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The local sharp maximal function and BMO on locally homogeneous spaces

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Abstract

We prove a local version of Fefferman-Stein inequality for the local sharp maximal function, and a local version of John-Nirenberg inequality for locally BMO functions, in the framework of locally homogeneous spaces, in the sense of Bramanti-Zhu [3].

1 Introduction

Real analysis and the theory of singular integrals have been developed first in the Euclidean setting and then in more general contexts, in view of their applications to harmonic analysis, partial differential equations, and complex analysis. Around 1970 the theory of *spaces of homogeneous type*, that is quasi-metric doubling measure spaces, started to be systematically developed in the monograph by Coifman-Weiss [10] and was successfully applied to several fields. Much more recently, some problems arising from the quest of a-priori estimates for PDEs suggested that real analysis would be a more flexible and useful tool if its concepts and results were stated also in a *local* version. The meaning of this localization is, roughly speaking, the following: we want an abstract theory which, when applied to the concrete setting of a bounded domain $\Omega \subset \mathbb{R}^n$ endowed with a local quasidistance ρ and a locally doubling measure $d\mu$ (typically, the Lebesgue measure), brings to integrals over metric balls $B_r(x)$ *properly contained* in Ω , and never requires to compute integrals over sets of the kind $B_r(x) \cap \Omega$, as happens when we apply the standard theory of spaces of homogeneous type to a bounded domain Ω . These versions, however, are not easily obtained *a posteriori* from the well established theory; instead, they require a careful analysis which often poses nontrivial new problems.

For instance, global L^p estimates for certain operators of Ornstein-Uhlenbeck type were proved in [2] using results from a theory of nondoubling spaces, developed in [1], which in particular applies to certain locally doubling spaces.

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Bramanti-Zhu in [3] developed instead a theory of singular and fractional integrals in locally doubling spaces; these results were applied in [4] to the proof of L^p and Schauder estimates for nonvariational operators structured on Hörmander's vector fields.

The aim of this paper is to continue the theory of locally doubling spaces, as started in [3], with two main results: a local version of Fefferman-Stein's theorem regarding the sharp maximal function, and a local version of John-Nirenberg inequality about BMO functions. The first of these results, in its Euclidean version, is a key ingredient of a novel approach to the proof of L^p estimates for nonvariational elliptic and parabolic operators with possibly discontinuous coefficients, first devised by Krylov in [16]; the present extension can open the way to the application of these techniques to more general differential operators, as shown in [5].

Comparison with the existent literature and main results. The “sharp maximal function” was introduced in the Euclidean context by Fefferman-Stein in [13], where the related L^p inequality was proved. In some spaces of generalized homogeneous type, this operator has been introduced and studied by Lai in [17]. *Local* sharp maximal functions have been studied by several authors (with different definitions). Here we follow the approach of Iwaniec [14] who proves, in the Euclidean context, a version of local sharp maximal inequality. His proof relies on a clever adaptation of Calderón-Zygmund decomposition, and the striking fact is that this construction can be adapted quite naturally to the abstract context of locally homogeneous spaces, exploiting the properties of the “dyadic cubes” abstractly constructed in this framework in [3]. Our first main result is the sharp maximal inequality stated in Theorem 3.4. This statement involves dyadic cubes and the dyadic local sharp maximal function (see Definition 3.3); since, however, these “cubes” are abstract objects which in the concrete application of the theory are not easily visualized, it is convenient to derive from Theorem 3.4 some consequences formulated in the language of metric balls and the local sharp maximal function (defined by means of balls, instead of dyadic cubes, see Definition 3.5). These results, more easily applicable, are Corollaries 3.7, 3.8, 3.9.

The space BMO of functions with bounded mean oscillation was introduced in [15], where the famous “John-Nirenberg inequality” is proved. Versions of this space and this inequality in spaces of homogeneous type have been given by several authors, starting with Buckley [6] (see also Caruso-Fanciullo [7] and Dafni-Yue [11]). To adapt this result to our context, we follow the approach contained in Mateu, Mattila, Nicolau, Orobitg [18, Appendix], see also Castillo, Ramos Fernández, Trousselot [8]. Our main result is Theorem 4.2, with its useful consequence, Theorem 4.5, stating that we can equivalently compute the mean oscillation of a function or its L^p version for any $p \in (1, \infty)$, always computing averages over small balls.

Plan of the paper. Section 2 contains some basic facts about locally doubling spaces; in Section 3 the local sharp maximal function is studied and several L^p

inequalities are proved about it; in section 4 the local John-Nirenberg inequality is proved.

2 Preliminaries about locally homogeneous spaces

We start recalling the abstract context of locally homogeneous spaces, as introduced in [3].

(H1) Let Ω be a set, endowed with a function $\rho : \Omega \times \Omega \rightarrow [0, \infty)$ such that for any $x, y \in \Omega$:

- (a) $\rho(x, y) = 0 \Leftrightarrow x = y$;
- (b) $\rho(x, y) = \rho(y, x)$.

For any $x \in \Omega, r > 0$, let us define the ball

$$B(x, r) = \{y \in \Omega : \rho(x, y) < r\}.$$

These balls can be used to define a topology in Ω , saying that $A \subset \Omega$ is open if for any $x \in A$ there exists $r > 0$ such that $B(x, r) \subset A$. Also, we will say that $E \subset \Omega$ is bounded if E is contained in some ball.

Let us assume that:

- (H2) (a) the balls are open with respect to this topology;
- (H2) (b) for any $x \in \Omega$ and $r > 0$ the closure of $B(x, r)$ is contained in $\{y \in \Omega : \rho(x, y) \leq r\}$.

It can be proved (see [3, Prop. 2.4]) that the validity of conditions (H2) (a) and (b) is equivalent to the following:

- (H2') $\rho(x, y)$ is a continuous function of x for any fixed $y \in \Omega$.
- (H3) Let μ be a positive regular Borel measure in Ω .
- (H4) Assume there exists an increasing sequence $\{\Omega_n\}_{n=1}^{\infty}$ of bounded measurable subsets of Ω , such that:

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega \tag{2.1}$$

and such for, any $n = 1, 2, 3, \dots$:

- (i) the closure of Ω_n in Ω is compact;
- (ii) there exists $\varepsilon_n > 0$ such that

$$\{x \in \Omega : \rho(x, y) < 2\varepsilon_n \text{ for some } y \in \Omega_n\} \subset \Omega_{n+1}; \tag{2.2}$$

We also assume that:

- (H5) there exists $B_n \geq 1$ such that for any $x, y, z \in \Omega_n$

$$\rho(x, y) \leq B_n (\rho(x, z) + \rho(z, y)); \tag{2.3}$$

- (H6) there exists $C_n > 1$ such that for any $x \in \Omega_n, 0 < r \leq \varepsilon_n$ we have

$$0 < \mu(B(x, 2r)) \leq C_n \mu(B(x, r)) < \infty. \tag{2.4}$$

(Note that for $x \in \Omega_n$ and $r \leq \varepsilon_n$ we also have $B(x, 2r) \subset \Omega_{n+1}$).

Definition 2.1 We will say that $(\Omega, \{\Omega_n\}_{n=1}^\infty, \rho, \mu)$ is a locally homogeneous space if assumptions (H1) to (H6) hold.

Dependence on the constants. The numbers ε_n, B_n, C_n will be called “the constants of Ω_n ”. It is not restrictive to assume that B_n, C_n are nondecreasing sequences, and ε_n is a nonincreasing sequence. Throughout the paper our estimates, for a fixed Ω_n , will often depend not only on the constants of Ω_n , but also (possibly) on the constants of $\Omega_{n+1}, \Omega_{n+2}, \Omega_{n+3}$. We will briefly say that “a constant depends on n ” to mean this type of dependence.

In the language of [10], ρ is a *quasidistance* in each set Ω_n ; we can also say that it is a *local quasidistance* in Ω . We stress that the two conditions appearing in (H2) are logically independent each from the other, and they do not follow from (2.3), even when ρ is a quasidistance in Ω , that is when $B_n = B > 1$ for all n . If, however, ρ is a *distance* in Ω , that is $B_n = 1$ for all n , then (H2) is automatically fulfilled. The continuity of ρ also implies that (2.3) still holds for $x, y, z \in \overline{\Omega}_n$. Also, note that $\mu(\Omega_n) < \infty$ for every n , since $\overline{\Omega}_n$ is compact.

