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NONCOMMUTATIVE POTENTIAL THEORY: A SURVEY

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Abstract. The aim of these notes is to provide an introduction to Noncommutative Potential Theory as given at I.N.D.A.M.-C.N.R.S. "Noncommutative Geometry and Applications" Lectures, Villa Mondragone-Frascati June 2014.

Contents.

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1. Introduction.

The aim of these notes is to provide an overview of the potential theory on operator algebras based on Dirichlet forms, as illustrated at Villa Mondragone-Frascati Lectures in June 2014.

Some of the main basic results of the theory are accompanied by sketches of their proofs, some others are given together with detailed verifications and the main subjects are illustrated by explicit examples. This part includes the generalized Beurling-Deny correspondence between Dirichlet forms and Markovian semigroups, the differential calculus underlying Dirichlet spaces, potentials, finite energy states and multipliers of Dirichlet spaces and the generalized Deny's inequality.
In particular, the construction of a natural Dirac operator and spectral triple, by the differential calculus associated to a Dirichlet form, opens the possibility to use the methods of A. Connes to explore the Noncommutative Geometry underlying Dirichlet spaces.

On the other hand, efforts have been done to provide contacts with other fields of operator algebras and to illustrate recent applications to other sectors of mathematics. In this respect we include, the construction and analysis of quantum Lévy processes on compact quantum groups, the connections between approximation properties of von Neumann algebras and spectra of Dirichlet forms, potential theory on the Clifford algebra of Riemannian manifolds and positive curvature, Dirichlet forms in Free Probability and the noncommutative approach to the potential theory of fractal sets.

To keep the length of the notes contained, some important subjects are not included such as the role of Dirichlet spaces in the K-theory of certain Banach algebras appearing toward mod Hilbert-Schmidt extension of the BDF-theory (see [V3]), the use of Dirichlet forms in Quantum Statistical Mechanics and infinite dimensional settings (see [LOZ]) and the subgaussian behavior of random variables in noncommutative probability (see [JZ]).

I wish to thank warmly the organizers and the audience for attending the lectures and taking part brightly to numerous discussions.

2. A review of classical potential theory

Classical potential theory (see [Ca], [Do]) concerns properties of the Dirichlet integral

$$\mathcal{D} : L^2(\mathbb{R}^d) \to [0,+\infty] \qquad \mathcal{D}[u] := \int_{\mathbb{R}^d} |\nabla u|^2 \, dm \,.$$

This is a lower semicontinuous quadratic form on the Hilbert space $L^2(\mathbb{R}^d, m)$ which is finite on the Sobolev space $H^{1,2}(\mathbb{R}^d)$. It is thus a closed quadratic form whose associated nonnegative, self-adjoint operator is the Laplace operator

$$\Delta := -\sum_{k=1}^d \partial_k^2 \qquad \mathcal{D}[u] = \|\sqrt{\Delta} u\|_2^2 \,.$$

It generates the heat semigroup $e^{-t\Delta} : L^2(\mathbb{R}^d, m) \to L^2(\mathbb{R}^d, m)$, whose heat kernel

$$e^{-t\Delta}(x,y) = (4\pi t)^{-d/2} e^{-\frac{|x-y|^2}{4t}}$$

is the fundamental solution of the heat equation $\partial_t u + \Delta u = 0$. The contraction property called Markovianity

$$\mathcal{D}[u \wedge 1] \le \mathcal{D}[u] \quad u = \overline{u} \in L^2(\mathbb{R}^d, m)$$

is responsible for the Maximum Principle for harmonic functions, i.e. solutions of the Laplace equation $\Delta u = 0$, the Maximum Principle for solution of the heat equation, continuity, contractivity and positive preserving properties of the heat semigroup $e^{-t\Delta}$ on the spaces $L^2(\mathbb{R}^d, m)$, $L^\infty(\mathbb{R}^d, m)$, $L^1(\mathbb{R}^d, m)$.
The Brownian motion (Ω, P_x, X_t) is the Markovian stochastic processes on $\mathbb{R}^d$ associated to the heat semigroup through

$$(e^{-t\Delta}u)(x) = \int_\Omega u(X_t(\omega))\, P_x(d\omega)\,.$$

The polar sets of the Brownian motion, i.e. the subsets of $\mathbb{R}^n$ almost surely avoided by the process, can be identified with the $\mathrm{Cap}\,(B) = 0$ sets for the electrostatic capacity Cap defined by $\mathcal{D}$. This is the set function defined initially on open sets $A \subset \mathbb{R}^n$ by

$$\mathrm{Cap}(A) := \inf\{\mathcal{D}[u] : u \in H^{1,2}(\mathbb{R}^n)\,,\ u \geq 1_A \ \mathrm{m-a.e.}\}$$

and then extended to arbitrary Borel sets as

$$\mathrm{Cap}(B) := \inf\{\mathrm{Cap}(A) : B \subseteq A\,,\ \ A \subset \mathbb{R}^n \text{ open}\}\,.$$

The capacity is strongly subadditive in the sense that

$$\mathrm{Cap}(A \cup B) + \mathrm{Cap}(A \cap B) \leq \mathrm{Cap}(A) + \mathrm{Cap}(B)$$

for all Borel sets $A, b \subseteq \mathbb{R}^n$ and by Choquet's capacity theory it can be extended to arbitrary analytic subsets of the Euclidean space.

The above properties are proved by the explicit knowledge of the Green function

$$(\Delta^{-1}u)(x) = \int_{\mathbb{R}^d} G(x,y)u(y)\, m(dy) \qquad G(x,y) = |x-y|^{2-d} \qquad d \geq 3\,.$$

A. Beurling and J. Deny developed in the late fifthes [BeDe 1,2] a *kernel free* potential theory on measured, locally compact Hausdorff spaces (X, m), generalizing the notion of Dirichlet integral as a closed quadratic form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ which is Markovian in the sense

$$\mathcal{E}[u \wedge 1] \leq \mathcal{E}[u] \qquad u = \overline{u} \in \mathcal{F}\,.$$

Their extension of potential theory was based on the notion of regularity of Dirichlet forms: one requires that there are sufficiently many finite energy continuous function in the sense that $C_0(X) \cap \mathcal{F}$ is a form core uniformly dense in $C_0(X)$. M. Fukushima achieved in the middle sixties [F 1,2] the construction of the Hunt Markovian process on X associated to a given Dirichlet form. The construction is based on the potential theory of a Dirichlet space and especially on the notion of capacity associated to $(\mathcal{E}, \mathcal{F})$. The work establishes a complete correspondence between Dirichlet forms, Markovian semigroups and symmetric Hunt processes on locally compact spaces (see [F 1,2], [FOT]).

3. Noncommutative Potential Theory

The need to extend the notion of Markovian semigroup to non commutative settings, where recognized by L. Gross in his works [G 1,2] about the construction of Hamiltonians for interacting bosons and fermions systems in Quantum Field Theory (see also [SU]). Markovian semigroups play also an important role to study Open Quantum Systems (see [Dav1]). Subsequently S. Albeverio and R. Hoegh-Krohn extended the Beurling-Deny theory of Dirichlet forms to the setting of a C*-algebra with trace ([AHK1], [AHK2]). This theory and applications were limited to the *tracial* case; it was studied and applied by various authors: among them, E.B. Davies - O.S. Rothaus [DR1,2] investigated the spectral properties of

the Bochner Laplacian on Riemannian manifolds, J.-L. Sauvageot constructed the transverse heat semigroup of a Riemannian foliation [S 2,3,4], E.B. Davies - J.M. Lindsay [DL] deepened the connections between Dirichlet forms and Markovian semigroups and D. Guido - T. Isola - S. Scarlatti extended these notions to the non-symmetric case [GIS]. In particular, J.L. Sauvageot [S1] discovered that every dissipation on a C*-algebra A with trace is canonically represented by a closed derivation in a C*-Hilbert A-bimodule.
To deal with Markovian semigroups and Dirichlet forms with respect to not necessarily tracial states, the framework was later extended to von Neumann algebras with separable predual, in the Haagerup standard form by S. Goldstein and J.M. Lindsay in [GL 1,2] and to general standard forms in [C 1,2]. The extension to von Neumann algebras with non separable predual was realized in [GL3]. See also the recent [Ri2] for a survey of applications.

3.1. **Dirichlet forms and Markovian semigroups on von Neumann algebras.** To handle the situation of a von Neumann algebra $\mathcal{M}$ with separable predual and normal faithful state $\omega \in \mathcal{M}_{*+}$, one exploits the structure of its standard form $(\mathcal{M}, L^2(\mathcal{M}), L^2_+(\mathcal{M}), J)$. Denoting by $\xi_\omega \in L^2_+(\mathcal{M})$ the cyclic vector representing the state, the nonlinear contraction $u \mapsto u \wedge 1$, on which the Markovianity of Dirichlet forms in the commutative case is based, is understood as the projection $\xi \mapsto \xi \wedge \xi_\omega$ of real vector $\xi = J\xi \in L^2(\mathcal{M})$ onto the closed and convex set $\xi_\omega - L^2_+(\mathcal{M}) = \{\xi \in L^2(\mathcal{M}) : \xi = J\xi \leq \xi_\omega\}$. Notice that, despite the notation adopted, the subspace of real vectors is a lattice if and only if $\mathcal{M}$ is commutative.
By the self-polarity of the positive cone $L^2_+(\mathcal{M})$, any vector $\zeta \in L^2(\mathcal{M})$ can be can written $\zeta = \xi + i\eta$ in terms of its real and imaginary parts defined as $\xi := (\zeta + J\zeta)/2$, $\eta := (\zeta - J\zeta)/2$. The positive part ζ_+ of a real vector $\zeta = J\zeta \in L^2(\mathcal{M})$ is defined as its Hilbert projection onto the positive cone $L^2_+(\mathcal{M})$. The properties of the Hilbert projection, ensure that the negative part, defined as $\zeta_- := \zeta_+ - \zeta$, is positive $\zeta_- \in L^2_+(\mathcal{M})$ and orthogonal to ζ_+ so that the real vector can be written uniquely as the difference $\zeta = \zeta_+ - \zeta_-$ of two positive, orthogonal vectors. The modulus of the real vector ζ is defined as $|\zeta| := \zeta_+ + \zeta_-$.

The unique trace state on the full matrix algebra $\mathbb{M}_n(\mathbb{C})$ will be denoted by $\mathrm{tr_n}$.

Definition 3.1. (Markovian semigroups) A self-adjoint C_0-semigroup $\{T_t : t \geq 0\}$ on $L^2(\mathcal{M})$ is Markovian with respect to $\omega \in \mathcal{M}_{*+}$ if

- $T_t \circ J = J \circ T_t \qquad t \geq 0$ (*reality*)
- $\xi \leq \xi_\omega \quad \Rightarrow \quad T_t\xi \leq \xi_\omega \qquad t \geq 0$ (*Markovianity*)
- $\{T_t : t \geq 0\}$ on $L^2(\mathcal{M})$ is *completely Markovian* if its matrix expansions

$$T^n_t([\xi_{ij}]_{ij}) := [T_t\xi_{ij}]_{ij}$$

are Markovian semigroups on $L^2(\mathcal{M} \otimes \mathbb{M}_n(\mathbb{C}))$ with respect to the state $\omega \otimes \mathrm{tr_n}$.

It can be proved that Markovian semigroups are necessarily contractive on $L^2(\mathcal{M})$ and positive preserving in the sense that $T_t(L^2_+(\mathcal{M})) \subseteq L^2_+(\mathcal{M})$ for all $t > 0$. Moreover, if $T_t\xi_\omega = \xi_\omega$ for all $t > 0$, then Markovianity is equivalent to the positive preserving property.

Using the properties of the symmetric embedding $\quad i_\omega : \mathcal{M} \to L^2(\mathcal{M}) \qquad i_\omega(x) := \Delta^{1/4}_\omega x\xi_\omega$ one can relate Markovian semigroups on the Hilbert space $L^2(\mathcal{M})$ to semigroups on the von Neumann algebra $\mathcal{M}$. In the following, $\{\sigma^\omega_t : t > 0\}$ will denote the modular group of ω and $\mathcal{M}_{\sigma^\omega}$ the subalgebra of its analytic elements.

Theorem 3.2. *(Modular ω-symmetry) Markovian semigroups on $L^2(\mathcal{M})$ with respect to ω are in one to one correspondence with C^*_0-continuous, positive preserving, contractive semigroups*

$\{S_t : t \geq 0\}$ *on* $\mathcal{M}$ *which are modular* ω*-symmetric in the sense that*

$$\omega(S_t(x)\sigma^{\omega}_{-i/2}(y)) = \omega(\sigma^{\omega}_{-i/2}(x)S_t(y)) \qquad x, y \in \mathcal{M}_{\sigma^\omega}\,, \quad t > 0\,,$$

through the relation

$$i_0(S_t(x)) = T_t(i_0(x)) \qquad x \in \mathcal{M}\,, \quad t > 0\,.$$

Moreover, $\{T_t : t > 0\}$ *is completely Markovian if and only if* $\{S_t : t > 0\}$ *is completely Markovian, i.e. it is a completely positive, normal, contractive semigroup.*

Remark 3.3. The weak*-continuity of semigroups on von Neumann algebras is in many respect the natural one. By a result due to G. A. Elliott [E], strongly continuous semigroups on W*-algebras are automatically norm continuous, thus with bounded generators.

Symmetric Markovian semigroups on the von Neumann algebra $\mathcal{M}$ or on the Hilbert space $L^2(\mathcal{M})$ can be characterized in terms of the following class of quadratic forms.

Definition 3.4. (Dirichlet forms) A Dirichlet form $\mathcal{E} : L^2(\mathcal{M}) \to (-\infty, +\infty]$ with respect to $\omega \in \mathcal{M}_{*+}$ is a lower semibounded, lower semicontinuous, quadratic form such that

- the domain $\mathcal{F} := \{\xi \in L^2(\mathcal{M}) : \mathcal{E}[\xi] < +\infty\}$ is dense in $L^2(\mathcal{M})$
- $\mathcal{E}[J\xi] = \mathcal{E}[\xi]$ for all $\xi \in L^2(\mathcal{M})$ *(reality)*
- $\mathcal{E}[\xi \wedge \xi_0] \leq \mathcal{E}[\xi]$ for all $\xi = J\xi \in L^2(\mathcal{M})$ *(Markovianity)*
- $(\mathcal{E}, \mathcal{F})$ is a *complete Dirichlet form* if its matrix expansions for $n \geq 1$

$$\mathcal{E}_n[(\xi_{ij})_{ij}] := \sum_{ij} \mathcal{E}[\xi_{ij}]$$

are Dirichlet forms on $\mathcal{M} \otimes \mathbb{M}_n(\mathbb{C})$ with respect to the normal state $\omega \otimes \mathrm{tr_n}$.

The domain $\mathcal{F}$ is called *Dirichlet space* when endowed with the graph norm

$$\|\xi\|_{\mathcal{F}} := \sqrt{\mathcal{E}[\xi] + \|\xi\|^2_{L^2(\mathcal{M})}}\,.$$

It can be proved that Dirichlet forms are necessarily nonnegative. In case $\mathcal{E}[\xi_\omega] = 0$, Markovianity is equivalent to the contraction property $\mathcal{E}[|\xi|] \leq \mathcal{E}[\xi]$ for $\xi = J\xi \in L^2(\mathcal{M})$. This property is in general weaker than Markovianity and corresponds to the positive preserving property of the semigroup. The following result connects the dynamical aspects of the theory to the infinitesimal ones (see [C 1,2], [GL 1,2]).

Theorem 3.5. *(Generalized Beurling-Deny correspondence) Dirichlet forms are in one to one correspondence with Markovian semigroups by*

$$\mathcal{E}[\xi] = \lim_{t\to 0} \frac{1}{t}(\xi | a - T_t\xi) \qquad a \in \mathcal{F}$$

or through the self-adjoint generator $(L, dom\,(L))$

$$T_t = e^{-tL} \qquad \mathcal{E}[a] = \|\sqrt{L}a\|^2_{L^2(A,\tau)} \quad a \in \mathcal{F} = dom\,(\sqrt{L})\,.$$

Completely Dirichlet forms on $L^2(\mathcal{M})$ *are in one to one correspondence with modular symmetric, completely Markovian semigroups on* $L^2(\mathcal{M})$ *and with completely Markovian semigroups on* $\mathcal{M}$.

Remark 3.6. i) By duality and interpolation, Markovian semigroups extend to C_0-semigroups on noncommutative $L^p(\mathcal{M})$ spaces, $p \in [1, +\infty]$. The extension to the predual $\mathcal{M}_* = L^1(\mathcal{M})$ is particularly important in ceratin applications to Quantum Information Theory (channels).

ii) Extending Markovian semigroups from $\mathcal{M}$ to $L^2(\mathcal{M})$ via *non symmetric* embeddings

$$i_\alpha(x) := \Delta^\alpha_{\xi_0} x\xi_0 \qquad \alpha \in [0,1/2] \qquad \alpha \neq 1/4\,,$$

one gets semigroups on $L^2(\mathcal{M})$ which automatically commute with the modular operator Δ_ω.

The structure of the self-dual positive cone $L^2_+(\mathcal{M})$ allows to approach ergodic properties of Markovian semigroups from different perspectives. For example, faces F of the self-polar cone $L^2_+(\mathcal{M})$ are in one to one correspondence with *Peirce projections* $P_e = eJeJ$ associated to projections $e \in \mathrm{Proj}\,(\mathcal{M})$

$$F = P_e(L^2_+(\mathcal{M},\omega))\,.$$

The following characterization (see [C3]) was proved in the trace case by L. Gross and applied to establish uniqueness of physical ground states for interacting bosons and fermions systems. Notice that the cyclicity of a positive vector $\xi \in L^2_+(\mathcal{M})$ is equivalent to the fact that $(\xi|\eta) > 0$ for all $\eta \in L^2_+(\mathcal{M})$.

Theorem 3.7. *(Ergodic Markovian semigroups) The following properties are equivalent:*

- *the Markovian semigroup $\{T_t : t \geq 0\}$ on $L^2(\mathcal{M})$ is ergodic: for $\xi, \eta \in L^2_+(\mathcal{M},\omega)$ there exists $t > 0$ such that $(\xi|T_t\eta)_2 > 0$*
- *the Markovian semigroup $\{T_t : t \geq 0\}$ on $L^2(\mathcal{M})$ is indecomposable: for some $t > 0$, T_t leaves invariant no proper face of the cone $L^2_+(\mathcal{M})$*
- $\lambda := \inf\{\mathcal{E}[\xi] : \|\xi\|_2 = 1\}$ *is a Perron-Frobenius eigenvalue: it is a simple eigenvalue with cyclic eigenvector $\xi_\lambda \in L^2_+(\mathcal{M})$.*

3.2. **KMS symmetric semigroups on C*-algebras.** Dirichlet forms may be used to study properties of completely positive, contractive semigroups on C*algebras in case they posses the following symmetry with respect to a fixed KMS state (see [C5]).
Let $\{\alpha_t : t \in \mathbb{R}\}$ be a strongly continuous automorphisms group on the C*-algebra A, A_α the algebra of its analytic elements and let $\omega \in A^*_+$ be a KMS$_\beta$-state for $\beta \in \mathbb{R}$.

Definition 3.8. (KMS symmetric semigroups on C*-algebras) A C$_0$-semigroup $\{S_t : t \geq 0\}$ on A is *KMS$_\beta$ ω-symmetric* if

$$\omega(bS_t(a)) = \omega(\alpha_{-\frac{i\beta}{2}}(a)S_t(\alpha_{+\frac{i\beta}{2}}(b))) \qquad a, b \in B$$

for some a dense, α-invariant, *-subalgebra $B \subseteq A_\alpha$ or, equivalently,

$$\omega(\alpha_{-\frac{i\beta}{2}}(b)S_t(a)) = \omega(\alpha_{-\frac{i\beta}{2}}(a)S_t(b)) \qquad a, b \in B\,.$$

Remark 3.9. i) KMS symmetry is a deformation of the KMS condition, in fact for $t = 0$ we get

$$\omega(ba) = \omega(\alpha_{-\frac{i\beta}{2}}(a)\alpha_{+\frac{i\beta}{2}}(b)) = \omega(a\alpha_{+i\beta}(b)) \qquad a, b \in B\,.$$

ii) In case $\{\alpha_t : t \in \mathbb{R}\}$ and $\{S_t : t \geq 0\}$ commute, KMS symmetry reduces to *GNS symmetry*

$$\omega(bS_t(a)) = \omega(S_t(b)a) \qquad a, b \in A\,,$$

a property that has been mostly used to formulate *detailed balance conditions* for open quantum systems.

In the spirit of KMS theory, the above symmetry may formulated in different guises, as for example

Proposition 3.10. *Consider the open strip $D_\beta := \{z \in \mathbb{C} : \mathrm{Im}(z) \in (0,\beta)\}$. The following conditions are equivalent*

- *a C_0-semigroup $\{S_t : t \geq 0\}$ on A is KMS$_\beta$ ω-symmetric*
- *for any $a, b \in A$ and on the KMS-strip $D_\beta \subset \mathbb{C}$ there exists a bounded continuous function $F_{a,b} : \overline{D_\beta} \to A$, analytic in D_β such that for $s \in \mathbb{R}$, $t \geq 0$*

$$F_{a,b}(s) = \omega(\alpha_{-s}(a) S_t(\alpha_{+s}(b))) , \qquad F_{a,b}(s + i\beta) = \omega(\alpha_{+s}(b) S_t(\alpha_{-s}(a))) .$$

Let $\omega \in A^*_+$ be a KMS$_\beta$-state for $\{\alpha_t : t \in \mathbb{R}\} \subset \mathrm{Aut}\,(A)$ and consider the cyclic GNS representation $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ of A, the von Neumann algebra $\mathcal{M} := \pi_\omega(A)''$ and the normal extension of ω to $\mathcal{M}$ given by $\omega(x) := (\xi_\omega | \pi_\omega(x)\xi_\omega)_2$, $x \in \mathcal{M}$ and associated modular automorphisms group $\{\sigma^\omega_t : t \in \mathbb{R}\}$ of $\mathcal{M}$. A standard form on the space $L^2(\mathcal{M}, \omega) \simeq \mathcal{H}_\omega$ is determined by $L^2_+(\mathcal{M}, \omega) = \overline{\{\Delta^{1/4}_\omega \pi_\omega(A_+)\xi_\omega\}}$.

The virtue of KMS symmetry is to force the semigroup to leave globally invariant the kernel of the cyclic representation of ω and thus to allows the study of the semigroup on the von Neumann algebra and on the standard Hilbert space $L^2(\mathcal{M}, \omega)$.

