A variational principle for plastic hinges in a beam

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Abstract. We focus the minimization of 1D free discontinuity problem with second order energy dependent on jump integrals but not on the cardinality of the discontinuity set, in the framework of L^∞ load. The related energies are not lower semi continuous in BH. Nevertheless we show that if a safe load condition is fulfilled then minimizers exist and they belong actually to SBH, say their second derivative has no Cantor part. If in addition a stronger condition on load is fulfilled then minimizer is unique and belongs to H^2. Moreover we can always select one minimizer whose number of plastic hinges does not exceed 2 and is the limit of minimizers of penalized problems.
When the load stays in the gap between safe load and regularity condition then minimizers with hinges are allowed; if in addition the load is symmetric and strictly positive then there is uniqueness of minimizer, the hinges of such minimizer are exactly two and they are located at the endpoints.

1. Introduction

Given f in $L^\infty(\mathbb{R})$, $\gamma > 0$, $L > 0$, we study the functional

$$F(w) \;=\; \int_{\mathbb{R}} \left(\frac{E\,J}{2}\,|\ddot{w}|^2 - f w \right) dx \;+\; \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| \tag{1.1}$$

dependent on real-valued functions w with $\operatorname{spt} w \subset [0, L]$ and w in $SBH(\mathbb{R})$ (e.g. w is an $L^1(\mathbb{R})$ function whose second derivative w'' is a Radon measure in $\mathbb{R}$ without Cantor part). For any w in $SBH(\mathbb{R})$, $\ddot{w}$ denotes the absolutely continuous part of w'', $S_{\dot{w}}$ the singular set of $\dot{w} = w'$ and $[\dot{w}] = \dot{w}_+ - \dot{w}_-$.
Functional (1.1) describes the total energy associated to deformation of an elastic-plastic beam which is clamped at both endpoints and whose reference configuration is the horizontal interval $[0, L]$; w is the vertical displacement of the beam under the action of the vertical load f.
The crease points set $S_{\dot{w}}$ of a minimizer w may be interpreted as location of plastic hinges in the beam at equilibrium: functional (1.1) takes into account that the energy released in the deformation of a clamped elastic plastic beam is the sum of elastic bending energy and

of energy concentrated at plastic hinges. Jump points are not allowed (say $S_w = \emptyset$) for admissible displacements w which must be continuous.
The flexural rigidity EJ of the beam is given by the product of Young modulus E times the beam cross section polar momentum of inertia J. The constant γ takes into account the resistance of the material to rotation at plastic hinges.
Unfortunately sequential w^*BH lower semicontinuity of functional (1.1) fails in SBH since absolutely continuous and jump part of w'' can merge in the limit (see Remark 2.3).
We notice that (1.1) is convex, nevertheless compactness of minimizing sequences "a priori" may fail since the jump set $S_{\dot w}$ may be an infinite set even if $F(w) < \infty$.
We extend F to the whole BH with value $+\infty$ if $w \notin SBH$ or $\operatorname{spt} w \not\subset [0, L]$, by defining $\mathcal{F}(w) : BH(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$

$$\mathcal{F}(w) \;=\; \begin{cases} \displaystyle\int_{\mathbb{R}} \left(\frac{EJ}{2} \, |\ddot w|^2 - f w \right) dx + \gamma \sum_{S_{\dot w}} |[\dot w]|, & w \in SBH(\mathbb{R}), \operatorname{spt} w \subset [0, L] \\ +\infty & \text{else}\,. \end{cases} \tag{1.2}$$

and we find a completely equivalent minimization problem (also $\mathcal{F}$ fails to be lower semicontinuous with respect to the w^*BH topology).
The relaxed seq. w^*BH l.s.c. envelope of $\mathcal{F}$ is difficult to handle, since it has an extra term containing the Cantor part of second derivative and it takes into account of the interplay between absolutely continuous and concentrated part of energy.
The strategy to overcome this difficulty consists in three steps: first we introduce a sequence of penalized functionals $\mathcal{F}^\varepsilon$, which depends on parameter $\varepsilon > 0$ and are defined for every $w \in BH$:

$$\mathcal{F}^\varepsilon(w) \;=\; \begin{cases} \mathcal{F}(w) \,+\, \varepsilon\, \sharp(S_{\dot w}) & \text{if } w \in SBH(\mathbb{R}), \operatorname{spt} w \subset [0, L] \\ +\infty & \text{else}\,; \end{cases} \tag{1.3}$$

then we study nonconvex functionals $\mathcal{F}^\varepsilon$ which are coercive and l.s.c. on BH but finite in SBH; eventually we jettison the parameter ε by showing that, provided the following safe load assumption on f holds true

$$\|f\|_{L^\infty} \,<\, 16\, \frac{\gamma}{L^2}\;, \tag{1.4}$$

the minimizing sequences for $\mathcal{F}$ are relatively compact in the w^*BH topology.
Theorem 2.1 shows that F, $\mathcal{F}$, its l.s.c envelope $sc^-\mathcal{F}$ and $\mathcal{F}^*$ (the Γ limit of $\mathcal{F}^\varepsilon$) all achieve their minimum among w having support contained in $[0, L]$ and all their minima coincide; moreover all minimizers w of F fulfil the following estimate for (absolutely continuous part of) bending moment $EJ\,\ddot w$

$$\|\ddot w\|_{L^\infty} \;\le\; \gamma \,/\, (EJ)\,,$$

and are balanced at creased points:

$$\ddot w_{\pm} \;=\; \gamma \;\mathrm{sign}[\dot w] \,/\, (EJ)\,.$$

The uniqueness of minimizer for F (or equivalently for $\mathcal{F}$) seems hard to tackle in general. Nevertheless we can always select minimizers of $\mathcal{F}$ which have no more than two hinges (Theorem 2.1). Strict sign of load without symmetry entails that all minimizers exhibit no more than 2 hinges (Theorem 3.7). Under additional sign assumption (e.g. symmetry and strict sign of load) also uniqueness for minimizer of F holds true (Theorem 3.8, and Remark 3.13) together with an explicit representation formula of minimizer (see (3.34) and (3.66)). The penalized functional (1.3) takes into account the total energy related to deformation of an elastic plastic beam: the four terms correspond (in their order, referring to (1.2)) to the elastic bending energy, potential energy and concentrated plastic yielding together with a minimal threshold cost ε for the formation of any plastic hinge: functional (1.3) was deduced as a Gamma limit by 2D or 3D thick approximation of the beam (see [12],[13],[14],[11]) starting from classic models of damage ([3],[9]) .

In a different framework, allowing for L^1 (or even Radon measure) load f, we showed a safe load condition ($\|f\|_{L^1} < 8\gamma/L$) and a regularity load condition ($\|f\|_{L^1} \leq 27\gamma/(4L)$) respectively entailing existence and H^2 regularity for minimizers of $\mathcal{F}_\varepsilon$ (see [15],[16]): such gap between the safe and regularity load condition is very narrow and makes difficult to check wether creased minimizers exist (actually they do exist: for an explicit construction of a load in this gap, exhibiting creased minimizers, see Section 4.1 of [15]).

Here we deal with L^∞ load and in this framework we prove a sharp L^∞ safe load condition (i.e. $\|f\|_{L^\infty} < 16\gamma/L^2$, Theorem 2.1) and also a sharp L^∞ regularity load condition (i.e. $\|f\|_{L^\infty} \leq 12\gamma/L^2$, Theorem 3.5), entailing respectively existence and regularity for minimizers of both $\mathcal{F}^\varepsilon$ and F: in this context we can prove that for any symmetric load f which stays in the gap and has a strict sign, then the minimizer is unique and has exactly two hinges located at the endpoints of the beam. The result is obtained by sharp estimates on the Green function and careful comparison between candidate minimizers.

Our analysis proves that the structure do not develop plastic hinges if the resistance γ fulfils

$$\gamma \;\geq\; \frac{L^2}{12}\,\|f\|_{L^\infty}\,,$$

say a condition which entails (by Theorem 3.5) that maximum bending moment of the purely elastic solution ([18]) does not exceed γ (see (3.10).

