^{-\star} CE. \tag{8.3}$$

We have for $x, y \in \rho_S(A) \cap \mathbb{R}$

$$E - \Theta(x) E \Theta(y)^* = (x - y) C^* (A - xI)^{-1} (A - yI)^{-*} C. \tag{8.4}$$

Slice hyperholomorphic extension gives:

$$E - \Theta(p) E \Theta(q)^* = pM(p, q) - M(p, q)\overline{q}, \tag{8.5}$$

where $M(p, q)$ denotes the positive definite kernel

$$M(p, q) = C^* \star (A - pI)^{-\star} (A^* - \overline{q} I)^{-\star_r} \star_r C.$$

8.2 Properties of the characteristic operator function

In the setting of Definition 1.3.1 we have:

Proposition 8.2.1. *The characteristic function is J-contractive.*

Proof. Define

$$K(p,q) = C^* \star (I - pA^*)^{-\star}(I - A\overline{q})^{-\star_r} \star_r C. \tag{8.6}$$

Then, $K(p,q)$ is positive definite in a neighborhood of the origin. Furthermore (see [14, Theorem 7.7])

$$J - S(p)JS(q)^* = pK(p,q) + K(p,q)\overline{q}. \tag{8.7}$$

The claim follows from Proposition 7.2.2 □

When $p = q$ this equation becomes

$$J - S(p)JS(p)^* = pK(p,p) + K(p,p)\overline{p}. \tag{8.8}$$

Note that the matrix $J - S(p)JS(p)^*$ need not be non-negative. It will be non-negative for $x > 0$ since then we have

$$J - S(x)JS(x)^* = 2xK(x,x).$$

Example 8.2.2. We take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then (1.19) is met with

$$C = -J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$S(p) = \begin{pmatrix} 1+p^2 & p \\ p & 1 \end{pmatrix}. \tag{8.9}$$

Furthermore, for $x, y \in \mathbb{R}$ such that $x + y \neq 0$ we have:

$$\frac{J - S(x)JS(y)^*}{x+y} = -\begin{pmatrix} 1+xy & x \\ y & 1 \end{pmatrix}. \tag{8.10}$$

8.3 Examples

Example 8.3.1. We first consider the finite dimensional case. As proved in [18], the reproducing kernel Hilbert space $\mathcal{H}(\Theta)$ associated to Θ is finite dimensional if and only if Θ is rational, i.e. $\Theta(x)$ is a rational function of the real function x and with values in $\mathbb{H}^{n\times n}$.

Example 8.3.2. Here we follow [45, pp. 76-79]. Let $J_0 \in \mathbb{H}^{n\times n}$ be a signature matrix, let $\xi(t), t \in [0,\infty)$ be a continuous $\mathbb{H}^{1\times n}$-valued function, fix $\ell > 0$, and consider the integral operator A_ℓ defined on $\mathbf{L}_2([0,\ell], dx, \mathbb{H})$ by

$$(A_\ell f)(x) = \int_x^\ell \xi(x)J\xi(y)^* f(y)dy. \tag{8.11}$$

The operator is continuous since ξ is, and as in the complex-valued case, its adjoint is given by

$$(A_\ell^* f)(x) = \int_0^x \xi(x)J\xi(y)^* g(y)dy. \tag{8.12}$$

Indeed, with $f, g \in \mathbf{L}_2([0,\ell], dx, \mathbb{H})$, we can write:

$$\begin{aligned}
\langle A_\ell f, g\rangle &= \int_0^\ell g(x)^*(A_\ell f)(x)dx \\
&= \int_0^\ell \int_0^\ell 1_{[x,\ell](y)} g(x)^*\xi(x)J\xi(y)^* f(y)dydx \\
&= \int_0^\ell \left(\int_0^\ell 1_{[x,\ell](y)} g(x)^*\xi(x)J\xi(y)^* dx\right) f(y)dy,
\end{aligned}$$

so that

$$(A_\ell^* g)(y) = \left(\int_0^\ell 1_{[x,\ell](y)} g(x)^*\xi(x)J\xi(y)^* dx\right)^* = \int_0^y \xi(y)J\xi(x)^* g(x)dx,$$

which is (8.12) after interchanging x and y.

We have (1.19) with $J = -J_0$ and

$$Cf = \int_0^\ell \xi(y)^* g(y)dy.$$

Indeed

$$(A_\ell + A_\ell^*)(f) = \xi(x)J\left(\int_0^\ell \xi(y)^* g(y)dy\right).$$

The characteristic operator function of A is given by $JW(\ell, 1/p)J$ where W is the solution to the canonical differential equation

$$\frac{\mathrm{d}}{\mathrm{dt}}W(t,p) = \frac{1}{p}\xi(t)^*\xi(t)(t) \star W(t,p). \tag{8.13}$$

Indeed, we have for real x,

$$W(t,x) = I_n + \frac{J}{x}\int_0^t \xi(u)^*\xi(u)W(u,x)du. \tag{8.14}$$

Furthermore, by definition of A_ℓ^*

$$\begin{aligned}((A_\ell^*-xI)\xi W)(t,x) &= \xi(t)\left(\int_0^t J\xi(u)^*\xi(u)W(u)du - x\xi(t)W(t)\right)\\ &= \xi(t)\left(\int_0^t J\xi(u)^*\xi(u)W(u)du - xW(t)\right)\\ &= x\xi(t)\left(\frac{1}{x}\int_0^t J\xi(u)^*\xi(u)W(u)du - W(t)\right)\\ &= -x\xi(t).\end{aligned}$$

Hence

$$\frac{1}{x}\xi(t)W(t,x) = \left((A_\ell^*-xI)^{-1}\xi\right)(t,x),$$

that is

$$\frac{\xi W}{x} = -(xI-A_\ell^*)^{-1}C.$$

Thus

$$\int_0^\ell \xi^*(t)\xi(t)W(t,x)dt = -C^*(xI-A_\ell^*)^{-1}C,$$

or, equivalently

$$J(W(\ell,x)-I_n) = -C^*(xI-A_\ell^*)^{-1}C,$$

and so

$$JW(\ell,1/x)J = I_n - xC^*(I-xA_\ell^*)^{-}CJ.$$

8.4 Inverse problems

Theorem 8.4.1. *Let Θ be a $\mathbb{H}^{n\times n}$-valued function, slice hyperholomorphic and J-contractive in an axially symmetric domain $\Omega\subset\mathbb{H}_+$, which is a neighborhood of infinity, and assume that $\Theta(\infty)=I_n$. Then $\Theta(1/p)$ is the characteristic operator function of a right linear continuous operator.*

Proof. The idea of the proof is to consider the reproducing kernel Hilbert space $\mathcal{H}(\Theta)$ associated to K_Θ and to build the operator in $\mathcal{H}(\Theta)$. We note that Ω contains the set $(-\infty,-x_0)\cup(x_0,\infty)$ for some $x_0>0$ and that, for $|x|\geq x_0$ we have

$$\Theta(-x)J\Theta(x)^* = J. \tag{8.15}$$

We proceed in a number of steps.

STEP 1: *It holds that (see [20, Proof of Theorem 2.3 p. 598] for the complex case)*

$$R_aK_\Theta(x,y) = \frac{-1}{a+y}\left(K_\Theta(x,y) - K_\Theta(x,-a)J\Theta(a)J\Theta(y)^*\right) \tag{8.16}$$

Making use of (8.15) we have

$$K_\Theta(x,y)=\frac{\Theta(-y)J\Theta(y)^*-\Theta(x)J\Theta(y)^*}{x+y}=-(R_{-y}\Theta)(x)J\Theta(y)^*$$

from which (8.16) follows from the resolvent identity

$$R_a-R_b=(a-b)R_aR_b. \tag{8.17}$$

STEP 2: *Let $a,b\in\mathbb{R}\cap\Omega$. Then $R_a\mathcal{H}(\Theta)\subset\mathcal{H}(\Theta)$, R_a is bounded and*

$$\langle R_af,g\rangle+\langle f,R_bg\rangle+(a+b)\langle R_af,R_bg\rangle=-g(b)^*Jf(a),\quad \forall f,g\in\mathcal{H}(\Theta). \tag{8.18}$$

We follow [22], where the complex setting is considered. Equality (8.18) is first proved on finite linear combinations of kernels, using (8.16). For $f=K(\cdot,y)c$ and $g=K(\cdot,z)d$, this amounts to prove:

$$\begin{aligned}-K(z,b)JK(a,y)=&\frac{-1}{a+y}\left(K(z,y)-K(z,-a)J\Theta(a)J\Theta(y)^*\right)-\\&+\frac{-1}{a+y}\left(K(z,y)-\Theta(z)J\Theta(b)^*JK(-b,y)\right)+\\&+\frac{a+b}{(a+y)(b+z)}\{(K(z,y)-K(z,-a)J\Theta(a)J\Theta)(y)^*)-\\&\quad-\Theta(z)J\Theta(b)^*JK(-b,z)+\Theta(z)J\Theta(b)^*JK(-b,-a)\Theta(a)J\Theta(y)^*\}\,.\end{aligned} \tag{8.19}$$

The computation is simple but quite long and is omitted; it is maybe well to remark that, in the case $\Theta=0$, (8.19) reduces to the easily verified equality

$$-\frac{1}{(a+y)(x+y)}-\frac{1}{(b+z)(x+y)}+\frac{a+b}{(a+y)(b+z)(x+y)}+\frac{1}{(b+z)(a+y)}=0. \tag{8.20}$$

Since point evaluations are bounded, there exists $C_a>0$ such that

$$\|f(a)\|\le C_a\cdot\|f\|,\quad f\in\mathcal{H}(\Theta). \tag{8.21}$$

Setting $a=b$ we get then from (8.18) the inequality of the form

$$\|R_af\|^2\le\frac{1}{2a}\left(2\|R_af\|\cdot\|f\|+2C_a^2\|f\|^2\right), \tag{8.22}$$

for f which are linear combinations of functions of the form $K(\cdot,x)\xi$, where $x\in\Omega\cap\mathbb{R}$ and $\xi\in\mathbb{H}^n$. It follows that

$$\frac{\|R_af\|^2}{\|f\|^2}\le\frac{\|R_af\|}{a\|f\|}+C_a^2$$

for such f. It follows that R_a has a bounded extension to all of $\mathcal{H}(\Theta)$.

STEP 3: $\ker R_a = \{0\}$ *for* $a \in \mathbb{R}_+ \cap \Omega$, *and in particular for* $f \in \mathcal{H}(\Theta)$ *there exists at most one* $c_f \in \mathbb{H}^n$ *such that the function*

$$p \star f(p) + c_f \tag{8.23}$$

belongs to $\mathcal{H}(\Theta)$.

By the resolvent identity (8.17), we have $\ker R_a = \ker R_b$ for any choice of $a, b \in \mathbb{R} \cap \Omega$. Let $\xi \neq 0 \in \ker R_a$ and let $F \in \mathcal{H}(\Theta)$. Then, by Proposition 7.2.4 we have $\lim_{x\to\infty} F(x) = 0$, contradicting $\xi \neq 0$ when $\xi \in \mathcal{H}(\Theta)$.