The basic concepts about Vitali covering lemma, the local maximal function and its L^p bound can be easily adapted to this context:

Lemma 2.2 (Vitali covering Lemma) *Let E be a measurable subset of Ω_n and let $\{B(x_\lambda, r_\lambda)\}_{\lambda \in \Lambda}$ be a family of balls with centers $x_\lambda \in \Omega_n$ and radii $0 < r_\lambda \leq r_n \equiv 2\varepsilon_n / (2B_{n+1} + 3B_{n+1}^2)$, such that $E \subset \bigcup_{\lambda \in \Lambda} B(x_\lambda, r_\lambda)$. Then one can select a countable subcollection $\{B(x_{\lambda_j}, r_{\lambda_j})\}_{j=1}^\infty$ of mutually disjoint balls so that*

$$E \subset \bigcup_{j=1}^\infty B(x_{\lambda_j}, K_n r_{\lambda_j}) \quad (2.5)$$

with $K_n = (2B_{n+1} + 3B_{n+1}^2)$ and, for some constant c depending on Ω ,

$$\sum_{j=1}^\infty \mu(B(x_{\lambda_j}, r_{\lambda_j})) \geq c\mu(E). \quad (2.6)$$

We can then give the following

Definition 2.3 Fix Ω_n, Ω_{n+1} and, for any $f \in L^1(\Omega_{n+1})$ define the local maximal function

$$M_{\Omega_n, \Omega_{n+1}} f(x) = \sup_{\substack{B(\bar{x}, r) \ni x \\ r \leq r_n}} \frac{1}{\mu(B(\bar{x}, r))} \int_{B(\bar{x}, r)} |f(y)| d\mu(y) \text{ for } x \in \Omega_n$$

where $r_n = 2\varepsilon_n / (2B_{n+1} + 3B_{n+1}^2)$ is the same number appearing in Vitali Lemma. (Actually, the following theorem still holds if this number r_n is replaced by any smaller number).

Then (see [3, Thm. 8.3]):

Theorem 2.4 *Let f be a measurable function defined on Ω_{n+1} . The following hold:*

- (a) *If $f \in L^p(\Omega_{n+1})$ for some $p \in [1, \infty]$, then $M_{\Omega_n, \Omega_{n+1}} f$ is finite almost everywhere in Ω_n ;*
- (b) *if $f \in L^1(\Omega_{n+1})$, then for every $t > 0$,*

$$\mu(\{x \in \Omega_n : (M_{\Omega_n, \Omega_{n+1}} f)(x) > t\}) \leq \frac{c_n}{t} \int_{\Omega_{n+1}} |f(y)| d\mu(y);$$

- (c) *if $f \in L^p(\Omega_{n+1})$, $1 < p \leq \infty$, then $M_{\Omega_n, \Omega_{n+1}} f \in L^p(\Omega_n)$ and*

$$\|M_{\Omega_n, \Omega_{n+1}} f\|_{L^p(\Omega_n)} \leq c_{n,p} \|f\|_{L^p(\Omega_{n+1})}.$$

By standard techniques, from the above theorem one can also prove the following:

Theorem 2.5 (Lebesgue differentiation theorem) *For every $f \in L^1_{loc}(\Omega_{n+1})$ and a.e. $x \in \Omega_n$ there exists*

$$\lim_{r \rightarrow 0^+} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} f(y) d\mu(y) = f(x).$$

In particular, for every $f \in L^1_{loc}(\Omega_{n+1})$ and a.e. $x \in \Omega_n$,

$$|f(x)| \leq M_{\Omega_n, \Omega_{n+1}} f(x).$$

A deep construction which is carried out in [3, Thm. 8.3], adapting to our local context an analogous construction developed in doubling spaces by Christ [9] is that of *dyadic cubes*. Their relevant properties are collected in the following:

Theorem 2.6 (Dyadic cubes) *(See [3, Thm. 3.1]). Let $(\Omega, \{\Omega_n\}_{n=1}^\infty, \rho, \mu)$ be a locally homogeneous space. For any $n = 1, 2, 3, \dots$ there exists a collection*

$$\Delta_n = \{Q_\alpha^k \subset \Omega, k = 1, 2, 3, \dots, \alpha \in I_k\}$$

(where I_k is a set of indices) of open sets called “dyadic cubes subordinated to Ω_n ”, positive constants $a_0, c_0, c_1, c_2, \delta \in (0, 1)$ and a set $E \subset \Omega_n$ of zero measure, such that for any $k = 1, 2, 3, \dots$ we have:

- (a) $\forall \alpha \in I_k$, each Q_α^k contains a ball $B(z_\alpha^k, a_0 \delta^k)$;
- (b) $\bigcup_{\alpha \in I_k} Q_\alpha^k \subset \Omega_{n+1}$;
- (c) $\forall \alpha \in I_k, 1 \leq l \leq k$, there exists $Q_\beta^l \supseteq Q_\alpha^k$;
- (d) $\forall \alpha \in I_k$, $\text{diam}(Q_\alpha^k) < c_1 \delta^k$ and $\overline{Q_\alpha^k} \subset B(z_\alpha^k, c_1 \delta^k) \subset \Omega_{n+2}$;
- (e) $\ell \geq k \implies \forall \alpha \in I_k, \beta \in I_\ell, Q_\beta^\ell \subset Q_\alpha^k$ or $Q_\beta^\ell \cap Q_\alpha^k = \emptyset$;

$$\begin{aligned}
(f) \quad & \Omega_n \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k \subset E; \\
(g) \quad & \forall \alpha \in I_k, x \in Q_\alpha^k \setminus E, j \geq 1 \text{ there exists } Q_\beta^j \ni x; \\
(h) \quad & \mu(B(x, 2r) \cap Q_\alpha^k) \leq c_2 \mu(B(x, r) \cap Q_\alpha^k)
\end{aligned} \tag{2.7}$$

for any $x \in Q_\alpha^k \setminus E, r > 0$. More precisely, for these x and r we have:

$$\mu(B(x, r) \cap Q_\alpha^k) \geq \begin{cases} c_0 \mu(B(x, r)) & \text{for } r \leq \delta^k \\ c_0 \mu(Q_\alpha^k) & \text{for } r > \delta^k \end{cases} \tag{2.8}$$

The sets Q_α^k can be thought as *dyadic cubes of side length δ^k* . Note that k is a *positive* integer, so we are only considering *small* dyadic cubes. The cubes $\{Q_\alpha^k\}$ are *subordinated to* a particular Ω_n , meaning that they essentially cover Ω_n (that is, their union covers Ω_n up to a set of zero measure) and are contained in Ω_{n+1} . Note that the cubes Q_α^k and all the constants depend on n , so we should write, more precisely

$$\left\{ Q_\alpha^{(n),k} \right\}_{\alpha \in I_k^{(n)}}; \delta^{(n)}; a_{0,(n)}, c_{0,(n)}, c_{1,(n)}, c_{2,(n)},$$

but we will usually avoid this heavy notation.

In the proof of the above theorem, δ is chosen *small enough*, so it is not restrictive to assume $c_1 \delta < 2\varepsilon_{n+1}$, which implies that the ball $B(z_\alpha^k, c_1 \delta^k)$ appearing in point (d) is $\subset \Omega_{n+2}$. (We remark this fact because in [3] the inclusion $B(z_\alpha^k, c_1 \delta^k) \subset \Omega_{n+2}$ is not stated).

Point (h) contains a crucial information: the triple $(Q_\alpha^k, \rho, d\mu)$ is a space of homogeneous type in the sense of Coifman-Weiss, that is the measure μ of ρ -balls *restricted to* Q_α^k is *doubling*. Note that in our context this property could fail to be true, instead, for the measure μ of ρ -balls *restricted to a fixed ρ -ball*.

We will also need the following:

Lemma 2.7 (Covering Lemma) *For every n and every positive integer k large enough, the set Ω_n can be essentially covered by a finite union of dyadic cubes Q_α^k (subordinated to Ω_{n+1}) with the following properties:*

- (i) $Q_\alpha^k \subset B(z_\alpha^k, c_1 \delta^k) \subset \Omega_{n+1}$
- (ii) $B(z_\alpha^k, c_1 \delta^k) \subset F_\alpha^k$ (essentially), $F_\alpha^k \subset B(z_\alpha^k, c' \delta^k) \subset \Omega_{n+1}$, where the set F_α^k is a finite union of dyadic cubes $Q_{\beta_\alpha}^k$, hence F_α^k is a space of homogeneous type, that is satisfies (h) of the previous theorem.

Proof. Since the whole Ω_{n+1} can be essentially covered by the union of the dyadic cubes Q_α^k subordinated to Ω_{n+1} , Ω_n is essentially covered by a subfamily of these. By (d) in the previous theorem, for each Q_α^k of these cubes, there exists a ball $B(z_\alpha^k, c_1 \delta^k)$ such that $Q_\alpha^k \subset B(z_\alpha^k, c_1 \delta^k) \subset \Omega_{n+2}$. However, since Q_α^k , and then $B(z_\alpha^k, c_1 \delta^k)$, contains a point of Ω_n , for k large enough $B(z_\alpha^k, c_1 \delta^k) \subset \Omega_{n+1}$, that is (i) holds.