For generic data f in L^∞, we show Euler equations (Theorem 3.1) and a Compliance Identity (Theorem 3.2) fulfilled by extremals of F : they provide the essential tools in the comparison between competing functions with the aim of selecting minimizers with relevant qualitative properties, without quantitative knowledge about their derivative jumps.
We show an explicit formula (Theorem 2.8) for the Gamma limit $\mathcal{F}^*$ of $\mathcal{F}^\varepsilon$ and show that the same L^∞ safe and regularity condition above (valid for $F,\ \mathcal{F},\ \mathcal{F}^\varepsilon$) apply also to $\mathcal{F}^*$:

$$\mathcal{F}^*(w) \;:=\; \Gamma\,(w^*BH)\lim{}_{\varepsilon\to 0}\;\mathcal{F}^\varepsilon(w) \;=$$

$$= \begin{cases} \displaystyle\int_{\mathbb{R}} \big(\varphi^{**}(\ddot{w}) \,-\, fw\big)\,dx \;+\; \gamma\left(\sum_{S_{\dot{w}}} |[\dot{w}]| \,+\, \|(w'')^c\|_T\right) & \\ \qquad\qquad\qquad\qquad \forall w\in BH :\ \operatorname{spt} w\subset[0,L]\,, & \\ +\infty \qquad\qquad\qquad \text{else.} & \end{cases} \tag{1.5}$$

where $(w'')^c$ is the Cantor part of w'', $\|\cdot\|_T$ denotes the total variation in $\mathbb{R}$ and

$$\varphi^{**}(s) \;=\; \begin{cases} (EJ/2)\, s^2 & \text{if } |s| \le \gamma/(EJ) \\ \gamma\,|s| \,-\, \gamma^2/(2EJ) & \text{if } |s| > \gamma/(EJ)\,. \end{cases} \tag{1.6}$$

We emphasize that the estimate $|\ddot{w}| \le \gamma/(EJ)$ a.e. in $(0,L)$ (proven for any minimizer w of $\mathcal{F}$) entails:

$$\varphi^{**}(\ddot{w}) = \varphi(\ddot{w}) \qquad \text{and} \qquad \mathcal{F}^*(w) = \mathcal{F}(w) = F(w) \qquad \forall w \in \operatorname{argmin} \mathcal{F}. \tag{1.7}$$

All functionals $F,\ \mathcal{F},\ \mathcal{F}^\varepsilon,\ \mathcal{F}^*$ refer to relaxed homogeneous Dirichlet boundary conditions (the beam is clamped at both endpoints); nevertheless minimizers with hinges located at the boundary are not excluded: if this phenomenon takes place then also boundary creases add a positive cost in the energy.

The structure of minimizers under symmetric load with a strict sign is described by main results (Theorem 3.8, Remarks 3.13, 4.9). In the simple case of constant load $f \equiv -\lambda,\ \lambda > 0$, this analysis provides the following complete picture as long as λ increases:

- for $0 \le \lambda \le 12\,\gamma/L^2$, F has exactly one minimizer which turns out to be $C^3(\mathbb{R})$, say we are in the elastic regime;
- for $12\,\gamma/L^2 < \lambda < 16\,\gamma/L^2$, F still has exactly one minimizer but there is the development of 2 plastic hinges at the boundary;
- for $\lambda > 16\,\gamma/L^2$ there is collapse: the infimum of F is $-\infty$;
- in all the range $0 \le \lambda < 16\,\gamma/L^2$ the minimizer is given (see (3.66)) by

$$z_\lambda(x) \;=\; -\,\lambda\, x^2\,(x-L)^2/(24EJ) \;-\; \frac{1}{(2EJ)}\,(\lambda\, L^2/12 - \gamma)^+\, x(L-x)\,,$$

the bending moment $EJ\,\ddot{z}_\lambda$ never exceeds γ, say $|\ddot{z}_\lambda(x)| \le \gamma/(EJ)$, and

$$\min F = F(z_\lambda) = -\frac{EJ}{2}\int_0^L |\ddot{z}_\lambda|^2 = -\frac{1}{1440\,EJ}\,\lambda^2\, L^5 - L\left((\lambda L^2/12 - \gamma)^+\right)^2/(2EJ)\,;$$