STEP 4: *The operator* A *defined by* (8.23) *and with domain*

$\operatorname{Dom} A = \{f \in \mathcal{H}(\Theta) : \exists\, c_f \in \mathbb{H}^n$ *for which the function* (8.23) *belongs to* $\mathcal{H}(\Theta)\}$

is closed.

Functions whose restriction to $\mathbb{R} \cap \Omega$ is of the form

$$f(x) = \sum_{u=1}^{U} K_\Theta(x, x_u)\xi_u \quad \text{with} \quad \sum_{u=1}^{U} \Theta(x_u)^*\xi_u = 0$$

belong to the domain of A since

$$\begin{aligned}
xf(x) &= \sum_{u=1}^{U} x\frac{J - \Theta(x)J\Theta(x_u)^*}{x + x_u}\xi_u \\
&= \sum_{u=1}^{U} \frac{x + x_u - x_u}{x + x_u}\left(J - \Theta(x)J\Theta(x_u)^*\right)\xi_u \\
&= -\sum_{u=1}^{U} x_u\frac{J - \Theta(x)J\Theta(x_u)^*}{x + x_u}\xi_u + J\left(\sum_{u=1}^{U}\xi_u\right) - \\
&\quad - \Theta(x)J\left(\sum_{u=1}^{U}\Theta(x_u)^*\xi_u\right).
\end{aligned}$$

It follows that the domain of A is dense. Indeed, let $f \in \mathcal{H}(\Theta)$ orthogonal to the domain of A. In particular

$$\langle f, K_\Theta(\cdot, x)\xi - K(\cdot, y)\Theta(y)^{-*}\Theta(x)^*\xi\rangle_{\mathcal{H}(\Theta)} = 0$$

for all $x, y \in \mathbb{R} \cap \Omega$ and $\xi \in \mathbb{H}^n$. Thus, for all such x, y and ξ we have

$$\xi^* f(x) \equiv \xi^*\Theta(x)\Theta(y)^{-1}f(y)$$

and so the function $x \mapsto \Theta(x)^{-1}f(x)$ is constant, i.e. $f(x) = \Theta(x)\eta$ for some $\eta \in \mathbb{H}^n$. Letting $x \longrightarrow \infty$, and since $\Theta(\infty) = I_n$, we get a contradiction with (7.12).

Let now $a \in \mathbb{R} \cap \Omega$. Since A is one-to-one and densely defined, the operator $(A - aI)^{-1}$ exists as a possibly unbounded densely defined operator. Let $f \in \operatorname{Ran}(A - aI)$. Then $f = (A - aI)g$ for some $g \in \mathcal{H}(\Theta)$. Thus for $p = x \in \mathbb{R} \cap \Omega$ we have

$$f(x) = xg(x) + c_g - ag(x),$$

so that

$$g(x) = \frac{f(x) + c_g}{x - a}. \tag{8.24}$$

Since g is continuous across $\mathbb{R} \cap \Omega$ we have $c_g = -f(a)$ and $g = R_a f$. Since we proved that R_a is continous we have $\mathbb{R} \cap \Omega \subset \rho(A)$. Here the S spectrum reduces to the regular spectrum since a is real.

Take now $a = -b$ in (8.18) and $f = (A - a)F$ and $g = (A + a)G$. Then

$$f(a) = c_F \quad \text{and} \quad g(a) = c_G,$$

and (8.18) becomes (with K defined by (7.2.6))

$$\langle F, AG\rangle + \langle AF, G\rangle = -\langle f, KJK^*g\rangle.$$

It follows from this identity that A has an adjoint and is a bounded operator, and that, moreover,

$$A + A^* = -KJK^* \tag{8.25}$$

We can now conclude the proof.

Let $d \in \mathbb{N}$ and $x \in \mathbb{R} \cap \Omega$. We write

$$K^*(T - xI)^{-1}KJd = K^*(T - x)^{-1}(J - \Theta(\cdot)J)Jd = K^*G,$$

where

$$G(t) = \frac{J - \Theta(t)J - (J - \Theta(x)J)}{t - x}Jd = \frac{\Theta(x) - \Theta(t)}{t - x}d.$$

Computing

$$K^*G = \lim_{t\to\infty} tG(t) = (\Theta(x) - I_n)d.$$

Thus

$$\Theta(x)d = d + K^*G = d + K^*(T - xI)^{-1}KJ$$

and so

$$\Theta(x) = I_n - K^*(xI - A)^1KJ \quad \text{and so } \Theta(1/x) = I_n - xK^*(xI - A)^{-1}KJ \tag{8.26}$$

and so Θ is the characteristic operator function of A^*.

$\square$

Chapter 9

$\mathcal{L}(\Phi)$ spaces and linear fractional transformations

In the complex setting case (and in the related time-varying case), $\mathcal{L}(\Phi)$ spaces and $\mathcal{H}(\Theta)$ (and the related $\mathcal{H}(A, B)$ spaces) are closely related by linear fractional transformations; this originates with the works of de Branges and de Branges and Rovnyak. In this section we outline the counterpart of such relations in the quaternionic setting. We work in the framework of the slice hyperholomorphic weights, introduced in Sections 3.4 and 3.5.

Definition 9.0.2. *A $\mathbb{H}^{n\times n}$-valued functions, say Φ, slice hyperhomorphic in an axially symmetric open Ω subset of the right half-space or of the ball respectively, and such that the kernel*

$$K_\Phi(x,y) = \begin{cases} \dfrac{\Phi(x)+\Phi(y)^*}{x+y} & (\textit{half space case}) \quad \textit{or} \\ \dfrac{\Phi(x)+\Phi(y)^*}{1-xy} & (\textit{quaternionic unit ball case}) \end{cases}$$

is positive definite in $\Omega \cap \mathbb{R}$ is called a Carathéodory function (resp. a Herglotz function).

9.1 $\mathcal{L}(\Phi)$ spaces associated to analytic weights

We focus on the case of Carathéodory functions, and will only mention briefly some of the results pertaining to Herglotz functions.

Theorem 9.1.1. *Let W_+ be a $\mathbb{H}^{n\times n}$-valued slice hyperholomorphic weight in Ω such that $\int_{\mathbb{R}} \|W_+(it)\|^2 dt < \infty$. The function*

$$\Phi(a) = \int_{\mathbb{R}} W_+(it)^*(W_+(p) \star (a-p)^{-\star})_{|p=it} dt \tag{9.1}$$

extends uniquely to the right half-space to an Herglotz function, still denoted by Φ.

Proof. For $a, b \in \Omega \cap \mathbb{R}$ we have to compute

$$\begin{aligned}
&\Phi(a) + \Phi(b)^* \\
&= \int_{\mathbb{R}} W_+(it)^*[W_+(p) \star (a-p)^{-\star}]_{|p=it}\, dt + \int_{\mathbb{R}} [(b+\bar{p})^{-\star_r} \star_r W_+(p)^*]_{|p=it} W_+(it)\, dt \\
&= \int_{\mathbb{R}} W_+(it)^*[(a-p)^{-1} W_+(p)]_{|p=it}\, dt + \int_{\mathbb{R}} [W_+(p)^*(b+\bar{p})^{-1}]_{|p=it} W_+(it)\, dt \\
&= \int_{\mathbb{R}} W_+(it)^*(a-it)^{-1} W_+(it)\, dt + \int_{\mathbb{R}} W_+(it)^*(b-it)^{-1} W_+(it)\, dt \\
&= \int_{\mathbb{R}} W_+(it)^*[(a-it)^{-1} + (b+it)^{-1}] W_+(it)\, dt \\
&= (a+b) \int_{\mathbb{R}} W_+(it)^*[(a-it)^{-1}(b+it)^{-1}] W_+(it)\, dt
\end{aligned}$$

which expresses that $\frac{\Phi(a)+\Phi(b)^*}{a+b}$ is positive definite in $\Omega \cap \mathbb{R}$. The result follows by slice hyperholomorphic extension. $\square$

Theorem 9.1.2. *The associated reproducing kernel Hilbert space is given by the closure of the set of functions of the form*

$$F(a) = \int_{\mathbb{R}} W_+(it)^* \left(f(p) \star (a-p)^{-\star} \star W_+(p)\right)_{|p=it} dt \tag{9.2}$$

where f belongs to the right linear span of the functions $p \mapsto (p+a)^{-\star}$, and with norm

$$\|F\|^2_{W_+} = \int_{\mathbb{R}} \|(f(p) \star W_+(p))_{|p=it}\|^2 dt.$$

Proof. Let $N \in \mathbb{N}$, $x_1, \ldots, x_N \in \Omega \cap \mathbb{R}$ and $c_1, \ldots, c_N \in \mathbb{H}^n$. The claim follows from the formula

$$\sum_{n=1}^{N} K_\Phi(x, x_n) c_n = \int_{\mathbb{R}} W_+(it)^* \left(f(p) \star (a-p)^{-\star} \star W_+(p)\right)_{|p=it} dt,$$

where $f(p) = \sum_{n=1}^{N} (x_n + p)^{-\star} c_n$ is such that

$$\|\sum_{n=1}^{N} K_\Phi(\cdot, x_n) c_n\|^2_{W_+} = \int_{\mathbb{R}} \|((f \star W_+)(p))_{|p=it}\|^2 dt.$$

$\square$

Proposition 9.1.3. *The space $\mathcal{L}(\Phi)$ is R_b-invariant for $b \in \Omega \cap \mathbb{R}$. Furthermore R_b is bounded and it holds that*

$$\langle R_a F, G\rangle_{W_+} + \langle F, R_b G\rangle_{W_+} + (a+b)\langle R_a F, R_b G\rangle_{W_+} = 0, \quad \forall F, G \in \mathcal{L}(\Phi). \tag{9.3}$$

Proof. Let F be of the form (9.2). Letting $a \in \mathbb{R}$ going to infinity shows that the only constant function is 0. Furthermore we have

$$\frac{F(a)-F(b)}{a-b} = \int_{\mathbb{R}} W_+(it)^* \left(f(p)\star(b-p)^{-\star}\right)\star(a-p)^{-\star})_{|p=it}dt$$

and so

$$\begin{aligned}\|R_bF\|^2 &= \int_{\mathbb{R}} \|(f(p)\star(b-p)^{-\star})_{|p=it}W_+(it)\|^2dt\\ &= \int_{\mathbb{R}} \frac{\|(f(p)\star(b-p)^{-\star})_{|p=it}W_+(it)\|^2}{b^2+t^2}dt \le \frac{1}{b^2}\|F\|^2.\end{aligned}$$