Let F_α^k the union of all the dyadic cubes $Q_{\beta_\alpha}^k$ intersecting $B(z_\alpha^k, c_1 \delta^k)$. Since $B(z_\alpha^k, c_1 \delta^k) \subset \Omega_{n+1}$ which is essentially covered by the union of all the dyadic

cubes Q_β^k , then $B(z_\alpha^k, c_1\delta^k)$ is essentially covered by F_α^k . Since, by (d) in the previous theorem, $\text{diam}(Q_\beta^k) < c_1\delta^k$, and each Q_β^k contained in F_α^k intersects $B(z_\alpha^k, c_1\delta^k)$, $\text{diam} F_\alpha^k$ is comparable to δ^k , so $F_\alpha^k \subset B(z_\alpha^k, c'\delta^k)$, which is again contained in Ω_{n+1} , for k large enough, since the ball contains a point of Ω_n . Finally, any finite union of dyadic cubes satisfies the doubling condition, by (2.7) and [3, Corollary 3.9]. ■

3 The local sharp maximal function

As we have explained in the Introduction, the proof of the sharp maximal inequality will be achieved following the approach in [14], which exploits a suitable version of Calderón-Zygmund decomposition. We start proving in the context of locally homogeneous spaces the following decomposition lemma.

For any measurable set E and function $f \in L^1(E)$, let

$$f_E = \frac{1}{|E|} \int_E f.$$

Lemma 3.1 *For fixed Ω_n, Ω_{n+1} we consider the family Δ_n of dyadic cubes built in Theorem 2.6. Let $Q_{\alpha_1}^1$ be a fixed dyadic cube (“of first generation”) and let $f \in L^1(Q_{\alpha_1}^1)$. For any $\lambda \geq a \equiv |f|_{Q_{\alpha_1}^1}$ there exists a countable family $\mathcal{C}_\lambda = \{Q_{\lambda,j}\}_{j=1,2,\dots}$ of pairwise disjoint dyadic subcubes of $Q_{\alpha_1}^1$ such that:*

- (i) $\lambda < |f|_{Q_{\lambda,j}} \leq c_n\lambda$ for $j = 1, 2, \dots$;
- (ii) if $\lambda \geq \mu \geq a$ then each cube $Q_{\lambda,j}$ is a subcube of one from the family \mathcal{C}_μ ;
- (iii) $|f(x)| \leq \lambda$ for a.e. $x \in Q_{\alpha_1}^1 \setminus \bigcup_j Q_{\lambda,j}$;

$$(iv) \sum_j |Q_{\lambda,j}| \leq \left| \left\{ x \in Q_{\alpha_1}^1 : Mf(x) > \frac{\lambda}{c'_n} \right\} \right|;$$

$$(v) \left| \left\{ x \in Q_{\alpha_1}^1 : Mf(x) > c''_n\lambda \right\} \right| \leq c'''_n \sum_j |Q_{\lambda,j}|$$

where c_n, c'_n, c''_n, c'''_n are constants > 1 only depending on n and we let for simplicity

$$Mf = M_{\Omega_{n+1}, \Omega_{n+2}} \left(f \chi_{Q_{\alpha_1}^1} \right)$$

(i.e., the local maximal function is computed after extending f to zero outside $Q_{\alpha_1}^1$).

Remark 3.2 *As will be apparent from the proof, this lemma still holds if instead of a fixed dyadic cube $Q_{\alpha_1}^1$ of the first generation we fix a cube $Q_{\alpha_{k_0}}^{k_0}$ of some fixed generation $k_0 > 1$. Throughout this section we will stick to the convention of considering $Q_{\alpha_1}^1$ a cube of first generation, just to simplify notation, however we must keep in mind that the results still hold under the more general assumption on $Q_{\alpha_{k_0}}^{k_0}$. Or, saying this with other words, we can think that the cube $Q_{\alpha_1}^1$ appearing in Theorem 3.4 and Corollaries 3.7 and 3.8 has diameter as small as we want.*

Proof. By point (c) in Thm. 2.6 for every dyadic cube $Q \subset Q_{\alpha_1}^1$ there exists an increasing chain of dyadic cubes

$$Q = Q_{\alpha_k}^k \subset Q_{\alpha_{k-1}}^{k-1} \subset \dots \subset Q_{\alpha_1}^1. \quad (3.1)$$

For a fixed $\lambda \geq a = |f|_{Q_{\alpha_1}^1}$, in order to define the family \mathcal{C}_λ we say that $Q \in \mathcal{C}_\lambda$ if, with the notation (3.1),

$$\lambda < |f|_Q \text{ and } |f|_{Q_{\alpha_s}^s} \leq \lambda \text{ for } s = 1, 2, \dots, k-1.$$

Note that any two cubes in \mathcal{C}_λ are disjoint, otherwise by point (e) in Thm. 2.6 one should be contained in the other, so they would be two different steps in the same chain (3.1), which contradicts our rule of choice. Let us show that \mathcal{C}_λ satisfies properties (i)-(iv).

(i). For $Q = Q_{\alpha_k}^k \in \mathcal{C}_\lambda$, by construction, $\lambda < |f|_Q$ and $|f|_{Q_{\alpha_{k-1}}^{k-1}} \leq \lambda$ hence

$$|f|_Q \leq \frac{1}{|Q_{\alpha_k}^k|} \int_{Q_{\alpha_{k-1}}^{k-1}} |f| \leq \frac{|Q_{\alpha_{k-1}}^{k-1}|}{|Q_{\alpha_k}^k|} |f|_{Q_{\alpha_{k-1}}^{k-1}} \leq c_n \lambda$$

since by points (a) and (d) in Thm. 2.6,

$$B(z_{\alpha}^k, a_0 \delta^k) \subset Q_{\alpha_k}^k \subset Q_{\alpha_{k-1}}^{k-1} \subset B(z_{\alpha_{k-1}}^{k-1}, c_1 \delta^{k-1}),$$

hence by the locally doubling condition

$$\frac{|Q_{\alpha_{k-1}}^{k-1}|}{|Q_{\alpha_k}^k|} \leq c_n$$

for some c_n only depending on n (in particular, independent of k). Hence (i) is proved.

(ii). For $Q = Q_{\alpha_k}^k \in \mathcal{C}_\lambda$, $\lambda \geq \mu \geq a$ we have

$$|f|_{Q_{\alpha_k}^k} > \lambda \geq \mu$$

hence in the chain (3.1) there is an l such that $|f|_{Q_{\alpha_l}^l} > \mu$, $|f|_{Q_{\alpha_{l-1}}^{l-1}} \leq \mu$. This means that in the chain (3.1) there is a cube $Q' = Q_{\alpha_l}^l \in \mathcal{C}_\mu$, and $Q' \supset Q$.

(iii). Let $x \in Q_{\alpha_1}^1 \setminus \bigcup_j Q_{\lambda, j}$ and let Q be any dyadic cube such that $x \in Q \subset Q_{\alpha_1}^1$. Consider again the chain (3.1) starting with Q . By our choice of x , none of the cubes $Q_{\alpha_l}^l$ in this chain belongs to \mathcal{C}_λ , and this means that $|f|_{Q_{\alpha_l}^l} \leq \lambda$. Then by point (g) in Thm. 2.6, for a.e. $x \in Q_{\alpha_1}^1 \setminus \bigcup_j Q_{\lambda, j}$ there exists a decreasing sequence of dyadic cubes $\{Q_{\alpha_l}^l\}$ such that $|f|_{Q_{\alpha_l}^l} \leq \lambda$ and $\bigcap Q_{\alpha_l}^l = \{x\}$. By Lebesgue's differentiation theorem, (iii) follows.

(iv). Let

$$f^*(x) = \sup_{x \in Q \in \Delta_n} |f|_Q.$$

In the previous point we have proved that for a.e. $x \in Q_{\alpha_1}^1 \setminus \bigcup_j Q_{\lambda,j}$ and dyadic cube Q such that $x \in Q \subset Q_{\alpha_1}^1$, we have $|f|_Q \leq \lambda$. Hence

$$f^*(x) \leq \lambda \text{ for a.e. } x \in Q_{\alpha_1}^1 \setminus \bigcup_j Q_{\lambda,j}.$$

Conversely, if $x \in \bigcup_j Q_{\lambda,j}$ then $|f|_{Q_{\lambda,j}} > \lambda$ hence $f^*(x) > \lambda$. These two facts mean that, up to a set of zero measure,

$$\bigcup_j Q_{\lambda,j} = \{x \in Q_{\alpha_1}^1 : f^*(x) > \lambda\}$$

hence, since the $\{Q_{\lambda,j}\}_j$ are pairwise disjoint,

$$\sum_j |Q_{\lambda,j}| = |\{x \in Q_{\alpha_1}^1 : f^*(x) > \lambda\}|.$$

However, again by points (a) and (d) in Thm. 2.6 and the locally doubling condition,

$$f^*(x) \leq c_n M_{\Omega_{n+1}, \Omega_{n+2}} \left(f \chi_{Q_{\alpha_1}^1} \right) (x) \equiv c_n M f(x),$$

hence (iv) follows.