Theorem 3.11. *A KMS$_\beta$ ω-symmetric, C_0-semigroup $\{S_t : t \geq 0\}$ on A*

- *leaves globally invariant the kernel of the cyclic representation*

$$S_t(\ker(\pi_{\omega_0})) \subseteq \ker(\pi_{\omega_0}) \qquad t > 0;$$

- *it extends to a modular ω-symmetric, C^*_0-semigroup $\{T_t : t \geq 0\}$ on the von Neumann algebra $\mathcal{M}$ by*

$$T_t \circ \pi_\omega = \pi_\omega \circ S_t \qquad t > 0;$$

- *it extends to a Markovian semigroup on $L^2(\mathcal{M}, \omega)$;*
- *it determines a Dirichlet form on $L^2(\mathcal{M}, \omega)$.*

The generators of norm continuous, completely positive, contractive semigroup on the hyperfinite type I factor, has been classified by G. Lindblad [L]. Using the properties of standard forms however, one may consider on any von Neumann algebras semigroups and their generators of similar type (see [C 1,2]).

Example 3.12. (Bounded Dirichlet forms)
On a standard form $(\mathcal{M}, L^2(\mathcal{M}), L^2_*(\mathcal{M}), J)$ consider the normal state $\omega \in \mathcal{M}_{*+}$ and define $j(x) := Jx^*J$ for $x \in \mathcal{M}$.
For fixed, finite subsets $\{a_k : k = 1, \ldots, n\} \subset \mathcal{M}$, $\{\mu_k, \nu_k : k = 1, \ldots, n\} \subset (0, +\infty)$, define the bounded operators

$$d_k : L^2(\mathcal{M}) \to L^2(\mathcal{M}) \qquad d_k := i(\mu_k a_k - \nu_k j(a_k))$$

and the bounded quadratic form on $L^2(\mathcal{M})$ by $\mathcal{E}[\xi] := \sum_{k=1}^n \|d_k\xi\|^2_{L^2(\mathcal{M})}$. Then $(\mathcal{E}, L^2(\mathcal{M})$ is
i) J-real iff $\sum_{k=1}^n [\mu_k^2 a_k^* a_k - \nu_k^2 a_k a_k^*] \in \mathcal{M} \cap \mathcal{M}'$;
ii) Markovian if moreover $\sum_{k=1}^n [\mu_k^2 a_k^* a_k - \mu_k\nu_k(a_k j(a_k) + a_k^* j(a_k^*)) + \nu_k^2 a_k a_k^*]\xi_\omega \geq 0$;
iii) the associated Markovian semigroup is conservative, $T_t\xi_\omega = \xi_\omega$ for all $t \geq 0$, if moreover the numbers $(\mu_k/\nu_k)^2$ are eigenvalues of the modular operator Δ_ω with eigenvectors $a_k\xi_\omega$;
iv) the generator has the form $L = \sum_{k=1}^n [\mu_k^2 a_k^* a_k - \mu_k\nu_k(a_k j(a_k) + a_k^* j(a_k^*)) + \nu_k^2 a_k a_k^*]$.

Condition i) is typically ensured in the framework of q-deformed Fock spaces and related factors [BKS].

Example 3.13. (Quantum Ornstein-Uhlenbeck semigroups)
Consider the canonical base $\{e_k : k \in \mathbb{N}\}$ of Hilbert space $h := l^2(\mathbb{N})$, the C*-algebra of compact operators $\mathcal{K}(h)$, the von Neumann algebra of bounded operators $\mathcal{B}(h)$ and the Hilbert-Schmidt standard form $(\mathcal{B}(h), \mathcal{L}^2(h), \mathcal{L}^2_+(h), J)$.

Fix parameters $\mu > \lambda > 0$, set $\nu := (\lambda/\mu)^2$ and consider the state

$$\omega_\nu(x) := (1-\nu)\sum_{k\geq 0}\nu^k(e_k|xe_k) \qquad x\in\mathcal{K}(h)$$

with cyclic vector $\xi_\nu := (1-\nu)^{1/2}\sum_{k\geq 0}\nu^{k/2}e_k\otimes\overline{e_k}$. The creation/annihilation operators, defined as

$$a^*(e_k) := \sqrt{k+1}e_{k+1} \quad a(e_k) := \sqrt{k}e_{k-1} \quad a(e_0) = 0$$

satisfy the *Canonical Commutation Relation*: $aa^* - a^*a = I$. Then the closure of the quadratic form $(\mathcal{E},\mathcal{F})$ on $\mathcal{L}^2(h)$

$$\mathcal{E}[\xi] := \|\mu a\xi - \lambda\xi a^*\|^2 + \|\mu a\xi^* - \lambda\xi^* a^*\|^2 \qquad \mathcal{F} := \text{linear span}\{e_k\otimes\overline{e_l} : k,l\in\mathbb{N}\}$$

is a Dirichlet form and the associated Markovian semigroup induces an ergodic, Markovian, C_0-semigroup on $\mathcal{K}(h)$ leaving the state ω_ν invariant.
Semigroups corresponding to this class of Dirichlet forms reduces to interesting classical Markovian semigroups on suitable maximal abelian subalgebras of $B(h)$ both atomic and diffuse.

4. Quantum Lévy Processes on Compact Quantum Groups

In classical potential theory probably the most studied class of Markovian semigroups is the one corresponding to Lev'y processes on Lie groups. These are processes and semigroups that commute with the left translations provided by the action of the group on itself. In this section we will describe an extension of that commutative situation on Compact Quantum Groups (see [CFK] for details).

Let us recall that a compact quantum group $\mathbb{G} = (A,\Delta)$ is a unital C*-algebra $A =: C(\mathbb{G})$ together with
i) a *coproduct* $\Delta : A\to A\otimes A$, a unital, $*$-homomorphism which is
ii) *coassociative* $(\Delta\otimes\mathrm{id}_A)\circ\Delta = (\mathrm{id}_A\otimes\Delta)\circ\Delta$ and satisfies
iii) *cancelation rules* $\overline{\mathrm{Lin}}((1\otimes A)\Delta(A)) = \overline{\mathrm{Lin}}((A\otimes 1)\Delta(A)) = A\otimes A$.

Here $A\otimes A$ denotes the projective tensor product of Banach spaces. A *unitary co-representation* of $\mathbb{G}$ is a unitary matrix $U = (u_{jk})\in M_n(A)$ such that $\Delta(u_{jk}) = \sum_{p=1}^n u_{jp}\otimes u_{pk}$ for all $j,k = 1,\dots,n$.

A theorem due to Woronowicz states that, if $\{U^s : s\in\widehat{\mathbb{G}}\}$ is a complete family of inequivalent irreducible, unitary co-representations of $\mathbb{G}$, the algebra of *polynomials*, defined by

$$\mathrm{Pol}(\mathbb{G}) := \mathrm{Span}\{u^s_{jk}; s\in\widehat{\mathbb{G}}, 1\leq j,k\leq n_s\}$$

is a dense *Hopf $*$-algebra* with *counit* $\varepsilon(u^s_{jk}) := \delta_{jk}$ and *antipode* $S(u^s_{jk}) := (u^s_{kj})^*$ satisfying the rules (m_A being the product in A)

$$(\varepsilon\otimes\mathrm{id})\Delta(a) = a \quad (\mathrm{id}\otimes\varepsilon)\Delta(a) = a \quad m_A(S\otimes\mathrm{id})\Delta(a) = \varepsilon(a)I = m_A(\mathrm{id}\otimes)\Delta(a)\,.$$

A result due to Woronowicz [...] ensures that the C*-algebra of a compact quantum group $A = C(\mathbb{G})$ is commutative if and only if it is of the form $A = C(G)$ for some compact group G with unit $e\in G$. Noticing that $A\otimes A = C(G\times G)$, in this case the co-multiplication is defined dualizing the product operation in G

$$(\Delta f)(s,t) = f(st) \qquad f\in C(G)\,,\quad s,t\in G\,.$$

Counit and antipode are defined by $\varepsilon(f) := f(e)$ and $S(f)(s) := f(s^{-1})$, for $f\in C(G)$ and $s\in G$.

Combining the tensor product operation with the co-multiplication, one may introduces new operations that in the case of compact group reduce to well known classical ones.

Definition 4.1. (Convolution)
i) Convolution $\xi * \xi' \in A^*$ of functionals $\xi\,, \xi' \in A^*$ is defined by

$$\xi * \xi' := (\xi \otimes \xi') \circ \Delta\,;$$

ii) convolution $\xi * a \in A$ of a functional $\xi \in A^*$ and an element $a \in A$ is defined by

$$\xi * a := (\mathrm{id} \otimes \xi)(\Delta a) \qquad a * \xi := (\xi \otimes \mathrm{id}\,)(\Delta a)\,.$$

Again by a result of Woronowicz, on a compact quantum group $\mathbb{G} = (A, \Delta)$ there exists a unique (Haar) state $h \in A^*_+$ which is both left and right invariant in the sense that

$$a^* h = h^* a = h(a) 1_A \qquad a \in A\,.$$

It is a $(\sigma\,, -1)$-KMS state with respect to a suitable $*$-automorphisms group of A

$$\{\sigma_t : t \in \mathbb{R}\} \qquad h(ab) = h(\sigma_{-i}(b)a) \qquad a, b \in \mathcal{A}\,.$$

Notice that, in general, the Haar state is not a trace. Compact quantum groups for which the Haar state is a trace are called Kac quantum groups.

By a result of Woronowicz, the antipode S is a closable operator on A and its closure $\overline{S}$ admits the polar decomposition a

$$\overline{S} = R \circ \tau_{\frac{i}{2}}\,,$$

where
i) $\tau_{\frac{i}{2}}$ generates a $*$-automorphisms group $\{\tau_t : t \in \mathbb{R}\}$ of the C^*-algebra A and
ii) R is a linear, anti-multiplicative, norm preserving involution on A such that $\tau_t \circ R = R \circ \tau_t$ for all $t \in \mathbb{R}$, called *unitary antipode*.

Example 4.2. The compact quantum group $SU_q(2) = (A, \Delta)$, $0 < q \leq 1$, is given by the universal C^*-algebra A generated by the coefficients of the matrix

$$U = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

with relations on α and γ ensuring unitarity $UU^* = U^*U = 1$. The other relation are
i) *co-multiplication* $\Delta(\alpha) := \alpha \otimes \alpha + \gamma \otimes \gamma, \quad \Delta(\gamma) := \gamma \otimes \alpha + \alpha^* \otimes \gamma$
ii) *counit* $\quad \varepsilon(\alpha) = 1 \qquad \varepsilon(\gamma) = 0$
iii) *antipode* $\quad S(\alpha) := \alpha^*\,, \qquad S(\gamma) := -q\gamma\,, \qquad S(u^s_{jk}) = (-q)^{(j-k)} u^s_{-k,-j}$
iv) *Haar state* $\quad h(u^s_{jk}) = \delta_{s,0}$
v) *automorphisms group* $\quad \sigma_z(u^s_{jk}) = q^{2iz(j+k)} u^s_{jk} \qquad z \in \mathbb{C}$
vi) *unitary antipode* $\quad R(u^s_{jk}) = q^{k-j}(u^s_{kj})^*$.

4.0.1. *Quantum Lévy Processes.* Here we recall the some basic definitions of Quantum Probability concerning stochastic processes in the noncommutative setting.

Let $\mathcal{A} = \mathrm{Pol}\,(\mathbb{G})$ be the Hopf $*$-agebra of a compact quantum group and $(\mathcal{P}, \Phi)$ a noncommutative probability space, i.e. a von Neumann algebra with a normal state on it.

i) A *Random variable* on $\mathcal{A}$ is a $*$-algebra homomorphism $j : \mathcal{A} \to \mathcal{P}$;
ii) the *distribution* of the random variable $j : \mathcal{A} \to \mathcal{P}$ is the state $\varphi_j = \Phi \circ j$ on $\mathcal{A}$;
iii) the *convolution* of the random variables $j_1, j_2 : \mathcal{A} \to \mathcal{P}$ is the random variable

$$j_1^* j_2 = m_{\mathcal{P}} \circ (j_1 \otimes j_2) \circ \Delta\,,$$

where $m_{\mathcal{P}}$ denotes the product in $\mathcal{P}$.

A *Quantum Stochastic Process* on $\mathcal{A}$ is a family of random variables $(j_{s,t})_{0\leq s\leq t\leq T}$ satisfying

i) $j_{tt} = \varepsilon \mathbf{1}_{\mathcal{P}}$ for all $0 \leq t \leq T$
ii) the *increment property*: $j^*_{rs} j_{st} = j_{rt}$ for all $0 \leq r \leq s \leq t \leq T$
iii) *weak continuity*: j_{st} converges to j_{ss} in distribution for $t \searrow s$ and all $0 \leq s \leq T$.

Definition 4.3. (Lévy Process) A quantum stochastic process on the Hopf algebra $\mathcal{A}$ is called a *Lévy Process* provided it has

- **independent** increments, i.e. for disjoint intervals $(t_i, s_i]$, $i = 1, \cdots, n$
$$\Phi\big(j_{s_1t_1}(a_1)...j_{s_nt_n}(a_n)\big) = \Phi\big(j_{s_1t_1}(a_1)\big)...\Phi\big(j_{s_nt_n}(a_n)\big)$$
and $[j_{s_i,t_i}(a_1), j_{s_j,t_j}(a_2)] = 0$ for $i \neq j$;
- **stationary** increments, i.e. $\varphi_{st} := \Phi \circ j_{st}$ depends only on $t - s$.

The following result establishes a bridge allowing the study the probabilistic subject of Lévy Processes from an analytic point of view.

Theorem 4.4. *Lévy processes* $\{j_{st} : 0 \leq s \leq t < +\infty\}$ *on a the Hopf* $*$*-algebra* $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$ *are in one to one correspondence with Markovian semigroups* $\{T_t : t > 0\}$ *on* $A = \mathrm{C}(\mathbb{G})$ *which are translation invariant in the sense that the following identity holds true*
$$\Delta \circ T_t = (\mathrm{id} \otimes T_t) \circ \Delta \qquad t \geq 0\,.$$

To sketch the construction of the semigroup from the process, consider first that the distributions $\varphi_t := \varphi_{0,t} = \Phi \circ j_{0,t}$ form a continuous convolution semigroup of states on $\mathcal{A}$:
$$\varphi_0 = \varepsilon \qquad \varphi_s * \varphi_t = \varphi_{s+t} \qquad \lim_{t\to 0} \varphi_t(b) = \varepsilon(b) \qquad b \in \mathcal{A}\,.$$

Its *generating functional* defined as $G = \frac{d}{dt}\varphi_t\big|_{t=0}$, pointwise on a suitable domain, allows to reconstruct the distributions a convolution exponential $\varphi_t = \exp_* tG$ for all $t > 0$. A *semigroup* $T_t : \mathcal{A} \to \mathcal{A}$ is then defined by convolution
$$T_t a := ((\mathrm{id} \otimes \varphi_t) \circ \Delta)(a) = \varphi_t * a, \quad t \geq 0\,, \quad a \in \mathcal{A}$$
and its *formal infinitesimal generator* $L : \mathcal{A} \to \mathcal{A}$ results as the convolution operator associated to the generating functional
$$L(a) = (\mathrm{id} \otimes G) \circ \Delta(a) = G * a \quad a \in \mathcal{A}\,.$$
The semigroup then extends to a translation invariant, Markovian semigroup $\{T_t : t > 0\}$ on A and its generator is the closure of L. Moreover, the relations
$$G = \varepsilon \circ L\,, \qquad \varphi_t = \varepsilon \circ T_t \qquad t > 0$$
allow to get the generating functional and the distributions directly from the generator and the semigroup.

The generating functionals of Lévy processes or Lévy semigroups can also be described cohomologically by 1-cocycles of the Hopf $*$-algebra .

Definition 4.5. A *Schürmann triple* $((\pi, K), \eta, G)$ on the Hopf $*$-algebra $\mathcal{A}$ consists of:

(1) a unital $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(K)$ of $\mathcal{A}$ on some pre-Hilbert space K,
(2) a 1-cocycle $\eta : \mathcal{A} \to K$ of the representation (π, K), i.e. a linear map verifying
$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\varepsilon(b) \quad \text{for all } a, b \in \mathcal{A},$$

(3) a hermitian linear functional $G : \mathcal{A} \to \mathbb{C}$ satisfying

$$G(ab) = \langle \eta(a^*), \eta(b) \rangle_K \quad \text{for} \quad a, b \in \ker \varepsilon.$$

Theorem 4.6. *In a Schürmann triple $((\pi, D), \eta, G)$, the functional G is the generating functional of a Lévy process on the Hopf $*$-algebra $\mathcal{A}$.*
Conversely, the generating functional G of any Lévy process on $\mathcal{A}$ arises from a 1-cocycle of a Schürmann triple (uniquely modulo unitary equivalence on the range of the cocycle).

The KMS symmetry of a Lévy semigroup with respect to the Haar state, may be detected through its generating functional by the use of the unitary antipode.

Theorem 4.7. *Let e^{-tL} be a Lévy semigroup on A with generating functional $G = \varepsilon \circ L$. The following properties are then equivalent*

- *the semigroup is KMS$_{-1}$ symmetric with respect to the Haar state*
- *the generator is KMS$_{-1}$ symmetric with respect to the Haar state*
- *the generating functional is invariant by the action of the unitary antipode R*

$$G = G \circ R \quad \textit{on the Hopf $*$-algebra} \quad \mathcal{A} = \mathrm{Pol}\,(\mathbb{G})\,.$$

Later in these notes, we will see the connection between the Dirichlet form of a Lévy process and the associated Schürmann triple.

We end this section with a useful description of the spectrum of the generator of a Lévy semigroup in terms of a complete family $\{u^s : s \in \widehat{\mathbb{G}}\}$ of irreducible, unitary co-representations.

Proposition 4.8. *The standard Hilbert space of the Haar state $L^2(A, h)$ decomposes as an orthogonal sum of the finite dimensional subspaces*

$$L^2(A, h) = \bigoplus_{s \in \widehat{\mathbb{G}}} E_s \qquad E_s := \mathrm{Span}\,\{u^s_{jk}\xi_h : j, k = 1, \cdots, n_s\} \qquad s \in \widehat{\mathbb{G}}\,.$$

Correspondingly, the generator L of a KMS$_{-1}$ symmetric Lévy semigroup decomposes as a direct sum $L = \bigoplus_{s \in \widehat{\mathbb{G}}} L_s$ of its restrictions L_s to the L-invariant subspaces E_s so that its spectrum has the following structure $\sigma(L) = \overline{\bigcup_{s \in \widehat{\mathbb{G}}} \sigma(L_s)}$.

Example 4.9. (Free orthogonal quantum group O_N^+). The universal C*-algebra $C_u(O_N^+)$ of the *free orthogonal quantum group* O_N^+ is generated by elements $\{v_{jk} = v^*_{jk} : i, k = 1, \cdots, N\}$ subject to the relations

$$\sum_{l=1}^N v_{lj} v_{lk} = \delta_{jk} = \sum_{l=1}^N v_{jl} v_{kl} \qquad \Delta v_{jk} = \sum_{l=1}^N v_{lj} \otimes v_{lk}\,.$$

The Haar state h is a trace which is faithful on $\mathrm{Pol}(O_N^+)$ but not on $C_u(O_N^+)$, thus the Lévy semigroup e^{-tL} is constructed on the reduced C*-algebra $C_r(O_N^+)$ (determined by the GNS representation of the Haar state). The classes of irreducible, unitary co-representations are parameterized by $\widehat{O_N^+} \cong \mathbb{N}$. Denote by $U_s \in \mathrm{Pol}[-N, N]$ the Chebyshev polynomial of the second kind, defined recursively as follows

$$U_0(x) = 1, \quad U_1(x) = x, \quad U_s(x) = xU_{s-1}(x) - U_{s-1}(x), \qquad x \in [-N, N], \quad s \in \mathbb{N}\,.$$

A generating functional is then defined by

$$G(u^{(s)}_{jk}) := \delta_{jk} \frac{U_s'(N)}{U_s(N)}\,, \quad s \in \mathbb{N}\,, \quad j, k = 1, \cdots, U_s(N)\,.$$

The generator L has discrete spectrum with eigenvectors $u_{jk}^{(s)}$ corresponding to eigenvalues and multiplicities given by

$$\lambda_s = \frac{U_s'(N)}{U_s(N)}\,, \qquad m_s = (U_s(N))^2\,.$$

In particular, spectral dimensions are given by $d_N = 3$ for $N = 2$ and by $d_N = +\infty$ for $N \geq 3$.

5. Approximation Properties of von Neumann algebras and spectrum of Dirichlet forms

In this section we describe two recent results concerning the connection between the spectrum of Dirichlet forms and approximation properties of von Neumann algebras, namely the Haagerup Approximation Property and Amenability (see [CS], [CS6]).

A second countable, locally compact group G has the *Haagerup Approximation Property HAP* if there exists a sequence of normalized, positive definite functions $\varphi_n \in C_0(G)$, converging to the constant function 1, uniformly on compact subsets. Equivalently, G has the HAP if it admits a proper, continuous, negative definite function. For example, by a famous result of U. Haagerup [Haa1], the free groups $\mathbb{F}_n$ with $n \geq 2$ generators have the HAP since their length functions are negative definite. The HAP has deep relations with several aspects of group theory, in particular, it plays a fundamental role in Higson-Kasparov [HiK] work on the Baum-Connes conjecture (see [CCJJV] for details).
A long research initiated by A. Connes and Jones (see [CJ]) culminated with various equivalent definitions of the HAP for general von Neumann algebras (see [CS]). The HAP for von Neumann algebras features prominently in S. Popa deformation rigidity program for II_1 factors [Po].

Definition 5.1. (Haagerup Approximation Properties for von Neumann algebras)
1. (Caspers-Skalski) A von Neumann algebra $\mathcal{M}$ has the *Haagerup approximation property with respect to a normal, semifinite, faithful weight* φ on it if there exists a sequence $S_n : \mathcal{M} \to \mathcal{M}$ of completely positive, φ-non-increasing maps on $\mathcal{M}$ such that their induced maps $T_n : L^2(\mathcal{M}, \varphi) \to L^2(\mathcal{M}, \varphi)$ on the standard space $L^2(\mathcal{M}, \varphi)$ are compact and strongly convergent to the identity map.
2. (Okayasu-Tomatsu) A von Neumann algebra $\mathcal{M}$ has the *standard form Haagerup approximation property* if there exists a sequence $T_n : L^2(\mathcal{M}) \to L^2(\mathcal{M})$ of contractive, completely positive, compact operators on a standard space $L^2(\mathcal{M})$, strongly convergent to the identity map.

These properties has been shown to be equivalent and they are collectively referred as *Haagerup approximation property.* In view of the generalized Beurling -Deny correspondence between Markovian semigroups and Dirichlet forms on standard forms of von Neumann algebras, it is not surprising that the HAP can be characterized spectrally from an infinitesimal point of view. In fact we have the nice result

Theorem 5.2. *(Caspers-Skalski) The following properties are equivalent*
- *The von Neumann algebra $\mathcal{M}$ has the Haagerup Approximation Property;*
- *there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on some standard form of $\mathcal{M}$, having discrete spectrum.*

As an application Caspers-Skalski obtained in [CS] an independent proof of the following result of M. Brannan [Br].