□

Proposition 9.1.4. *The space $\mathcal{L}(\Phi)$ contains no non-zero constant element. To see that, let a go to $+\infty$ in (9.2). Hence the operator*

$$\mathsf{M}_pF = pF + c_F$$

where $c_F \in \mathbb{H}^n$ is uniquely defined, is anti-selfadjoint in $\mathcal{L}(\Phi)$ and satisfies

$$R_b = (\mathsf{M}_p - bI)^{-1}, \quad b \in \Omega\cap\mathbb{R}. \tag{9.4}$$

Proof. Let F be of the form (9.2). We have

$$\begin{aligned}aF(a) &= a\int_{\mathbb{R}} W_+(it)^*\left(f(p)\star(a-p)^{-\star}\star W_+(p)\right)_{|p=it}\,dt\\ &= \int_{\mathbb{R}} W_+(it)^*\left(f(p)\star(a-p+p)\star(a-p)^{-\star}\star W_+(p)\right)_{|p=it}\,dt\\ &= \int_{\mathbb{R}} W_+(it)^*\left(f(p)\star(p)\star(a-p)^{-\star}\star W_+(p)\right)_{|p=it}\,dt\\ &\quad \underbrace{+\int_{\mathbb{R}} W_+(it)^*\left(f(p)\star W_+(p)\right)_{|p=it}\,dt}_{-c_F},\end{aligned}$$

and so M_p is unitarily equivalent to multiplication by it in $\mathbf{L}_2(W_+^*W_+, dt)$, and hence anti selfadjoint, since the above equality extends by slice hyperholomorphic extension. Furthermore, let F and G in $\mathcal{L}(\Phi)$ be such that $(\mathsf{M}_p - bI)^{-1}F = G$. Then, for real $a\in\Omega$ we have

$$F(a) = aG(a) + c_G - bG(a),$$

so that

$$G(a) = \frac{F(a)-c_G}{a-b}.$$

It follows that $c_G = F(a)$ and that (9.4) holds. □

We now turn to the case of the quaternionic unit ball. We define

$$\Phi(x) = \int_0^{2\pi} W_+(e^{it})^* \left(\left(\frac{p+x}{p-x} \right) \star W_+(p) \right)_{|p=e^{it}} dt, \quad t \in (-1,1). \tag{9.5}$$

The proof of the following proposition is similar to that of Proposition 9.1.3 and is omitted.

Proposition 9.1.5. *The function $\Phi(x)$ extends to a uniquely defined Carathéodory function in $\mathbb{B}_1$. The associated reproducing kernel Hilbert space $\mathcal{L}(\Phi)$ is R_b-invariant for $b \in (-1,1)$. The operators R_b are bounded in $\mathcal{L}(\Phi)$ and satisfy*

$$\langle F, G\rangle + a\langle R_aF, G\rangle + b\langle F, R_bG\rangle + (1-ab)\langle R_aF, R_bG\rangle = 0 \tag{9.6}$$

where $F, G \in \mathcal{L}(\Phi)$ and $a, b \in (-1,1)$.

It follows from (9.6) that the corresponding space $\mathcal{L}(\Phi)$ contains no non-zero constant, and one can define the operator M_p as in the half-plane case. It is also useful to note that

$$f_-(a) = 2\int_{\mathbb{R}} W_+(e^{it})^* \left(f(e^{it}) \star (e^{it}-a)^{-\star} \star W_+(e^{it}) \right) - \Phi(a) \star f(a),$$

where the function Φ is given by (9.5).

9.2 Linear fractional transformations and inverse problem

As in the complex setting case, and as one can see from our previous works [1, 8, 18], linear fractional transformations play a key role in Schur analysis and related problems in operator theory. The following result was first proved by de Branges and Rovnyak in [43]. The argument is essentially the same.

Theorem 9.2.1. *Let J_1 be given by (1.8), and let Θ and Φ be respectively J_1-contractive and of Carathéodory class, with corresponding reproducing kernel Hilbert spaces $\mathcal{H}(\Theta)$ and $\mathcal{L}(\Phi)$. Then the map τ*

$$\tau : \quad F \mapsto \begin{pmatrix} \Phi & I_n \end{pmatrix} \star F$$

is a contraction from $\mathcal{H}(\Theta)$ into $\mathcal{L}(\Phi)$ if and only if there is a Schur multiplier S such that

$$\Phi \star (\Theta_{11} - \Theta_{12} - (\Theta_{11}+\Theta_{12}) \star S) = (\Theta_{21}+\Theta_{22}) \star S + (\Theta_{22} - \Theta_{21}) \tag{9.7}$$

Proof. Assuming τ contractive (and hence bounded). As we know from [18], its adjoint is given by

$$\tau^* \left(K_\Phi(\cdot, b)c \right) = K_\Theta(\cdot, b) \star_r \begin{pmatrix} \Phi(b)^*c \\ c \end{pmatrix}$$

The fact that τ is contractive is equivalent to the fact that the kernel

$$K_\Phi(p,q) - \begin{pmatrix}\Phi & I_n\end{pmatrix} \star K_\Theta(p,q) \star_r \begin{pmatrix}\Phi(q)^*c \\ c\end{pmatrix}$$

is positive definite in Ω. To finish the proof note that this kernel can be rewritten as

$$\begin{pmatrix}\Phi(x) & I_n\end{pmatrix} \frac{\Theta(x)J_1\Theta(y)^*}{x+y} \begin{pmatrix}\Phi(y)^*c \\ c\end{pmatrix}$$

is positive definite for $x, y \in \Omega \cap \mathbb{R}$. This kernel in turn can be written as

$$\begin{aligned}&\frac{(\Phi\Theta_{11}+\Theta_{21})(\Phi\Theta_{12}+\Theta_{22})^* + (\Phi\Theta_{12}+\Theta_{22})(\Phi\Theta_{11}+\Theta_{21})}{x+y} = \\ &\frac{(\Phi\Theta_{11}+\Theta_{21}+\Phi\Theta_{12}+\Theta_{22})(\Phi\Theta_{11}+\Theta_{21}+\Phi\Theta_{12}+\Theta_{22})^*}{2(x+y)} \\ &- \frac{(\Phi\Theta_{11}+\Theta_{21}-\Phi\Theta_{12}-\Theta_{22})(\Phi\Theta_{11}+\Theta_{21}-\Phi\Theta_{12}-\Theta_{22})^*}{2(x+y)}.\end{aligned}$$

To conclude one uses the slice hyperholomorphic version of Leech's theorem; see [18, Section 11.6]. To prove the converse just read backwards. $\square$

In the complex setting, the question of finding all the linear fractional representations of a given Carathéodory (or Herglotz) function allows to gather under a common framework a wide range of problems, ranging from interpolation problems to inverse spectral problems. It bears the name *lossless inverse scattering problem* because of its connections with linear network theory; see Definition 1.2.1. See [49, 50, 81, 4] for the links with network theory. A systematic study of this problem was done in [20, 21]. The following result follows [20, Theorem 3.1, p. 600] and gives a family of solutions to the corresponding problem in the quaternionic setting. Recall that the space $\mathcal{M}^\square$ was introduced in the proof of Theorem 3.4.3. See Definition 3.4.4.

Theorem 9.2.2. *Let W_+ be an analytic weight and let $\mathcal{M}$ be a resolvent-invariant subspace of $\mathbf{L}_2(W_+^*W_+)$, with extension $\mathcal{M}^\square$. The map $F \mapsto \begin{pmatrix}\Phi & I_n\end{pmatrix} F$ is an isometry from $\mathcal{M}^\square$ into $\mathcal{L}(\Phi)$.*

Proof. By definition of f_- we have

$$\begin{aligned}\begin{pmatrix}\Phi & I_n\end{pmatrix} F &= \Phi(a)f(a) + f_-(a) \\ &= \int_{\mathbb{R}} W_+(it)^*W_+(it) \star \left\{(f(it)-f(a)) \star (it-a)^{-\star} + (it-a)^{-\star}f(a)\right\} dt \\ &= \int_{\mathbb{R}} W_+(it)^*W_+(it) \star f(it) \star (it-a)^{-\star} dt.\end{aligned}$$

$\square$

It follows from Theorem 3.4.3 that $\mathcal{M}^\square$ is a $\mathcal{H}(\Theta)$ space and from Theorem 9.2.1 that Φ and Θ are related by a linear fractional transformation.

Definition 9.2.3. *For a given analytic weight W_+ we will call lossless inverse scattering problem the question of describing the associated function Φ as a linear fractional transformation of the form* (9.2.1).

In a future publication we will consider this problem in greater details.

Chapter 10

Canonical differential systems

Canonical differential systems and their connections to operators have a long history; see e.g. [2, 36, 57, 87, 88]. In this section we consider such systems in the quaternionic setting. in particular in the case of rational spectral data.

10.1 The matrizant

Let J be a signature matrix with real entries. Canonical differential systems are differential equations of the form (8.13), or of the more particular form

$$J\frac{\mathrm{d}}{\mathrm{dt}}F(t,p) = (pI + H(t)) \star F(t,p), \tag{10.1}$$

where p is a quaternionic variable, and have been thoroughly studied in the complex setting; see [86]. They provide a convenient unifying framework to discuss a number of questions pertaining to inverse scattering, non-linear partial differential equations, and other topics. The solution of either of these equations subject to the initial condition I_n is called the matrizant.

Of particular interest is the case where $J = J_0$ (with J_0 as in(1.21)) and H is of the form

$$H(t) = \begin{pmatrix} 0 & v(t) \\ -v(t)^* & 0 \end{pmatrix},$$

that is

$$J_0\frac{\mathrm{d}}{\mathrm{dt}}F(t,p) = \left(pI + \begin{pmatrix} 0 & v(t) \\ -v(t)^* & 0 \end{pmatrix}\right) \star F(t,p). \tag{10.2}$$

Proposition 10.1.1. *The matrizant is an entire function of p.*

Proof. It suffices to apply the map ω (see (3.18)) to (10.2), and use the corresponding complex-variable result (see e.g. [57] for the latter). $\square$

In [30] a class of potentials, called pseudo-exponential potentials, were introduced; these potentials correspond to rational characteristic spectral functions, and in the present setting are given by the formula

$$v(t) = -2ce^{-2ta}\left(I_m + \Omega Y - \Omega e^{-2ta^*} Y e^{-2ta}\right)^{-1}(b + \Omega c^*). \tag{10.3}$$

In this expression, $(a, b, c) \in \mathbb{H}^{m\times m} \times \mathbb{H}^{m\times n} \times \mathbb{H}^{n\times m}$, with the property that the S spectra of a and of $a^\times = a - bc$ are in the right half-space and Ω and Y belong to $\mathbb{H}^{m\times m}$ and are the unique solutions of the Lyapunov equations

$$\begin{aligned} Ya + a^*Y &= c^*c && (10.4)\\ \Omega a^{\times *} + a^\times \Omega &= bb^*. && (10.5)\end{aligned}$$

Because of the hypothesis on the spectra of a and $a^\times$ we note that Ω and Y are strictly positive and thus $(I_m + Y\Omega)$ is invertible. For future reference we note that

$$\Omega(I_m + Y\Omega)^{-1} = \sqrt{\Omega}(I_m + \sqrt{\Omega} Y \sqrt{\Omega})^{-1}\sqrt{\Omega} > 0, \tag{10.6}$$

where $\sqrt{\Omega}$ denotes the positive squareroot of Ω.