- for $12\,\gamma/L^2 < \lambda < 16\,\gamma/L^2$ the (unique and creased) equilibrium fulfils also

$$EJ\,\ddot{z}_\lambda(0) = \gamma\ \mathrm{sign}[\dot{z}_\lambda](0) \;=\; EJ\,\ddot{z}_\lambda(L) \;=\; \gamma\ \mathrm{sign}[\dot{z}_\lambda](L) \;=\; -\gamma$$

which are generic properties of minimizers with hinges at endpoints.

As far as it concerns penalized functionals $\mathcal{F}^\varepsilon$ we remark that non uniqueness phenomena may occur even for constant load (see Theorem 3.15 and related Example 3.16): both smooth and creased minimizers may appear for suitable choice of constant load and parameter ε.

In the fourth section we apply our techniques to an hinged-hinged elastic-plastic beam with cost-free hinges at both endpoints: say hinges at the endpoints are assumed "a priori" existing. The related energy functional

$$\Lambda(w) = \begin{cases} \displaystyle\int_0^L \left(\frac{EJ}{2}|\ddot{w}|^2 - fw\right)\, dx + \gamma \sum_{S_{w'}} |[w']| \\ \qquad\qquad\qquad\qquad \text{if } w \in SBH(0,L) :\ w(0) = w(L) = 0\,, \\ +\infty \quad \text{otherwise for } w \text{ in } BH(0,L)\,, \end{cases} \tag{1.8}$$

takes into account only "internal" hinges while rotations at the boundary of the beam don't pay additional energy. About hinged-hinged elastic-plastic beam we prove in Theorem 4.1 that any minimizer of (1.8) belongs to $H^2(0,L)$ and coincides with the solution ω of the hinged-hinged purely elastic beam:

$$\omega \in H^2(0,L) \cap H^1_0(0,L)\,, \qquad \text{s.t.} \qquad \omega'''' = f \quad (0,L)\,.$$

We emphasize that SBH functions may have creases (i.e. plastic hinges) but cannot have jumps: all internal creases pay in both functionals (1.1) and (1.8) while boundary creases pay only in functionals (1.1) and they are cost free in functional (1.8).

About coupling of elastic and plastic energies for plates and partial regularity of related minimizers we refer to [13],[4],[5],[6],[7],[10].

In the following we assume $EJ = 1$ without loss of generality, possibly by re-scaling γ and f : anyway we emphasize that the main results of the paper (Theorems 2.1, 3.5, 4.1) hold true unchanged even if $EJ \neq 1$ (see Remark 4.9) while Theorem 3.8 about structure of minimizers hold true up to suitable back-scaling of constant (shown in Remark 4.9).

Outline of the paper

2. Clamped-clamped elastic-plastic beam: existence of minimizer

We denote by $\mathcal{M}(\mathbb{R})$ the space of Radon measures on $\mathbb{R}$.
We denote by $\|\mu\|_T$ the total variation in $\mathbb{R}$ of μ and by $\|\mu\|_{T(E)}$ the total variation in E for any $\mu \in \mathcal{M}(\mathbb{R})$ and any Borel set $E \subset \mathbb{R}$.
Any $\mu \in \mathcal{M}(\mathbb{R})$ can be split into three parts, say $\mu = \mu^a + \mu^j + \mu^c$ where μ^a is the absolutely continuous part, μ^j is the purely atomic part and μ^c is the diffuse singular one (the Cantor part of μ): the decomposition is unique. Analogously, if I is an interval, then any $w \in BV(I)$ can be represented by $w = w_a + w_j + w_c$ where w_a has an absolutely continuous distributional derivative $(w_a)' = (w')^a \in L^1(I)$, w_j is a piece-wise constant function and $(w_j)' = (w')^j$ is purely atomic), w_c is a Cantor-type function (i.e. $(w_c)' = (w')^c$: for any $w \in BV(I)$ these three functions are uniquely defined up to additive constants ([1], Corollary 3.33), the constants are 0 when the support of w is a compact subset of I.
We label $\dot{w} = (w_a)'$ the absolutely continuous part of distributional derivative w', hence we write as follows the unique decomposition of the derivative for a BV function with compact support: $w' = \dot{w} + (w_j)' + (w_c)'$.
The set of approximate discontinuity of $\dot{w}$ (see [1]) is labelled by $S_{\dot{w}}$ and will be shortly referred to as the singular set of $\dot{w}$.