Theorem 5.3. *The von Neumann algebras $L^\infty(C_r(O_N^+), h)$ of the free orthogonal quantum groups O_N^+ in the cyclic representation of the Haar state h on $L^2(C_r(O_N^+), h)$, have Haagerup Approximation Property.*

The result follows from the Caspers-Skalski equivalence and the construction of a Dirichlet form with discrete spectrum illustrated in Example 4.9.

By the previous result, it is natural to inquire if more specific discreteness conditions, about the spectrum of a Dirichlet form on a von Neumann algebra $\mathcal{M}$, may provide stronger approximation properties for $\mathcal{M}$.
As a guiding result we may recall the one concerning the amenability of a finitely generated group Γ (see for example [CCJJV]): if there exists a negative type function ℓ having sub-exponential growth

$$\sum_{s\in\Gamma} e^{-t\ell(s)} < +\infty \qquad \text{for all } t > 0\,,$$

then Γ is amenable. Notice that the functions $e^{-t\ell}$ are positive definite by a classical Schoenberg's theorem and that they give rise to the Markovian semigroup $T_t a := e^{-t\ell}a$ on GNS space $L^2(\lambda(\Gamma)'', \tau)$ of the von Neumann algebra $\lambda(\Gamma)''$ generated by the left regular representation endowed with its normalized trace. Hence the quadratic form of its generator $\mathcal{E}_\ell[a] = \sum_{s\in\Gamma} \ell(s)|a(s)|^2$ is a Dirichlet form whose spectrum coincides with $\{\ell(s) \in [0, +\infty) : s \in \Gamma\}$.

Here we illustrate some results from [CS6].

Definition 5.4. (Spectral growth of Dirichlet forms) Let us consider now a von Neumann algebra with separable predual $\mathcal{M}$ and a faithful, normal state $\omega \in M_{*,+}$ on it. Suppose that $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(\mathcal{M}, \omega)$ having discrete spectrum $\sigma(\mathcal{E}, \mathcal{F}) = \{\lambda_k \geq 0 : k \in \mathbb{N}\}$. Let us define the *spectral growth rate* $\omega(\mathcal{E}, \mathcal{F})$ as follows

$$\omega(\mathcal{E}, \mathcal{F}) := \limsup_{n\in\mathbb{N}} \sqrt[n]{m_n}\,, \quad m_n := \sharp(\Lambda_n)\,, \quad \Lambda_n := \{k \in \mathbb{N} : \lambda_k \in [n, n+1)\}\,.$$

The Dirichlet form is said to have

- *exponential growth* if $\omega(\mathcal{E}, \mathcal{F}) > 1$;
- *sub-exponential growth* if $\omega(\mathcal{E}, \mathcal{F}) \leq 1$.

Theorem 5.5. *Let $\mathcal{M}$ be a von Neumann algebra with separable predual. If there exists a Dirichlet form $(\mathcal{E}, \mathcal{F})$ with sub-exponential growth $\omega(\mathcal{E}, \mathcal{F}) \leq 1$, then $\mathcal{M}$ is amenable.*

Corollary 5.6. *Let $\mathcal{M}$ be a non amenable von Neumann algebra with separable predual. Then any Dirichlet form $(\mathcal{E}, \mathcal{F})$, with respect to any faithful, normal state $\omega \in M_{*,+}$, has exponential growth rate $\omega(\mathcal{E}, \mathcal{F}) > 1$, i.e. there exists a sequence of eigenvalues having exponentially growing distribution.*

Example 5.7. Applying the theorem above to the Dirichlet form considered in Example 4.9, we have an independent proof of the result of M. Brannan [Br] by which the von Neumann algebras $L^\infty(C_r(O_N^+), h)$ of the free orthogonal quantum groups O_N^+ in the cyclic representation of the Haar state h on $L^2(C_r(O_N^+), h)$.

6. Dirichlet forms and differential calculus on C*-algebras

In this section we focus the attention on a C*-algebra with semifinite, faithful, lower semi-continuous, positive trace (A, τ) and denote $\mathcal{M} = L^\infty(A, \tau)$ the von Neumann algebra acting on the space $L^2(A, \tau)$, generated by the GNS representation of (A, τ). Our goal is to show that there exists a differential calculus on the C*-algebra A underlying a Dirichlet form. A pivotal role will be played by the following property. We refer to [CS1] for details.

Definition 6.1. (Regular Dirichlet forms) A Dirichlet form $(\mathcal{F},\mathcal{F})$ on $L^2(A,\tau)$ is said to be *regular* if $\mathcal{B} := A\cap\mathcal{F}$ is a form core, dense in the C*-algebra A.

Notice that, even in the commutative setting, *regularity* was the key condition, introduced by Beurling-Deny, to develop a rich potential theory in Dirichlet spaces over locally compact spaces; regularity was also central in M. Fukushima work to associate (uniquely in a precise sense) a symmetric Hunt-Markov process, on locally compact space, to a Dirichlet form.

From an algebraic point of view, the importance of regularity, both in the commutative and in the noncommutative setting, is suggested by the following result, due to Beurling-Deny [BeDe2] in the first case and to Davies-Lindsay [DL] in the second one. Here we provide an alternative proof from [C5] emphasizing the role of the lower semicontinuity of the Dirichlet form.

We remark that J.L. Sauvageot [S1] discovered that every dissipation on a C*-algebra A with trace is canonically represented by a closed derivation in a C*-Hilbert A-bimodule. The regularity assumption above, on which the development of the differential calculus associated to a Dirichlet form is based, should be seen as a weakening of the Feller property for the corresponding Markovian semigroup, i.e. the possibility to extend it from $L^2(A,\tau)$ to a strongly continuous, Markovian semigroup on the C*-algebra A.

Lemma 6.2. *(Dirichlet algebra) Let $(\mathcal{F},\mathcal{F})$ be a Dirichlet form on $L^2(A,\tau)$. Then*

- *$\mathcal{B} := A\cap\mathcal{F}$ is a $*$-subalgebra of A, called Dirichlet algebra;*
- *$\mathcal{B}_e := \mathcal{M}\cap\mathcal{F}$ is a weakly*-dense $*$-subalgebra of $\mathcal{M}$, called weak Dirichlet algebra.*

Proof. By convexity, lower semicontinuity and Markovianity, for $a = a^*\in\mathcal{B}$, $\|a\| = 1$

$$\mathcal{E}\Big[\int_0^1 dt\, a\wedge t\Big] \le \int_0^1 dt\,\mathcal{E}[a\wedge t] \le \mathcal{E}[a]\,.$$

Since $\frac{a^2}{2} = a - \int_0^1 dt\, a\wedge t$, it results $a^2\in\mathcal{B}$. By scaling, the same it is true for all $a = a^*\in\mathcal{B}$. Hence, if $b = b^*\,, c = c^*\in\mathcal{B}$, then $(b+c) = (b+c)^*\,, (b-c) = (b-c)^*$ so that

$$bc + cb = (b+c)^2 - b^2 - c^2 \in\mathcal{B} \qquad b^2 - c^2 = \frac{(b+c)(b-c)+(b-c)(b+c)}{2}\in\mathcal{B}\,,$$

$$(b+ic)^2 = (b^2-c^2) + i(bc+cb)\in\mathcal{B}\,.$$

Decomposing a generic $a\in\mathcal{B}$ as $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$ we conclude that $a^2\in\mathcal{B}$.

If $a, b\in\mathcal{B}$, considering the matrix $\begin{bmatrix} 0 & a\\ b & 0\end{bmatrix}\in M_2(\mathcal{B})$, and applying the above result to the extension of the Dirichlet form $\mathcal{E}$ on $M_2(A)$, we obtain $\begin{bmatrix} ab & 0\\ 0 & ba\end{bmatrix} = \begin{bmatrix} 0 & a\\ b & 0\end{bmatrix}^2\in M_2(\mathcal{B})$ so that $ab\in\mathcal{B}$ and $\mathcal{B}$ is an algebra. An analogous proof holds true for $\mathcal{B}_e$. The weakly* density of $\mathcal{B}_e$ in $\mathcal{M}$ follows from the weakly*-continuity of the Markovian semigroup associated to the Dirichlet form. □

In case the Dirichlet form is regular, the Dirichlet algebra $\mathcal{B}$ is a norm dense, involutive sub-algebra of the C*-algebra A. As a by-product of the differential calculus we are going to associate to $(\mathcal{E},\mathcal{F})$, we will see that $\mathcal{B}$ retains some topological features of A: these algebras have the same K-theory (see [C4]).

6.1. **Dirichlet forms and derivations on C*-algebras.** The structure of the Dirichlet integral on a Euclidean space suggests that Dirichlet forms could be constructed through a differential calculus on the C*-algebra. Specifically, we will deal with the following one.

Definition 6.3. (Derivations on C*-algebras) A derivation $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ on (A, τ) is given by

- a norm dense *-subalgebra $\mathcal{B} \subseteq A \cap L^2(A, \tau)$
- a symmetric, Hilbert A-bimodule $(\mathcal{H}, \mathcal{J})$

$$\mathcal{J} : \mathcal{H} \to \mathcal{H} \quad \text{anti-unitary} \quad \mathcal{J}(a\xi b) = b^* \mathcal{J}(\xi) a^* \qquad a, b \in \mathcal{B}\,,\ \xi \in \mathcal{H}$$

- a linear, symmetric map $\partial : \mathcal{B} \to \mathcal{H}$ satisfying

$$\partial(a^*) = \mathcal{J}(\partial a) \qquad a \in \mathcal{B}$$

and the *Leibniz rule*

$$\partial(ab) = (\partial a)b + a(\partial b) \qquad a, b \in \mathcal{B}\,.$$

Example 6.4. Let (V, g) be a Riemannian manifold without boundary, $A := C_0(V)$, $\mathcal{B} := C_c^\infty(V)$ and let $\mathcal{H} := L^2(T^{\mathbb{C}}V)$ be the Hilbert space of square integrable sections of the complexified tangent bundle $T^{\mathbb{C}}V := TV \otimes \mathbb{C}$ acted on by continuous functions by pointwise multiplication (here left and right actions coincide so that $\mathcal{H}$ is a mono-module). A derivation $\partial := \nabla^{\mathbb{C}}$ is thus defined as the complexified gradient operator with involution given by $\mathcal{J}(\xi \otimes z) := \xi \otimes \bar{z}$ for $\xi \otimes z \in TV \otimes \mathbb{C}$.

In the following we will exploit the fact that a *closed* derivation $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ on (A, τ) not only satisfies the Leibniz rule, by definition, but also a *chain rule*. To properly state this fact, consider, for a fixed $a = a^* \in A$, the representations L_a and R_a of the algebra of continuous functions $\mathrm{C}(sp(a))$ on the spectrum $sp(a)$, uniquely defined for $f \in C(sp(a))$ and $\xi \in \mathcal{H}$ by

$$L_a(f)\xi = \begin{cases} f(a)\xi & \text{if} \quad f(0) = 0 \\ \xi & \text{if} \quad f \equiv 1 \end{cases} \qquad R_a(f)\xi = \begin{cases} \xi f(a) & \text{if} \quad f(0) = 0 \\ \xi & \text{if} \quad f \equiv 1\,, \end{cases}$$

and the representation $L_a \otimes R_a$ of $C(sp(a)) \otimes C(sp(a)) = C(sp(a) \times sp(a))$.

For closed interval $I \subseteq \mathbb{R}$ and $f \in \mathrm{C}^1(I)$, denote by $\tilde{f} \in \mathrm{C}(I \times I)$ its *difference quotient*

$$\tilde{f}(s, t) = \begin{cases} \frac{f(s) - f(t)}{s - t} & \text{if} \quad s \neq t \\ f'(s) & \text{if} \quad s = t. \end{cases}$$

Proposition 6.5. *(Chain rule for derivations) Let $(D(\partial), \partial, \mathcal{H}, \mathcal{J})$ be a norm closed derivation, densely defined on A. Then for $a = a^* \in D(\partial)$, a closed interval $sp(a) \subseteq I$ and $f \in C^1(I)$ such that $f(0) = 0$, one has*

$$f(a) \in D(\partial)\,, \qquad \partial(f(a)) = (L_a \otimes R_a)(\tilde{f})\, \partial(a)\,,$$

which implies

$$\|\partial(f(a))\|_{\mathcal{H}} \leq \|f'\|_{\mathrm{C}(I)} \cdot \|\partial(a)\|_{\mathcal{H}}\,.$$

In the present tracial case, the projection of an element $a = a^* \in A \cap L^2(A, \tau)$, onto the closed, convex subset $1_{\mathcal{M}} - L^2(A, \tau) \subset L^2(A, \tau)$, involved in the Markovian property of Dirichlet forms, can be described by functional calculus $a \wedge 1_{\mathcal{M}} = f(a)$ in terms of the *unit contraction* $f(t) := t \wedge 1 \in \mathbb{R}$ defined for $t \in \mathbb{R}$. Approximating the unit contraction by C^1-maps and using the chain rule, one obtains

Theorem 6.6. *(Dirichlet forms from derivations) Let $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ be a derivation on (A, τ) closable on $L^2(A, \tau)$. Then a Dirichlet form is obtained as closure of the quadratic form $(\mathcal{E}, \mathcal{B})$*

$$\mathcal{E}[\xi] := \|\partial a\|^2_{\mathcal{H}} \qquad a \in \mathcal{B}\,.$$

Hence derivations gives rise to Dirichlet forms analogously to the case where the gradient operator provides the Dirichlet integral on a Riemannian manifold. As a first noncommutative example of application of the the above result, we may consider the following

Example 6.7. (Noncommutative tori) This is a pivotal family of spaces since the appearance of Noncommutative Geometry [Co]. Let $\mathbb{T}^2_\theta$ be the rotation C*-algebra associated to the parameter $\theta \in \mathbb{R}$. It is the universal C*-algebra generated by two unitaries U and V, satisfying the relation

$$VU = e^{2i\pi\theta} UV\,.$$

For $\theta = 0$ it is isomorphic to the commutative C*-algebra $C(\mathbb{T}^2)$ of continuous functions on the 2-torus. If $\theta \in \mathbb{Q}$ then $\mathbb{T}^2_\theta$ is isomorphic to the algebra of continuous sections of bundles of C*-algebras where fibers are full matrix algebras. If however $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then $\mathbb{T}^2_\theta$, called *noncommutative torus*, is a simple C*-algebra with unique tracial state $\tau : \mathbb{T}^2_\theta \to \mathbb{C}$ specified by

$$\tau(U^n V^m) = \delta_{n,0}\delta_{m,0} \qquad n, m \in \mathbb{Z}\,.$$

The standard space of the trace can be described as

$$L^2(\mathbb{T}^2_\theta, \tau) = \{ \sum_{n,m\in\mathbb{Z}} \alpha_{n,m} U^n V^m : \sum_{n,m\in\mathbb{Z}} |\alpha_{n,m}|^2 < +\infty \}$$

where the cyclic vector representing the trace is the identity I. An orthonormal base is given by $\{U^n V^m : (n, m) \in \mathbb{Z} \times \mathbb{Z}\}$ and the left and right actions by

$$U\Big(\sum_{n,m\in\mathbb{Z}} \alpha_{n,m} U^n V^m\Big) := \sum_{n,m\in\mathbb{Z}} \alpha_{n,m} U^{n+1} V^m \qquad \Big(\sum_{n,m\in\mathbb{Z}} \alpha_{n,m} U^n V^m\Big)V := \sum_{n,m\in\mathbb{Z}} \alpha_{n,m} U^n V^{m+1}\,.$$

The (completely positive, unital), *heat semigroup* $\{T_t : t \geq 0\}$ on $\mathbb{T}^2_\theta$ is characterized by

$$T_t(U^n V^m) = e^{-t(n^2+m^2)} U^n V^m \qquad n, m \in \mathbb{Z}$$

and the same formula define its extension to $L^2(\mathbb{T}^2_\theta, \tau)$. The latter is τ-symmetric and the associated regular Dirichlet form is given by

$$\mathcal{E}\Big[\sum_{n,m\in\mathbb{Z}} \alpha_{n,m} U^n V^m\Big] = \sum_{n,m\in\mathbb{Z}} (n^2 + m^2)|\alpha_{n,m}|^2\,.$$

The spectrum is independent upon θ and thus coincides with the spectrum of the ordinary Laplacian on the torus $\mathbb{T}^2$. The derivation associated to $\mathcal{E}$ is the direct sum

$$\partial(a) = \partial_U(a) \oplus \partial_V(a)$$

of the (partial) derivations ∂_U and ∂_V defined by

$$\partial_U(U^n V^m) = inU^n V^m\,, \qquad \partial_V(U^n V^m) = imU^n V^m \qquad n, m \in \mathbb{Z}\,.$$

The heat semigroup is clearly conservative, and the $\mathbb{T}^2_\theta$–bimodule $\mathcal{H}_0$ associated with $\mathcal{E}$, as in Corollary 4.17, is a sub–bimodule of $L^2(\mathbb{T}^2_\theta, \tau) \oplus L^2(\mathbb{T}^2_\theta, \tau)$.

Example 6.8. (Clifford Dirichlet form of free fermion systems) The following one is, historically, the first example of noncommutative Dirichlet form, studied by L. Gross in Quantum Field Theory [G1,2].
Let h be an infinite dimensional, separable, real Hilbert space and $Cl(h)$ the complexification of the Clifford algebra over h. It is a simple C*–algebra with a unique trace state τ whose associated von Neumann algebra being the hyperfinite type II$_1$ factor.
By the Chevalley–Segal isomorphism, here denoted by D, $L^2(A,\tau)$ can be canonically identified with (the complexification of) the antisymmetric Fock space $\Gamma(h)$ over h. L. Gross [G1,2] showed that the Second Quantization $\Gamma(I_h)$ of the identity operator I_h on h is isomorphic to the Number operator $N = D^{-1}\Gamma(I_h)D$, which is the generator of a completely positive, conservative, C_0-semigroup over $Cl(h)$. To describe the structure of N, let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal base of h and let $\{A_i : i \in \mathbb{N}\}$ be the corresponding family of *annihilation* operators on $\Gamma(h)$. For each $i \in \mathbb{N}$ the operator $D_i := D^{-1}A_iD$, defined on the domain $D(\sqrt{N})$, is a densely defined, closed operator with values in $L^2(A,\tau)$ and

$$N = \sum_{i\in\mathbb{N}} D_i^*D_i = D^{-1}(\sum_{i\in\mathbb{N}} A_i^*A_i)D\,.$$

Moreover, $D(\sqrt{N})$ is a sub–algebra of $Cl(h)$ and on it the following Leibniz rules hold true:

$$D_i(ab) = D_i(a))b + \gamma(a)(D_i(b))\,.$$

Here γ is the extension to $L^2(A,\tau)$ of the canonical involution of $Cl(h)$ which is the unique extension of the map $v \mapsto -v$ on h. This shows that, by considering on $L^2(A,\tau)$ the GNS right action of $Cl(h)$ and the new left action given by γ, we obtain a closed derivation on $Cl(h)$ with values in $L^2(A,\tau)$. The Dirichlet form then is given by the formula

$$\mathcal{E}[a] = ||N^{1/2}(a)||^2_{\mathcal{H}} = \sum_{i\in\mathbb{N}} ||D_i(a)||^2_{L^2(A,\tau)}\,,$$

where the tangent bimodule is a sub-bimodule of $\bigoplus_{i\in\mathbb{N}} L^2(A,\tau)$ and the derivation is $\bigoplus_{i\in\mathbb{N}} D_i$.

We are now going to sketch the construction by which any regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on a C*-algebra A with trace, gives rise to a differential calculus on A. The construction relies on the Stinspring Theorem on the structure of completely positive maps and on the regularity of Dirichlet forms.

Theorem 6.9. *(Derivations from Dirichlet forms) Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form on $L^2(A,\tau)$ and $\mathcal{B} := A\cap\mathcal{F}$ its Dirichlet algebra. Then there exists a derivation $(\mathcal{B},\partial,\mathcal{H},\mathcal{J})$ such that*

$$\mathcal{E}[\xi] := \|\partial a\|^2_{\mathcal{H}} \qquad a \in \mathcal{B}\,.$$

To acquaint about the proof of the result, let us describe the main steps:

- a sesquilinear from on $\mathcal{B}\otimes\mathcal{B}$ is defined by the sesquilinear form $\mathcal{E}$

$$(c\otimes d|\, a\otimes b) := \frac{1}{2}\big(\mathcal{E}(c,abd^*) + \mathcal{E}(cdb^*,a) - \mathcal{E}(db^*,c^*a)\big);$$

- by the Stinspring representation of the resolvent on the von Neumann algebra

$$(I+\varepsilon L)^{-1}(a) = W_\varepsilon^*\pi_\varepsilon(a)W_\varepsilon \qquad a\in\mathcal{M}\,,$$

- the sesquilinear form above is shown to be positive definite by the following identity

$$\begin{aligned}(c\otimes d|\, a\otimes b) &= \lim_{\varepsilon\to 0}\frac{1}{2}\tau\Big(d^*\frac{L}{I+\varepsilon L}(c^*)ab + d^*c^*\frac{L}{I+\varepsilon L}(a)b - d^*\frac{L}{I+\varepsilon L}(c^*a)b\Big)\\ &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\tau\Big(d^*(W_\varepsilon c - \pi_\varepsilon(c)W_\varepsilon)^*(W_\varepsilon a - \pi_\varepsilon(a)W_\varepsilon)b + d^*c^*(I - W_\varepsilon^*W_\varepsilon)ab\Big);\end{aligned}$$

- denote by $\mathcal{H}_0$ the Hilbert space obtained from $\mathcal{B}\otimes\mathcal{B}$ by separation and completion
- and prove the bound $\|a\otimes b\|^2_{\mathcal{H}_0}\le \|b\|^2_A\cdot\mathcal{E}[a]$ for $a,b\in\mathcal{B}$
- a right A-module structure on $\mathcal{H}_0$ is obtained setting

$$(a\otimes b)c := a\otimes bc \qquad a,b,c\in\mathcal{B}$$

- a left A-module structure on $\mathcal{H}_0$ is obtained setting

$$d(a\otimes b) := da\otimes b - d\otimes ab \qquad a,b,c,d\in\mathcal{B}$$

- a derivation $\partial_0:\mathcal{B}\to\mathcal{H}_0$ is obtained setting

$$(\partial_0(a)|b\otimes c) := \frac{1}{2}(\mathcal{E}(a,bc)+\mathcal{E}(b^*,ca^*)-\mathcal{E}(b^*a,c)) \qquad a,b,c\in\mathcal{B}$$

- and the following identity

$$\mathcal{E}[a]-\|\partial_0(a)\|^2_{\mathcal{H}_0} = \lim_{\varepsilon\to 0}\frac{1}{2}\tau\Big(\frac{L}{I+\varepsilon L}(a^*a)\Big)$$

  allows to prove the result in the conservative case where $T_t1_{\mathcal{M}}=1_{\mathcal{M}}$, for all $t\ge 0$ and the right hand side of the above identity vanishes identically.