An explicit formula for the matrizant is given in the following theorem; see [32, Theorem 2.1, p.9]; we follow the notation of that paper, but the reader should bear in mind differences in signs and variables because of the present setting. In a way similar to [32] we set $f = (b^* + c\Omega)(I_m + Y\Omega)^{-1}$,

$$F = \begin{pmatrix} c & 0\\ 0 & f\end{pmatrix}, \quad T = \begin{pmatrix} -a & 0\\ 0 & -a^*\end{pmatrix},$$

$$G = \begin{pmatrix} 0 & -f^*\\ -c^* & 0\end{pmatrix}, \quad Z = \begin{pmatrix} 0 & \Omega(I_m + Y\Omega)^{-1}\\ Y & 0\end{pmatrix},$$

and

$$Q(t, s) = Fe^{tT}\left(I_{2m} - e^{tT}Ze^{tT}\right)^{-1} e^{sT}G. \tag{10.7}$$

The invertibility of $X(t) = I_{2m} - e^{tT}Ze^{tT}$ for $t > 0$ will be proved in Theorem 10.1.2. Furthermore we note that

$$TJ_0 = J_0T, \quad FJ_0 = J_0F, \quad GJ_0 = -J_0G, \quad \text{and} \quad ZJ_0 = -J_0Z. \tag{10.8}$$

To ease the notation, we do not precise the size the identity matrices in J_0. These will always be understood from the context.

Theorem 10.1.2. *Let J_0 be as in* (1.21) *and let Q be given by* (10.7)*. Then, the function*

$$U(t, q) = e^{tqJ_0} + \int_t^\infty Q(t, s) \star e^{sqJ_0}\,ds \tag{10.9}$$

is the unique solution to the canonical system (10.2) *subject to the boundary condition*

$$\lim_{\substack{q = x\in\mathbb{R}\\ t\to\infty}} e^{-txJ_0}U(t, x) = I_{2n}. \tag{10.10}$$

where the potential $v(t)$ in (10.2) *is given by* (10.3)*.*

Remark 10.1.3. In the framework of complex numbers, the factor e^{-txJ_0} is replaced by e^{-itxJ_0}, allowing an interpretation in terms of incoming and outgoing wave, hence the terminology terms such as scattering function and the like. Here, as we already noticed, the boundary is made of the purely imaginary quaternions. We do not have a wave interpretation of the various functions associated in Section 10.2 to U.

Proof of Theorem 10.1.2. We follow the arguments of [32, Proof of Theorem 2.1, p.9], and proceed in a number of steps.

STEP 1: *Z is the (unique) solution of the Lyapunov equation*

$$TZ + ZT = GF \tag{10.11}$$

Indeed,

$$TZ + ZT = -\begin{pmatrix} 0 & a\Omega(I_m + Y\Omega)^{-1} + \Omega(I_m + Y\Omega)^{-1}a^* \\ a^*Y + Ya & 0 \end{pmatrix}.$$

The $(2,1)$ block is equal to c^*c in view of (10.4). Multiplying the $(1,2)$ block by (-1) we can write (using (10.4) and (10.5) to go from the third line to the fifth one)

$$\begin{aligned}
&a\Omega(I_m + Y\Omega)^{-1} + (I_m + \Omega Y)^{-1}\Omega a^* = \\
&\quad = (I_m + \Omega Y)^{-1}\left(a^*(I_m + Y\Omega) + (I_m + \Omega Y)a\Omega\right)(I_m + Y\Omega)^{-1} \\
&\quad = (I_m + \Omega Y)^{-1}\left(\Omega a^* + a\Omega + \Omega(a^*Y + Ya)\Omega\right)(I_m + Y\Omega)^{-1} \\
&\quad = (I_m + \Omega Y)^{-1}\left(\Omega a^{\times *} + a^{\times}\Omega + \Omega c^*b^* + bc\Omega + \Omega(a^*Y + Ya)\Omega\right)(I_m + Y\Omega)^{-1} \\
&\quad = (I_m + \Omega Y)^{-1}\left(bb^* + \Omega c^*b^* + bc\Omega + \Omega c^*c\Omega\right)(I_m + Y\Omega)^{-1} \\
&\quad = (I_m + \Omega Y)^{-1}\left((b + \Omega c^*)(b + \Omega c^*)^*\right)(I_m + Y\Omega)^{-1} \\
&\quad = f^*f,
\end{aligned}$$

and hence the result since

$$GF = \begin{pmatrix} 0 & -f^*f \\ -c^*c & 0 \end{pmatrix}.$$

STEP 2: *The matrix function $X(t) = I_{2m} - e^{tT}Ze^{tT}$ is invertible for every real t.*

To prove the claim we write

$$X(t) = \begin{pmatrix} I_m & -e^{-ta}\Omega(I_m + Y\Omega)^{-1}e^{-ta^*} \\ -e^{-ta^*}Ye^{-ta} & I_m \end{pmatrix}$$

By Schur complement $X(t)$ is invertible if and only if

$$I_m + e^{-ta}\Omega(I_m + Y\Omega)^{-1}e^{-ta^*}e^{-ta^*}Ye^{-ta}$$

is invertible. But the matrices

$$e^{-ta}\Omega(I_m+Y\Omega)^{-1}e^{-ta^*} \quad \text{and} \quad e^{-ta^*}Ye^{-ta}$$

are positive (in fact strictly positive since Y and $\Omega(I_m+Y\Omega)^{-1}$ are strictly positive; see (10.6) for the latter). So $X(t)$ is invertible. In view of later computation we note that

$$X^{-1}(t) = \begin{pmatrix} I_m & -e^{-ta}\Omega(I_m+Y\Omega)^{-1}e^{-ta^*} \\ e^{-ta^*}Ye^{-ta} & I_m \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} (I_m+e^{-ta}\Omega(I_m+Y\Omega)^{-1}e^{-2ta^*}Ye^{-ta})^{-1} & 0 \\ 0 & (I_m+e^{-ta^*}Ye^{-2ta}\Omega(I_m+Y\Omega)^{-1}e^{-ta^*})^{-1} \end{pmatrix}. \tag{10.12}$$

STEP 3: *$Q(t,s)$ satisfies the differential equation*

$$J_0\frac{\partial Q}{\partial t}+\frac{\partial Q}{\partial s}J_0 = (J_0Q(t,t)-Q(t,t)J_0)Q(t,s). \tag{10.13}$$

With $X(t)=I_{2m}-e^{tT}Ze^{tT}$ for $t>0$, and using the Lyapunov equation (10.11), we have

$$X'(t) = -e^{tT}TZe^{tT}-e^{tT}ZTe^{tT} = -e^{tT}GFe^{tT}.$$

Hence we can write:

$$\begin{aligned}\frac{\partial Q}{\partial t} &= Fe^{Tt}X^{-1}e^{sT}G - Fe^{tT}X^{-1}X'X^{-1}e^{st}G \\ &= Fe^{Tt}TX^{-1}e^{sT}G + Fe^{tT}X^{-1}e^{tT}GFe^{tT}X^{-1}e^{st}G \\ &= Fe^{Tt}TX^{-1}e^{sT}G + Q(t,t)Q(t,s).\end{aligned}$$

On the other hand,

$$\frac{\partial Q}{\partial s} = Fe^{Tt}X^{-1}e^{sT}TG.$$

Hence, using (10.8),

$$\begin{aligned}J_0\frac{\partial Q}{\partial t}+\frac{\partial Q}{\partial s}J_0 &= Fe^{Tt}TJ_0X^{-1}e^{sT}G + J_0Q(t,t)Q(t,s) - Fe^{Tt}X^{-1}J_0e^{sT}TG \\ &= J_0Q(t,t)Q(t,s) + Fe^{tT}X^{-1}(-J_0TX+XTJ_0)X^{-1}e^{sT}G \\ &= J_0Q(t,t)Q(t,s) + Fe^{tT}X^{-1}e^{tT}(TJ_0Z-ZTJ_0)e^{tT}X^{-1}e^{sT}G \\ &= J_0Q(t,t)Q(t,s) - Fe^{tT}X^{-1}e^{tT}(TZJ_0+ZTJ_0)e^{tT}X^{-1}e^{sT}G \\ &= J_0Q(t,t)Q(t,s) - Fe^{tT}X^{-1}e^{tT}GFJ_0e^{tT}X^{-1}e^{sT}G \\ &= (J_0Q(t,t)-Q(t,t)J_0)Q(t,s).\end{aligned}$$

STEP 4: *We have*

$$J_0Q(t,t)-Q(t,t)J_0 = \begin{pmatrix} 0 & v(t) \\ v(t)^* & 0 \end{pmatrix}$$

with v given by (10.3).

Let $(Q_{ij})_{i,j=1,2}$ denote the the block matrix decomposition of Q. We first note that

$$J_0Q(t,t) - Q(t,t)J_0 = \begin{pmatrix} 0 & 2Q_{12}(t,t) \\ -2Q_{21}(t,t) & 0 \end{pmatrix}.$$

We have

$$\left(I_{2m} - e^{tT}Ze^{tT}\right) = \begin{pmatrix} I_m & -e^{-ta}\Omega(I_m+Y\Omega)^{-1})e^{-ta^*} \\ -e^{-ta^*}Ye^{-ta} & I_m \end{pmatrix},$$

and so

$$\left(I_{2m} - e^{tT}Ze^{tT}\right)^{-1} = \begin{pmatrix} I_m & e^{-ta}\Omega(I_m+Y\Omega)^{-1}e^{-ta^*} \\ e^{-ta^*}Ye^{-ta} & I_m \end{pmatrix} \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & \Delta_1^{-*} \end{pmatrix}$$

with

$$\Delta_1 = I_m - e^{-ta}\Omega(I_m+Y\Omega)^{-1}e^{-2ta^*}Ye^{-ta}.$$

Hence

$$\begin{aligned} Q(t,t) = -&\begin{pmatrix} ce^{-ta} & 0 \\ 0 & fe^{-ta^*} \end{pmatrix} \begin{pmatrix} I_m & e^{-ta}\Omega(I_m+Y\Omega)^{-1}e^{-ta^*} \\ e^{-ta^*}Ye^{-ta} & I_m \end{pmatrix} \times \\ &\times \begin{pmatrix} \Delta_1^{-1} & 0 \\ 0 & \Delta_1^{-*} \end{pmatrix} \begin{pmatrix} 0 & e^{-ta}f^* \\ e^{-ta^*}c^* & 0 \end{pmatrix}, \end{aligned}$$

so that

$$Q_{12}(t) = Q_{21}(t)^* = -ce^{-ta}\Delta_1^{-1}e^{-ta}f^*.$$

To conclude we recall that $f = (b^* + c\Omega)(I_m + Y\Omega)^{-1}$; we write (since $\Omega(I_m + Y\Omega)^{-1} = (I_m + \Omega Y)^{-1}\Omega$)

$$\begin{aligned} -ce^{-ta}\Delta_1^{-1}e^{-ta}f^* &= -ce^{-2ta}(I_m + \Omega(I_m - Y\Omega)^{-1}e^{-2ta^*}Ye^{-2ta})^{-1}(I_m+\Omega Y)^{-1}(b+\Omega c^*) \\ &= -ce^{-2ta}(I_m + \Omega Y - \Omega e^{-2ta^*}Ye^{-2ta})^{-1}(b+\Omega c^*). \end{aligned}$$

We note that the matrix function

$$I_m + \Omega Y - \Omega e^{-2ta^*}Ye^{-2ta} = \sqrt{\Omega}\left(I_m + \sqrt{\Omega}(Y - \Omega e^{-2ta^*}Ye^{-2ta})\sqrt{\Omega}\right)\sqrt{\Omega}^{-1}$$

is invertible for all $t \geq 0$ since $Y \geq e^{-2ta^*}Ye^{-2ta}$ for all such $t \geq 0$.