We fix the beam length L and the load f

$$L > 0\,, \qquad f \,\in\, L^\infty\,, \tag{2.1}$$

introduce two function spaces

$$BH(I) = \{w \in L^1(I) : w'' \in \mathcal{M}(I)\}$$

$$SBH(I) = \{w \in L^1(I) : w'' \in \mathcal{M}(I),\ (w'')^c \equiv 0\}$$

and formalize homogeneous Dirichlet boundary condition by introducing the admissible sets

$$\mathcal{K} = \{w \in SBH(\mathbb{R}) : \operatorname{spt} w \subset [0, L]\} \tag{2.2}$$

$$\mathcal{K}^* = \{w \in BH(\mathbb{R}) : \operatorname{spt} w \subset [0, L]\}. \tag{2.3}$$

In this section we study the existence of minimizers for functional $\mathcal{F} : BH(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{F}(w) \ = \begin{cases} \displaystyle\int_{\mathbb{R}} \left(\frac{1}{2} |\ddot{w}|^2 - f w \right) dx \, + \, \gamma \sum_{S_{\dot{w}}} \ |[\dot{w}]| & \text{if } w \in \mathcal{K} \\ +\infty & \text{otherwise in } BH(\mathbb{R}). \end{cases} \tag{2.4}$$

The interval $[0, L]$ represents the reference configuration of an elastic plastic beam, f is the vertical dead load acting on the beam, w is the vertical displacement while points in the singular set $S_{\dot{w}}$ are the plastic hinges of the beam. Here $\gamma > 0$ is a constant depending on the material and the functional $\mathcal{F}$ describes the total energy related to deformation of a clamped elastic-plastic beam with unitary flexural rigidity EJ.
We emphasize that there are sequences $\{w_n\} \subset \mathcal{K}$ such that $\mathcal{F}(w_n)$ is bounded but $\{w_n\}$ is not compact in $\mathcal{K}$ with respect to any topology which renders $\mathcal{F}$ lower semicontinuous; therefore existence of minimizers for (2.4) cannot be proven by standard direct methods in the Calculus of Variations.
From now on if w is a minimum point of $\mathcal{F}$ we shall briefly write $w \in \operatorname{argmin} \mathcal{F}$. The main result of this section is the following statement showing that $\operatorname{argmin} \mathcal{F}$ is not the empty set.

Theorem 2.1. *(*L^∞ **safe load condition for clamped beam***)*
Assume that $f \in L^\infty(\mathbb{R})$ *satisfies*

$$\|f\|_{L^\infty(0,L)} \ < \ \frac{16\,\gamma}{L^2} \tag{2.5}$$

Then $\mathcal{F}$ *achieves a finite minimum and*

$$\|\ddot{w}\|_{L^\infty} \ \le \ \gamma \qquad \forall w \in \operatorname{argmin} \mathcal{F}\,. \tag{2.6}$$

Moreover there is at least one minimizer w *of* $\mathcal{F}$ *such that* $\sharp(S_{\dot{w}}) \le 2$.

The proof of Theorem 2.1 will be achieved at the end of this section through several steps, the first of which is the following Poincaré-type inequality. In the next section we show that in many relevant cases there is also uniqueness for minimizers of $\mathcal{F}$.