(v). Let $Q_{\lambda,j} \in \mathcal{C}_\lambda$. For some $k = 2, 3, \dots$, we will have $Q_{\lambda,j} = Q_{\alpha_k}^k$ and by points (a) and (d) in Thm. 2.6,

$$B(z_\alpha^k, a_0 \delta^k) \subset Q_{\alpha_k}^k \subset B(z_{\alpha_k}^k, c_1 \delta^k)$$

for some z_α^k . For a $K > 1$ to be chosen later, let $KQ_{\lambda,j} = B(z_{\alpha_k}^k, Kc_1 \delta^k)$. For any $x \notin \bigcup_j KQ_{\lambda,j}$ and any ball $B = B_r(\bar{x})$ such that $x \in B_r(\bar{x})$ and $r \leq r_0$ we

have, extending f to zero outside $Q_{\alpha_1}^1$ if $B \not\subset Q_{\alpha_1}^1$,

$$\int_B |f| = \int_{B \setminus \bigcup_j Q_{\lambda,j}} |f| + \sum_j \int_{B \cap Q_{\lambda,j}} |f|$$

by point (iii)

$$\leq \lambda |B| + \sum_{j: B \cap Q_{\lambda,j} \neq \emptyset} \int_{Q_{\lambda,j}} |f|$$

by point (i)

$$\leq \lambda |B| + c_n \lambda \sum_{j: B \cap Q_{\lambda,j} \neq \emptyset} |Q_{\lambda,j}|. \quad (3.2)$$

Next, we need the following

Claim. There exists $K, H > 1$ (only depending on n) such that if $x \notin \bigcup_i KQ_{\lambda,i}$ and $B_r(\bar{x}) \cap Q_{\lambda,j} \neq \emptyset$ then $Q_{\lambda,j} \subset B_{Hr}(\bar{x})$.

Proof of the Claim. Recall that

$$\begin{aligned} B(z_\alpha^k, a_0\delta^k) &\subset Q_{\lambda,j} \subset B(z_{\alpha_k}^k, c_1\delta^k) \\ KQ_{\lambda,j} &= B(z_{\alpha_k}^k, Kc_1\delta^k). \end{aligned}$$

Since $B_r(\bar{x}) \cap Q_{\lambda,j} \neq \emptyset$, in particular $B_r(\bar{x}) \cap B(z_\alpha^k, a_0\delta^k) \neq \emptyset$ hence

$$\rho(\bar{x}, z_\alpha^k) \leq B_{n+1}(r + a_0\delta^k).$$

Since $x \in B_r(\bar{x})$,

$$\rho(x, z_\alpha^k) \leq B_{n+2}(r + \rho(\bar{x}, z_\alpha^k)) \leq B_{n+2}(r + B_{n+1}(r + a_0\delta^k))$$

and since $x \notin KQ_{\lambda,j}$,

$$B_{n+2}(r + B_{n+1}(r + a_0\delta^k)) > Kc_1\delta^k$$

which, picking $K = \frac{2B_{n+1}B_{n+2}a_0}{c_1}$, gives

$$\delta^k < \frac{(1 + B_{n+1})}{B_{n+1}a_0}r.$$

Then $Q_{\lambda,j} \subset B(z_{\alpha_k}^k, c_1\delta^k) \subset B_{Hr}(\bar{x})$ for a suitable H depending on n , namely for $z \in Q_{\lambda,j}$,

$$\begin{aligned} \rho(z, \bar{x}) &\leq B_{n+1}(c_1\delta^k + \rho(\bar{x}, z_\alpha^k)) \leq B_{n+1}(c_1\delta^k + B_{n+1}(r + a_0\delta^k)) \\ &\leq \delta^k(B_{n+1}c_1 + B_{n+1}^2a_0) + B_{n+1}^2r \\ &\leq \frac{(1 + B_{n+1})}{B_{n+1}a_0}r(B_{n+1}c_1 + B_{n+1}^2a_0) + B_{n+1}^2r \equiv Hr, \end{aligned}$$

which proves the Claim.

Let us come back to the proof of (v). By (3.2) and the Claim we have (since the dyadic cubes in the sum are disjoint)

$$\int_{B_r(\bar{x})} |f| \leq \lambda |B_r(\bar{x})| + c_n \lambda |B_{Hr}(\bar{x})|$$

and, by the locally doubling condition, for $r \leq r_n$ small enough,

$$|f|_B \leq c'_n \lambda$$

for every $B \ni x \in Q_{\alpha_1}^1 \setminus \bigcup_j KQ_{\lambda,j}$, that is

$$Mf(x) \leq c'_n \lambda$$

for any such x , so that

$$\{x \in Q_{\alpha_1}^1 : Mf(x) > c'_n \lambda\} \subset \bigcup_j KQ_{\lambda,j}$$

and

$$|\{x \in Q_{\alpha_1}^1 : Mf(x) > c'_n \lambda\}| \leq \sum_j |KQ_{\lambda,j}| \leq c''_n \sum_j |Q_{\lambda,j}|.$$

■

We can now prove the following local analog of Fefferman-Stein inequality. Let us first define a version of *dyadic sharp maximal function*:

Definition 3.3 For $f \in L^1(Q_{\alpha_1}^1)$, $x \in Q_{\alpha_1}^1$, let

$$f_{\Delta}^{\#}(x) = \sup_{\substack{x \ni Q \in \Delta_n \\ Q \subset Q_{\alpha_1}^1}} \frac{1}{|Q|} \int_Q |f - f_Q|.$$

Note that this definition involves only the values of f in $Q_{\alpha_1}^1$ (there is no need of extending f outside that cube).

Theorem 3.4 (Local Fefferman-Stein inequality) Let $f \in L^1(Q_{\alpha_1}^1)$ and assume $f_{\Delta}^{\#} \in L^p(Q_{\alpha_1}^1)$ for some $p \in [1, +\infty)$. Then $Mf \in L^p(Q_{\alpha_1}^1)$ and

$$\left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} (Mf)^p \right)^{1/p} \leq c_{n,p} \left\{ \left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} (f_{\Delta}^{\#})^p \right)^{1/p} + \left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} |f| \right) \right\} \quad (3.3)$$

for some constant $c_{n,p}$ only depending on n, p , where, as above,

$$Mf = M_{\Omega_{n+1}, \Omega_{n+2}}(f \chi_{Q_{\alpha_1}^1}).$$

Proof. Let $a = |f|_{Q_{\alpha_1}^1}$. We start proving the following estimate: for every $\lambda \geq 2c_n a$ and every $A > 0$,

$$\sum_j |Q_{\lambda,j}| \leq \left| \left\{ x \in Q_{\alpha_1}^1 : f_{\Delta}^{\#}(x) > \frac{\lambda}{A} \right\} \right| + \frac{2}{A} \sum_{Q \in \mathcal{C}_{\lambda/2c_n}} |Q|. \quad (3.4)$$

Fix $\lambda \geq 2c_n a$. By point (i) in Lemma 3.1, for any $Q_{\lambda,k} \in \mathcal{C}_{\lambda}$,

$$\lambda < |f|_{Q_{\lambda,j}} \leq c_n \lambda;$$

also, for any $Q \in \mathcal{C}_{\lambda/2c_n}$,

$$|f|_Q \leq c_n \frac{\lambda}{2c_n} = \frac{\lambda}{2}$$

so that, for such $Q_{\lambda,j}$ and Q ,

$$\begin{aligned} \frac{1}{|Q_{\lambda,j}|} \int_{Q_{\lambda,j}} |f - f_Q| &\geq |f|_{Q_{\lambda,j}} - |f|_Q > \lambda - \frac{\lambda}{2} = \frac{\lambda}{2} \\ |Q_{\lambda,j}| &< \frac{2}{\lambda} \int_{Q_{\lambda,j}} |f - f_Q|. \end{aligned} \quad (3.5)$$

By point (ii) in Lemma 3.1, since $\lambda > \lambda/2c_n$, any $Q_{\lambda,j} \in \mathcal{C}_\lambda$ is contained in some $Q \in \mathcal{C}_{\lambda/2c_n}$; also, the cubes $Q \in \mathcal{C}_{\lambda/2c_n}$, like the cubes $Q_{\lambda,j} \in \mathcal{C}_\lambda$ are pairwise disjoint, hence we can write

$$\sum_j |Q_{\lambda,j}| = \sum_{Q \in \mathcal{C}_{\lambda/2c_n}} \sum_{\substack{Q_{\lambda,j} \in \mathcal{C}_\lambda \\ Q_{\lambda,j} \subset Q}} |Q_{\lambda,j}|. \quad (3.6)$$