STEP 5: *The function U defined by* (10.9) *is a solution of the canonical system* (10.2).

We first take $p = x$ real. We can write:

$$\begin{aligned}
J_0\frac{\mathrm{d}}{\mathrm{dt}}U(t,x) &= xe^{txJ_0}I_n - J_0Q(t,t)e^{txJ_0} + \int_t^\infty J_0\frac{\partial Q}{\partial t}(t,s)e^{sxJ_0}ds\\
&= xe^{txJ_0}I_{2n} - J_0Q(t,t)e^{txJ_0} +\\
&\quad + \int_t^\infty \left(-\frac{\partial Q}{\partial s}(t,s)J_0 + (J_0Q(t,t) - Q(t,t)J_0)Q(t,s)\right)e^{sxJ_0}ds\\
&= xe^{txJ_0}I_{2n} - J_0Q(t,t)e^{txJ_0} + Q(t,t)J_0e^{txJ_0} -\\
&\quad - x\int_t^\infty Q(t,s)J_0e^{sxJ_0}ds + (J_0Q(t,t) - Q(t,t)J_0)Q(t,s))\,e^{sxJ_0}ds\\
&= (xI_{2n} + (J_0Q(t,t) - Q(t,t)J_0))\,U(t,x),
\end{aligned}$$

where the various integrals converge since the integral

$$\int_t^\infty e^{sT}e^{xsJ_0}ds = \int_0^\infty \begin{pmatrix} e^{-s(a-x)}I_m & 0\\ 0 & e^{-s(a^*+x)}I_m\end{pmatrix}ds \tag{10.14}$$

converges for all x not in the spectral sets of a or $-a^*$.

STEP 6: *The function* (10.9) *is the only solution of* (10.2) *with the asserted asymptotics.*

We first check that (10.9) satisfies (10.10). Because of the spectra condition on a we have

$$\lim_{t\to\infty} e^{-xtJ_0}e^{tT} = 0.$$

On the other hand,

$$\begin{aligned}
\int_t^\infty e^{sT}Ge^{sJ_0x}ds &= \int_t^\infty \begin{pmatrix} 0 & -e^{-s(a+xI_n)}f^*\\ e^{-s(a^*-xI_n)}c^* & 0\end{pmatrix}ds\\
&= \begin{pmatrix} 0 & -(a+xI_n)^{-1}e^{-t(a+xI_n)}f^*\\ (a^*-xI_n)^{-1}e^{-t(a^*-xI_n)}c^* & \end{pmatrix}\\
&\longrightarrow 0_{2n\times 2n}
\end{aligned}$$

as $t\to\infty$. It follows that (10.10) is in force. To prove uniqueness consider U_1 and U_2 two solutions. Then $U_1(t,p)\star(U_1(0,p))^{-\star}$ and $U_2(t,p)\star(U_2(0,p))^{-\star}$ have the same initial condition at $t=0$ and thus coincide. Taking now into account the asymptotic condition (10.10) we get

$$\lim_{t\to\infty} e^{-txJ_0}U_k(t,x)U_k(0,x))^{-1} = U_k(0,x)^{-1}, \quad k = 1,2,$$

and hence $U_1(0,x)^{-1} = U_2(0,x)^{-1}$. Thus U_1 and U_2 have the same initial condition at $t=0$, and so $U_1(t,x) = U_2(t,x)$. The result for x replaced by a quaternionic variable follows by slice hyperholomorphic extension. $\square$

Proposition 10.1.4. *Let $\Theta(t,p)$ be the solution of* (10.1) *with initial condition $\Theta(0,p)=I_n$. Then, for $x,y\in\mathbb{R}\cap\Omega$ we have:*

$$\int_0^a \Theta(u,x)^*\Theta(u,y)du=\frac{\Theta(a,x)^*J_0\Theta(a,y)-J_0}{x+y}. \tag{10.15}$$

Proof. We have

$$\begin{aligned}
x\int_0^a \Theta(u,x)^*\Theta(u,y)du+\int_0^a \Theta(u,x)^*\Theta(u,y)ydu=&\\
&=\int_0^a\left(\frac{\mathrm{d}}{\mathrm{du}}\Theta(u,x)^*J_0+\Theta(u,x)^*V(u)\right)\Theta(u,y)du+\\
&\quad+\int_0^a\Theta(u,x)^*\left(\frac{\mathrm{d}}{\mathrm{du}}J_0\Theta(u,y)-V(u)\Theta(u,y)\right)du\\
&=\int_0^a\frac{\mathrm{d}}{\mathrm{du}}\left(\Theta(u,x)^*J_0\Theta(u,y)\right)du\\
&=\Theta(a,x)^*J_0\Theta(a,y)-J_0,
\end{aligned}$$

and hence the result. □

Multiplying on the left by $\begin{pmatrix}I_n & I_n\end{pmatrix}$ and by its transpose on the right, and setting

$$\begin{pmatrix}I_n & I_n\end{pmatrix}\Theta(t,x)=\begin{pmatrix}E_+(t,x) & E_-(t,x)\end{pmatrix}, \tag{10.16}$$

we get

$$\int_0^a\begin{pmatrix}I_n & I_n\end{pmatrix}\Theta(u,x)^*\Theta(u,y)\begin{pmatrix}I_n\\ I_n\end{pmatrix}du=\frac{E_+(a,x)E_+(a,y)^*-E_-(a,x)E_-(a,y)^*}{x+y}. \tag{10.17}$$

We set $S(a,p)=E_+(a,p)^{-\star}E_-(a,p)$. Then the kernel on the right side of (10.17) can be rewritten as

$$E_+(a,x)\frac{I_n-S(a,x)S(a,y)^*}{x+y}E_+(a,y)^*. \tag{10.18}$$

We set

$$K_{S_a}(x,y)=\frac{I_n-S(a,x)S(a,y)^*}{x+y}. \tag{10.19}$$

Theorem 10.1.5. *For every $f\in\mathbf{L}_2([0,a],dx,\mathbb{H}^n)$ there exists a $Tf\in\mathbf{H}_2(\Pi_+)\ominus S_a\star\mathbf{H}_2(\Pi_+)$ such that*

$$\int_0^a\begin{pmatrix}I_n & I_n\end{pmatrix}\Theta(u,p)f(u)du=E_+(a,p)(Tf)(p) \tag{10.20}$$

and the map $f\mapsto Tf$ is unitary.

Proof. We now study the image of the operator $J_0\frac{\mathrm{d}}{\mathrm{dt}}f - Vf$ under T. From (10.17) we see that the function

$$\Theta(u,y)\begin{pmatrix} I_n \\ I_n \end{pmatrix}c, \quad c \in \mathbb{H}^n,$$

is send isometrically to the function $K_{S_a}(u,y)E_+(a,y)^*c$, and similarly for finite linear combinations of such functions. For a finite linear combination

$$x(u) = \sum_{m=1}^{M} \Theta(u,y_m)\begin{pmatrix} I_n \\ I_n \end{pmatrix}c_m$$

the image is

$$\begin{aligned}\sum_{m=1}^{M} y_m K_{S_a}(u,y_m)E_+(a,y_m)^*c_m = x\left(\sum_{m=1}^{M} y_m K_{S_a}(u,y_m)E_+(a,y_m)^*c_m\right) + \\ + \sum_{m=1}^{M} E(a,y_m)^*c_m - \\ - S_a(a,x)\left(\sum_{m=1}^{M} yS_a(a,y_m)^*E(a,y_m)^*c_m\right),\end{aligned}$$

that is

$$Tx(u) = ux(u) + c_x + S_a(u,a)d_x$$

with

$$\begin{aligned} c_x &= \sum_{m=1}^{M} E(a,y_m)^*c_m, \\ d_x &= -\sum_{m=1}^{M} yS_a(a,y_m)^*E(a,y_m)^*c_m. \end{aligned}$$

□

10.2 The characteristic spectral functions

The function $U(0,p)$ plays a key role in direct and inverse problems associated to (10.2); following the complex case we will call it the *asymptotic equivalence matrix function*.

Theorem 10.2.1. *The asymptotic equivalence matrix function associated to a potential of the form* (10.3) *is rational and* J_0*-unitary,*

$$U(0,x)J_0U(0,-x)^* = J_0, \tag{10.21}$$

at all real points where it is defined, with a realization given by

$$U(0,p) = I_{2m} + C \star (pI_{2m} - A)^{-\star} B \tag{10.22}$$

where

$$\begin{aligned} A &= \begin{pmatrix} a^* & 0 \\ 0 & -a \end{pmatrix} & (10.23)\\ B &= \begin{pmatrix} c^* & 0 \\ 0 & (I_m + \Omega Y)^{-1}(b + \Omega c^*) \end{pmatrix} & (10.24)\\ C &= \begin{pmatrix} -c\Omega & c(I_m + \Omega Y) \\ -(b^* + c\Omega) & (b^* + c\Omega)Y \end{pmatrix}, & (10.25) \end{aligned}$$

that is,

$$\begin{aligned} U_{11}(0,p) &= I_n - c\Omega(pI_m - a^*)^{-\star}c^* & (10.26)\\ U_{12}(0,p) &= c(I_m + \Omega Y)(a + pI_m)^{-\star}(I_m + \Omega Y)^{-1}(b + \Omega c^*) & (10.27)\\ U_{21}(0,p) &= -(b^* + c\Omega)(pI_m - a^*)^{-\star}c^* & (10.28)\\ U_{22}(0,p) &= I_n + (b^* + c\Omega)Y(a + pI_m)^{-\star}(I_m + \Omega Y)^{-1}(b + \Omega c^*) & (10.29) \end{aligned}$$

Finally the associated Hermitian matrix to the realization (10.23)-(10.25) *is*

$$H = \begin{pmatrix} -\Omega & I_m + \Omega Y \\ I_m + Y\Omega & -(Y + Y\Omega Y) \end{pmatrix}. \tag{10.30}$$

Proof. We have

$$\begin{aligned} U(0,p) &= I_{2n} + F(I_{2m} - Z)^{-1}\int_0^\infty e^{sT}Ge^{sxJ_0}ds \\ &= I_{2n} + \begin{pmatrix} c & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} I_m & -\Omega(I_m + Y\Omega)^{-1} \\ -Y & I_m \end{pmatrix}^{-1} \times \\ &\quad \times \int_0^\infty \begin{pmatrix} e^{-sa} & 0 \\ 0 & e^{-sa^*} \end{pmatrix}\begin{pmatrix} 0 & -f^* \\ -c^* & 0 \end{pmatrix}\begin{pmatrix} e^{sx}I_n & 0 \\ 0 & e^{-sx}I_n \end{pmatrix} ds. \end{aligned}$$

We have

$$\begin{aligned} &\int_0^\infty \begin{pmatrix} e^{-sa} & 0 \\ 0 & e^{-sa^*} \end{pmatrix}\begin{pmatrix} 0 & -f^* \\ -c^* & 0 \end{pmatrix}\begin{pmatrix} e^{sx}I_n & 0 \\ 0 & e^{-sx}I_n \end{pmatrix} ds \\ &\qquad = \begin{pmatrix} 0 & (a + xI_m)^{-1}f^* \\ (a^* - xI_m)^{-1}c^* & \end{pmatrix} \end{aligned}$$

Finally we check that (4.5) and (4.6) are satisfied with H given by (10.30). $\square$

The following result is the counterpart of [32, Theorem 3.1, p. 14].