Lemma 2.2. *(*L^1*–*BH **Poincaré Inequality***) Let* $v \in BH(\mathbb{R})$, *s.t.* $\operatorname{spt} v \subset [0, L]$ *then*

$$\|v\|_{L^1} \ \le \ \frac{L^2}{16} \, \|v''\|_{T(\mathbb{R})} \ . \tag{2.7}$$

The equality in (2.7) holds true iff $v = r_s$

$$r_s(x) \;=\; s\left(\frac{L}{2} - \left|x - \frac{L}{2}\right|\right)^+ \qquad \forall s \in \mathbb{R} \tag{2.8}$$

Proof -
Fix $v \in \mathcal{K}^* = \{v \in BH(\mathbb{R})\ s.t.\ \operatorname{spt} v \subset [0, L]\}$. Without loss of generality we assume $v \not\equiv 0$. Then define

$$\widetilde{v}(x) \;=\; \begin{cases} \text{convex envelope of } -|v| \text{ evaluated at } x & \text{if } x \in [0, L] \\ 0 & \text{if } x \notin [0, L]\,. \end{cases}$$

We claim that $\widetilde{v}$ fulfils

$$\begin{cases} \widetilde{v} \in BH(\mathbb{R})\,, \quad \operatorname{spt}\widetilde{v} \subset [0, L]\,, \quad v \le 0\,, \quad \widetilde{v} \text{ convex in } [0, L]\,, \\ \displaystyle\int_0^L |\widetilde{v}|\,dx \;=\; \int_0^L -\widetilde{v}\,dx \;\ge\; \int_0^L |v|\,dx\,, \qquad \|\widetilde{v}\,''\|_T \;\le\; \|v''\|_T\;. \end{cases} \tag{2.9}$$

The only non trivial point in (2.9) is the estimate of total variation: $\|\widetilde{v}\,''\|_T \;\le\; \|v''\|_T\,$, which we prove below.
Set $\psi(s) = -|s|$, $\varphi(x) = -|v(x)| = \psi \circ v$, so that $v \in BH(\mathbb{R})$, $\psi \in BH(\mathbb{R})$, ψ is Lipschitz and $\psi(0) = 0$. Hence, by Theorems 1 and 4 and Lemma 3.1 of [17], $-|v| = \psi \circ v$ belongs to $BH(\mathbb{R})$, and we can evaluate its second derivative by suitable chain-rule for superposition of BH functions (here $\operatorname{sign}(0) = 0$, $\operatorname{sign}(s) = s/|s|$, $s \ne 0$):

$$(-|v|)^{\ddot{}} \;=\; -\operatorname{sign}(v)\,\ddot{v} \tag{2.10}$$

$$((-|v|)'')^j \;=\; -\operatorname{sign}(v)\,(v'')^j \;-\; \sum_{t\,:\,v(t)=0} (\,|\dot{v}_+(t)| + |\dot{v}_-(t)|\,)\ \delta_t \tag{2.11}$$

$$((-|v|)'')^c \;=\; -\operatorname{sign}(v)\,(v'')^c \tag{2.12}$$

The three measures in (2.10)-(2.12) are mutually singular.
Moreover the absolutely continuous (2.10) and Cantor part (2.12) obviously do not increase their total variation with respect to the corresponding part of v'', and the respective inequalities still hold true after taking the convex envelope:

$$\|(\widetilde{v}\,'')^{\ddot{}}\|_T \;\le\; \|\ddot{v}\|_T\,,$$
$$\|(\widetilde{v}\,'')^c\|_T \;\le\; \|(v)^c\|_T\,.$$

On the other hand, total variation of (2.11) could be bigger than total variation of $(v'')^j$ due to sign changes of v. Nevertheless, since $\widetilde{v}$ is strictly negative in $(0, L)$, the terms $(\,|\dot{v}_+(t)| + |\dot{v}_-(t)|\,)\ \delta_t$ disappear in the convex envelope for any $t \ne 0, L$. So

$$\|(\widetilde{v}\,'')^j\|_T \;\le\; \|(v'')^j\|_T \,+\, |\dot{v}_+(0)| \,+\, |\dot{v}_-(L)|\,,$$
$$\|\widetilde{v}\,''\|_{T(J)} \;\le\; \|v''\|_{T(J)} \qquad \forall \text{ open interval } J \subset\subset (0, L)\,.$$