For any $Q \in \mathcal{C}_{\lambda/2c_n}$, by (3.5) and since the $Q_{\lambda,j}$ are disjoint

$$\sum_{\substack{Q_{\lambda,j} \in \mathcal{C}_\lambda \\ Q_{\lambda,j} \subset Q}} |Q_{\lambda,j}| \leq \sum_{\substack{Q_{\lambda,j} \in \mathcal{C}_\lambda \\ Q_{\lambda,j} \subset Q}} \frac{2}{\lambda} \int_{Q_{\lambda,j}} |f - f_Q| \leq \frac{2}{\lambda} \int_Q |f - f_Q|.$$

Let now fix a number $A > 0$ and distinguish two cases:

a. If $\frac{1}{|Q|} \int_Q |f - f_Q| \leq \frac{\lambda}{A}$, then

$$\sum_{\substack{Q_{\lambda,j} \in \mathcal{C}_\lambda \\ Q_{\lambda,j} \subset Q}} |Q_{\lambda,j}| \leq \frac{2}{\lambda} \frac{\lambda}{A} |Q| = \frac{2}{A} |Q|.$$

b. If $\frac{1}{|Q|} \int_Q |f - f_Q| > \frac{\lambda}{A}$, then for every $x \in Q$

$$f_\Delta^\#(x) > \frac{\lambda}{A},$$

that is

$$Q \subset \left\{ x \in Q_{\alpha_1}^1 : f_\Delta^\#(x) > \frac{\lambda}{A} \right\}$$

and

$$\sum_{\substack{Q_{\lambda,j} \in \mathcal{C}_\lambda \\ Q_{\lambda,j} \subset Q}} |Q_{\lambda,j}| \leq \left| Q \cap \left\{ x \in Q_{\alpha_1}^1 : f_\Delta^\#(x) > \frac{\lambda}{A} \right\} \right|.$$

In any case we can write

$$\sum_{\substack{Q_{\lambda,j} \in \mathcal{C}_\lambda \\ Q_{\lambda,j} \subset Q}} |Q_{\lambda,j}| \leq \left| Q \cap \left\{ x \in Q_{\alpha_1}^1 : f_\Delta^\#(x) > \frac{\lambda}{A} \right\} \right| + \frac{2}{A} |Q|.$$

Adding up these inequalities for $Q \in \mathcal{C}_{\lambda/2c_n}$, recalling (3.6) and the fact that the cubes Q are pairwise disjoint, we get (3.4).

By (3.4) and points (v), (iv) in Lemma 3.1, we have

$$\begin{aligned}
& |\{x \in Q_{\alpha_1}^1 : Mf(x) > c_n''\lambda\}| \\
& \leq c_n''' \left(\left| \left\{ x \in Q_{\alpha_1}^1 : f_{\Delta}^{\#}(x) > \frac{\lambda}{A} \right\} \right| + \frac{2}{A} \sum_{Q \in \mathcal{C}_{\lambda/2c_n}} |Q| \right) \\
& \leq c_n''' \left(\left| \left\{ x \in Q_{\alpha_1}^1 : f_{\Delta}^{\#}(x) > \frac{\lambda}{A} \right\} \right| + \frac{2}{A} \left| \left\{ x \in Q_{\alpha_1}^1 : Mf(x) > \frac{\lambda}{2c_n c_n'} \right\} \right| \right)
\end{aligned} \tag{3.7}$$

for any $A > 0, \lambda \geq 2c_n a$.

We now want to compute integrals using the identity (for any $F \in L^p(Q_{\alpha_1}^1), 1 \leq p < \infty$)

$$\int_{Q_{\alpha_1}^1} |F(y)|^p dy = \int_0^{+\infty} pt^{p-1} |\{x \in Q_{\alpha_1}^1 : |F(x)| > t\}| dt.$$

Letting

$$\mu(t) = |\{x \in Q_{\alpha_1}^1 : |Mf(x)| > t\}|$$

and integrating (3.7) for $\lambda \in (2c_n a, N)$ and any fixed $N > 2c_n |f|_{Q_{\alpha_1}^1}$ after multiplying by $p\lambda^{p-1}$ we have

$$\begin{aligned}
& \int_{2c_n a}^N p\lambda^{p-1} \mu(c_n''\lambda) d\lambda \\
& \leq c_n''' \left(\int_{2c_n a}^N p\lambda^{p-1} \left| \left\{ x \in Q_{\alpha_1}^1 : f_{\Delta}^{\#}(x) > \frac{\lambda}{A} \right\} \right| d\lambda + \frac{2}{A} \int_{2c_n a}^N p\lambda^{p-1} \mu\left(\frac{\lambda}{2c_n c_n'}\right) d\lambda \right)
\end{aligned} \tag{3.8}$$

Changing variable in each of the three integrals in (3.8) we get:

$$\begin{aligned}
& \int_{2c_n c_n'' a}^{c_n'' N} pt^{p-1} \mu(t) dt \\
& \leq (c_n'')^p c_n''' \left(A^p \int_0^{+\infty} pt^{p-1} |\{x \in Q_{\alpha_1}^1 : f_{\Delta}^{\#}(x) > t\}| dt + \right. \\
& \quad \left. \frac{2}{A} (2c_n c_n')^p \int_0^{\frac{N}{2c_n c_n'}} pt^{p-1} \mu(t) dt \right).
\end{aligned} \tag{3.9}$$

Using also the elementary inequality

$$\int_0^{2c_n c_n'' a} pt^{p-1} \mu(t) dt \leq \int_0^{2c_n c_n'' a} pt^{p-1} |Q_{\alpha_1}^1| dt = |Q_{\alpha_1}^1| (2c_n c_n'' a)^p,$$

together with (3.9), since $\frac{N}{2c_n c'_n} < N < c''_n N$ we get

$$\begin{aligned} & \int_0^{c''_n N} p t^{p-1} \mu(t) dt \\ & \leq (c''_n)^p c'''_n \left(A^p \int_0^{+\infty} p t^{p-1} \left| \left\{ x \in Q_{\alpha_1}^1 : f_{\Delta}^{\#}(x) > t \right\} \right| dt \right. \\ & \quad \left. + \frac{2}{A} (2c_n c'_n)^p \int_0^{c''_n N} p t^{p-1} \mu(t) dt \right) + |Q_{\alpha_1}^1| (2c_n c''_n a)^p. \end{aligned}$$

Letting finally $A = 4(2c_n c'_n c''_n)^p c'''_n$ we conclude

$$\begin{aligned} & \int_0^{c''_n N} p t^{p-1} \mu(t) dt \\ & \leq c_{n,p} \left(\int_0^{+\infty} p t^{p-1} \left| \left\{ x \in Q_{\alpha_1}^1 : f_{\Delta}^{\#}(x) > t \right\} \right| dt + |Q_{\alpha_1}^1| |f|_{Q_{\alpha_1}^1}^p \right) \end{aligned}$$

which implies that $Mf \in L^p(Q_{\alpha_1}^1)$ and

$$\|Mf\|_{L^p(Q_{\alpha_1}^1)}^p \leq c_{n,p} \left(\|f_{\Delta}^{\#}\|_{L^p(Q_{\alpha_1}^1)}^p + |Q_{\alpha_1}^1| |f|_{Q_{\alpha_1}^1}^p \right)$$

that is (3.3). ■

We now want to reformulate the above theorem in terms of balls, instead of dyadic cubes, to make it more easily applicable to concrete situations. This reformulation can be done in several ways. First of all, we introduce the local sharp maximal functions defined by balls instead of cubes.

Definition 3.5 For $f \in L^1_{loc}(\Omega_{n+1})$, $x \in \Omega_n$, let

$$f_{\Omega_n, \Omega_{n+1}}^{\#}(x) = \sup_{\substack{B(\bar{x}, r) \ni x \\ \bar{x} \in \Omega_n, r \leq \varepsilon_n}} \frac{1}{|B(\bar{x}, r)|} \int_{B(\bar{x}, r)} |f - f_{B(\bar{x}, r)}|.$$

Let us compare this function with its dyadic version $f_{\Delta}^{\#}$:

Lemma 3.6 With the above notation, for any $x \in Q_{\alpha_1}^1$,

$$f_{\Delta}^{\#}(x) \leq c_n f_{\Omega_{n+1}, \Omega_{n+2}}^{\#}(x)$$

for some constant c_n only depending on n . Here the function f can be assumed either in $L^1_{loc}(\Omega_{n+2})$ or in $L^1(Q_{\alpha_1}^1)$ and extended to zero outside $Q_{\alpha_1}^1$.