To handle the general case, a long *detour*, based on the norm closability of the derivation ∂_0, is needed to handle the left hand side of the above identity.

Example 6.10. (Dirichlet forms and derivations on group C*-algebras) Let us describe more carefully the content of the structure theorem in a particularly interesting situation. Let G be a locally compact, unimodular group with identity $e\in G$. Denote by $\lambda,\rho:G\to\mathcal{B}(L^2(G))$ the left, right regular representations and by $\mathrm{C}^*_r(G)$ the reduced group C*-algebra. A trace is determined by $\tau(a)=a(e)$ for $a\in C_c(G)$, its GNS space $L^2(A,\tau)$ coincides with $L^2(G)$ and the standard cone is the one of positive definite functions (see [Dix]).
For any continuous, negative definite function $\ell:G\to[0,+\infty)$, the function $e^{-t\ell}$ is positive definite for any $t>0$ (see [deH]). Since the pointwise product of positive definite functions is positive definite too, we have that

$$(T_ta)(t) = e^{-t\ell(g)}a(g) \qquad a\in L^2(G)$$

is a Markovian semigroup with generator determined by

$$(La)(g)=\ell(g)a(g) \qquad a\in C_c(G)\,.$$

and associated Dirichlet form given by

$$\mathcal{E}_\ell[a]=\int_G \ell(g)|a(g)|^2\,dg \qquad a\in L^2(G)\,.$$

To construct the associated derivation, one uses the orthogonal representation $\pi:G\to B(\mathcal{K})$ and the 1-cocycle

$$c:G\to\mathcal{K} \qquad c(gh)=c(g)+\pi(g)c(h) \qquad g,h\in G$$

representing the negative definite function as $\ell(g)=\|c(g)\|^2_{\mathcal{K}}$. Thus a Hilbert $\mathrm{C}^*_r(G)$-bimodule is set by $L^2(G,\mathcal{K}_{\mathbb{C}})\simeq L^2(G)\otimes\mathcal{K}_{\mathbb{C}}$ acted on the left by $\lambda\otimes\pi_{\mathbb{C}}$ and on the right by $id\otimes\rho$. Using the cocycle identity, an easy computation allows to shows that

$$\partial_\ell: C_c(G)\to L^2(G,\mathcal{K}_{\mathbb{C}}) \qquad (\partial_\ell a)(g)=c(g)a(g) \qquad g\in G$$

defines a derivation which represent the Dirichlet as $\mathcal{E}_\ell[a]=\|\partial_\ell a\|^2_{L^2(G,\mathcal{K}_{\mathbb{C}})}$.

Remark 6.11. Notice that the class of groups C*-algebras $C^*_r(\Gamma)$ of countable, discrete groups Γ, coincides with the class *co-commutative* compact quantum groups. Moreover, the Markovian semigroups associated to negative definite functions are exactly the Lévy semigroups in this subclass. In particular, one can describe the Schurmann triple in terms of the 1-cocycle representing the negative type function (see [CFK] for the details). Notice also that when $\Gamma = \mathbb{Z}^d$, the group C*-algebra is isomorphic to the algebra of continuous functions on the torus $C^*_r(\mathbb{Z}^d) \simeq C(\mathbb{T}^d)$. Moreover, if negative type function is given by

$$\ell : \mathbb{Z}^d \to [0, +\infty) \qquad \ell(z) := |z|^2 \,,$$

then the corresponding Dirichlet space and Markovian semigroup are equivalent, by Fourier series, to the Dirichlet integral and heat semigroup on $\mathbb{T}^n$, respectively.

6.2. **Dirichlet forms and K-theory.** Several consequences can be derived from the structure of regular Dirichlet forms in terms of derivations. On the topological side we have the following ones (see [C4]).

Proposition 6.12. *(K-theory of Dirichlet spaces) Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on a C*-algebra with trace (A, τ). Then the following properties holds true:*

- *the form domain $\mathcal{F}$ is closed under Lipschiz functional calculus*

$$a = a^* \in \mathcal{F} \text{ and } f \in \mathrm{Lip}_0(\mathbb{R}) \quad \Rightarrow \quad f(a) \in \mathcal{F} \text{ and } \mathcal{E}[f(a)] \leq \|f\|^2_{\mathrm{Lip}_0(\mathbb{R})} \cdot \mathcal{E}[a] \,;$$

- *the Dirichlet algebra and the C*-algebra have equivalent K-groups: $K_*(\mathcal{B}) = K_*(A)$;*
- *if the trace τ is finite, the Dirichlet algebra $\mathcal{B}$ is a semisimple, involutive, Banach algebra when endowed with the norm*

$$\|a\|_{\mathcal{B}} := \|a\|_{\mathcal{M}} + \sqrt{\mathcal{E}[a]} \qquad a \in \mathcal{B} \,,$$

  so that, in particular, it has a unique Banach algebra topology.

Corollary 6.13. *The triple $(A, \tau, \mathcal{F})$ determines the quasi conformal class of the Dirichlet form in the sense that two Dirichlet forms $(\mathcal{E}_1, \mathcal{F})$, $(\mathcal{E}_2, \mathcal{F})$ on $L^2(A, \tau)$ with common domain are comparable as follows*

$$\frac{1}{k}(\mathcal{E}_1[a] + \|a\|_2^2) \leq \mathcal{E}_2[a] + \|a\|_2^2 \leq k(\mathcal{E}_1[a] + \|a\|_2^2) \qquad a \in \mathcal{F}$$

for some constant $k > 0$.

As projective modules over a C*-algebra A can be described by projections in matrix ampliations of A, the complete Markovianity of a Dirichlet form allows to introduce *Dirichlet structures* on projective modules. Here are some consequences, in a commutative situation.

Proposition 6.14. *(Dirichlet structures on vector bundles) Let (X, m) be compact Haudorff space endowed with a finite, positive, Borel measure and let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(X, m)$. Then we have:*

- *any finite dimensional, locally trivial vector bundle $E \to X$ acquires a canonical Dirichlet structure, i.e. a class of compatible atlases with transition matrices having (finite energy) entries in $\mathcal{B}$;*
- *the space of sections $\mathcal{B}(E, X)$ of the Dirichlet structure has a canonical Banach module structure over $\mathcal{B}$.*

6.3. **Decomposition of derivations and Dirichlet forms.** Since the derivations associated to regular Dirichlet forms take values in Hilbert bimodules over the the C*-algebra A, one has at disposal the tools of representation theory to study them. In this section we will briefly describe the influence of decomposition theory (we refer to [CS2]).

Definition 6.15. Consider a derivation $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ on A and the von Neumann algebra $\mathcal{L}_{A-A}(\mathcal{H})$ of operators commuting both with the left and right actions of A.

- $T \in \mathcal{L}_{A-A}(\mathcal{H})$ is *∂-bounded* if $\mathcal{B} \ni b \mapsto T(\partial b) \in \mathcal{H}$ is bounded from A to $\mathcal{H}$;
- a projection $p \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ is *approximately ∂- bounded* if increasing limit $p = \lim_\alpha p_\alpha$ of a net of ∂-bounded projections $p_\alpha \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$;
- *equivalently*, $p \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ is *approximately ∂- bounded* if the A-bimodule $p\mathcal{H}$ splits as direct sum $\bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$ and the derivation $p \circ \partial$ decomposes as a direct sum $\bigoplus_{n\in\mathbb{N}} \partial_n$ of bounded derivations;
- a projection $p \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ is *completely ∂-unbounded* if 0 is the only ∂-bounded projection smaller than p;
- a projection $p \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ is *bounded, approximately bounded, completely unbounded* if the identity $1_\mathcal{H}$ is a *∂-bounded, approximately ∂-bounded, completely ∂-unbounded* projection.

Theorem 6.16. *(Decomposition of Derivations and Dirichlet forms) There exists a greatest approximately ∂-bounded projection $P_{ab} \in \mathrm{Proj}(\mathcal{L}_{A-A}(\mathcal{H}))$ such that any ∂-bounded operator $T \in \mathcal{L}_{A-A}(\mathcal{H})$ satisfies $TP_{ab} = P_{ab}$.*
Consequently, any derivation $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ on A decomposes canonically as

$$\partial = \partial_u \oplus \partial_{ab} : \mathcal{B} \to \mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_{ab}$$

where

- $\partial_u := (I - P_{ab}) \circ \partial, \quad \mathcal{H}_u := P_u\mathcal{H}$ *is the completely unbounded part*
- $\partial_{ab} := P_{ab} \circ \partial, \quad \mathcal{H}_{ab} := P_{ab}\mathcal{H}$ *is the approximately bounded part.*
- *A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a C*-algebra with trace (A, τ), thus decomposes accordingly*

$$\mathcal{E} = \mathcal{E}_u + \mathcal{E}_{ab} \qquad \mathcal{E}_*[a] = \|\partial_* a\|_*^2 \qquad * = u, ab\,, \qquad a \in \mathcal{B}\,.$$

Let us compare below the decomposition just obtained with the one discovered by Beurling-Deny, in the commutative situation.

Example 6.17. (*Decompositions of Dirichlet forms on commutative C*-algebras*)
In the commutative situation $(C_0(X), m)$ of a locally compact, separable Hausdorff space X endowed with a positive Borel measure m, Beurling-Deny [BD2] proved that a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ decomposes

$$\mathcal{E}[a] = \mathcal{E}_c[a] + \mathcal{E}_j[a] + \mathcal{E}_k[a] \qquad a \in \mathcal{B}$$

in terms of the *diffusive* $\mathcal{E}_c$, the *jumping* $\mathcal{E}_j$ and the *killing* $\mathcal{E}_k$ parts. The latter has the representation

$$\mathcal{E}_k[a] = \int_X k(dx)\, |a(x)|^2$$

for a unique positive Radon measure k on X; the jumping part appears as

$$\mathcal{E}_j[a] = \int_{X\times X\setminus\Delta_X} j(dx, dy)\, |a(x) - a(y)|^2$$

for a unique positive Radon measure j on $X \times X$ supported off the diagonal $\Delta_X := \{(x,x) \in X \times X : x \in X\}$; the continuous or diffusive part $\mathcal{E}_c$ is characterized by the following strong local property:

$$\mathcal{E}_c[a+b] = \mathcal{E}_c[a] + \mathcal{E}_c[b]$$

for all $b \in \mathcal{B}$ constant in a neighborhood of the support of $a \in \mathcal{B}$. One of the beautiful results of the theory of Dirichlet forms on locally compact case is that each of the three parts of the Beurling-Deny splitting has a precise probabilistic interpretation in terms of the Hunt process associated to the Dirichlet form: strongly local Dirichlet forms give rise to diffusions, i.e. processes with continuous sample paths with no killing inside X; the measure j indicates the rate of jumps of the sample paths of the process and the measure k indicates the killing probability of the sample paths of the process inside X (see [FOT Part II Chapter 4.5] for a nice exposition of these and other results connecting the potential theory of Dirichlet forms to properties of the associated process).

Using the tools of representation theory and in particular the notion of *support of a representation*, the Beurling-Deny decomposition can be obtained from the description of the Dirichlet form in terms of the derivation $(\mathcal{B}, \partial, \mathcal{H}, J)$ such that $\mathcal{E}[a] = \|\partial a\|^2_{\mathcal{H}}$ for $a \in \mathcal{B}$.
The tangent bimodule $\mathcal{H}$ of a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is a representation of the C*-algebra $C_0(X) \otimes C_0(X) \simeq C_0(X \times X)$ and thus can be canonically decomposed as the sum $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_j \oplus \mathcal{H}_k$ of the part $\mathcal{H}_c$ supported by the diagonal Δ_X of $X \times X$, the part $\mathcal{H}_j$ supported off the diagonal Δ_X and the completely degenerate part $\mathcal{H}_k$ of the $C_0(X)$-bimodule $\mathcal{H}$. The submodules $\mathcal{H}_c$, $\mathcal{H}_j$ and $\mathcal{H}_k$ are images of suitable, orthogonal projections $p_c\,, p_j\,, p_k \in \mathcal{L}_{C_0(X)-C_0(X)}(\mathcal{H})$, commuting with the left and right actions of $C_0(X)$.
Then $p_c \circ \partial$, $p_j \circ \partial$, $p_k \circ \partial$ are symmetric derivations from $\mathcal{B}$ to $\mathcal{H}$ giving rise to three forms that coincide with the three parts of the Beurling-Deny decomposition

$$\mathcal{E}_c[a] = \|p_c(\partial a)\|^2_{\mathcal{H}}\,, \quad \mathcal{E}_j[a] = \|p_j(\partial a)\|^2_{\mathcal{H}}\,, \quad \mathcal{E}_k[a] = \|p_k(\partial a)\|^2_{\mathcal{H}}\,, \qquad a \in \mathcal{B}\,.$$

In addition, a more detailed description of the strongly local part can be obtained. In fact, as the support of $\mathcal{H}_c$ is contained in the diagonal Δ_X, the left and right actions of $C_0(X)$ coincide and $\mathcal{H}_c$ is a $C_0(X)$-mono-module, i.e. a representation of the C*-algebra $C_0(X)$. Since, by I.M. Gel'fand's theory, the irreducible representations of $C_0(X)$ are in one to one correspondence with the points of X, one may represent $\mathcal{H}_c$ as a direct integral

$$\mathcal{H}_c = \int_X^{\oplus} \mu(dx)\, \mathcal{H}_x$$

of a measurable family of Hilbert spaces $\{\mathcal{H}_x : x \in X\}$ carrying the actions $(a \cdot \xi) := a(x)\xi$ for $a \in C_0(X)$ and $\xi \in \mathcal{H}_x$. The Hilbert space $\mathcal{H}_x$ have no other meaning that to represent the multiplicity of the irreducible representation associated to $x \in X$ by their dimension $\dim(\mathcal{H}_x)$. Correspondingly, the derivation is represented as a direct integral

$$\partial = \int_X^{\oplus} \mu(dx)\, \partial_x$$

of derivations $\partial_x : \mathcal{B} \to \mathcal{H}_x$ satisfying the Leibniz rules

$$\partial_x(ab) = (\partial_x a)b(x) + a(x)(\partial_x b) \qquad a, b \in \mathcal{B}\,, \quad x \in X$$

and the strongly local part of the Dirichlet form appears as "superposition"

$$\mathcal{E}_c[a] = \int_X \mu(dx)\ \|\partial_x a\|^2_{\mathcal{H}_x} \qquad a \in \mathcal{B}\,.$$

On the other hand, comparing the Beurling-Deny decomposition with the decomposition obtained in Theorem 6.16 above, one gets the identifications $\mathcal{E}_u = \mathcal{E}_c$ of the completely unbounded part with the diffusive part of $\mathcal{E}$ and the identification $\mathcal{E}_{ab} = \mathcal{E}_j + \mathcal{E}_k$ of the approximately bounded part with the sum of the jumping and killing parts.
Notice that, while the Beurling-Deny *continuous/jumping/killing* decomposition has a meaning only in commutative settings, the decomposition *completely unbounded/approximately bounded* holds true in a general framework.

7. Dirac Laplacian and positive curvature of Riemannian manifolds

In this section we will apply the completely unbounded/approximately bounded decomposition theory outlined above to a narural class of Dirichlet form arising in Riemannian Geometry (see [CS2]).
We will see how the Markovian property of the quadratic form of the Dirac Laplacian on Riemannian manifold exactly measures the positivity of the curvature operator. The result follows from the decomposition theory of Dirichlet forms outlined above and the specific structure of the Clifford (finite dimensional) C*-algebras of Euclidean spaces.

On a Riemannian manifold (V, g), the Euclidean tangent spaces (T_xV, g_x) gives rise to complexified Clifford algebras with traces $(Cl(T_xV), \tau_x)$, for any $x \in V$. Gluing together this vector spaces one may consider the complexified Clifford bundle $Cl(V, g)$ and the Clifford C*-algebra $C_0^*(V, g) := C_0(Cl(V, g))$ of continuous sections of it, vanishing at infinity.
Coupling the traces on fibers with the integral with respect to the Riemannian measure m, one obtains a trace $\tau = \int_V \tau_x \, m(dx)$ on $C_0^*(V, g)$. Denoting by $\nabla : C^\infty(Cl(V, g)) \to C^\infty(Cl(V, g) \otimes T^*V)$ Levi-Civita connection of (V, g), we provide a alternative proof of the following result by E.B. Davies and O.Rothaus [DR 1,2].

Theorem 7.1. *The closure of the quadratic form given by the Bochner integral*

$$\mathcal{E}_B[\sigma] := \int_V |\nabla\sigma(x)|^2 \, dx \qquad \sigma \in C^\infty(Cl(V, g))$$

is Dirichlet form on $L^2(C_0^*(V), \tau)$.

The associated self-adjoint operator $\nabla^*\nabla$ is called the *Bochner* or *connection Laplacian.*

To outline the proof of the result, consider that

- the Hilbert space $L^2(Cl(V, g) \otimes T^*V)$ is a $C_0^*(V, g)$-bimodule

$$\sigma_1 \cdot (\sigma_2 \otimes \omega) \cdot \sigma_3 := (\sigma_1 \cdot \sigma_2 \cdot \sigma_3) \otimes \omega$$

  where the "$\cdot$" denote the Clifford product among sections of the Clifford bundle;
- a symmetry $\mathcal{J} : L^2(Cl(V, g) \otimes T^*V) \to L^2(Cl(V, g) \otimes T^*V)$ is defined by

$$\mathcal{J}(\sigma \otimes \omega) := \sigma^* \otimes \bar{\omega}$$

  where σ^* denotes the involution in the Clifford algebra and $\bar{\omega}$ the obvious involution of complexified 1-forms;
- by definition, the Levi-Civita connection satisfies the metric property: for any smooth vector field X

$$\nabla(f\sigma) = \sigma \otimes df + f\nabla\sigma \qquad X(\sigma|\sigma) = (\nabla_X\sigma|\sigma) + (\sigma|\nabla_X\sigma)\,;$$

- since the contraction i_X commutes with the actions of the Clifford algebra and $\nabla_X = i_X \circ \nabla$ we have
$$i_X(\nabla(\sigma\cdot\sigma)) = i_X((\nabla\sigma)\cdot\sigma + \sigma(\nabla\sigma))\,;$$
- as this is true for any vector field X we have
$$\nabla(\sigma\cdot\sigma) = (\nabla\sigma)\cdot\sigma + \sigma(\nabla\sigma)$$
from which the Leibniz property follows by polarization;

Hence, the covariant derivative of a Riemannian manifold, once considered on the Clifford algebra, is a closed derivation defined on the Sobolev space $H^1(Cl(V,g))$ of sections of the Clifford bundle and taking values in the $C_0^*(V,g)$-bimodule $L^2(Cl(V,g)\otimes T^*V)$. Since
$$\mathcal{E}_B[\sigma] = \|\nabla\sigma\|^2_{L^2(Cl(V,g)\otimes T^*V)}$$
for all $\sigma\in H^1(Cl(V,g))$, we have that the Bochner integral is a Dirichlet form with $C^\infty(Cl(V,g))$ as a form core.

Notice that, while the above result for the Bochner-Laplacian is independent upon the sign of the curvature of (V,g), a parallel result for the Dirac-Laplacian reflects the sign of the curvature, as we are going to inspect.
Recall that the Dirac operator of a Riemannian manifold (V,g) is defined (locally) as
$$(D\sigma)(x) := \sum_{i=1}^n e_i(x)\cdot(\nabla_{e_i}\sigma)(x) \qquad x\in V\,,$$
where the vector fields $\{e_i\}_{i=1}^n$ fthonormal base at $x\in V$. orm an or an orthonormal base (see [LM] for the details of the construction). Under the canonical isomorphism of vector spaces between the Clifford algebra $Cl(V,g)$ and the exterior algebra $\Lambda^*(V,g)$, the Dirac operator D transforms into the operator $d+d^*$ and the Dirac Laplacian D^2 transform into the Hodge-de Rham Laplacian $\Delta_{\mathrm{HdR}} = d^*\circ d + d\circ d^*$. The difference is that Δ_{HdR} depends only upon the differential structure of V, while D^2 is constructed through the metric g. The Dirac operator D is closable on $L^2(Cl(V,g))$ and the domain of the closure is the first Sobolev space $H^1(Cl(V,g))$.

Theorem 7.2. *The following properties are equivalent:*
- *the quadratic form of the Dirac Laplacian*
$$\mathcal{E}_D[\sigma] := \|D\sigma\|^2_{L^2(Cl(V,g)\otimes T^*V)} = \int_V |D\sigma(x)|^2\,dx \qquad \sigma\in H^1(Cl(V,g))$$
is a regular Dirichlet form on $L^2(C_0^*(V,g),\tau)$;
- *the heat semigroup e^{-tD^2} is a Markovian, C_0-semigroup on $C_0^*(V,g)$;*
- *the curvature operator of (V,g) is nonnegative: $\widehat{R}\geq 0$.*

Example 7.3. On a compact, connected, orientable surface Σ, there exists a metric g such that $\mathcal{E}_D$ is a Dirichlet form if and only if Σ is homeomorphic to the sphere S^2 or the torus T^2.

To outline the proof of the theorem, let us start to recall the basic ingredient. The curvature endomorphisms $R_x(v_1,v_2): T_xV\to T_xV$
$$R_x(v_1,v_2)v := -(\nabla_{v_1}\nabla_{v_2}v - \nabla_{v_2}\nabla_{v_1}v - \nabla_{[v_1,v_2]}v)(x) \qquad v,v_1,v_2\in C_c^\infty(TV),\quad x\in V$$
defines the curvature tensor $R_x\in\bigotimes^4 T_x^*V$
$$R_x(v_1,v_2,v_3,v_4) = (R_x(v_1,v_2)v_3|v_4)_{T_xV} \qquad v_1,v_2.v_3,v_4\in C_c^\infty(TV),\quad x\in V$$

and the curvature operators $\widehat{R}_x : \bigwedge_x^2 V \to \bigwedge_x^2 V$

$$(\widehat{R}_x v_1 \wedge v_2 | v_3 \wedge v_4)_{\bigwedge_x^2 V} = R_x(v_1, v_2, v_3, v_4) \qquad v_1, v_2.v_3, v_4 \in C_c^\infty(TV), \quad x \in V\,.$$

The curvatures identities, excluding the Bianchi's ones, implies that $\widehat{R}_x$ is symmetric and thus self-adjoint when extended on the complexification of $\bigwedge_x^2$.
The proof uses Bochner Identity $D^2 = \Delta_B + \frac{1}{4}\Theta_R$ in terms of quadratic forms

$$\mathcal{E}_D = \mathcal{E}_B + \frac{1}{4} Q_R\,, \qquad Q_R[\sigma] = \int_V Q_R(x)[\sigma_x]\, dx$$

where the curvature part can be written

$$Q_R(x)[\sigma_x] = \sum_{\alpha=1}^{n(n-1)/2} \mu_\alpha \|[\eta_\alpha(x), \sigma_x]\|_2^2$$

in terms of a basis of orthonormal eigenvectors $\{\eta_\alpha(x) : \alpha = 1, \ldots, n(n-1)/2\} \subset \bigwedge_x^2$ of $\widehat{R}_x$ corresponding to the eigenvalues $\{\mu_\alpha : \alpha = 1, \ldots, n(n-1)/2\}$.
As commutators are bounded derivations, if $\widehat{R} \geq 0$ then all eigenvalues are nonnegative $\mu_\alpha \geq 0$ and Q_R is a (superposition) of Dirichlet forms. Moreover, as $\mathcal{E}_B$ is a Dirichlet form by the result of Davies-Rothaus, then $\mathcal{E}_D$ results, by the Bochner Identity, as a (superposition of) Dirichlet forms.