To tame the total variation at the boundary of the interval we set $z(x) = -|v(x)|$ and we observe that, either

• $\dot{\widetilde{v}}_+(0) = \dot{z}_+(0)$, hence $\|\widetilde{v}''\|_{T(\{0\})} = \|v''\|_{T(\{0\})}$, $\|\widetilde{v}''\|_{T([0,L])} < \|v''\|_{T([0,L])}$;
or
• $\dot{\widetilde{v}}_+(0) \neq \dot{z}_+(0)$, hence $\dot{\widetilde{v}}_+(0) < \dot{z}_+(0)$, $\widetilde{v}$ is strictly less than z in an open interval $(0,\bar{x})$ (where $\bar{x}$ is chosen such that the interval is the maximal one fulfilling this property), so $\widetilde{v}(\bar{x}) = z(\bar{x})$; then by convexity $\dot{z}_-(\bar{x}) \leq \dot{\widetilde{v}}_-(\bar{x}) \leq \dot{\widetilde{v}}_+(\bar{x}) \leq \dot{z}_+(\bar{x})$, $0 \leq [\dot{\widetilde{v}}](\bar{x}) \leq [\dot{z}](\bar{x})$ so that

$$\|\widetilde{v}\,''\|_{T(\{\bar{x}\})} \leq \|v''\|_{T(\{\bar{x}\})},$$

moreover, by taking into account that $\dot{z}_\pm(\bar{x}) = -\operatorname{sign}(v(\bar{x}))\,\dot{v}_\pm$, $\operatorname{spt} v'' \subset [0,L]$ and $[\dot{\widetilde{v}}](0)$ is the slope of $\widetilde{v}$ in the interval $(0,\bar{x})$ we deduce

$$0 > -|\widetilde{v}\,''|_{T(\{0\})} = [\dot{\widetilde{v}}](0) = \dot{\widetilde{v}}_+(0) = \dot{\widetilde{v}}_-(\bar{x}) \geq \dot{z}_-(\bar{x}) = -\operatorname{sign}(v(\bar{x}))\,\dot{v}_-(\bar{x}) =$$

$$= -\operatorname{sign}(v(\bar{x}))\,(v'')\big([0,\bar{x})\big) \geq -|v''|\big([0,\bar{x})\big),$$

and since $\widetilde{v}$ is affine linear in $(0,\bar{x})$

$$\|\widetilde{v}\,''\|_{T([0,\bar{x}))} < \|v''\|_{T([0,\bar{x}))};$$

the behavior around L can be dealt exactly as the one around 0, so we achieve the inequality $\|\widetilde{v}''\|_{T([0,L])} < \|v''\|_{T([0,L])}$ involving total variations in second case too.
Then claim (2.9) is proven in any case. By (2.9) we get

$$\inf\left\{\frac{\|v''\|_T}{\|v\|_{L^1}} : v \in \mathcal{K}^*\right\} \;=\; \inf\left\{\frac{\|v''\|_T}{\|v\|_{L^1}} : v \in \mathcal{K}^*,\ v \text{ convex in } [0,L]\right\}. \qquad (2.13)$$

If we take $v \in \mathcal{K}^*$, v convex in $[0,L]$ and $v \not\equiv 0$, then

$$-\infty < v'_+(0) \leq 0, \qquad 0 \leq v'_-(L) < +\infty$$

and we can define

$$\check{v}(x) \;=\; (v'_+(0)x) \vee \big((v'_-(L)(x-L)\big) \quad \text{if } x \in [0,L] \qquad \text{and} \quad \check{v}(x) \equiv 0 \quad \text{otherwise}.$$