Proof. For any dyadic cube $Q = Q_{\alpha_k}^k \subset Q_{\alpha_1}^1 \subset \Omega_{n+1}$ we have (see points (a), (d) in Theorem 2.6)

$$B(z_{\alpha}^k, a_0 \delta^k) \subset Q \subset B(z_{\alpha_k}^k, c_1 \delta^k) \subset \Omega_{n+2}.$$

Let us briefly write B_1, B_2 in place of $B(z_\alpha^k, a_0\delta^k), B(z_{\alpha_k}^k, c_1\delta^k)$. Then by the locally doubling condition

$$\frac{|B_2|}{|Q|} \leq \frac{|B_2|}{|B_1|} \leq c_n$$

and we can write

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f - f_Q| &\leq \frac{1}{|Q|} \int_{B_2} |f - f_Q| \\ &\leq c_n \frac{1}{|B_2|} \int_{B_2} |f - f_Q| \\ &\leq c_n \left(\frac{1}{|B_2|} \int_{B_2} |f - f_{B_2}| + |f_Q - f_{B_2}| \right). \end{aligned}$$

Also,

$$\begin{aligned} |f_Q - f_{B_2}| &= \left| \frac{1}{|Q|} \int_Q (f - f_{B_2}) \right| \leq \frac{1}{|Q|} \int_Q |f - f_{B_2}| \\ &\leq c_n \frac{1}{|B_2|} \int_{B_2} |f - f_{B_2}| \end{aligned}$$

hence

$$\frac{1}{|Q|} \int_Q |f - f_Q| \leq c_n (1 + c_n) \frac{1}{|B_2|} \int_{B_2} |f - f_{B_2}|$$

and the assertion follows. ■

Exploiting the previous Lemma and Theorem 2.5 we can now rewrite the statement of Theorem 3.4 as follows:

Corollary 3.7 *Let $f \in L^1(Q_{\alpha_1}^1)$ and assume $f_{\Omega_{n+1}, \Omega_{n+2}}^\# \in L^p(Q_{\alpha_1}^1)$ for some $p \in [1, +\infty)$. Then $f \in L^p(Q_{\alpha_1}^1)$ and*

$$\begin{aligned} &\left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} |Mf|^p \right)^{1/p} \\ &\leq c_{n,p} \left\{ \left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} (f_{\Omega_{n+1}, \Omega_{n+2}}^\#)^p \right)^{1/p} + \left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} |f| \right) \right\} \end{aligned}$$

for some constant $c_{n,p}$ only depending on n, p . Here M is defined as in Theorem 3.4 and, again, the function f can be assumed either in $L_{loc}^1(\Omega_{n+2})$ or in $L^1(Q_{\alpha_1}^1)$ and extended to zero outside $Q_{\alpha_1}^1$.

The following is also useful:

Corollary 3.8 *Let $B_1 \subset Q_{\alpha_1}^1 \subset B_2$ with B_1, B_2 concentric balls of comparable radii (like in the proof of Lemma 3.6) and assume that $f \in L^1(B_2)$ with*

$f_{\Omega_{n+1}, \Omega_{n+2}}^\# \in L^p(Q_{\alpha_1}^1)$ for some $p \in [1, +\infty)$ and $f_{B_2} = 0$ (where the function f can be assumed either in $L^1_{loc}(\Omega_{n+2})$ or in $L^1(B_2)$ and extended to zero outside B_2). Then $f \in L^p(Q_{\alpha_1}^1)$ and we have

$$\begin{aligned} \left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} |f|^p \right)^{1/p} &\leq c_{n,p} \left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} \left(f_{\Omega_{n+1}, \Omega_{n+2}}^\# \right)^p \right)^{1/p} \\ \left(\frac{1}{|B_1|} \int_{B_1} |f|^p \right)^{1/p} &\leq c'_{n,p} \left(\frac{1}{|B_2|} \int_{B_2} \left(f_{\Omega_{n+1}, \Omega_{n+2}}^\# \right)^p \right)^{1/p} \end{aligned}$$

for some constants $c_{n,p}, c'_{n,p}$ only depending on n, p . Also, removing the assumption $f_{B_2} = 0$ we can write

$$\left(\frac{1}{|B_1|} \int_{B_1} |f - f_{B_2}|^p \right)^{1/p} \leq c'_{n,p} \left(\frac{1}{|B_2|} \int_{B_2} \left(f_{\Omega_{n+1}, \Omega_{n+2}}^\# \right)^p \right)^{1/p}.$$

Proof. We can write

$$\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} |f| = \frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} |f - f_{B_2}| \leq c_n \frac{1}{|B_2|} \int_{B_2} |f - f_{B_2}| \leq c_n f_{\Omega_{n+1}, \Omega_{n+2}}^\#(x)$$

for every $x \in B_2$. Averaging this inequality on $Q_{\alpha_1}^1$ we get

$$\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} |f| \leq c_n \frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} f_{\Omega_{n+1}, \Omega_{n+2}}^\# \leq c_n \left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} \left(f_{\Omega_{n+1}, \Omega_{n+2}}^\# \right)^p \right)^{1/p}$$

so that, by recalling Corollary 3.7

$$\left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} |f|^p \right)^{1/p} \leq c_{n,p} \left(\frac{1}{|Q_{\alpha_1}^1|} \int_{Q_{\alpha_1}^1} \left(f_{\Omega_{n+1}, \Omega_{n+2}}^\# \right)^p \right)^{1/p},$$

which also implies the second inequality, by the locally doubling condition and the comparability of the radii of B_1, B_2 . ■

Although, in the previous Corollary, the second inequality has the pleasant feature of involving balls instead of dyadic cubes (however, note the two different balls appearing at the left hand side of the last inequality), remember that we cannot choose these balls as we like, since they are related to dyadic cubes.

In concrete applications of this theory, we could use this result to bound $\|f\|_{L^p(\Omega_n)}$. To this aim, recall that the domain Ω_n can be covered (up to a zero measure set) by a finite union of dyadic cubes of the kind $Q_{\alpha_1}^1$, but Ω_n is not covered by the union of the corresponding *smaller* balls B_1 . We then need to improve the previous corollary, replacing the dyadic cube Q on the left hand side with a *larger* ball:

Corollary 3.9 *For any n and every k large enough, the set Ω_n can be covered by a finite union of balls $B_R(x_i)$ of radii comparable to δ^k such that for any*

such ball B_R and every f supported in B_R such that $f \in L^1(B_R)$, $\int_{B_R} f = 0$, and $f_{\Omega_{n+2}, \Omega_{n+3}}^\# \in L_{loc}^p(\Omega_{n+1})$ for some $p \in [1, \infty)$, one has

$$\|f\|_{L^p(B_R)} \leq c_{n,p} \left\| f_{\Omega_{n+2}, \Omega_{n+3}}^\# \right\|_{L^p(B_{\gamma R})}$$

with $\gamma > 1$ absolute constant.

Proof. Applying Lemma 2.7, let us (essentially) cover Ω_n with a finite union of dyadic cubes

$$Q_\alpha^k \subset B(z_\alpha^k, c_1 \delta^k) \subset F_\alpha^k \subset B(z_\alpha^k, c' \delta^k)$$

(where the inclusion $B(z_\alpha^k, c_1 \delta^k) \subset F_\alpha^k$ is only essential). We claim that the balls $B(z_\alpha^k, c_1 \delta^k)$ are the required covering of Ω_n . To see this, let f be supported in $B(z_\alpha^k, c_1 \delta^k)$ and with vanishing integral. Then the same is true for f with respect to the larger ball $B(z_\alpha^k, c' \delta^k)$. We can then apply Corollary 3.8 to each dyadic cube Q_β^k which constitutes F_α^k , writing:

$$\left(\frac{1}{|Q_\beta^k|} \int_{Q_\beta^k} |f|^p \right)^{1/p} \leq c_{n,p} \left(\frac{1}{|Q_\beta^k|} \int_{Q_\beta^k} \left(f_{\Omega_{n+2}, \Omega_{n+3}}^\# \right)^p \right)^{1/p}$$

(note that the local sharp function is $f_{\Omega_{n+2}, \Omega_{n+3}}^\#$ because we are using dyadic balls related to Ω_{n+1}), that is

$$\int_{Q_\beta^k} |f|^p \leq c_{n,p}^p \int_{Q_\beta^k} \left(f_{\Omega_{n+2}, \Omega_{n+3}}^\# \right)^p.$$

Adding these inequalities for all the cubes Q_β^k in F_α^k we get

$$\begin{aligned} \left(\int_{B(z_\alpha^k, c_1 \delta^k)} |f|^p \right)^{1/p} &\leq \left(\int_{F_\alpha^k} |f|^p \right)^{1/p} \\ &\leq c_{n,p} \left(\int_{F_\alpha^k} \left(f_{\Omega_{n+2}, \Omega_{n+3}}^\# \right)^p \right)^{1/p} \leq c_{n,p} \left(\int_{B(z_\alpha^k, c' \delta^k)} \left(f_{\Omega_{n+2}, \Omega_{n+3}}^\# \right)^p \right)^{1/p} \end{aligned}$$

which is our assertion, with $R = c_1 \delta^k$, $\gamma = c'/c_1$. ■

4 Local BMO and John-Nirenberg inequality

We start defining the space of functions with locally bounded mean oscillation in a locally homogeneous space:

Definition 4.1 Let $f \in L^1(\Omega_{n+1})$. We say that f belongs to $BMO(\Omega_n, \Omega_{n+1})$ if

$$[f]_n \equiv \sup_{x \in \Omega_n, r \leq 2\epsilon_n} \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f(y) - f_{B(x, r)}| d\mu(y) < \infty.$$

The main result in this section is the following.