In the opposite direction, the strategy is in two moves: the first is prove that, given a Euclidean space E with orthonormal base $\{e_i\}_{i=1}^{n=\dim E}$ and a symmetric operator $T : \bigwedge^2(E) \to \bigwedge^2(E)$, a form on $L^2(Cl(E), \tau)$ of type

$$Q_T(x)[\xi] = \sum_{1\leq i<j\leq n} \sum_{1\leq k<l\leq n} \langle e_k \wedge e_l | T(e_i \wedge e_j)\rangle \langle [e_k \cdot e_l, \xi] | [e_i \cdot e_j, \xi]\rangle$$

is a Dirichlet form if and only if $(\xi|T\xi)_{\bigwedge^2(E)} \geq 0$ for all $\xi \in \bigwedge^2(E)$. This part uses again the correspondence Dirichlet form/derivations and the ideal structure of finite dimensional Clifford algebras; in the second part one has to disentangle the role of connection and curvature in the left hand side of the Bochner identity $\mathcal{E}_D = \mathcal{E}_B + \frac{1}{4}Q_R$ and prove that if $\mathcal{E}_D$ is a Dirichlet form a fortiori Q_R has to be a Dirichlet form too. The conclusion of the proof uses repeatedly the above completely unbounded/approximately bounded decomposition to realize that the approximately bounded part of the quadratic form of the Dirac operator is proportional to the curvature term of the Bochner Identity $(\mathcal{E}_D)_j = \frac{1}{4}Q_R$ so that Q_R is, a fortiori, a Dirichlet form and then the curvature operator has to be nonnegative $\widehat{R} \geq 0$.

Corollary 7.4. *If the curvature operator of a Riemannian manifold V is nonnegative $\widehat{R} \geq 0$, then the space of harmonic forms has the structure of a finite dimensional C^*-algebra, hence a finite sum of full matrix algebras.*
In particular, the sum of all Betti numbers $b_0(V) + \cdots + b_n(V)$ is a sum of squares of natural numbers.

8. Dirichlet forms in Free Probability

In this section we describe aspects of noncommutative potential theory appearing in Free Probability Theory discovered by Dan Virgil Voiculescu (see [V 1,2,3]).
Let (M, τ) be a noncommutative probability space, i.e. a von Neumann algebra endowed with a faithful, normal, finite and normalized trace.

Let us fix a unital *-subalgebra $1 \in B \subset M$ and a finite set $X := \{X_1, \cdots, X_n\} \subset M$ of noncommutative random variables, i.e. self-adjoint elements of M, algebraically free with respect to B.
Let us consider the *-subalgebra $B[X] \subset M$ generated by X and B (regarded as the algebra of noncommutative polynomials in the variables X with coefficients in the algebra B) and the von Neumann subalgebra $W \subset M$ generated by $B[X]$.

Let $HS(L^2(W,\tau)) \simeq \overline{L^2(W,\tau)} \otimes L^2(W,\tau)$ be the Hilbert W-bimodule of Hilbert-Schmidt operators on $L^2(W,\tau)$ and $1 \otimes 1 \in HS(L^2(W,\tau))$ the rank one projection onto the multiples of the unit $1 \in M \subset L^2(M,\tau)$.

Within this franmework, D.V. Voiculescu introduced a natural differential calculus and an associated Dirichlet form.

Theorem 8.1. *There exist unique derivations $\partial_{X_i} : B[X] \to HS(L^2(W,\tau))$ such that*

- $\partial_{X_i} X_j = \delta_{ij} 1 \otimes 1\,, \qquad i,j = 1, \cdots, n;$
- $\partial_{X_i} b = 0 \qquad i = 1, \cdots, n, \quad b \in B\,.$

Under the assumption $1 \otimes 1 \in \mathrm{dom}\,(\partial^*_{X_i})$ for all $i = 1, \cdots, n$, it follows that

- *$(\partial_{X_i}, B[X])$ is densely defined and closable in $L^2(W,\tau)$ for all $i = 1, \cdots, n$,*
- *the closure of the densely defined quadratic form*

$$\mathcal{E}_X[a] := \sum_{i=1}^n \|\partial_{X_i} a\|^2_{\mathrm{HS}} \qquad a \in \mathcal{F}_X := B[X]$$

is a Dirichlet form on $L^2(W,\tau)$.

Under the same assumption of the previous result, Voiculescu defined two relevant notions in Free Probability.

Definition 8.2. Under the assumption $1 \otimes 1 \in \mathrm{dom}\,(\partial^*_{X_i}) \quad i = 1, \cdots, n$, the *Noncommutative Hilbert Transform* of X with respect to B is defined as

$$\mathcal{J}(X : B) := \Big(\sum_{i=1}^n \partial^*_{X_i} \partial_{X_i}\Big)(X_1 + \cdots + X_n) \in L^2(W,\tau)$$

and the *Relative Free Fisher information* of X with respect to B is defined as

$$\Phi^*(X : B) := \|\mathcal{J}(X : B)\|^2\,.$$

In the commutative case where $M = L^\infty(\mathbb{R}, m)$, $B = \mathbb{C}1$ is the subalgebra of constant functions and the random variable $X \in M$ has distribution μ_X, one has that $W = L^\infty(\mathbb{R}, \mu_X)$, $B[X]$ is the algebra of polynomials on $\mathbb{R}$ and the derivation $\partial_X f$ coincides with the difference quotient. In case $p := \frac{d\mu_X}{dm} \in L^3(\mathbb{R}, m)$, then $\mathcal{J}(X : B)$ is the usual Hilbert transform

$$Hp(t) := \mathrm{p.v.}\, \frac{1}{\pi} \int_{\mathbb{R}} \frac{p(s)}{t-s}\, ds\,.$$

It has been proved by Ph. Biane [Bi] that the Hessian of the Free Fischer information coincides with the Dirichlet form $\mathcal{E}_X$ on the domain where the Free Fischer information if finite.

Since the derivations ∂_{X_i} $i = 1, \cdots, n$ annihilate B, the kernel of the Voiculescu's Dirichlet form introduced above, contains B and its spectrum has zero as eigenvalue. Moreover

Theorem 8.3. *A Free Poincaré inequality holds true for some* $c > 0$

$$c\|Y - \tau(Y)\|_2^2 \leq \mathcal{E}_X[Y] \qquad Y \in \mathcal{F}_X := \bigcap_{i=1}^{n} dom\,(\partial_{X_i})$$

or, equivalently, the spectrum of the Dirichlet form $(\mathcal{E}_X, \mathcal{F}_X)$ *is contained in* $\{0\} \cup [c, +\infty)$*, if and only if the random variable* X *is centered, it has unital covariance and it has semicircular distribution.*

A nice application has been derived from the results above, by Y. Dabrowski [Da]

Theorem 8.4. *If the free Fisher information is finite* $\Phi^*(X : \mathbb{C}) < +\infty$ *then* W *is a factor.*

To sketch the proof consider that a consequence of the assumption is that the X_j are diffuse operators. If $Z \in W \cap W'$ is also in the domain of the Dirichlet form, $Z \in \mathcal{F}_X$, then

$$0 = \partial_{X_i}([Z, X_j]) = [\partial_{X_i}(Z), X_j] \qquad i \neq j\,.$$

Thus each $\partial_{X_i}(Z) \in HS(L^2(W, \tau))$ is a compact operator commuting with a diffuse operator X_j and then vanishes $\partial_{X_i}(Z) = 0$. The Free Poincaré inequality allows to conclude that Z is a multiple of the identity. A standard resolvent regularization allows to remove the extra assumption $Z \in \mathcal{F}_X$.

9. L^2-rigidity in von Neumann algebras and Markovian semigroups

In the framework of Sorin Popa deformation/rigidity theory of inclusions of von Neumann algebras, a fruitful approach has been undertaken by J. Peterson [Pe 1,2] in terms of Markovian semigroups and their (relative) continuity properties.

Definition 9.1. (L^2-rigid inclusions) Let $N \subset M$ be an inclusion of finite von Neumann algebras and τ a fixed normal, faithful trace on M.

- An L^2-deformation of N is a Markovian semigroup $\{T_t : t > 0\}$ on $L^2(M, \tau)$;
- an inclusion $B \subset N$ is said to be L^2-rigid if any L^2-deformation for N converges uniformly on the unit ball of B

$$\lim_{t \to 0^+} \sup_{\|b\|_B = 1} \|b - T_t b\|_2 = 0\,.$$

Example 9.2. Let $\Lambda \subset \Gamma$ be countable discrete groups and $L(\Lambda) \subset L(\Gamma)$ the inclusion of the corresponding (left) von Neumann algebras. We have seen in Example 6.10 that a function $\ell : \Gamma \to [0, +\infty)$ of negative type gives rise to a corresponding Markovian semigroup $T_t : l^2(\Gamma) \to l^2(\Gamma)$, by the multiplication operators $T_t a = e^{-t\ell} a$. Then $\{T_t : t > 0\}$ is an L^2-deformation of $L(\Lambda)$ if and only if ℓ is inner, i.e.

$$\ell(t) = \|\xi - \pi(t)\xi\|_{\mathcal{K}}^2 \qquad t \in \Gamma$$

for some orthogonal representation $\pi : \Gamma \to \mathbb{B}(\mathcal{K})$ and a unit vector $\xi \in \mathcal{K}$.

Several interesting rigidity results may be derived in [Pe 1,2] by the construction of derivations in suitable bimodules.

Theorem 9.3. *Let* N *be a finite von Neumann algebras with normal, finite, faithful trace* τ*.*

- *if* $B \subset N$ *is a subalgebra with no non-zero amenable summands then the inclusion* $B' \cap N$ *is* L^2*-rigid;*
- *if* N *is a non-amenable* II_1 *factor which is non-prime or has property* Γ*, then* N *is* L^2*-rigid;*

- *(Ozawa Theorem) if a countable discrete group* Γ *has a proper cocycle* $c : \Gamma \to (\ell(\Gamma))^{\oplus\infty}$*, then* $L(\Gamma)$ *is solid, i.e.* $B' \cap L(\Gamma)$ *is amenable for any diffuse subalgebra* $B \subset L(\Gamma)$*.*
  Last result applies, in particular, to free group factors $\Gamma = \mathbb{F}_n\,, \quad 2 \leq n \leq +\infty$*.*

10. Dirac operator and Fredholm module of Dirichlet spaces

On a regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on a C*-algebra with trace (A, τ), there exists a natural Dirac operator D that identifies, by its commutator properties with the actions of elements of A, two sub-algebras of $\mathcal{M}$: the Lipschiz algebra and the algebra of finite energy multipliers. Both of them can be equivalently described in terms of the *carré du champ* or energy distributions of $(\mathcal{E}, \mathcal{F})$. We will show how they gives rise to a spectral triple and to a Fredholm module. These constructions have been applied to fractals in [CS3], [CGIS2] and to compact quantum groups in [CFK]. Finer aspects may found in [CS4]. See also the recent [Ri2] where the relationships between Dirichlet forms and corresponding Lipschiz seminorms as well as between Dirac operators, quotient energy seminorms and resistance distances is investigated, in the finite dimensional situations.

10.1. **Carré du champ.** One of the main subject of potential theory of regular Dirichlet spaces $(\mathcal{E}, \mathcal{F})$ on C*-algebras with trace (A, τ), is the following class of functionals. Here we adopt a definition based on the derivation associated to the Dirichlet form (compare with the classical one as in[LJ]).

Definition 10.1. (Carré du champ) The carré du champ of $a \in \mathcal{F}$ is the positive functional $\Gamma[a] \in A^*_+$

$$\Gamma[a] : A \to \mathbb{C} \qquad \langle \Gamma[a], b\rangle := (\partial a|(\partial a)b)_{\mathcal{H}} \qquad b \in A$$

defined using the derivation $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ representing $(\mathcal{E}, \mathcal{F})$. It easy to see that

$$\langle \Gamma[a], b\rangle := \frac{1}{2}\{\mathcal{E}(ab^*|a) + \mathcal{E}(a|ab) - \mathcal{E}(a^*a|b)\} \qquad a, b \in \mathcal{B}\,.$$

In particular, $\mathcal{E}[a] = \langle \Gamma[a], 1_{A^{**}}\rangle$ for all $a \in \mathcal{F}$.

When $\mathcal{E}[a]$ represents the total energy of a system in a configuration $a \in \mathcal{F}$, then $\Gamma[a]$ may be interpreted as the energy distribution.

Example 10.2. In case of the Dirichlet integral on $\mathbb{R}^n$, the carré du champ are absolutely continuous with respect to the Lebesgue measure m and reduces to

$$\Gamma[a] = |\nabla a|^2 \cdot m \qquad a \in H^1(\mathbb{R}^n)\,.$$

Example 10.3. Consider a continuous, negative definite function $\ell : G \to [0, +\infty)$ on a locally compact, unimodular group and the associated Dirichlet form

$$\mathcal{E}_\ell[a] = \int_G \ell(g)|a(g)|^2\, dg \qquad a \in L^2(G)\,.$$

The carré du champ can be explicitly computed

$$\langle \Gamma[a], b\rangle = \int_G dg \int_G dh\, \gamma_\ell(g, h)\overline{a(g)}a(h)b(h^{-1}g)$$

in terms of the *energy kernel* γ_ℓ given by

$$\gamma_\ell(g, h) := \frac{1}{2}\{\ell(g) + \ell(h) - \ell(h^{-1}g)\} = (c(g)|c(h))_{H_\ell}$$

or through the 1-cocycle (π_ℓ, H_ℓ, c) associated to the negative type function. The 1-cocycle $c : G \to H_\ell$ is an isometric embedding of the metric space $(G, d_{\sqrt{\ell}})$ with metric $d_{\sqrt{\ell}}(s,t) := \sqrt{\ell(s^{-1}t)}$ into the real Hilbert space H_ℓ

$$\|c(s) - c(t)\|_{H_\ell} = \sqrt{\ell(s^{-1}t)} = d_{\sqrt{\ell}}(s,t) \qquad s,t \in G\,.$$

The energy kernel can thus be written

$$\gamma_\ell(g,h) = \frac{1}{2}\{d_{\sqrt{\ell}}(g,e)^2 + d_{\sqrt{\ell}}(h,e)^2 - d_{\sqrt{\ell}}(g,h)^2\} \qquad h,g \in G$$

and interpreted as deviation from orthogonality. If the negative type function gives rise to a metric $(g,h) \mapsto d_\ell(g,h) := \ell(h^{-1}g)$ on G, then the energy kernel γ_ℓ is the Gromov product in the metric space (G, d_ℓ) based on the identity $e \in G$

$$\gamma_\ell(g,h) = (g|h)_e := \frac{1}{2}\{d_\ell(g,e) + d_\ell(h,e) - d_\ell(g,h)\} \qquad g,h \in G\,.$$

In applications the trace τ, finite or not, may represent the volume distribution of a system. In general the energy distribution $\Gamma[a]$ is not comparable with the volume distribution, as it happens for example on fractals.

In these lectures the carré du camp of a Dirichlet space will be important to discuss its metric and conformal properties.

10.2. **Dirac operator on the Lipschiz algebra.** Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on (A,τ), $(\mathcal{F}, \partial, \mathcal{H}, \mathcal{J})$ its differential square root and $(\mathcal{F}^*, \partial^*, \mathcal{H}, \mathcal{J})$ its Hilbert space adjoint. As $(\mathcal{B}, \partial, \mathcal{H}, \mathcal{J})$ is a derivation on A, it is justified to refer to $(\mathcal{F}^*, \partial^*, \mathcal{H}, \mathcal{J})$ as the *divergence* of $(\mathcal{E}, \mathcal{F})$.

Definition 10.4. (Dirac operator) *The Dirac operator $(D, \mathcal{H}_D)$ of the Dirichlet space is the densely defined, self-adjoint operator acting on $\mathcal{H}_D := L^2(A,\tau) \oplus \mathcal{H}$ given by*

$$D := \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix} \qquad \mathrm{dom}(D) := \mathcal{F} \oplus \mathcal{F}^* \subseteq \mathcal{H}_D$$

or more explicitly

$$D\begin{pmatrix} a \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}\begin{pmatrix} a \\ \xi \end{pmatrix} = \begin{pmatrix} \partial^*\xi \\ \partial a \end{pmatrix}, \qquad \begin{pmatrix} a \\ \xi \end{pmatrix} \in \mathcal{F} \oplus \mathcal{F}^*\,.$$

By definition, the Dirac operator anti-commutes with involution $\gamma := \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$:

$$D\gamma + \gamma D = 0\,.$$

By the Spectral Theorem, the operators $\partial^*\partial$ and $\partial\partial^*$ are self-adjoint operators on the Hilbert spaces $L^2(A,\tau)$ and $\mathcal{H}$, respectively, which are unitarely equivalent one each other on the orthogonal complement of their kernels. The Dirac operator of a Dirichlet space is a differential square root of the generator of the Markovian semigroup in the sense that

$$D^2 = \begin{pmatrix} \partial^*\partial & 0 \\ 0 & \partial\partial^* \end{pmatrix}.$$

We summarize in the lemma below some obvious spectral propertiy of the Dirac operator.

Lemma 10.5. *The kernel of the Dirac operator is given by* $\ker(D) = \ker(\partial) \oplus \ker(\partial^*)$ *and its dimension by* $\dim\ker(D) = \dim\ker(\partial) + \dim\ker(\partial^*) = \dim L^2(A,\tau)$*. Away from zero, the spectrum of the Dirac operator* $(D, dom(D))$ *on the Hilbert space* $L^2(A,\tau) \oplus \mathcal{H}$ *is given by*

$$\sigma(D) \setminus \{0\} = [(\sigma(\sqrt{\Delta}) \setminus \{0\})] \cup [-(\sigma(\sqrt{\Delta}) \setminus \{0\})] .$$

Moreover, if the spectrum $\sigma(\Delta)$ *of the generator is discrete then, away from zero, the spectrum* $\sigma(D)$ *of the Dirac operator is discrete too. In particular, if* $\lambda \in \sigma(\Delta)$ *is a non vanishing eigenvalue of the generator, with associated normalized eigenvector* $a_\lambda \in \mathrm{dom}(\Delta)$*, then* $\pm\sqrt{\lambda} \in \sigma(D)$ *are the corresponding non vanishing eigenvalues of* D *and the associated normalized eigenvectors are given by* $a_\lambda \oplus (\pm\lambda^{-\frac{1}{2}}\partial a_\lambda) \in \mathrm{dom}(D)$*.*

In A. Connes' Nonncommutative Geometry [Co], a primary rôle of a Dirac operator is to single out, through its interplay with the action of elements of A, a subalgebra to gives a Spectral Triple from which topological and geometric properties can be derived. We are going to show that for the Dirac operator of a Dirichlet space, this algebra can be naturally described in terms of the carré du champ, i.e. through energy distributions.

Proposition 10.6. *(Bounded commutators and Lipschiz seminorms) Consider* $L^2(A,\tau)$*,* $\mathcal{H}$ *and* $\mathcal{H}_D$ *as left A-modules. For* $a \in \mathcal{B}$*, the following properties are equivalent*

- $[D, a]$ *is bounded on* $\mathcal{H}_D$*;*
- $[\partial, a]$ *is bounded from* $L^2(A,\tau)$ *to* $\mathcal{H}$*;*
- $\Gamma[a]$ *is absolutely continuous w.r.t.* τ *with bounded Radon-Nikodym derivative*

$$h_a \in L^\infty(A,\tau) \qquad \langle\Gamma[a], b\rangle = \tau(h_a b) \qquad b \in L^1(A,\tau).$$

In particular $\|[D, a]\|_{\mathcal{H}_D\to\mathcal{H}_D} = \|[\partial, a]\|_{L^2\to\mathcal{H}} = \|h_a\|_{\mathcal{M}}$ *for all a satisfying the above conditions.*

- *For* $a \in \mathcal{B} \cap \mathrm{dom}_{\mathcal{M}}(L)$*, the above properties are also equivalent to* $a^*a \in \mathrm{dom}_{\mathcal{M}}(L)$*.*

Definition 10.7. (Lipschiz algebra of a Dirichlet space) The *-subalgebra $\mathcal{L}(\mathcal{F}) \subseteq \mathcal{B}$ of elements satisfying the first three properties above, is called the *Lipschiz algebra* of the Dirichlet space. It will be assumed to be seminormed by

$$\mathcal{L}(\mathcal{F}) \ni a \mapsto \|[D, a]\|_{\mathcal{H}_D\to\mathcal{H}_D} = \|[\partial, a]\|_{L^2\to\mathcal{H}} = \|h_a\|_{\mathcal{M}} .$$

In particular, $\mathcal{L}(\mathcal{F}) \cap \mathrm{dom}_{\mathcal{M}}(L)$ is an involutive subalgebra contained in $\mathcal{L}(\mathcal{F})$.

Example 10.8. In case of the Dirichlet integral $\mathcal{D}$ on $\mathbb{R}^n$, the derivation ∂ coincides with the gradient operator ∇ and then the commutator $[\partial, a]$ multiplies a function b by the gradient ∇a, $[\partial, a]b = (\nabla a)b$. Its operator norm $\|[\partial, a]\|_\infty$ thus coincides with the uniform norm of the gradient $\||\nabla a|\|_\infty$ and then the Dirichlet algebra $\mathcal{L}(H^1(\mathbb{R}^n))$ is given by the algebra $\mathrm{Lip}(\mathbb{R}^n)$ of Lipschiz functions of the Euclidean metric.
In case of the Bochner integral of a Riemannian manifold (V, g), the derivation ∂ coincides with the exterior differential d, the commutator $[\partial, \sigma]$ is the pointwise Clifford left multiplication by $d\sigma$ so that the Dirichlet algebra coincides with the algebra of Lipschiz sections, with respect to the metric g, of the Clifford algebra.

In the first case, the metric induced by the Dirac operator through the Connes' formula

$$
\begin{aligned}
d_D(x,y) :&= \sup\{|a(x)-a(y)| : \|[D,a]\|_{\mathcal{H}_D\to\mathcal{H}_D} \le 1\} \\
&= \sup\{|a(x)-a(y)| : a \in \mathcal{L}(\mathcal{F}), \quad \|[\partial,a]\|_{L^2\to\mathcal{H}_D} \le 1\} \\
&= \sup\{|a(x)-a(y)| : a \in \mathcal{L}(\mathcal{F}), \quad \||\nabla|\|_\infty \le 1\} \\
&= |x-y|_{\mathbb{R}^n} \qquad x,y\in\mathbb{R}^n
\end{aligned} \tag{10.1}
$$

is exactly the Euclidean metric on $\mathbb{R}^n$. In the second case, the Connes' formula provides the original metric g of the Riemannian manifold.