Theorem 10.2.2. *There exists a unique $\mathbb{H}^{2n\times n}$-valued solution $X(t,p)$ to the canonical system* (10.2) *subject to the boundary conditions*

$$\begin{aligned} \begin{pmatrix} I_n & -I_n \end{pmatrix} X(0,p) &\equiv 0, & (10.31)\\ \lim_{\substack{q=x\in\mathbb{R}\\ t\to\infty}} \begin{pmatrix} 0 & e^{tx}I_n \end{pmatrix} U(t,x) &= I_n. & (10.32) \end{aligned}$$

Then the limit

$$\lim_{\substack{q=x\in\mathbb{R}\\ t\to\infty}} \begin{pmatrix} e^{-tx}I_n & 0 \end{pmatrix} U(t,x) \tag{10.33}$$

exists.

Proof. The proof is easily adapted from the of [32, Theorem 3.1, p. 14]. We look for a solution of the form $X(t,p) = U(t,p) \star \begin{pmatrix} S(p) \\ L(p) \end{pmatrix}$, where S and L are slice hyperholomorphic. Condition (10.31) gives

$$(U_{11}(0,x) - U_{21}(0,x))S(x) = (U_{22}(0,x) - U_{12}(0,x))L(x).$$

Rewriting (10.32) as

$$\lim_{\substack{q=x\in\mathbb{R}\\ t\to\infty}} \begin{pmatrix} 0 & I_n \end{pmatrix} e^{-txJ_0} U(t,x) = 0,$$

and taking into account (10.10) we then obtain $B(x) = I_n$, and hence $L(p) = I_n$, and

$$S(p) = (U_{11}(0,p) - U_{21}(0,p))^{-\star} \star (U_{22}(0,p) - U_{12}(0,p)). \tag{10.34}$$

□

The function (10.34) is called the *scattering function* associated to the system (10.2).

Theorem 10.2.3. *The scattering function is rational, unitary in the sense that*

$$S(x)S(-x)^* = I_n \tag{10.35}$$

at all real points where it is defined, and admits a Wiener-Hopf factorization $S(p) = S_+(p) \star S_-(p)$ where

$$\begin{aligned} S_-(p) &= I_n - b^* \star (pI_m - a^*)^{-\star} c^*, & (10.36)\\ S_+(p) &= I_n - (b^*Y - c)(I_m + \Omega Y)^{-1} \star (pI_m + a^\times)^{-\star}(b + \Omega c^*), & (10.37) \end{aligned}$$

with inverses

$$\begin{aligned} S_-(p)^{-\star} &= I_n + b^* \star (pI_m - a^{\times *})^{-\star} c^*, & (10.38)\\ S_+(p)^{-\star} &= I_n + (b^*Y - c) \star (pI_m + a)^{-\star}(I_m + Y\Omega)^{-1}(b + \Omega c^*). & (10.39) \end{aligned}$$

Proof. Equation (10.35) is obtained after multiplying both sides of (10.21) by $\begin{pmatrix} I_n & -I_n \end{pmatrix}$ on the left and by its transpose on the right.

The formulas for S_- and $S_+^{-\star}$ follow directly from (10.26)-(10.29). To obtain S_+ we use formula (4.2) for the realization of the inverse; taking into account the Lyapunov equations (10.4) and (10.5) and the definition of $a^\times$ (that is, $a^\times = a - bc$) we first compute

$$\begin{aligned}
-a - (I_m + \Omega Y)^{-1}&(b+\Omega c^*)(b^*Y - c) \\
&= -a - (I_m + \Omega Y)^{-1}(bb^*Y + \Omega c^*b^*Y - bc - \Omega cc^*) \\
&= -a - (I_m + \Omega Y)^{-1}\left\{(\Omega a^{\times *} + a^\times\Omega)Y + \right. \\
&\quad \left. +\Omega(a^* - a^{\times *})Y + a^\times - a - \Omega(Ya + a^*Y)\right\} \\
&= -a - (I_m + \Omega Y)^{-1}\left\{a^\times(I_m + \Omega Y) - (I_m + \Omega Y)a\right\} \\
&= -(I_m + \Omega Y)^{-1}a^\times(I_m + \Omega Y).
\end{aligned}$$

□

We note that S also admits a factorization of the type (4.7).

The inverse scattering problem (as opposed to the lossless inverse scatteing problem defined earlier) consists in finding the potential associated to a function satisfying the hypothesis of the previous theorem.

We now define two other characteristic spectral functions, namely the spectral function and the Weyl function.

Definition 10.2.4. *The function $S_-^{-*}S_-^{-1}$ is called the spectral function.*

Definition 10.2.5. *Let $\Theta(t,p)$ be the matrizant, defined as the solution to (10.2) with initial condition $\Theta(0,p) = I_{2n}$. The uniquely defined $\mathbb{H}^{n\times n}$-valued function N such that*

$$\int_0^\infty \begin{pmatrix} N(x)^* & I_n \end{pmatrix} \Theta(u,x)^*\Theta(u,x) \begin{pmatrix} N(x) \\ I_n \end{pmatrix} du < \infty \tag{10.40}$$

is called the Weyl function.

10.3 Canonical differential systems associated to an operator

We follow closely the papers [32, 66], and [83, 84, 85], and associate to an operator A satisfying (1.19) a canonical differential expression of the type (10.1).

Let A be a right quaternionic operator in the right quaternionic Hilbert space $\mathcal{H}$, with finite dimensional part, and write as in (1.19) above:

$$A + A^* = -C^*JC.$$

Lemma 10.3.1. *Consider the differential equation*

$$\begin{aligned} D'(t) &= -JD(t)A, \quad t > 0 && (10.41)\\ D(0) &= C, && (10.42)\end{aligned}$$

where the unknown $D(t)$ is $\mathbf{L}(\mathcal{H}, \mathbb{H}^n)$-valued and let $\Sigma(t)$ be the $\mathbf{L}(\mathcal{H}, \mathcal{H})$-valued function defined by

$$\Sigma(t) = I_{\mathcal{H}} + \int_0^t D(u)^* D(u) du, \quad t > 0. \tag{10.43}$$

Then

$$\Sigma(t)A + A^*\Sigma(t) = -D(t)^* JD(t), \quad t > 0. \tag{10.44}$$

Proof. Differentiating both sides of (10.44) we obtain

$$D(t)^*D(t)A + A^*D(t)^*D(t) = -D'(t)^*JD(t) - D(t)^*JD'(t),$$

which holds since $D'(t) = -JD(t)A$ and $J^2 = I_n$. Equation (10.44) reduces to (1.19) for $t = 0$, and hence (10.44) holds fo all $t \geq 0$. □

Define

$$S(t, p) = I_n - pD(t)\Sigma(t)^{-1} \star (I_n - pA^*)^{-\star} D(t)^* J \tag{10.45}$$

Proposition 10.3.2. *The function* (10.45) *is J-contractive for every $t \geq 0$.*

Proof. We first recall that the inverse mapping theorem holds in the quaternionic setting; as in the complex case this follows from the open mapping theorem, still valid in the quaternionic setting; see [18, p. 73]. Hence $\Sigma(t)$ is boundedly invertible since $\Sigma(t) \leq I_{\mathcal{H}}$. We also note that any bounded positive quaternionic operator has a unique positive squareroot; this follows from the spectral theorem for quaternionic self-adjoint operators; see [12]. Rewrite now (10.44) as

$$A(t) + A(t)^* = -C(t)^* JC(t)$$

with

$$A(t) = (\Sigma(t))^{1/2} A(\Sigma(t))^{-1/2} \quad \text{and} \quad C(t) = D(t)(\Sigma(t))^{-1/2}.$$

Then, (10.45) becomes

$$S(t, p) = I_n - pC(t)\Sigma(t) \star (I_n - pA(t)^*)^{-\star} C(t)^* J, \tag{10.46}$$

and the result follows then from Proposition 8.2.1. □

The following is an adaptation of [66, Proposition 2.2].

Proposition 10.3.3. *With the notation of Lemma 10.3.1, the function $W(t, p) = S(t, p) \star e^{\frac{Jt}{p}}$ is a solution of the canonical differential system*

$$\frac{\mathrm{d}}{\mathrm{dt}} W(t, p) = \left(JH(t) + \frac{J}{p} \right) \star W(t, p) \tag{10.47}$$

where

$$H(t) = D(t)\Sigma^{-1}(t)D(t)^*J - JD(t)\Sigma^{-1}(t)D(t)^*. \tag{10.48}$$

Proof. We follow the proof of [66, Proposition 2.2].