Theorem 4.2 (Local John-Nirenberg inequality) *There exist positive constants b_n, R_n such that $\forall f \in BMO(\Omega_n, \Omega_{n+1})$ and for any ball $B(a, R)$, with $a \in \Omega_n$ and $R \leq R_n$, the following inequality holds true*

$$\mu(\{x \in B(a, R) : |f(x) - f_{B(a, R)}| > \lambda\}) \leq 2e^{-\frac{b_n \lambda}{[f]_n}} \mu(B(a, R)) \quad \forall \lambda > 0. \quad (4.1)$$

Remark 4.3 *As will appear from the proof, the constant R_n is strictly smaller than the number $2\varepsilon_n$ appearing in the definition of $BMO(\Omega_n, \Omega_{n+1})$. Explicitly, we will see that*

$$R_n = \frac{2\varepsilon_n}{B_{n+1} \left(\frac{9}{2} B_{n+1}^2 + 3B_{n+1} + 1 \right)}. \quad (4.2)$$

Proof. We can assume $[f]_n = 1$, since (4.1) does not change dividing both f and λ for a constant.

Let

$$K_n = 2B_{n+1} + 3B_{n+1}^2$$

be the constant appearing in Vitali covering Lemma 2.2,

$$\alpha_n = B_{n+1} \left(\frac{3}{2} K_n + 1 \right)$$

$$R_n = \frac{2\varepsilon_n}{\alpha_n}.$$

Let $a \in \Omega_n$, $R \leq R_n$ and let $S = B(a, R)$ (since $R < \varepsilon_n$, $S \subseteq \Omega_{n+1}$). The proof consists in an iterative construction.

Step 1. We will prove that there exists a family of balls $\{S_j\}_{j=1}^\infty \subset S$ and constants $c, \lambda_0 \geq 1$ depending on n such that:

i)

$$\{x \in S : |f(x) - f_S| > \lambda_0\} \subset \bigcup_{j=1}^\infty S_j \subset S;$$

ii)

$$\sum_{j=1}^\infty \mu(S_j) \leq \frac{1}{2} \mu(S);$$

iii)

$$|f_S - f_{S_j}| \leq c\lambda_0$$

To prove this we start defining the maximal operator associated to S letting, for any $x \in S$,

$$M_S f(x) = \sup \left\{ \frac{1}{\mu(B)} \int_B |f(y) - f_S| d\mu(y) : B \text{ ball}, x \in B, B \subseteq \alpha_n S \right\}$$

where $\alpha_n S = B(a, \alpha_n R) \subseteq \Omega_{n+1}$ since $\alpha_n R \leq 2\varepsilon_n$.

We claim that there exists $A = A(n) > 0$ such that for all $t > 0$

$$\mu(\{x \in S : M_S f(x) > t\}) \leq \frac{A}{t} \mu(S). \quad (4.3)$$

To show this, let $t > 0$ and let

$$U_t = \{x \in S : M_S f(x) > t\}.$$

For every $x \in U_t$ there exists a ball B_x such that $x \in B_x \subseteq \alpha_n S$ and

$$\mu(B_x) < \frac{1}{t} \int_{B_x} |f(y) - f_S| d\mu.$$

Now, by Vitali Lemma 2.7 there exists a countable subcollection of disjoint balls $\{B(x_i, r_i)\}$ such that

$$U_t \subseteq \bigcup_{i=1}^{\infty} B(x_i, K_n r_i).$$

Then, since by definition of S and $M_S f$, $\bigcup_{i=1}^{\infty} B(x_i, r_i) \subseteq \alpha_n S \subset \Omega_{n+1}$, for some constant $A = A(n)$ which can vary from line to line we have

$$\begin{aligned} \mu(U_t) &\leq \mu\left(\bigcup_{i=1}^{\infty} B(x_i, K_n r_i)\right) \leq \sum_{i=1}^{\infty} \mu(B(x_i, K_n r_i)) \leq A \sum_{i=1}^{\infty} \mu(B(x_i, r_i)) \\ &\leq \sum_{i=1}^{\infty} \frac{A}{t} \int_{B(x_i, r_i)} |f - f_S| d\mu = \frac{A}{t} \int_{\bigcup_{i=1}^{\infty} B(x_i, r_i)} |f - f_S| d\mu \\ &\leq \frac{A}{t} \int_{\alpha_n S} |f - f_S| d\mu \leq \frac{A}{t} \left\{ \int_{\alpha_n S} |f - f_{\alpha_n S}| d\mu + \int_{\alpha_n S} |f_{\alpha_n S} - f_S| d\mu \right\} \\ &\leq \frac{A}{t} \{ \mu(\alpha_n S) [f]_n + \mu(\alpha_n S) |f_S - f_{\alpha_n S}| \} \\ &\leq \frac{A}{t} \mu(\alpha_n S) [f]_n \\ &\leq \frac{A}{t} \mu(S) \end{aligned}$$

where we exploited the assumption $[f]_n = 1$. Hence (4.3) is proved.

Let now $\lambda_0 > A$, we consider the following open set

$$U = \{x \in S : M_S f(x) > \lambda_0\}.$$

We have, by (4.3),

$$\mu(U \cap S) = \mu(U) \leq \frac{A}{\lambda_0} \mu(S) < \mu(S), \quad (4.4)$$

from which

$$S \cap U^c \neq \emptyset.$$

Then for any $x \in S$ we set

$$r(x) = \frac{1}{2K_n} \rho(x, U^c) \quad \forall x \in U.$$

If $x, y \in S$ we have $\rho(x, y) \leq 2B_{n+1}R$. Then $\forall x \in U$ (taking a point $y \in U^c \cap S$ in the following inequality)

$$r(x) \leq \frac{1}{2K_n} \rho(x, y) \leq \frac{1}{2B_{n+1}(2 + 3B_{n+1})} 2B_{n+1}R \leq \frac{R}{5}$$

If $y \in B(x, K_n r(x))$ for some $x \in U$, we have

$$\rho(y, x) < K_n r(x) = \frac{K_n}{2K_n} \rho(x, U^c) < \rho(x, U^c)$$

then $y \in U$, from which

$$B(x, K_n r(x)) \subseteq U. \quad (4.5)$$

On the other hand

$$U \subset \bigcup_{x \in U} B(x, r(x)) \quad (4.6)$$

and from the Vitali Lemma there exists a countable sequence of disjoint balls $\{B(x_j, r_j)\}$ ($r_j = r(x_j)$) such that

$$U \subset \bigcup_{j=1}^{\infty} B(x_j, K_n r_j)$$

which by the inclusion (4.5) means that

$$U = \bigcup_{j=1}^{\infty} B(x_j, K_n r_j). \quad (4.7)$$

Moreover $B(x_j, 3K_n r_j) \cap U^c \neq \emptyset \quad \forall j \in \mathbb{N}$ and $B(x_j, 3K_n r_j) \subseteq \alpha_n S$ since $\alpha_n = B_{n+1} \left(\frac{3K_n}{2} + 1 \right)$.

If $y \in B(x_j, 3K_n r_j) \cap U^c$, then $M_s f(y) \leq \lambda_0$ and

$$\frac{1}{\mu(B(x_j, 3K_n r_j))} \int_{B(x_j, 3K_n r_j)} |f - f_S| d\mu \leq \lambda_0. \quad (4.8)$$

We now set

$$S_j = B(x_j, K_n r_j)$$

and we have, by (4.8)

$$\begin{aligned} |f_S - f_{S_j}| &= \left| \frac{1}{\mu(S_j)} \int_{S_j} f d\mu - f_S \right| \leq \frac{1}{\mu(S_j)} \int_{S_j} |f - f_S| d\mu \\ &\leq \frac{c}{\mu(B(x_j, 3K_n r_j))} \int_{B(x_j, 3K_n r_j)} |f - f_S| d\mu \leq c\lambda_0 \end{aligned} \quad (4.9)$$

which is point iii).

By the differentiation theorem we have that for a.e. $x \in S \setminus \cup_j S_j$ (that by (4.7) implies that $x \in U^c$ so that $M_S f(x) \leq \lambda_0$)

$$|f(x) - f_S| \leq \lambda_0.$$

This means that

$$\{x \in S : |f(x) - f_S| > \lambda_0\} \subset \bigcup_{j=1}^{\infty} S_j \subset S$$

which is point i).