Define the phase $F_D := D|D|^{-1}$ of the Dirac operator to be zero on $\ker(D)$.

Theorem 10.9. *(Spectral triple and Fredholm module) Assume the spectrum of $(\mathcal{E},\mathcal{F})$ on $L^2(A,\tau)$ to be discrete. Then $(\mathcal{L}(\mathcal{F}), D, \mathcal{H}_D)$ is a spectral triple in the sense*

- *$[D,a]$ is bounded on $\mathcal{H}_D$ for all $a\in\mathcal{L}(\mathcal{F})$*
- *$\mathrm{sp}(D)$ is discrete away from zero.*

Moreover, setting $F := F_D + P_{\ker(D)}$, then $(\mathcal{L}(\mathcal{F}), F, \mathcal{H}_D)$ is a Fredholm module

- *$F = F^*$, $\qquad F^2 = I$*
- *$[F,a]$ is compact on $\mathcal{H}_D$ for all $a\in\mathcal{L}(\mathcal{F})$.*

The difference with Connes definition of spectral triple lies in the fact that the Dirac operator of a Dirichlet space with discrete spectrum, has discrete spectrum away from zero and zero is an eigenvalue of infinite degeneracy if the algebra A is infinite dimensional.

Example 10.10. (Ground State representations of Schrödinger operators) Let us consider a lower semibounded Hamiltonian $H := -\Delta + V$ with potential V, on the space $L^2(\mathbb{R}^n, m)$. Assume the spectrum to be discrete $\mathrm{sp}(H) = \{E_0 < E_1 < \dots\}$ and consider the ground state $\psi_0 \in L^2(\mathbb{R}^n, m)$ with lowest eigenvalue E_0: $H\psi_0 = E_0\psi_0$ and ground state measure $\nu_0 := |\psi_0|^2\cdot m$. As the energy form $\mathcal{E}_H[a] := \|\nabla a\|^2_{L^2(\mathbb{R}^n,\mathbb{R}^n)} + (a|Va)_{L^2(\mathbb{R}^n)}$ of the Hamiltonian H, on its domain satisfies $\mathcal{E}_H[|a|] = \mathcal{E}_H[a]$, we have that the ground state is strictly positive $\phi_0 > 0$. Using the ground state transformation

$$U : L^2(\mathbb{R}^n, m) \to L^2(\mathbb{R}^n, |\psi_0|^2\cdot m) \qquad U(f) := \psi_0^{-1} f\,, \qquad f\in L^2(\mathbb{R}^n, m)\,,$$

the ground state representation of H is defined as $H_{\phi_0} := U(H-E_0)U^{-1}$. Since the Schrödinger semigroup e^{-tH} is positivity preserving on $L^2(\mathbb{R}^n, m)$ and $e^{-tH}\psi_0 = e^{-tE_0}\psi_0$, then the semigroup $e^{-tH_{\psi_0}}$ is positive preserving on $L^2(\mathbb{R}^n,\nu_0)$ and leaves invariant $e^{-tH_{\psi_0}}1 = 1$ the constant function 1. It is thus a Markovian semigroup on the standard form $L^2(\mathbb{R}^n,\nu_0)$ of the von Neumann algebra $L^\infty(\mathbb{R}^n,\nu_0) = L^\infty(\mathbb{R}^n, m)$ with respect to the cyclic and separating vector represented by the constant function 1. More explicitly, the quadratic form of H_{ψ_0} is, on its natural domain $\mathcal{F}_{\psi_0}$, a Dirichlet form on $L^2(\mathbb{R}^n,\nu_0)$

$$\mathcal{E}_{\psi_0}[a] = \|\sqrt{H_{\psi_0}}a\|_2^2 = \int_{\mathbb{R}^n} |\nabla a|^2\cdot d\nu_0 \qquad a\in\mathcal{F}_{\psi_0}$$

which is regular with respect to the commutative, unital, nonseparable C*-algebra $C_b(\mathbb{R}^n)$ of all bounded continuous functions on the Euclidean space. The associated derivation, taking values in the $C_b(\mathbb{R}^n)$-monomodule $L^2(\mathbb{R}^n,\nu_0)\otimes\mathbb{C}^n$ of vector fields square integrable with respect to the ground state measure, is simply given by the gradient operator

$$\partial : \mathcal{F}_{\psi_0} \to L^2(\mathbb{R}^n,\nu_0)\otimes\mathbb{C}^n \qquad \partial a = \nabla a\,.$$

The Lipschiz algebra of the Dirichlet space thus coincides with the Lipschiz algebra of the Euclidean space $\mathcal{L}(\mathcal{F}_{\psi_0}) = \mathcal{L}(\mathbb{R}^n)$ and the distance induced on $\mathbb{R}^n$ coincides with Euclidean distance. Thus, as far as the distance function associated to the spectral triple is concerned, we are in the same situation we would have reached if we would have considered the Dirichlet integral $\mathcal{D}$ of $\mathbb{R}^n$. However, the fact that the latter is referred to the C*-algebra $C_0(\mathbb{R}^n)$ of continuous functions vanishing at infinity and the presently constructed spectral triple is referred on $C_b(\mathbb{R}^n)$ marks a difference at a topological level. To be more specific, consider the potential of an harmonic oscillator $V(x) := |x|^2$. Then spectral dimension of the triple $(C_b(\mathbb{R}^n)\cap\mathrm{Lip}(\mathbb{R}^n), D_{\psi_0}, L^2(\mathbb{R}^n,\nu_0)\oplus L^2(\mathbb{R}^n,\nu_0))$ is $2n$, twice the spectral dimension n obtained using the Dirichlet integral. As $C_b(\mathbb{R}^n)$ is the algebra of continuous function of the Stone-Cech compactification $\beta\mathbb{R}^n$ of $\mathbb{R}^n$, we conclude that our spectral triple assigns dimension $2n$ to $\beta\mathbb{R}^n$.

In situations where the Lipschiz algebra $\mathcal{L}(\mathcal{F})$ is not dense in the C*-algebra A, for example, on post critically finite fractals and for natural classes of Dirichlet structures on them, one may be interested in getting directly a Fredholm module bypassing the spectral triple.

10.3. **Fredholm module of a Dirichlet space.** To construct the Fredholm module associated to a Dirichlet form, we consider the image of the associated derivation and the symmetry with respect to it.

Definition 10.11. (Phase operator of a Dirichlet form) Let us consider a regular Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(A,\tau)$. Let $\partial : \mathcal{B}\to\mathcal{H}$ be the associated derivation, defined on the Dirichlet algebra $\mathcal{B} = A\cap\mathcal{F}$ with values in the symmetric Hilbert module $(\mathcal{H},\mathcal{J})$.
Let $P \in \mathrm{Proj}\,(\mathcal{H})$ be the projection onto the closure $\overline{\mathrm{Im}\partial}$ of the range of the derivation

$$P\mathcal{H} := \overline{\mathrm{Im}\,\partial}$$

and call $F := P - P^\perp : \mathcal{H}\to\mathcal{H}$ the *phase operator* associated to the regular Dirichlet space.

Theorem 10.12. *(Fredholm module of a Dirichlet form) Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(A,\tau)$ such that*

i) the spectrum $\mathrm{sp}(\mathcal{E},\mathcal{F}) = \{\lambda_k \geq 0 : k = 1,\cdots,+\infty\}$ *is discrete*

ii) the (normalized) eigenvectors $\{a_k : k = 1,\cdots,+\infty\}$ belongs to the C*-algebra A

iii) the Green function of $(\mathcal{E},\mathcal{F})$, defined as $G := \sum_{k=1}^\infty \lambda_k^{-1} a_k^* a_k$, belongs to A.

Then $(F,\mathcal{H})$ is a Fredholm module over A in the sense of [At] and a densely, 2-summable Fredholm module over A in the sense of [Co Chapter IV 1.γ Definition 8].

Proof. Clearly $F^* = F$, $F^2 = I$. Since, by the regularity of the Dirichlet form, the Dirichlet algebra $\mathcal{B}$ is involutive and dense in A, to prove that $[F,a]$ is compact for all $a\in A$, we may assume that $a = a^*\in\mathcal{B}$. Consider the identity

$$[P,a] = PaP^\perp - P^\perp aP \tag{10.2}$$

from which we have

$$|[P,a]|^2 = |PaP^\perp|^2 + |P^\perp aP|^2 \tag{10.3}$$

and then

$$\|[F,a]\|^2_{\mathcal{L}^2} = 4\|[P,a]\|^2_{\mathcal{L}^2} = 8\|P^\perp aP\|^2_{\mathcal{L}^2}\,. \tag{10.4}$$

By definition of P, the identities $P\circ\partial=\partial$ and $P^\perp\circ\partial=0$ hold true. Using the Leibniz rule we have

$$P^\perp aP(\partial b)=P^\perp(a\partial b)=P^\perp(\partial(ab)-(\partial a)b)=-P^\perp((\partial a)b)\qquad b\in\mathcal{B} \tag{10.5}$$

so that

$$\|P^\perp aP(\partial b)\|=\|P^\perp((\partial a)b)\|\leq\|(\partial a)b\|\,. \tag{10.6}$$

Denote $k_0:=\inf\{k\in\mathbb{N}:\lambda_k>0\}$. By assumption ii), the eigenvectors belong to the Dirichlet algebra $\mathcal{B}$ so that the vectors $\xi_k:=\lambda_k^{-1/2}\partial a_k$, $k\geq k_0$, form an orthonormal complete system in $P\mathcal{H}$. We may thus bound the Hilbert-Schmidt norm of the commutator $[F,a]$ as follows

$$\begin{aligned}\|[F,a]\|^2_{\mathcal{L}^2}&=8\|P^\perp aP\|^2_{\mathcal{L}^2}=8\sum_{k=k_0}^\infty\lambda_k^{-1}\|P^\perp aP(\partial a_k)\|^2_{\mathcal{H}}\leq 8\sum_{k=k_0}^\infty\lambda_k^{-1}\|a_k\partial a\|^2_{\mathcal{H}}\\&=8\sum_{k=k_0}^\infty\lambda_k^{-1}(a_k\partial a|a_k\partial a)=8(\sum_{k=k_0}^\infty\lambda_k^{-1}a_k^*a_k\partial a|\partial a)=8(G\partial a|\partial a)\\&=8(\mathcal{J}\partial a|\mathcal{J}(G\partial a))=8(\partial(a^*)|(\partial(a^*))G^*)=8(\partial a|(\partial a)G)\\&=8\langle\Gamma[a],G\rangle\\&\leq 8\cdot\|G\|_A\cdot\mathcal{E}[a]\end{aligned} \tag{10.7}$$

which is finite by assumption iii). □

The above result applies (see [CS3]) to the class of Dirichlet forms constructed by J. Kigami [Ki] on self-similar fractal spaces, associated to regular harmonic structures (a specific example will be discussed in Section 12 below).

Remark 10.13. The above result indicates a direct connection between the summability properties of the Fredholm module and two of the main objects of potential theory, namely, the Dirichlet form $\mathcal{E}$ and the Green function G.

11. Potential Theory in Dirichlet spaces

Finer properties of the differential calculus underlying a Dirichlet spaces rely on properties of the basic objects of the Potential Theory of Dirichlet forms. One of them, the carré du champ, has been already introduced in the previous section. In this section we introduce the others, finite energy states, potentials, multipliers, and illustrate some basic relations among them (see [CS4]).

11.1. **Potentials and finite-energy states.** Consider a regular Dirichlet space $(\mathcal{E},\mathcal{F})$ with its Hilbertian graph norm $\|a\|_{\mathcal{F}}:=\sqrt{\mathcal{E}[a]+\|a\|^2_{L^2(A,\tau)}}$.

Definition 11.1. (Potentials, Finite Energy Functionals)

- $p\in\mathcal{F}$ is called a *potential* if

$$(p|a)_{\mathcal{F}}\geq 0\qquad a\in\mathcal{F}_+:=\mathcal{F}\cap L^2_+(A,\tau)$$

  Denote by $\mathcal{P}\subset L^2(A,\tau)$ the closed convex cone of potentials.
- $\omega\in A^*_+$ has *finite energy* if for some $c_\omega\geq 0$

$$|\omega(a)|\leq c_\omega\cdot\|a\|_{\mathcal{F}}\qquad a\in\mathcal{F}\,.$$

$\mathcal{P}$ is the polar cone of $L^2_+(A,\tau)$ with respect to the energy scalar product. Finite energy functionals are not necessarily continuous with respect to the trace.

Example 11.2. In a d-dimensional Riemannian manifold (V,g), the volume measure μ_W of a $(d-1)$-dimensional compact submanifold $W\subset V$ has finite energy.

Potentials and finite energy functionals are in one to one correspondence.

Theorem 11.3. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on (A,τ). Then we have*

- *potentials are positive: $\mathcal{P}\subset L^2_+(A,\tau)$;*
- *for any fixed potential $p\in\mathcal{P}$, the functional*
$$\omega_p : A\to\mathbb{C}\qquad \omega_p(a):=(p|a)_{\mathcal{F}}\qquad a\in\mathcal{F}$$
has finite energy;
- *given a finite energy functional $\omega\in A^*_+$, there exists a unique potential representing it*
$$G(\omega)\in\mathcal{P}\qquad \omega(a)=(G(\omega)|a)_{\mathcal{F}}\qquad a\in\mathcal{F}\,.$$

Example 11.4. If $h\in L^2_+(A,\tau)\cap L^1(A,\tau)$ then the functional $\omega_h\in A^*_+$, defined by
$$\omega_h(a):=\tau(ha)\qquad a\in A\,,$$
is a finite energy functional whose potential is given by $G(\omega_h)=(I+L)^{-1}h$.

Example 11.5. Let $\mathcal{E}_\ell$ be the Dirichlet form on $A:=C^*_r(\Gamma)$, associated to a negative definite function ℓ on a countable, discrete group Γ. Then $\omega\in A^*_+$ is a finite energy functional if and only if
$$\sum_{t\in\Gamma}\frac{|\omega(\delta_s)|^2}{1+\ell(s)}<+\infty\quad\text{with potential given by}\quad G(\omega)(s)=\frac{\omega(\delta_s)}{1+\ell(s)}\qquad s\in\Gamma\,.$$
Since $\varphi_\ell:=(1+\sqrt{\ell})^{-1}$ is a positive definite, normalized function, there exists a state $\omega_\ell\in A^*_+$ such that $\varphi_\ell(s)=\omega_\ell(\delta_s)$ for all $s\in\Gamma$. Thus ω has finite energy if and only if
$$\sum_{s\in\Gamma}\frac{|\omega(\delta_s)|^2}{(1+\sqrt{\ell}(s))^2}=\sum_{s\in\Gamma}|\varphi_\ell(s)\cdot\varphi_\omega(s)|^2<+\infty\,.$$
Notice that $\varphi_\ell\cdot\varphi_\omega$ is a coefficient of a square integrable, sub-representation of the product $\pi_{\omega_\ell}\otimes\pi_\omega$ of the representations $(\pi_\ell,\mathcal{H}_\ell,\xi_\ell)$ and $(\pi_\omega,\mathcal{H}_\omega,\xi_\omega)$ associated to ω_ℓ and ω. Hence if ω has finite-energy, the representation $\pi_{\omega_\ell}\otimes\pi_\omega$ and the left regular representation λ_Γ are not disjoint. Moreover, as ω has finite energy simultaneously with respect to $\mathcal{E}_\ell$ and $\mathcal{E}_{\lambda^{-2}\ell}$ for $\lambda>0$, the family of normalized, positive definite functions
$$\varphi_\lambda(s)=\frac{\lambda}{\lambda+\sqrt{\ell(s)}}\cdot\varphi_\omega(s)\qquad s\in\Gamma\,,$$
generates a family of cyclic representations $\{\pi_\lambda:\lambda>0\}$ contained in λ_Γ, deforming the cyclic representation π_ω associated to the finite energy state ω to the left regular representation λ_Γ
$$\lim_{\lambda\to0^+}\varphi_\lambda=\delta_e\,,\qquad\lim_{\lambda\to+\infty}\varphi_\lambda=\varphi_\omega\,,$$
so that the representation π_ω is weakly contained in the regular representation λ_Γ: $\pi_\omega\preceq\lambda_\Gamma$.

The boundedness of the potential $G(\omega)$ of a finite energy functional ω is a regularity property that allows to promote the embedding $\mathcal{F}\to L^1(A,\omega)$ to an embedding $\mathcal{F}\to L^2(A,\omega)$.

Theorem 11.6. *(Deny's embedding) Let $\omega \in A^*_+$ be a finite energy functional with bounded potential*

$$G(\omega) \in \mathcal{P} \cap L^\infty(A, \tau)\,.$$

Then the following inequality holds true

$$\omega(b^*b) \le ||G(\omega)||_{\mathcal{M}}\, ||b||^2_{\mathcal{F}} \qquad b \in \mathcal{B}\,.$$

Denoting by $\xi_\omega \in L^2_+(A, \tau)$ the cyclic, positive vector representing ω, the map

$$D_\omega : \mathcal{B} \to L^2(A, \tau) \qquad D_\omega(b) := b\xi_\omega$$

extends to a bounded map from $\mathcal{F}$ to $L^2(A, \tau)$.

Example 11.7. Let $\mathcal{E}_\ell$ be the Dirichlet form associated to a negative type function ℓ on a countable discrete group Γ. Deny's embedding applies whenever

- $\sum_s \frac{1}{1+\ell(s)} |\omega(\delta_s)|^2 < +\infty$ $\qquad$ ω has finite energy
- $\displaystyle\sum_s \frac{\omega(\delta_s)}{1+\ell(s)} \lambda(s) \in \lambda(\Gamma)''$ $\qquad$ ω has bounded potential.

It is possible, in explicit examples, to find ω which is a coefficient of $C^*(G)$, but not a coefficient of the regular representation (i.e. ω is singular with respect to τ).

Even if the boundedness of the potential $G(\omega)$ of a finite energy functional ω is dropped, a bound similar to the one above persists. The previous and the following result (see [CS4]), proved by Jacques Deny in the commutative case (see [Den]), lay at the core of the potential theory of Dirichlet forms.

Theorem 11.8. *(Deny's inequality) For any finite energy functional $\omega \in A^*_+$ with potential $G(\omega) \in \mathcal{P}$, the following inequality holds true*

$$\omega\Big(b^* \frac{1}{G(\omega)} b\Big) \le ||b||^2_{\mathcal{F}} \qquad b \in \mathcal{F}\,.$$

The equality is attained for $b = G(\omega)$.

In the noncommutative setting, since, in general, the finite energy functional ω is not a trace, the proof of the above two results requires considerations of KMS symmetric Dirichlet forms on standard forms of von Neumann algebras.

A first consequence of the Deny embedding is the following result showing that, among the finite energy functionals of a Dirichlet space, there are the energy measures or carré du champ of bounded potentials.

Theorem 11.9. *Let $G \in \mathcal{P} \cap \mathcal{M}$ be a bounded potential. Then*

- $\langle G, b^*b\rangle_{\mathcal{F}} \le \|G\|_{\mathcal{M}} \cdot \|b\|^2_{\mathcal{F}} \qquad b \in \mathcal{B}$
- *$\Gamma[G] \in A^*_+$ is a finite energy functional.*

11.2. **Multipliers of Dirichlet spaces.** The following is another central subject of the potential theory of Dirichlet spaces whose properties reveal geometrical aspects (see [CS4]). The classical theory concerning multipliers of the Dirichlet integral of a Euclidean domain may be found in [MS].

Definition 11.10. (Dirichlet space multipliers) An element of the von Neumann algebra $b \in L^\infty(A,\tau)$ is a multiplier of $\mathcal{F}$ if

$$b \cdot \mathcal{F} \subseteq \mathcal{F}\,, \qquad \mathcal{F} \cdot b \subseteq \mathcal{F}\,.$$

By the Closed Graph Theorem, multipliers are bounded operators on $\mathcal{F}$. The *-algebra of multipliers $\mathcal{M}(\mathcal{F})$ is a subalgebra of the algebra off all bounded operators on $\mathcal{F}$: $\mathcal{M}(\mathcal{F}) \subset \mathbb{B}(\mathcal{F})$.

Example 11.11. Let $\mathcal{F}_\ell$ be the Dirichlet space associated to a negative type function ℓ on a discrete group Γ. Then the unitaries $\delta_t \in \lambda(\Gamma)''$ are multipliers and

$$\|\delta_t\|_{\mathbb{B}(\mathcal{F}_\ell)} = \sup_{s\in\Gamma} \sqrt{\frac{1+\ell(st)}{1+\ell(s)}} \leq \sqrt{2}\sqrt{1+\ell(t)} \qquad t \in \Gamma.$$

Example 11.12. (Sobolev algebra of multipliers on Riemannian manifolds) In case of the Dirichlet integral of a compact Riemannian manifold (V,g) having dimension $d \geq 3$

$$\mathcal{E}[a] = \int_V |\nabla a|^2 \, dm_g \qquad a \in \mathcal{F} := H^{1,2}(V)\,,$$

from the Sobolev embedding

$$\|b\|^2_{\frac{2d}{d-2}} \leq S(V,g) \cdot \|b\|^2_{\mathcal{F}} \qquad b \in \mathcal{F} := H^{1,2}(V,g)\,,$$

$S(V,g)$ being the Sobolev constant, one derives an embedding of the Sobolev algebra

$$H^{1,d}_\infty(V,g) := H^{1,d}(V,g) \cap L^\infty(V,m_g)$$

into the multipliers algebra

$$H^{1,d}_\infty(V,g) \to \mathcal{M}(\mathcal{F}) \qquad \|a\|_{\mathbb{B}(\mathcal{F})} \leq c \cdot \|a\|_{H^{1,d}_\infty(V,g)}\,.$$

Notice that the d-Dirichlet integral $\int_V |\nabla a|^d \, dm_g$ and the seminorm of the Sobolev algebra $H^{1,d}_\infty(V,g)$ are conformal invariants of (V,g).

Multipliers exist on any Dirichlet space with a certain wealth.