STEP 1: *We find a differential equation satisfied by the function $D(t)\Sigma(t)^{-1}$.*

Removing the dependence on t to lighten the notation we have:

$$\begin{aligned}(D\Sigma^{-1})' &= -JDA\Sigma^{-1} - D\Sigma^{-1}\Sigma'\Sigma^{-1}\\ &= -JD(-\Sigma^{-1}D^*JD\Sigma^{-1} - \Sigma^{-1}A^*) - D\Sigma^{-1}D^*D\Sigma^{-1}\\ &= J(D\Sigma^{-1})A^* + J(D\Sigma^{-1}D^*J - JD\Sigma^{-1}D^*)(D\Sigma^{-1})\end{aligned}$$

STEP 2: *We assume $p = x \in \Omega \cap (0, \infty)$ and show the derivative of $S(t, x)$ with respect to t satisfies*

$$\frac{\mathrm{d}}{\mathrm{dt}}S(t,x) = \left(JH(t) + \frac{J}{x}\right)S(t,x) - \frac{S(t,x)J}{x}. \tag{10.49}$$

The $\star$-product reduces to the pointwise product and we have (still removing the dependence on t in most of the instances)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}S(t,x) = &-xJ(D\Sigma^{-1})A^*(I - xA^*)^{-1}D^*J-\\ &- xJ(D\Sigma^{-1}D^*J - JD\Sigma^{-1}D^*)(D\Sigma^{-1})(I - xA^*)^{-1}D^*J+\\ &+ xD\Sigma^{-1}(I - xA^*)^{-1}A^*D^*.\end{aligned} \tag{10.50}$$

Writing $xA^*(I - xA^*)^{-1} = -I + (I - xA^*)^{-1}$ we can rewrite the first and third terms in the above sum as:

$$\begin{aligned}-xJ(D\Sigma^{-1})A^*(I - xA^*)^{-1}D^*J &= JD\Sigma^{-1}D^*J - JD\Sigma^{-1}(I - xA^*)^{-1}D^*J\\ xD\Sigma^{-1}(I - xA^*)^{-1}A^*D^* &= -D\Sigma^{-1}D^* + D\Sigma^{-1}(I - xA^*)^{-1}D^*.\end{aligned}$$

We further remark that

$$\begin{aligned}JD\Sigma^{-1}D^*J - D\Sigma^{-1}D^* &= JH,\\ -JD\Sigma^{-1}(I - xA^*)^{-1}D^*J &= \frac{J(S(x,p) - I_n)}{x},\\ -D\Sigma^{-1}(I - xA^*)^{-1}D^* &= \frac{(S(x,p) - I_n)J}{x}.\end{aligned}$$

Thus (10.50) can be rewritten as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}S(t,x) &= JH(S - I_n) + JH+\\ &\quad + \frac{J(S - I)}{x} - \frac{(S - I)J}{x}\\ &= (JH + \frac{J}{x})S - \frac{SJ}{x}.\end{aligned} \tag{10.51}$$

It is then clear that:

STEP 3: *The function* $W(t,x) = S(t,x)e^{\frac{Jt}{x}}$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{dt}}W(t,x) = \left(JH(t) + \frac{J}{x}\right)W(t,x)$$

$\square$

Remark 10.3.4. The operator A does not determine uniquely H. Indeed, take $a > 0$ and $c = \sqrt{2a}$ so that (1.19) is met with $J = -1$. Then (10.48) leads to $H = 0$ since J is a real scalar, but $S(x) = \frac{1+pa}{1-pa}$.

Remark 10.3.5. The matrix function H satisfies $H(t) + H(t)^* = 0$. When

$$J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{10.52}$$

it is of interest to find when H is of the form

$$H(t) = \begin{pmatrix} 0 & v(t) \\ -v(t)^* & 0 \end{pmatrix}. \tag{10.53}$$

The $\mathbb{H}^{n\times n}$-valued function v is then called the potential.

10.4 An example

We conclude this section with an example.

Example 10.4.1. With A, J, and C as in Example 8.2.2, the differential equation (10.41) has solution

$$D(t) = \begin{pmatrix} 0 & 1+t \\ 1 & 0 \end{pmatrix},$$

and the function (10.43) equals

$$\begin{aligned}\Sigma(t) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^t D(u)^*D(u)du = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 & 0 \\ 0 & (1+u)^2 \end{pmatrix} du \\ &= \begin{pmatrix} 1+t & 0 \\ 0 & \frac{(1+t)^3+2}{3} \end{pmatrix}.\end{aligned}$$

Hence, for real x

$$S(t,x) = \begin{pmatrix} \dfrac{(1+t)^3 + 2 + 3x^2(1+t)^2}{(1+t)^3+2} & \dfrac{3x(1+t)}{(1+t)^3+2} \\ x & 1 \end{pmatrix},$$

with associated potential

$$v(t) = \frac{3(1+t)}{(1+t)^3+2} - 1.$$

Bibliography

[1] Kh. Abu-Ghanem, D. Alpay, F. Colombo, D.P. Kimsey, and I. Sabadini. Boundary interpolation for slice hyperholomorphic Schur functions. *Integral Equations and Operator Theory*, 82:223–248, 2015.

[2] V. M. Adamjan. On the theory of canonical differential operators in Hilbert space. *Dokl. Akad. Nauk SSSR*, 178:9–12, 1968.

[3] D. Alpay. Some remarks on reproducing kernel Kreĭn spaces. *Rocky Mountain J. Math.*, 21:1189–1205, 1991.

[4] D. Alpay. *Algorithme de Schur, espaces à noyau reproduisant et théorie des systèmes*, volume 6 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 1998.

[5] D. Alpay. *An advanced complex analysis problem book. Topological Vector Spaces, Functional Analysis, and Hilbert spaces of analytic functions.* Birkhäuser/Springer Basel AG, Basel, 2015.

[6] D. Alpay, J. Ball, I. Gohberg, and L. Rodman. *Realization and factorization of rational matrix functions with symmetries*, volume 47 of *Operator Theory: Advances and Applications* , pages 1–60. Birkhäuser Verlag, Basel, 1990.

[7] D. Alpay, J. Ball, I. Gohberg, and L. Rodman. J–unitary preserving automorphisms of rational matrix functions: State space theory, interpolation and factorization. *Linear Algebra and its Applications*, 197–198:531–566, 1994.

[8] D. Alpay, V. Bolotnikov, F. Colombo, and I. Sabadini. Interpolation problems for certain classes of slice hyperholomorphic functions. *Integral Equations Operator Theory*, 86(2):165–183, 2016.

[9] D. Alpay, V. Bolotnikov, and Ph. Loubaton. On tangential H_2 interpolation with second order norm constraints. *Integral Equations and Operator Theory*, 24:156–178, 1996.

[10] D. Alpay, M. Bożejko, F. Colombo, D.P. Kimsey, and I. Sabadini. Boolean convolution in the quaternionic setting. *Linear Algebra Appl.*, 506:382–412, 2016.

[11] D. Alpay, P. Bruinsma, A. Dijksma, and H.S.V. de Snoo. *Interpolation problems, extensions of symmetric operators and reproducing kernel spaces I*, volume 50 of *Operator Theory: Advances and Applications*, pages 35–82. Birkhäuser Verlag, Basel, 1991.

[12] D. Alpay, F. Colombo, and D. Kimsey. The spectral theorem for quaternionic unbounded normal operators based on the S-spectrum. *J. Math. Phys.*, 57(2):023503, 27, 2016.

[13] D. Alpay, F. Colombo, D. Kimsey, and I. Sabadini. Wiener algebra for the quaternions. *Mediterr. J. Math.*, 13(5):2463–2482, 2016.

[14] D. Alpay, F. Colombo, I. Lewkowicz, and I. Sabadini. Realizations of slice hyperholomorphic generalized contractive and positive functions. *Milan Journal of Mathematics*, 83:91–144, 2015.

[15] D. Alpay, F. Colombo, and I. Sabadini. Generalized quaternionic Schur functions in the ball and half-space and Krein-Langer factorization. In *Hypercomplex Analysis: New perspectives and applications*, Trends Math., pages 19–41. Birkhäuser/Springer Basel AG, Basel, 2014.

[16] D. Alpay, F. Colombo, and I. Sabadini. Krein–Langer Factorization and Related Topics in the Slice Hyperholomorphic Setting. *J. Geom. Anal.*, 24(2):843–872, 2014.

[17] D. Alpay, F. Colombo, and I. Sabadini. Inner product spaces and Krein spaces in the quaternionic setting. In D. Alpay and B. Kirstein, editors, *Recent Advances in Inverse Scattering, Schur Analysis and Stochastic Processes*, volume 244 of *Operator Theory: Advances and Applications*, pages 33–65. Birkhäuser, 2015.

[18] D. Alpay, F. Colombo, and I. Sabadini. *Slice hyperholomorphic Schur analysis*, volume 256 of *Operator Theory: Advances and Applications*. Basel: Birkhäuser/Springer, 2016.

[19] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo. *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, volume 96 of *Operator theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1997.

[20] D. Alpay and H. Dym. Hilbert spaces of analytic functions, inverse scattering and operator models, I. *Integral Equation and Operator Theory*, 7:589–641, 1984.

[21] D. Alpay and H. Dym. Hilbert spaces of analytic functions, inverse scattering and operator models, II. *Integral Equation and Operator Theory*, 8:145–180, 1985.

[22] D. Alpay and H. Dym. On applications of reproducing kernel spaces to the Schur algorithm and rational J-unitary factorization. In I. Gohberg, editor, *I. Schur methods in operator theory and signal processing*, volume 18 of *Operator Theory: Advances and Applications*, pages 89–159. Birkhäuser Verlag, Basel, 1986.

[23] D. Alpay and H. Dym. Structured invariant spaces of vector valued functions, hermitian forms, and a generalization of the Iohvidov laws. *Linear Algebra Appl.*, 137/138:137–181, 1990.

[24] D. Alpay and H. Dym. Structured invariant spaces of vector valued functions, sesquilinear forms, and a generalization of the Iohvidov laws. *Linear Algebra Appl.*, 137/138:413–451, 1990.

[25] D. Alpay and H. Dym. On reproducing kernel spaces, the Schur algorithm, and interpolation in a general class of domains. In *Operator theory and complex analysis (Sapporo, 1991)*, volume 59 of *Oper. Theory Adv. Appl.*, pages 30–77. Birkhäuser, Basel, 1992.

[26] D. Alpay and H. Dym. On a new class of reproducing kernel spaces and a new generalization of the Iohvidov laws. *Linear Algebra Appl.*, 178:109–183, 1993.

[27] D. Alpay and H. Dym. On a new class of realization formulas and their applications. *Linear Algebra Appl.*, 241/243:3–84, 1996.

[28] D. Alpay and I. Gohberg. *On orthogonal matrix polynomials*, volume 34 of *Operator Theory: Advances and Applications*, pages 25–46. Birkhäuser Verlag, Basel, 1988.

[29] D. Alpay and I. Gohberg. Unitary rational matrix functions. In I. Gohberg, editor, *Topics in interpolation theory of rational matrix-valued functions*, volume 33 of *Operator Theory: Advances and Applications*, pages 175–222. Birkhäuser Verlag, Basel, 1988.

[30] D. Alpay and I. Gohberg. Inverse spectral problem for differential operators with rational scattering matrix functions. *Journal of differential equations*, 118:1–19, 1995.

[31] D. Alpay and I. Gohberg. Pairs of selfadjoint operators and their invariants. *Algebra i Analiz*, 16(1):70–120, 2004.

[32] D. Alpay, I. Gohberg, M.A. Kaashoek, and A.L. Sakhnovich. Direct and inverse scattering problem for canonical systems with a strictly pseudo-exponential potential. *Math. Nachr.*, 215:5–31, 2000.

[33] D. Alpay and I. Sabadini. Beurling-Lax type theorems in the complex and quaternionic setting. *Linear Algebra Appl.*, 530:15–46, 2017.

[34] D. Alpay, B. Schneider, M. Shapiro, and D. Volok. Fonctions rationnelles et théorie de la réalisation: le cas hyper–analytique. *Comptes Rendus Mathématiques*, 336:975–980, 2003.

[35] D. Alpay, M. Shapiro, and D. Volok. Rational hyperholomorphic functions in R^4. *J. Funct. Anal.*, 221(1):122–149, 2005.