Moreover, by the doubling property (H7) and (4.4),

$$\begin{aligned} \sum_{j=1}^{\infty} \mu(S_j) &\leq c \sum_{j=1}^{\infty} \mu(B(x_j, r_j)) = c\mu\left(\bigcup_{j=1}^{\infty} B(x_j, r_j)\right) \leq \\ &\leq c\mu(U) \leq c\frac{A}{\lambda_0}\mu(S) = \frac{1}{2}\mu(S), \end{aligned}$$

having finally chosen $\lambda_0 = 2cA$, so that also point ii) is proved and step 1 is completed.

Step 2 consists in doing the same construction on each ball S_j constructed in Step 1, which allows to conclude that, for every $j_1 = 1, 2, \dots$ there exists a sequence of balls $\{S_{j_1 j_2}\}_{j_2=1}^{\infty} \subset S_{j_1}$ such that (for the same constants c, λ_0 of Step 1)

i)

$$\{x \in S_{j_1} : |f(x) - f_{S_{j_1}}| > \lambda_0\} \subset \bigcup_{j_2=1}^{\infty} S_{j_1 j_2} \subset S_{j_1};$$

ii)

$$\sum_{j_2=1}^{\infty} \mu(S_{j_1 j_2}) \leq \frac{1}{2}\mu(S_{j_1});$$

iii)

$$|f_{S_{j_1}} - f_{S_{j_1 j_2}}| \leq c\lambda_0.$$

Point ii) of Step 2 and Step 1 imply

$$\sum_{j_1, j_2=1}^{\infty} \mu(S_{j_1 j_2}) \leq \frac{1}{2} \sum_{j_1=1}^{\infty} \mu(S_{j_1}) \leq \frac{1}{4}\mu(S).$$

Also, point i) of Step 2 and point iii) of Step 1, imply that for a.e. $x \in S_{j_1} \setminus \bigcup_{j_2=1}^{\infty} S_{j_1 j_2}$,

$$|f(x) - f_S| \leq |f(x) - f_{S_{j_1}}| + |f_{S_{j_1}} - f_S| \leq \lambda_0 + c\lambda_0 < 2c\lambda_0$$

which means that

$$\{x \in S_{j_1} : |f(x) - f_S| > 2c\lambda_0\} \subset \bigcup_{j_2=1}^{\infty} S_{j_1 j_2} \subset S_{j_1}. \quad (4.10)$$

However, point i) of Step 1 implies

$$\{x \in S : |f(x) - f_S| > 2c\lambda_0\} \subset \{x \in S : |f(x) - f_S| > \lambda_0\} \subset \bigcup_{j_1=1}^{\infty} S_{j_1}$$

hence (4.10) rewrites as

$$\{x \in S : |f(x) - f_S| > 2c\lambda_0\} \subset \bigcup_{j_1, j_2=1}^{\infty} S_{j_1 j_2}$$

and, letting $\lambda_1 = c\lambda_0$,

$$\mu(\{x \in S : |f(x) - f_S| > 2\lambda_1\}) \leq \frac{1}{4}\mu(S). \quad (4.11)$$

Relation (4.11) summarizes the joint consequences of Steps 1 and 2.

Proceeding this way the iterative construction, at *Step N* we will have that:

$$\mu(\{x \in S : |f(x) - f_S| > N\lambda_1\}) \leq \frac{1}{2^N}\mu(S). \quad (4.12)$$

Now, let $\lambda > 0$. If $\lambda \geq \lambda_1$, let N be the positive integer such that

$$N\lambda_1 < \lambda \leq (N+1)\lambda_1,$$

then

$$\begin{aligned} \mu(\{x \in S : |f(x) - f_S| > \lambda\}) &\leq \mu(\{x \in S : |f(x) - f_S| > N\lambda_1\}) \\ &\leq \frac{1}{2^N}\mu(S) = e^{-N \log 2}\mu(S) \leq 2e^{-\left(\frac{\log 2}{\lambda_1}\right)\lambda}\mu(S) \end{aligned}$$

Finally, if $0 < \lambda \leq \lambda_1$,

$$\mu(\{x \in S : |f(x) - f_S| > \lambda\}) \leq \mu(S) \leq 2e^{-\left(\frac{\log 2}{\lambda_1}\right)\lambda}\mu(S)$$

and the assertion follows (recall we are assuming $[f]_n = 1$), with $b_n = \frac{\log 2}{\lambda_1}$. ■

Definition 4.4 Let $p \in (1, +\infty)$. We say that f belongs to $BMO^p(\Omega_n, \Omega_{n+1})$ if $f \in L^p(\Omega_{n+1})$ and

$$[f]_{p,n} \equiv \sup_{x \in \Omega_n, r \leq R_n} \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p d\mu(y) \right)^{1/p} < \infty$$

where R_n is the constant appearing in (4.2), strictly smaller than $2\varepsilon_n$.

Now we compare the spaces $BMO(\Omega_n, \Omega_{n+1})$ and $BMO^p(\Omega_n, \Omega_{n+1})$.

Theorem 4.5 *For any $p \in (1, \infty)$ and n we have*

$$BMO^p(\Omega_n, \Omega_{n+1}) = BMO(\Omega_n, \Omega_{n+1}).$$

Moreover, there exists a positive constant $c_{n,p}$ such that for any $f \in BMO(\Omega_n, \Omega_{n+1})$,

$$[f]_{p,n} \leq c_{n,p} [f]_n. \quad (4.13)$$

In particular, $BMO(\Omega_n, \Omega_{n+1}) \subseteq L^p(\Omega_n)$ for every $p \in (1, \infty)$.

Remark 4.6 *Comparing this result with those about the local Fefferman-Stein function proved in the previous section (for instance, Corollary 3.9), we see that the present theorem is a “local” result in a different sense. Here, in the upper bound (4.13), there is not an enlargement of the domain, passing from the left to the right hand side; instead, the local seminorms $[f]_{p,n}$ are computed taking the supremum over balls of radii $r \leq R_n$, which is a stricter condition than the bound $r \leq 2\varepsilon_n$ defining the seminorm $[f]_n$.*

Proof. Let $f \in BMO^p(\Omega_n, \Omega_{n+1})$. By Hölder’s inequality we can write, for every $x \in \Omega_n, r \leq R_n$,

$$\begin{aligned} & \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu(y) \\ & \leq \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}|^p d\mu(y) \right)^{1/p} \leq [f]_{p,n} < \infty. \end{aligned}$$

On the other hand, if $R_n < r < 2\varepsilon_n$ we have

$$\begin{aligned} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y) - f_{B(x,r)}| d\mu(y) & \leq 2 \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y) \\ & \leq 2 \frac{1}{\mu(B(x,R_n))} \int_{\Omega_{n+1}} |f(y)| d\mu(y) \\ & \leq c_n \|f\|_{L^p(\Omega_{n+1})} \end{aligned}$$

because

$$\inf_{x \in \Omega_n} \mu(B(x, R_n)) \geq c_n > 0,$$

as can be easily proved as a consequence of the local doubling condition. Therefore $[f]_n < \infty$ and $f \in BMO(\Omega_n, \Omega_{n+1})$.

Conversely, to prove (4.13), let B be a ball centered in $x \in \Omega_n$ with radius $r \leq R_n$. Then by Theorem 4.2 we have

$$\begin{aligned} \int_B |f(y) - f_B|^p d\mu(y) & = \int_0^{+\infty} p\lambda^{p-1} \mu(\{z \in B : |f(z) - f_B| > \lambda\}) d\lambda \\ & \leq 2 \int_0^{+\infty} p\lambda^{p-1} e^{-b_n \lambda / [f]_n} \mu(B) d\lambda \\ & = \mu(B) [f]_n^p 2p \int_0^{+\infty} t^{p-1} e^{-b_n t} dt, \end{aligned}$$

from which

$$\left(\frac{1}{\mu(B)} \int_B |f(y) - f_B|^p d\mu(y) \right)^{1/p} \leq \left(2p \int_0^{+\infty} t^{p-1} e^{-b_n t} dt \right)^{1/p} [f]_n = c_{n,p} [f]_n$$

which gives (4.13) and the inclusion $BMO(\Omega_n, \Omega_{n+1}) \subseteq BMO^p(\Omega_n, \Omega_{n+1})$.

Finally, to show that $f \in L^p(\Omega_n)$ we can cover Ω_n with a finite collection of balls $B(x, R_n)$ with $x \in \Omega_n$, writing

$$\begin{aligned} \left(\frac{1}{\mu(B)} \int_B |f(y)|^p d\mu(y) \right)^{1/p} &\leq \left(\frac{1}{\mu(B)} \int_B |f(y) - f_B|^p d\mu(y) \right)^{1/p} + |f_B| \\ &\leq [f]_{p,n} + \frac{1}{\mu(B)} \|f\|_{L^1(\Omega_{n+1})} < \infty \end{aligned}$$

which implies the finiteness of $\|f\|_{L^p(\Omega_n)}$. ■

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