Theorem 11.13. *Let $\mathcal{I}(A,\tau) \subset L^\infty(A,\tau)$ be the norm closure of the ideal $L^1(A,\tau) \cap L^\infty(A,\tau)$. Then*

- *$(I+L)^{-1}h$ is a multiplier for any $h \in \mathcal{I}(A,\tau)$*

$$\|(I+L)^{-1}h\|_{\mathbb{B}(\mathcal{F})} \leq 2\sqrt{5}\|h\|_\infty \qquad h \in \mathcal{I}(A,\tau)$$

- *bounded L^p-eigenvectors of the generator L, are multipliers*

$$h \in L^p(A,\tau) \cap L^\infty(A,\tau) \qquad Lh = \lambda h \quad \Rightarrow \quad \|h\|_{\mathbb{B}(\mathcal{F})} \leq 2\sqrt{5}(1+\lambda)\|h\|_\infty$$

- *the algebra of* **finite energy multipliers** *$\mathcal{M}(\mathcal{F}) \cap \mathcal{F}$ is a form core*
- *the Dirichlet form is regular on the C^*-algebra $\overline{\mathcal{M}(\mathcal{F}) \cap \mathcal{F}}$, norm closure in $L^\infty(A,\tau)$*
- *$\overline{\mathcal{M}(\mathcal{F}) \cap \mathcal{F}} = A$ provided the resolvent is strongly continuous on A*

$$\lim_{\varepsilon\downarrow 0} \|(I+\varepsilon L)^{-1}a - a\|_{\mathcal{M}} = 0 \qquad a \in A\,.$$

Remark 11.14. The definition of multiplier of a Dirichlet space $\mathcal{F}$ does not involve properties of the quadratic form $\mathcal{E}$ other than that to be closed. Proofs of existence and large supply of multipliers are based on the properties of potentials and finite energy states developed in noncommutative potential theory.

11.3. **Finite energy multipliers and their seminorm.** In many situations at hand, where a Dirichlet form is naturally available on a C*-algebra A, as for examples on dual of discrete groups, compact quantum groups and Riemannian manifolds, the Dirichlet algebra is rich enough to make meaningful investigating whether or not the seminorm $a \mapsto \|[D,a]\|_\infty$ is a *Lipschiz seminorm* in the sense of Marc Rieffel [Ri1]. This is the key step to investigate A from a metric point of view. This means that the Connes' formula

$$d(\phi,\psi) := \sup\{\phi(a) - \psi(a) : a \in (\mathcal{E},\mathcal{F}), \|[D,a]\| \le 1\} \qquad \phi\,,\psi \in A^*_{+,1}$$

defines a metric on the space of states on A which induce the weak*-topology.
There are, however, other natural situations in which the Lipschiz algebra is very much reduced or even collapses to the multiples of the identity. These situations are those in which energy and volume are distributed singularly with respect to each other. Examples are furnished by the class of the Dirichlet forms coming from harmonic structures on post critically finite fractals (see [Ku], [Ki], [Hi]). Some of them will be described in the next section.
In such situations one may wonders how to replace the Lipschiz algebra by a richer one whose seminorm satisfies the Rieffel's conditions of a *Lipschiz seminorm*. We are going to see that a candidate is always available in any Dirichlet space.

Theorem 11.15. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(A,\tau)$. For fixed $a \in L^\infty(A,\tau)$, the following properties are equivalent:*

- *$a \in \mathcal{M}(\mathcal{F}) \cap \mathcal{F}$ is a finite energy multiplier;*
- *the commutator $[\partial, a]$ is a bounded operator from $\mathcal{F}$ to $\mathcal{H}$;*
- *the following inequality holds true*

$$\|(\partial a)b\|_{\mathcal{H}} \le c_a \cdot \|b\|_{\mathcal{F}} \qquad b \in \mathcal{B}\,,\ \text{for some } c_a \ge 0\,;$$

- *$\mathcal{F}$ is continuously embedded in $L^2(A,\Gamma[a])$ by the bounded extension of the map*

$$M_a : \mathcal{B} \to L^2(A,\Gamma[a]) \qquad M_a(b) := (\partial a)b\,.$$

Moreover, for $a \in \mathcal{M}(\mathcal{F}) \cap \mathcal{B}$, by the Leibniz rule we have

$$[\partial, a]b = \partial(ab) - a\partial b = (\partial a)b \qquad b \in \mathcal{B}$$

so that, since $\mathcal{B}$ is a form core,

$$\|[\partial,a]\|_{\mathcal{F}\to\mathcal{H}} = \sup_{a\in\mathcal{B}} \frac{\|(\partial a)b\|_{\mathcal{H}}}{\|b\|_{\mathcal{F}}} = \|M_a\|_{\mathcal{F}\to L^2(A,\Gamma[a])}\,.$$

The above properties suggest that the role of the Lipschiz algebra $\mathcal{L}(\mathcal{F})$ and its seminorm $a \mapsto \|D,a\|$ can be played by the algebra $\mathcal{M}(\mathcal{F}) \cap \mathcal{F}$ of finite energy multipliers endowed by its seminorm $a \mapsto \|[\partial,a]\|_{\mathcal{F}\to\mathcal{H}}$.

Next example suggests that the change could be from metric to conformal geometry.

Example 11.16. (Finite energy multipliers and conformal geometry) On a compact Riemannian manifold (V,g) having dimension $d \ge 3$ and for elements belonging to the Sobolev algebra $a \in H^{1,d}_\infty(V,g)$, by the Hölder inequality and the Sobolev embedding, we have that for any element of the Dirichlet algebra $b \in H^{1,2}(V,g) \cap L^\infty(A,\tau)$

$$\int_V |b|^2 \cdot |\nabla a|^2\, dm_g \le \Big(\int_V |b|^{\frac{2d}{d-2}}\, dm_g\Big)^{\frac{d-2}{d}} \cdot \Big(\int_V |\nabla a|^d\, dm_g\Big)^{\frac{2}{d}} \le S(V,g) \cdot \|b\|^2_{H^{1,2}(V,g)} \cdot \||\nabla a|\|^2_d$$

so that the multiplier seminorm is bounded above by the Sobolev seminorm

$$\|[\nabla,a]\|_{\mathcal{F}\to\mathcal{H}} \le \sqrt{S(V,g)} \cdot \||\nabla a|\|_d\,.$$

Since the latter seminorm is a conformal invariant of (V, g) we have

$$\sup\{\|[\nabla, a\circ\varphi]\|_{\mathcal{F}\to\mathcal{H}} : \varphi \in \mathrm{C\,o}\,(V,g)\} \le \sqrt{S(V,g)}\cdot |||\nabla a|||_d \qquad a \in H^{1;d}_\infty(V,g)\,,$$

where $\mathrm{C\,o}\,(V, g)$ is the group of all conformal transformations of the Riemannian manifold.

12. Noncommutative Potential theory on fractals

In this section we apply noncommutative potential theory to study a specific fractal set, the Sierpinski gasket, from the point of view of Noncommutative Geometry. The study of fractal sets in NCG, initiated by Connes [Co], has been pursued by Lapidus [La], Guido-Isola [GI1], [GI2], Cipriani-Sauvageot [CS3], Christensen-Ivan-Schrohe [CIS]. In particular, the results illustrated below concerning the Sierpinski gasket have been recently extended to the Vicsek square in [GI3].

Definition 12.1. (Sierpinski gasket) The Sierpinski gasket is the subset $K \subset \mathbb{C}$ of the plane, determined in the following way. Let $\{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle in the plane $\mathbb{C}$ and consider the contractions $F_i : \mathbb{C} \to \mathbb{C} \qquad F_i(z) := (z + p_i)/2$ for $i = 1, 2, 3$. A compact set $K \subset \mathbb{C}$ is uniquely determined by the equation

$$K = F_1(K) \cup F_2(K) \cup F_3(K)$$

as the fixed point of the map $C \mapsto F_1(C) \cup F_2(C) \cup F_3(C)$ with respect to the Hausdorff distance on compact subsets of $\mathbb{C}$.

The above equation is referred to as the *self-similarity* property of K. It has the consequence that, by iteration, K can be reconstructed as a whole from any arbitrary small part of it (synecdoche). Moreover, the gasket has some geometric and analytic features, as for example

- K is not a manifold
- K is not semi-locally simply connected hence
- K does not admit a universal cover
- volume and energy are distributed singularly on K
- there exist localized eigenfunctions,

that force one to consider K as *singular space*, if analyzed with the tools of classical Newton-Lipschiz differential calculus and Riemannian differential geometry. We notice also that, it was precisely the spectral properties to attract physicists's attentions on self-similar fractals.

12.1. **Harmonic structures and Dirichlet forms.** The measure theory of K is obviously *dominated*, to a certain extent, by the class of self-similar volume measures. These may be defined, for some fixed $(\alpha_1, \alpha_2, \alpha_3) \in (0, 1)^3$ such that $\sum_{i=1}^3 \alpha_i = 1$, as the unique solution of the self-similarity equation

$$\int_K f\,d\mu = \sum_{i=1}^3 \alpha_i \int_K (f\circ F_i)\,d\mu \qquad f \in C(K)$$

When $\alpha_i = 1/3$ for all $i = 1, 2, 3$ then μ is the normalized Hausdorff measure on K associated to the restriction of the Euclidean metric: its dimension is $d = \ln 3/\ln 2$.

The construction of an interesting class of Dirichlet forms K is based on the notion of *harmonic structure*, introduced by J. Kigami. To prepare the discussion we need to fix some notations:

- word spaces: $\quad \sum_0 := \emptyset\,, \quad \sum_m := \{1, 2, 3\}^m\,, \quad \sum := \bigcup_{m\ge 0} \sum_m$
- length of a word $\sigma \in \sum_m$: $\qquad |\sigma| := m$

- iterated contractions: $F_\sigma := F_{i_{|\sigma|}} \circ \dots F_{i_1}$ if $\sigma = (i_1, \dots, i_{|\sigma|})$
- vertices sets: $V_0 := \{p_1, p_2, p_3\}$, $V_m := \bigcup_{|\sigma|=m} F_\sigma(V_0)$

The quadratic form on the three-point set V_0 defined by

$$\mathcal{E}_0 : C(V_0) \to [0, +\infty) \qquad \mathcal{E}_0[a] := (a(p_1) - a(p_2))^2 + (a(p_2) - a(p_3))^2 + (a(p_3) - a(p_1))^2 ,$$

is a Dirichlet form with respect to any measure on V_0. It is the first of a sequence of Dirichlet forms $\mathcal{E}_m$ defined on vertices sets V_m which satisfy an energy minimization procedure typical of potential theory.

Theorem 12.2. *The sequence of quadratic forms on $C(V_m)$ $m \in \mathbb{N}$, defined by*

$$\mathcal{E}^m[a] := \sum_{|\sigma|=m} \Big(\frac{5}{3}\Big)^m \mathcal{E}_0[a \circ F_\sigma] \qquad a \in C(V_m)$$

is an harmonic structure in the sense that

$$\mathcal{E}^m[a] = \min\{\mathcal{E}^{m+1}[b] : b|_{V_m} = a\} \qquad a \in C(V_m) .$$

Theorem 12.3. *A well defined quadratic form $\mathcal{E} : C(K) \to [0, +\infty]$ is defined by*

$$\mathcal{E} : C(K) \to [0, +\infty] \qquad \mathcal{E}[a] := \lim_{m \to +\infty} \mathcal{E}^m[a|_{V_m}] .$$

It is a lower semicontinuous quadratic form on $C(K)$, which is Markovian in the sense that

$$\mathcal{E}[a \wedge 1] \leq \mathcal{E}[a] \qquad a \in C(K)$$

and self-similar in the sense that

$$\mathcal{E}[a] = \frac{5}{3} \sum_{i=1}^{3} \mathcal{E}[a \circ F_i] \qquad a \in C(K) .$$

On the domain $\mathcal{F} \subset C(K)$ where it is finite, $\mathcal{E}$ is densely defined and closed on $L^2(K, \mu)$, for any self-similar measure μ on K. $(\mathcal{E}, \mathcal{F})$ is thus a Dirichlet form and the associated self-adjoint operator $(H_\mu, D(H_\mu))$ has discrete spectrum.

Notice that, the quadratic form $(\mathcal{E}, \mathcal{F})$ and energy measures $\Gamma[a]$ are independent upon the choice of the volume measure μ, while the corresponding self-adjoint operator H_μ and then its spectrum are. In particular, the Dirichlet algebra coincides with the whole form domain: $\mathcal{B} := \mathcal{F} \cap C(K) = \mathcal{F}$. As a consequence, the derivation $(\mathcal{F}, \partial, \mathcal{H}, \mathcal{J})$ which represents the Dirichlet form is independent upon the choice of the reference volume measure μ. The divergence operator $(\mathcal{F}^*, \partial^*)$ however is dependent upon μ because it is defined as the Hilbert space adjoint of the densely defined operator $(\mathcal{F}, \partial)$ between the spaces $L^2(K, \mu)$ and $\mathcal{H}$. This properties, shared with the Dirichlet integral on a bounded interval of the real line, should be regarded as typical of low dimensional situations.

12.2. **Spectral reconstruction of the volume measure.** The classical connection between asymptotics of the spectrum of the Laplace-Beltrami operator and the volume measure of a Euclidean domain or a Riemannian manifold, referred to H. Weyl, has been generalized to post critically finite fractal sets and, in particular, to the Sierpinski gasket K, by Kigami-Lapidus [KiLa 1,2].

Theorem 12.4. *Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form associated to a harmonic structure on K and let $(H_\mu, D(H_\mu))$ be corresponding self-adjoint operator on space $L^2(K,\mu)$ of a self-similar measure μ on K.*
The unique positive number d_S such that $\sum_{i=1}^{3}(\frac{3}{5}\alpha_i)^{d_S/2} = 1$, called the spectral exponent, determines the asymptotic behavior of the eigenvalue counting function

$$0 < \liminf_{\lambda\to+\infty} \frac{\sharp\{eigenvalues\ of\ H_\mu \le \lambda\}}{\lambda^{d_S/2}} \le \limsup_{\lambda\to+\infty} \frac{\sharp\{eigenvalues\ of\ H_\mu \le \lambda\}}{\lambda^{d_S/2}} < +\infty\,.$$

When the measure μ is the most symmetric one so that $\alpha_i = 1/3$ for $i = 1,2,3$, the spectral exponent $d_S = \frac{\ln 9}{\ln 5}$ differs from the Hausdorff dimension $\frac{\ln 3}{\ln 2}$ of the gasket, endowed with the Euclidean metric of the plane. The spectral exponent is related by $d_S = \frac{2d_H}{d_H+1}$ to the Hausdorff dimension $d_H = \frac{\ln 3}{\ln 5/3}$ of K, determined by $\sum_{i=1}^{3}(\frac{3}{5})^{d_H} = 1$, when this set is endowed with a suitable distance, called *resistance metric*, associated to the energy form $\mathcal{E}$ (see [...]).

By a classical result of H. Weyl, the Riemannian measure of compact Riemannian manifold can be reconstructed by the asymptotic behavior of the spctrum of the Laplace-Beltrami operator. In a complete parallel way this can be done on post critically finite fractal sets: on the Sierpinski gasket K for example, one just replaces the Riemannian measure by one the self-similar measures and the Dirichlet integral by one the self-similar Dirichlet forms introduced above.

Let us denote by M_f the multiplication operator on $L^2(K,\mu)$ by a function $f \in C(K)$.

Theorem 12.5. *The self-similar measure μ with weights $\alpha_i = 1/3$ can be re-constructed as*

$$\int_K f\,d\mu = Trace_{Dix}(M_f \circ H_\mu^{-d_S/2}) = Res_{s=d_S} Trace(M_f \circ H_\mu^{-s/2}) \qquad f \in C(K)\,.$$

In the first equality $Trace_{Dix}$ denote the Dixmier's trace while, in the second one, the notation refers to the residue of the meromorphic extension of the analytic function $z \mapsto Trace(M_f \circ H_\mu^{-z/2})$ defined for $\mathfrak{Re}\, z > d_S$.

One of the main difficulties encountered in studying the geometry of the gasket rely on the following phenomenon discovered by Kusuoka [Ku] and investigated by [Hi].

Theorem 12.6. *Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form associated to a harmonic structure on K. Then the energy measures $\Gamma[a]$ of any nonconstant $a \in \mathcal{F}$, are singular with respect to all the self-similar measures on K.*

An consequence of the above result is that the Lipschiz algebra of the Dirichlet space $\mathcal{F}$ trivializes: $\mathcal{L}(\mathcal{F}) = \mathbb{C}$.

12.3. **A Fredholm module on Sierpinski gasket.** The choice of a volume measure μ on K, determines a Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(K,\mu)$ with domain $\mathcal{F} \subset C(K)$ and a corresponding differential calculus $(\mathcal{F},\partial,\mathcal{H},\mathcal{J})$. As the Dirichlet form is strongly local, the corresponding tangent bimodule is in fact a $C(K)$-monomodule, in the sense that left and right actions coincide.

By a direct application of Theorem 10.12 in Section 10.3, the differential calculus can be used to construct a natural K-theory invariant on the Sierpinski gasket. Notice that, as K is a compact subset of the plane $\mathbb{C}$, its K-theory can be calculated as $\mathbb{K}^1(K) = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}$ with one generator for each lacuna (connect components of the complement of the gasket).

Theorem 12.7. *Let F be the symmetry with respect to the range of the derivation $Im(\partial) \subset \mathcal{H}$, i.e. $F := P - P^\perp$ where $P \in Proj(\mathcal{H})$ the projection onto $Im(\partial)$.*
Then $(F, \mathcal{H})$ is a 2-summable (ungraded) Fredholm module over $C(K)$ and

$$Trace(|[F,a]|^2) \leq c_\mu \, \mathcal{E}[a] \qquad a \in \mathcal{F}$$

for some constant $c_\mu > 0$.

The constant function c_μ is proportional to the uniform norm of the Green function G_μ of the self-adjoint operator H_μ. Since the commutator $[F,a]$ vanishes precisely on the constant functions, the bound above can be interpreted as a kind of Poincaré inequality.

12.4. **Spectral triples on Sierpinski gasket.** We have seen that independently upon the choice of the self-similar, reference measure μ, the Lipschiz algebra is trivial $\mathcal{L}(\mathcal{F}) = \mathbb{C}$. Consequently, the construction of a meaningful spectral triple on the gasket K cannot rely on the Dirac operator associated to the self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$,as discussed in the Section 9..
To overcome this difficulty, the strategy is to exploit the fact that K is post critically finite and to imagine K as assembled from a countable sequence of pieces, smaller and smaller in size, glued together at some of their points.
The Dirac operator on K will be defined as the direct sum of the Dirac operators on these pieces. To be able to perform explicit calculations with eigenvalues and eigenfunctions, we will work with a homeomorphic version of the gasket K in which the pieces patched together are circles (instead of the downward triangles in K, called lacunas).

In this process two points should be noticed. The first is that the action-through-restrictions of $C(K)$ on the direct sum of the modules corresponding to the circles, encodes the topology of the gasket. The second is that the Dirac operators on each circle have to be chosen carefully: they will be the Dirac operators associated to Dirichlet forms on circles corresponding to exotic (non local) differential structures. This is due to a fundamental discovery by Johnson [Jo] who proved that the restrictions of finite energy functions $a \in \mathcal{F}$ to any segment in K, belong to a particular fractional Sobolev space.

12.4.1. *Spectral triples on quasi-circles.* We begin to consider on the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ differential structures different from the usual one. These are called *quasi-circles* and are associated to the following family of Dirichlet forms for $\alpha \in (0,1)$.

Let us consider on $\mathbb{T}$, the normalized Lebesgue measure dz and the Dirichlet integral on $L^2(\mathbb{T})$ with associated Laplace operator Δ. By Fourier transform we have the representation

$$\mathcal{E}_1[a] = \sum_{k \in \mathbb{Z}} |k|^2 \cdot |a_k|^2 \qquad a \in H^1(\mathbb{T}),$$

where $a_k = \int_{\mathbb{T}} a(z)\overline{z}^k \, dz$ are the Fourier coefficients. As the function $\mathbb{Z} \ni k \mapsto |k|^2$ is negative definite on the additive group of integers $\mathbb{Z}$, the functions $\mathbb{Z} \ni k \mapsto |k|^{2\alpha}$ are negative definite too for any $\alpha \in (0,1]$. Consequently, the quadratic forms

$$\mathcal{E}_\alpha[a] = \sum_{k \in \mathbb{Z}} |k|^{2\alpha} \cdot |a_k|^2 \qquad a \in H^\alpha(\mathbb{T}),$$

are regular Dirichlet forms defined on the subspaces $H^\alpha(\mathbb{T}) \subset L^2(\mathbb{T})$ where they are finite. The algebra $C^\gamma(\mathbb{T})$ of Hölder continuous functions of order $\gamma \in (\alpha, 1]$ is a form core contained in the Dirichlet algebra. Moreover, for $\alpha > \frac{1}{2}$, one has $H^\alpha(\mathbb{T}) \subset C(\mathbb{T})$ so that in these cases the Dirichlet algebra coincides with form domain itself.

As customary, we adopt the notation $|z-w|$ for the distance in $\mathbb{T}$ between $z, w \in \mathbb{T}$.