[36] D. Arov and H. Dym. J-inner matrix functions, interpolation and inverse problems for canonical systems. V. The inverse input scattering problem for Wiener class and rational $p \times q$ input scattering matrices. *Integral Equations Operator Theory*, 43(1):68–129, 2002.

[37] D.Z. Arov and H. Dym. *Bitangential direct and inverse problems for systems of integral and differential equations*, volume 145 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2012.

[38] H. Bart, I. Gohberg, and M.A. Kaashoek. *Minimal factorization of matrix and operator functions*, volume 1 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1979.

[39] T. Boros, A. Sayed, and T. Kailath. Structured matrices and unconstrained rational interpolation problems. *Linear Algebra Appl.*, 203/204:155–188, 1994.

[40] L. de Branges. Some Hilbert spaces of analytic functions I. *Trans. Amer. Math. Soc.*, 106:445–468, 1963.

[41] L. de Branges. The expansion theorem for Hilbert spaces of entire functions. In *Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966)*, pages 79–148. Amer. Math. Soc., Providence, R.I., 1968.

[42] L. de Branges. *Espaces Hilbertiens de fonctions entières.* Masson, Paris, 1972.

[43] L. de Branges and J. Rovnyak. Canonical models in quantum scattering theory. In C. Wilcox, editor, *Perturbation theory and its applications in quantum mechanics*, pages 295–392. Wiley, New York, 1966.

[44] L. de Branges and J. Rovnyak. *Square summable power series.* Holt, Rinehart and Winston, New York, 1966.

[45] M. S. Brodskiĭ. *Triangular and Jordan representations of linear operators.* American Mathematical Society, Providence, R.I., 1971. Translated from the Russian by J. M. Danskin, Translations of Mathematical Monographs, Vol. 32.

[46] F. Colombo, I. Sabadini, and D.C. Struppa. *Noncommutative functional calculus*, volume 289 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011. Theory and applications of slice hyperholomorphic functions.

[47] T. Constantinescu, A. Sayed, and T. Kailath. Displacement structure and completion problems. *SIAM J. Matrix Anal. Appl.*, 16(1):58–78, 1995.

[48] L. de Branges. *Hilbert spaces of entire functions*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1968.

[49] P. Dewilde and H. Dym. Lossless chain scattering matrices and optimum linear prediction: the vector case. *International Journal in Circuit Theory and Applications*, 9:135–175, 1981.

[50] P. Dewilde and H. Dym. Lossless inverse scattering, digital filters, and estimation theory. *IEEE Trans. Inform. Theory*, 30(4):644–662, 1984.

[51] W.F. Donoghue. *Monotone matrix functions and analytic continuation*, volume 207 of *Die Grundlehren der mathematischen Wissennschaften*. Springer–Verlag, 1974.

[52] N. Dunford and J.T. Schwartz. *Linear operators. Part III*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. Spectral operators, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1971 original, A Wiley-Interscience Publication.

[53] D. Dym, H.and Volok. Zero distribution of matrix polynomials. *Linear Algebra Appl.*, 425(2-3):714–738, 2007.

[54] H. Dym. *J–contractive matrix functions, reproducing kernel Hilbert spaces and interpolation*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1989.

[55] H. Dym. A Hermite theorem for matrix polynomials. In *Topics in matrix and operator theory (Rotterdam, 1989)*, volume 50 of *Oper. Theory Adv. Appl.*, pages 191–214. Birkhäuser, Basel, 1991.

[56] H. Dym. *Shift, realizations and interpolation, Redux*, volume 73 of *Operator Theory: Advances and Applications*, pages 182–243. Birkhäuser Verlag, Basel, 1994.

[57] H. Dym and A. Iacob. Positive definite extensions, canonical equations and inverse problems. In H. Dym and I. Gohberg, editors, *Proceedings of the workshop on applications of linear operator theory to systems and networks held at Rehovot, June 13–16, 1983*, volume 12 of *Operator Theory: Advances and Applications*, pages 141–240. Birkhäuser Verlag, Basel, 1984.

[58] H. Dym and H.P. McKean. *Gaussian processes, function theory and the inverse spectral problem*. Academic Press, 1976.

[59] J. Gantner. Operator Theory on One-Sided Quaternionic Linear Spaces: Intrinsic S- Functional Calculus and Spectral Operators. To appear in Mem. Amer. Math. Soc.

[60] Y. Genin, P. Van Dooren, T. Kailath, J.M. Delosme, and M. Morf. On Σ-lossless transfer functions and related questions. *Linear Algebra Appl.*, 50:251–275, 1983.

[61] G. Gentili, C. Stoppato, and D.C. Struppa. *Regular functions of a quaternionic variable*. Springer Monographs in Mathematics. Springer, Heidelberg, 2013.

[62] R. Ghiloni, V. Moretti, and A. Perotti. Continuous slice functional calculus in quaternionic Hilbert spaces. *Rev. Math. Phys.*, 25(4):1350006, 83, 2013.

[63] R. Ghiloni and A. Perotti. Slice regular functions on real alternative algebras. *Adv. Math.*, 226(2):1662–1691, 2011.

[64] I. Gohberg and G. Heinig. Inversion of finite Toeplitz matrices with entries from a noncommutative algebra. *Revue roumaine de mathématiques pures et appliquées*, 19:623–665, 1974.

[65] I. Gohberg and G. Heinig. Inversion of finite Toeplitz matrices consisting of elements of a noncommutative algebra [mr0353040]. In *Convolution equations and singular integral operators*, volume 206 of *Oper. Theory Adv. Appl.*, pages 7–46. Birkhäuser Verlag, Basel, 2010.

[66] I. Gohberg, M.A. Kaashoek, and A.L. Sakhnovich. Canonical systems with rational spectral densities: explicit formulas and applications. *Math. Nachr.*, 194:93–125, 1998.

[67] I. Gohberg and L. Lerer. *Matrix generalizations of M.G. Kreĭn theorems on orthogonal polynomials*, volume 34 of *Operator Theory: Advances and Applications*, pages 137–202. Birkhäuser Verlag, Basel, 1988.

[68] T. Kailath. A theorem of I. Schur and its impact on modern signal processing. In I. Gohberg, editor, *I. Schur methods in operator theory and signal processing*, volume 18 of *Oper. Theory Adv. Appl.*, pages 9–30. Birkhäuser, Basel, 1986.

[69] T. Kailath and A. Sayed. Displacement structure: theory and applications. *SIAM Rev.*, 37:297–386, 1995.

[70] M. Kaltenbäck and H. Woracek. Pontryagin spaces of entire functions. I. *Integral Equations Operator Theory*, 33(1):34–97, 1999.

[71] M. Kaltenbäck and H. Woracek. Pontryagin spaces of entire functions. II. *Integral Equations Operator Theory*, 33(3):305–380, 1999.

[72] M. Kaltenbäck and H. Woracek. Pontryagin spaces of entire functions. III. *Acta Sci. Math. (Szeged)*, 69(1-2):241–310, 2003.

[73] M.G. Kreĭn. Distribution of roots of polynomials orthogonal on the unit circle with respect to a sign-alternating weight. *Teor. Funkciĭ Funkcional. Anal. i Priložen. Vyp.*, 2:131–137, 1966. (In Russian).

[74] M.G. Kreĭn and H. Langer. Über die verallgemeinerten Resolventen und die charakteristische Funktion eines isometrischen Operators im Raume Π_k. In *Hilbert space operators and operator algebras (Proc. Int. Conf. Tihany, 1970)*, pages 353–399. North–Holland, Amsterdam, 1972. Colloquia Math. Soc. János Bolyai.

[75] G. Laville. On Cauchy-Kovalewski extension. *J. Funct. Anal.*, 101(1):25–37, 1991.

[76] X-J Li. The Riemann hypothesis for polynomials orthogonal on the unit circle. *Math. Nachr.*, 166:229–258, 1994.

[77] A. Lindquist and G. Picci. Infinite-dimensional stochastic realizations of continuous-time stationary vector processes. In *Topics in operator theory systems and networks (Rehovot, 1983)*, volume 12 of *Oper. Theory Adv. Appl.*, pages 335–350. Birkhäuser, Basel, 1984.

[78] M. S. Livšic. On spectral decomposition of linear nonself-adjoint operators. *Mat. Sbornik N.S.*, 34(76):145–199, 1954.

[79] I.V. Mikhailova. Weyl matrix circles as a tool for uniqueness in the theory of multiplicative representation of J-inner matrix functions [translated from *analysis in infinite-dimensional spaces and operator theory (russian)*, 101–117, "Naukova Dumka", Kiev, 1983]. In *Topics in interpolation theory (Leipzig, 1994)*, volume 95 of *Oper. Theory Adv. Appl.*, pages 397–417. Birkhäuser, Basel, 1997.

[80] V.P. Potapov. The multiplicative structure of J-contractive matrix–functions. *Trudy Moskow. Mat. Obs.*, 4:125–236, 1955. English translation in: American mathematical society translations (2), vol. 15, p. 131–243 (1960).

[81] P. Rao and P. Dewilde. *System theory for lossless wave scattering*, volume 19 of *Operator Theory: Advances and Applications*, pages 333–358. Birkhäuser Verlag, Basel, 1986.

[82] G.C. Rota. On models for linear operators. *Comm. Pure Appl. Math.*, 13:469–472, 1960.

[83] A. Sakhnovich. Exact solutions of nonlinear equations and the method of operator identities. *Linear Algebra Appl.*, 182:109–126, 1993.

[84] A. Sakhnovich. Canonical systems and transfer matrix-functions. *Proc. Amer. Math. Soc.*, 125(5):1451–1455, 1997.

[85] L.A. Sakhnovich. Factorization problems and operator identities. *Russian mathematical surveys*, 41:1–64, 1986.

[86] L.A. Sakhnovich. *Spectral theory of canonical differential systems. Method of operator identities*, volume 107 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1999. Translated from the Russian manuscript by E. Melnichenko.

[87] L.A. Sakhnovich. *Spectral theory of canonical differential systems. Method of operator identities*, volume 107 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1999. Translated from the Russian manuscript by E. Melnichenko.

[88] L.A. Sakhnovich. *Integral equations with difference kernels on finite intervals*, volume 84 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer, Cham, 2015. Second edition, revised and extended.

[89] A. Sayed, T. Constantinescu, and T. Kailath. Time-variant displacement structure and interpolation problems. *IEEE Trans. Automat. Control*, 39(5):960–976, 1994.

[90] A. Sayed and T. Kailath. Fast algorithms for generalized displacement structures and lossless systems. *Linear Algebra Appl.*, 219:49–78, 1995.

[91] L. Schwartz. Sous espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés (noyaux reproduisants). *J. Analyse Math.*, 13:115–256, 1964.

[92] Y. Shelah. Quaternionic Wiener Algebras, Factorization and Applications. *Adv. Appl. Clifford Algebr.*, 27(3):2805–2840, 2017.

[93] Y.L. Shmul'yan. Division in the class of J–expansive operators. *Math. Sb.*, 116:516—525, 1967.