Proposition 12.8. *(Fractional Dirichlet forms on a circle) The domain $H^\alpha(\mathbb{T})$ is the fractional Sobolev space*

$$H^\alpha(\mathbb{T}) = \{a \in L^2(\mathbb{T}) : \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|a(z)-a(w)|^2}{|z-w|^{2\alpha+1}}\, dzdw\} .$$

and $(\mathcal{E}^\alpha, H^\alpha(\mathbb{T}))$ is the quadratic form of the spectral α-root Δ^α of the Laplacian Δ, whose spectrum is given by $\mathrm{sp}(\Delta^\alpha) = \{|k|^{2\alpha} : k \in \mathbb{Z}\}$.
The differential calculus associated to $(\mathcal{E}^\alpha, H^\alpha(\mathbb{T}))$ is given by the derivation $(H^\alpha(\mathbb{T}), \partial_\alpha, \mathcal{H}, \mathcal{J})$ where $\mathcal{H} := L^2(\mathbb{T}\times\mathbb{T})$ is a symmetric Hilbert $C(K)$-bimodule with actions and involution given by

$$(a\xi)(z,w) := a(z)\xi(z,w)\,, \quad (\xi a)(z,w) := \xi(z,w)a(w)\,, \quad (\mathcal{J}\xi)(z,w) := \overline{\xi(w,z)}$$

and the derivation $\partial_\alpha : H^\alpha(\mathbb{T}) \to \mathcal{H}$ is given by

$$\partial_\alpha(a)(z,w) := \sqrt{\varphi_\alpha(z\overline{w})}(a(z)-a(w)) \qquad z,w \in \mathbb{T}$$

for a suitable function $\varphi_\alpha \in C(\mathbb{T})$ proportional to a specific Clausen cosine special function. In particular we have

$$\mathcal{E}_\alpha[a] = \|\Delta^{\alpha/2}a\|_2^2 = \int_{\mathbb{T}}\int_{\mathbb{T}} \varphi(z\overline{w})\cdot|a(z)-a(w)|^2 \qquad \alpha \in H^\alpha(\mathbb{T})\,.$$

Notice that the Dirichlet form $(\mathcal{E}^\alpha, H^\alpha(\mathbb{T}))$ and then the corresponding differential calculus are non local and in particular of jumping type. A *quasi-circle* is the topological space $\mathbb{T}$ endowed with one of the above non-local differential structures. Using the result of previous sections we have the following

Proposition 12.9. *(Spectral Triples on quasi-circles) Let $\alpha \in (0,1]$ and consider on the Hilbert space $\mathcal{K}_\alpha := L^2(\mathbb{T}\times\mathbb{T}) \bigoplus L^2(\mathbb{T})$, the left $C(\mathbb{T})$-module structure resulting from the sum of those of $L^2(\mathbb{T}\times\mathbb{T})$ and $L^2(\mathbb{T})$ and the operator*

$$D_\alpha := \begin{pmatrix} 0 & \partial_\alpha \\ \partial_\alpha^* & 0 \end{pmatrix} .$$

Then $\mathcal{A}_\alpha := \{a \in C(\mathbb{T}) : \sup_{z\in\mathbb{T}} \int_{\mathbb{T}} \frac{|a(z)-a(w)|^2}{|z-w|^{2\alpha+1}} < +\infty\}$ is a uniformly dense subalgebra of $C(\mathbb{T})$ and $(\mathcal{A}_\alpha, D_\alpha, \mathcal{K}_\alpha)$ is a densely defined Spectral Triple on $C(\mathbb{T})$.
Moreover, setting D_α^{-1} to be zero on $\ker D_\alpha$ we have
i) D_α^{-1} has discrete spectrum and its zeta function is given by

$$\zeta_{D_\alpha}(s) = 4\zeta(\alpha s) s \in \mathbb{C}$$

where ζ is the Riemann zeta function;
ii) the dimension of the triple is α^{-1} and the volume measure on $\mathbb{T}$ can be recovered by

$$\int_{\mathbb{T}} f(z)\, dz = \frac{\alpha}{4}\mathrm{Res}_{s=\alpha^{-1}}\mathrm{tr}(f|D_\alpha|^{-s}) \qquad f \in C(\mathbb{T});$$

iii) the Connes distance on $\mathbb{T}$ induces by the spectral triple satisfies, for some $c_1(\varepsilon)\,, c_2(\alpha) > 0$

$$c_1(\varepsilon)\cdot|z-w|^{\alpha+\varepsilon} \le d_{D_\alpha}(z,w) \qquad z,w \in \mathbb{T} \qquad \alpha \in (0,1]\,, \varepsilon > 0\,,$$

$$d_{D_\alpha}(z,w) \le c2(_\alpha)\cdot|z-w|^\alpha \qquad z,w \in \mathbb{T} \qquad \alpha \in [\frac{1}{2}, 1]\,;$$

iv) the Dirichlet form $(\mathcal{E}_\alpha, H^\alpha(\mathbb{T}))$ can be recovered as a residue of a functional as follows

$$\mathcal{E}_\alpha[a] = \frac{2}{\alpha}\lim_{s\to 1}(s-1)\mathrm{tr}(|D_\alpha|^{-s}[D_\alpha, a]|D_\alpha|^{-s}) \qquad a \in H^\alpha(\mathbb{T}).$$

12.4.2. *Dirac operators on Sierpinski gasket.* Gluing together spectral triples of quasi-circles of suitable sizes, it is possible to construct a spectral triple on the gasket itself.

The main lacuna $\ell_\emptyset$ of the gasket is the triangle with vertices $q_1 := (p_2+p_3)/2$, $q_2 := (p_3+p_1)/2$, $q_3 := (p_1+p_2)/2$. Identifying isometrically $\ell_\emptyset$ with the circle $\mathbb{T}$, we may consider, for any fixed $\alpha \in (0,1)$, the Dirac operator $(C(K), D_\emptyset, \mathcal{K}_\emptyset)$ where

- $\mathcal{K}_\emptyset := L^2(\ell_\emptyset \times \ell_\emptyset) \oplus L^2(\ell_\emptyset)$
- $D_\emptyset := D_\alpha$
- the action of $C(K)$ is given by restriction $\pi_\emptyset(a)b := a|_{\ell_\emptyset}$.

For any word $\sigma \in \sum$ consider the Dirac operators $(C(K), \pi_\sigma, D_\sigma, \mathcal{K}_\sigma)$ where

- $\mathcal{K}_\sigma := \mathcal{K}_\emptyset$
- $D_\sigma := 2^{|\sigma|}D_\alpha$
- the action of $C(K)$ is given by contraction/restriction $\pi_\sigma(a)b := (a \circ F_\sigma)|_{\ell_\emptyset}\, b$.

Finally, consider the Dirac operator $(\mathcal{A}, \pi, D, \mathcal{K})$ where

- $\mathcal{K} := \oplus_{\sigma\in\sum}\mathcal{K}_\sigma$
- $\pi := \oplus_{\sigma\in\sum}\pi_\sigma$
- $D := \oplus_{\sigma\in\sum}D_\sigma$

and $\mathcal{A}$ is the subalgebra of functions $f \in C(K)$ having bounded commutator $[D, \pi(f)]$ with the Dirac operator D. Notice that $\dim \mathrm{Ker}(D) = +\infty$ and that D^{-1} will be defined to be zero on $\mathrm{Ker}(D)$.

12.4.3. *Volume functionals and their spectral dimensions.* The first result about the above triple concerns the spectrum of the associated volume functional. As a reminiscence of the self-similarity of K, we will notice the appearance of a sequence of complex poles, i.e. complex dimensions, which are absent in case of a Riemannian manifold.

Theorem 12.10. *The volume zeta function $\mathcal{Z}_D$ of the Dirac operator $(C(K), \pi, D, \mathcal{K})$, i.e. the meromorphic extension of the function $\mathbb{C} \ni s \mapsto Trace(|D|^{-s})$ is given by*

$$\mathcal{Z}_D(s) = \frac{4\zeta(\alpha s)}{1 - 3\cdot 2^{-s}}$$

where ζ denotes the Riemann zeta function. The dimensional spectrum is given by

$$\mathcal{S}_{dim} = \{\frac{1}{\alpha}\} \cup \{\frac{\log 3}{\log 2}\Big(1 + \frac{2\pi i}{\log 3}k\Big) : k \in \mathbb{Z}\} \subset \mathbb{C}$$

and its abscissa of convergence, called volume dimension, is given by $d_D = \max(\alpha^{-1}, d_H)$ where $d_H = \frac{\log 3}{\log 2}$ is the Hausdorff dimension. When $\alpha > \frac{\log 2}{\log 3}$, then $d_D = d_H$ is a simple pole and the residue of the meromorphic extension of $\mathbb{C} \ni s \mapsto Trace(f|D|^{-s})$ gives the integral with respect to the d_H-dimensional Hausdorff measure H_{d_H}

$$Res_{s=d_D}Trace(\pi(f)|D|^{-s}) = \frac{4d_H}{\log 3}\frac{\zeta(d_H)}{(2\pi)^{d_H}}\int_K f\, dH_{d_H} \qquad f \in C(K).$$

12.4.4. *Connes metrics on Sierpinski gasket.* As far as the metric aspects of the Dirac operator D are concerned, we have

Theorem 12.11. *For any $\alpha \in (0,1]$, the algebra $\mathcal{A}$ contains the algebra $C^{0,1}(K)$ of Lipschiz functions on K with respect to the restriction of the Euclidean metric of the plane. It is thus a dense subalgebra of $C(K)$ and $(\mathcal{A}, D, \mathcal{K})$ is a Spectral Triple. In particular, $\mathcal{A} \ni f \mapsto [D, \pi(f)]$ is a Lip-seminorm in the sense of Rieffel [...] and the associated Connes' metric*

$$\rho_D(x,y) := \sup_{f\in\mathcal{A}} \frac{|f(x)-f(y)|}{\|[D,\pi(f)]\|} \qquad x,y \in K$$

is bi-Lipschiz w.r.t. the restriction of the geodesic metric on K.

12.4.5. *Energy functionals and their spectral dimensions.* By the spectral triple it is possible to recover, in addition to dimension, volume measure and metric, also the energy form of K. This is of particular significance because the geometry of the gasket and other self-similar fractal sets is intimately associated to the energy functional. Moreover, the Dirichlet form will be obtained as a residue of a corresponding energy functional at a pole (the energy dimension) which differs from the pole (the volume dimension) at which the residue restitutes the volume measure. This *dimension shift* has to be seen as a consequence of the fact that energy and volume are distributed singularly on K.

Theorem 12.12. *Let $\alpha_0 := \log(10/3)/\log 4$ and consider the spectral triple $(\mathcal{A}, D, \mathcal{K})$ for $\alpha \in (0, \alpha_0]$.*
i) Then for any $a \in \mathcal{F}$ and $\mathfrak{Re}(s) > \delta_D$, $|D|^{-s/2}|[D,\pi(a)]|^2|D|^{-s/2}$ is a trace-class operator.

ii) The abscissa of convergence of the meromorphic energy functional

$$\mathbb{C} \ni s \mapsto Z_{D,a}(s) := Trace(|D|^{-s/2}|[D,\pi(a)]|^2|D|^{-s/2})$$

is $\delta_D := \max(\alpha^{-1}, d_E)$ where $d_E := \frac{\log 12/5}{\log 2}$ is called the energy dimension.

iii) If $\delta_D = d_E$ then $s = \delta_D$ is a simple pole and the residue of the functional $Z_{D,a}$ at δ_D is proportional to the Dirichlet form

$$Res_{s=\delta_D} Trace(|D|^{-s/2}|[D,\pi(a)]|^2|D|^{-s/2}) = const.\,\mathcal{E}[a] \qquad a \in \mathcal{F}$$

by a constant independent on $a \in \mathcal{F}$.

In the above results concerning the volume and energy functionals associated to the spectral triple $(\mathcal{A}, D, \mathcal{K})$ on the algebra $C(K)$, the residues at the poles d_D and δ_D can be replaced by the noncommutative integrals given by Dixmier traces, even if with subtleties: for example, in case of the energy functional, using the Dixmier trace one can reproduces the Dirichlet form not on the whole form domain but rather on a suitable form core. See [CGIS2] for details.

REFERENCES

[AHK1] S. Albeverio, R. Hoegh-Krohn, Dirichlet Forms and Markovian semigroups on C^*–algebras, *Comm. Math. Phys.* **56** (1977), 173-187.

[AHK2] S. Albeverio, R. Hoegh-Krohn, Frobenius theory for positive maps on von Neumann algebras, *Comm. Math. Phys.* **64** (1978), 83-94.

[Ara] H. Araki, Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule, *Pacific J. Math.* **50** (1974), 309-354.

[At] M.F. Atiyah, Global theory of elliptic operators, *Proc. Internat. Conf. on Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)* (1970), 21–30 Univ. of Tokyo Press, Tokyo.

[BeDe1] A. Beurling and J. Deny, Espaces de Dirichlet I: le cas élémentaire, *Acta Math.* **99** (1958), 203-224.

[BeDe2] A. Beurling and J. Deny, Dirichlet spaces, *Proc. Nat. Acad. Sci.* **45** (1959), 208-215.

[Bi] P. Biane, Logarithmic Sobolev Inequalities, Matrix Models and Free Entropy, *Acta Math. Sinica, English Series.* **19** (2003), 497-506.

[Boz] M. Bozejko, Positive definite functions on the free group and the noncommutative Riesz product, *Bollettino U.M.I.* **5-A** (1986), 13-21.

[BKS] M. Bozejko, B. Kummerer, R. Speicher, q -Gaussian processes: non-commutative and classical aspects, *Comm. Math. Phys.* **195**, no 1, (1997), 129154.

[Br] M. Brannan, Approximation properties for free orthogonal and free unitary quantum groups, *J. Reine Angew. Math.* **672** (2012), 223-251.

[CS] M. Caspers, A. Skalski, The Haagerup approximation property for von Neumann algebras via quantum Markov semigroups and Dirichlet forms, *Comm. Math. Phys.* **336**, no 3, (2015), 16371664.

[Ca] H. Cartan, Sur les fondements de la thorie du potentiel, *Bull. Soc. Math. France* **69** (1941), 71–96.

[CCJJV] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, A. Valette, "Groups with the Haagerup property. Gromov's a-T-menability", Progress in Mathematics, 197, Birkhuser Verlag, Basel, 2001

[CIS] E. Christensen, C. Ivan, E. Schrohe, Spectral triples and the geometry of fractals *J. Noncommut. Geom.* **6** 2012, no. 2, 249274.

[C1] F. Cipriani, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, *PhD Thesis I.S.A.S - S.I.S.S.A. Trieste*, (1992).

[C2] F. Cipriani, Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras, *J. Funct. Anal.* **147** (1997), no. 1, 259–300.

[C3] F. Cipriani, Perron theory for positive maps and semigroups on von Neumann algebras, *CMS Conf. Proc. Amer. Math. Soc.*, Providence RI, **29** (2000), 115–123.

[C4] F. Cipriani, "Dirichlet forms as Banach algebras and applications", *Pacific Jour. of Math.* **223** (2006), no. 2, 229–249.

[C5] F. Cipriani, "Dirichlet forms on noncommutative spaces", in "Quantum Potential Theory", L.N.M. 1954(2008), 161–276.

[CFK] F. Cipriani, U. Franz, A. Kula, Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory, *J. Funct. Anal.* **266** (2014), no. 5, 2789–2844.

[CS1] F. Cipriani, J.-L. Sauvageot, Derivations as square roots of Dirichlet forms, *J. Funct. Anal.* **201** (2003), no. 1, 78–120.

[CS2] F. Cipriani, J.-L. Sauvageot, Noncommutative potential theory and the sign of the curvature operator in Riemannian geometry, *Geom. Funct. Anal.* **13** (2003), no. 3, 521–545.

[CS3] F. Cipriani, J.-L. Sauvageot, Fredholm modules on P.C.F. self-similar fractals and their conformal geometry, *Comm. Math. Phys.* **286** (2009), no. 2, 541–558.

[CS4] F. Cipriani, J.-L. Sauvageot, Variations in noncommutative potential theory: finite energy states, potentials and multipliers, *Trans. Amer. Math. Soc.* **367** (2015), no. 7, 4837–4871.

[CS5] F. Cipriani, J.-L. Sauvageot, Negative definite functions on groups with polynomial growth, *To appear in "Noncommutative Analysis, Operator Theory and Applications", Birkhauser ed.*,(2016).

[CS6] F. Cipriani, J.-L. Sauvageot, Amenability and sub-exponential spectral growth rate of Dirichlet forms on von Neumann algebras, *In preparation.*

[CGIS1] F. Cipriani, D. Guido, T. Isola, J.-L. Sauvageot, "Integrals and Potential of differential 1-forms on the Sierpinski Gasket", *Adv. in Math.* **239** (2013), 128–163.

[CGIS2] F. Cipriani, D. Guido, T. Isola, J.-L. Sauvageot, "Spectral triples for the Sierpinski Gasket", *J. Funct. Anal.* **266**, no. 8, (2014), 4809–4869.

[Co] A. Connes, "Noncommutative Geometry", Academic Press, 1994.

[CJ] A. Connes, V. Jones, Property T for von Neumann algebras, *Bull. London Math. Soc.* **17**, no. 1, (1985), 5762.

[Da] Y. Dabrowski, A note about proving non- under a finite non-microstates free Fisher information assumption, *Math. Z.* **258** (2010), 3662-3674.

[Dav1] E.B. Davies, "Quantum Open Systems", Academic Press, London U.K. 1976.

[DL] E.B. Davies, J.M. Lindsay, Non–commutative symmetric Markov semigroups, *Math. Z.* **210** (1992), 379-411.

[DR1] E.B. Davies, O.S. Rothaus, Markov semigroups on C*–bundles, *J. Funct. Anal.* **85** (1989), 264-286.

[DR2] E.B. Davies, O.S. Rothaus, A BLW inequality for vector bundles and applications to spectral bounds, *J. Funct. Anal.* **86** (1989), 390-410.

[Den] J. Deny, Méthodes hilbertien en thorie du potentiel, *Potential Theory (C.I.M.E., I Ciclo, Stresa)*, Ed. Cremonese Roma, 1970. **85**, 121-201.

[deH] P. de la Harpe, "Topics in Geometric Group Theory", Chicago Lectures in Mathematics, The University of Chicago Press, 2000.

[Dix] J. Dixmier, "Les C*–algèbres et leurs représentations", Gauthier–Villars, Paris, 1969.

[Do] J.L. Doob, "Classical potential theory and its probabilistic counterpart", Springer-Verlag, New York, 1984.

[E] G.A. Elliott, On the convergence of a sequence of completely positive maps to the identity, *J. Austral. Math. Soc. Ser. A* **68**, no. 3, (2000), 340348.

[F1] M. Fukushima, Regular representations of Dirichlet spaces, *Trans. Amer. Math. Soc.* **155** (1971), 455-473.

[F2] M. Fukushima, Dirichlet spaces and strong Markov processes, *Trans. Amer. Math. Soc.* **162** (1971), 185-224.

[FOT] M. Fukushima, Y. Oshima, M. Takeda, "Dirichlet Forms and Symmetric Markov Processes", de Gruyter Studies in Mathematics, 1994.

[GL1] S. Goldstein, J.M. Lindsay, Beurling-Deny conditions for KMS-symmetric dynamical semigroups, *C. R. Acad. Sci. Paris Ser. I* **317** (1993), 1053-1057.

[GL2] S. Goldstein, J.M. Lindsay, KMS-symmetric markov semigroups, *Math. Z.* **219** (1995), 591-608.

[GL3] S. Goldstein, J.M. Lindsay, Markov semigroups KMS-symmetric for a weight, *Math. Ann.* **313** (1999), 39-67.

[G1] L. Gross, Existence and uniqueness of physical ground states, *J. Funct. Anal.* **10** (1972), 59-109.

[G2] L. Gross, Hypercontractivity and logarithmic Sobolev inequalities for the Clifford–Dirichlet form, *Duke Math. J.* **42** (1975), 383-396.

[GI1] D. Guido, T. Isola, Dimensions and singular traces for spectral triples, with applications to fractals, *J. Funct. Anal.* **203** 2003, no. 2, 362400.

[GI2] D. Guido, T. Isola, Dimensions and spectral triples for fractals in R^N *Theta Ser. Adv. Math. "Advances in operator algebras and mathematical physics"*, **5**, 89108, Theta, Bucharest, 2005.

[GI3] D. Guido, T. Isola, New results on old spectral triples for fractals in R^N *in press in "Noncommutative Analysis, Operator Theory and Applications"*, Birkhauser Basel, 2016.

[GIS] D. Guido, T. Isola, S. Scarlatti, Non-symmetric Dirichlet forms on semifinite von Neumann algebras, *J. Funct. Anal.* **135** (1996), 50-75.

[Haa1] U. Haagerup, An example of a nonnuclear C*-algebra, which has the metric approximation property, *Invent. Math.* **50** (1978), no. 3, 279-293.

[Hi] M. Hino, On singularity of energy measures on self-similar sets, Probab. Theory Related Fields **132**, (2005), no. 2, 265-290.

[Jo] A. Jonsson, A trace theorem for the Dirichlet form on the Sierpinski gasket, Math. Z. **250** 2005, 599-609.

[Ki] J. Kigami, "Analysis on Fractals", Cambridge Tracts in Mathematics vol. **143**, Cambridge University Press, 2001.

[KiLa1] J. Kigami, M.L. Lapidus Weyls problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals Comm. Math. Phys. **158** 1993, no. 1, 93125.

[KiLa2] J. Kigami, M.L. Lapidus Self-similarity of volume measures for Laplacians on p.c.f. self-similar fractals Comm. Math. Phys. **217** 2001, no. 1, 165180.

[Ku] S. Kusuoka, Dirichlet forms on fractals and products of random matrices. Publ. Res. Inst. Math. Sci. **25**, (1989), 659-680.

[JZ] M. Junge, Q. Zeng, Subgaussian 1-cocycles on discrete groups, *J. London Math. Soc.* **92** (2015), , no 2, 242-264.

[La] M.L. Lapidus, Analysis on fractals, Laplacians on self-similar sets, noncommutative geometry and spectral dimensions, Topol. Methods Nonlinear Anal. **4** 1994, 137-195.

[LM] H.B. Lawson JR., M.-L. Michelson, "Spin Geometry", Princeton University Press, Princeton New Jersey, 1989.

[L] G. Lindblad, On the generators of quantum dynamical semigroups., *Comm. Math. Phys.* **48** (1976), 119-130.

[LJ] Y. Le Jan, Mesures associés a une forme de Dirichlet. Applications., *Bull. Soc. Math. France* **106** (1978), 61-112.

[LOZ] P. Lugiewicz, R. Olkiewicz, B. Zegarlinski, Nonlinear Markov semigroups on C*-algebras, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **16**, no. 1, (2013), 1350004, 1-21.

[MS] V.G. Maz'ya, T.O. Shaposhnikova, "Theory of Sobolev multipliers. With applications to differential and integral operators.", Grundlehren der Mathematischen Wissenschaften 337, Springer Verla 2009.

[Pe1] J. Peterson, L^2-rigidity in von Neumann algebras, *Invent. Math.* **175** (2009), 417-433.

[Pe2] J. Peterson, A 1-cohomology characterization of property (T) in von Neumann algebras, *Pacific J. Math.* **243** (2009), no. 1, 181-199.

[Po] S. Popa, On a class of type Π_1 factors with Betti numbers invariants, *Annals of Math.* **163** (2006), 809-899.

[Ri1] M. Rieffel, Metrics on state spaces, *Doc. Math. (electronic)* **4** (1999), 559-600.

[Ri2] M. Rieffel, Non-commutative resistance networks, *SIGMA Symmetry Integrability Geom. Methods Appl.* **10** (2014), Paper 064, 46 pp.

[S1] J.-L. Sauvageot, Tangent bimodule and locality for dissipative operators on C*–algebras, Quantum Probability and Applications IV, *Lecture Notes in Math.* **1396** (1989), 322-338.

[S2] J.-L. Sauvageot, Quantum Dirichlet forms, differential calculus and semigroups, Quantum Probability and Applications V, *Lecture Notes in Math.* **1442** (1990), 334-346.

[S3] J.-L. Sauvageot, Semi–groupe de la chaleur transverse sur la C*–algèbre d'un feulleitage riemannien, *C.R. Acad. Sci. Paris Sér. I Math.* **310** (1990), 531-536.

[S4] J.-L. Sauvageot, Semi–groupe de la chaleur transverse sur la C*–algèbre d'un feulleitage riemannien, *J. Funct. Anal.* **142** (1996), 511-538.

[SU] R. Schrader, D.A. Uhlenbrock, Markov structures on Clifford algebras, *Jour. Funct. Anal.* **18** (1975), 369-413.

[V1] D.V. Voiculescu, Lectures on Free Probability theory, *Lecture Notes in Math.* **1738** (2000), 279-349.

[V2] D.V. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, *Invent. Math.* **132** (1998), 189-227.

[V3] D.V. Voiculescu, Almost Normal Operators mod HilbertSchmidt and the K-theory of the Algebras $E\Lambda(\Omega)$, *J. Noncommut. Geom.*, **8**, no. 4, (2014), 11231145.