Then $\check{v} \leq v$ and $\|\check{v}\,''\|_{T(\mathbb{R})} = 2(v_-(L) - v'_+(0)) = \|v''\|_{T([0,L])}$. So

$$\begin{aligned} &\inf\{\ \|v''\|_T \,/\, \|v\|_{L^1}\ : v \in \mathcal{K}^*,\ v \text{ convex in } [0,L]\,\} \geq \\ &\geq \inf\{\ \|v''\|_T \,/\, \|v\|_{L^1}\ : v(x) = (-ax) \vee (b(x-L)),\ a>0,\ b>0\,\} = \qquad (2.14) \\ &\qquad \inf\left\{4(a+b)^2/(abL^2) : a>0,\ b>0\right\} = 16/L^2. \end{aligned}$$

Actually the infimum in (2.14) is a minimum and is achieved iff $a = b$ say only when v is a roof function. By summarizing (2.13),(2.14) prove (2.7). About the fact that only roof functions (2.8) achieve the equality in (2.7) we emphasize that also the transformation $v \to \widetilde{v}$ strictly reduces the relevant quotient $|v''|_T \,/\, \|v\|_{L^1}$ whenever $|v| \not\equiv |\widetilde{v}|$, since in such case inequality for $\int |v|$ in (2.9) is strict. ■

Remark 2.3. *There are sequences v_k in $SBH(\mathbb{R})$ s.t. $\mathcal{F}(v_k) \le A < +\infty$, for all k but $v_k \overset{w^*BH}{\rightharpoonup} v \in BH \setminus SBH$. Hence $\mathcal{F}$ is not seq. w^*BH l.s.c.*
On the other hand choosing alternative extension $\mathcal{G} = \int_{\mathbb{R}} \left(\frac{EJ}{2}|\ddot{w}|^2 - fw\right) dx + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]|$ instead of $\mathcal{F}$ given by (1.2) would be inappropriate since $\inf \mathcal{G} = -\infty$ *even for constant load. More presisely we can exhibit v, v_k as claimed above and in addition such that*

$$v'' = (v'')^c,\ v_k'' = (v_k'')^j \quad \text{and, if } f \equiv 1,\ \mathcal{G}(-tv) = -t\int_0^1 v \ \to -\infty,\ \text{as } t \to \infty\ :$$

let $C = C(x)$, $C_k = C_k(x)$, $x \in [0,1]$, be respectively the Cantor-Vitali function and its monotone, piece-wise constant approximation (related to subintervals of length 3^{-k}). Set

$$v(x) \ = \ \begin{cases} \displaystyle\int_0^x C(4t/L)\,dt & 0 \le x \le L/4 \\ \displaystyle 2\int_0^{L/4} C(4t/L)\,dt \ - \int_0^{L/2-x} C(4t/L)\,dt & L/4 \le x \le L/2 \\ v(L-x) & L/2 \le x \le L \end{cases}$$

$$v_k(x) \ = \ \begin{cases} \displaystyle\int_0^x C_k(4t/L)\,dt & 0 \le x \le L/4 \\ \displaystyle 2\int_0^{L/4} C_k(4t/L)\,dt \ - \int_0^{L/2-x} C_k(4t/L)\,dt & L/4 \le x \le L/2 \\ v_k(L-x) & L/2 \le x \le L\,; \end{cases}$$

since there is no crease at points $L/4, L/2, 3L/4$, then we deduce:
$v'' = (v'')^c \not\equiv 0\,,\ \ v_k'' = (v_k'')^j\,,\ \ \|v''\|_T = 4 = \|v_k''\|_T,\ \ v > 0$ *in* $(0,1),\ \int_0^1 v > 0,$
$\left|\int f v_k\right| \le \|v_k\|_{L^1}\|f\|_{L^\infty} \le 2\,L\,\int_0^{L/4} C_k(4t/L)\,dt\,\|f\|_{L^\infty} \ \le\ (L^2/2)\int_0^1 C dt\,.$

Motivated by the previous Remark, we introduce the following family of penalized functionals $\mathcal{F}^\varepsilon$ dependent on parameter $\varepsilon > 0$: these functionals are seq.w^*BH l.s.c. but non convex.

$$\mathcal{F}^\varepsilon(w) \ = \begin{cases} \displaystyle\int_{\mathbb{R}} \left(\frac{1}{2}|\ddot{w}|^2 - fw\right) dx + \ \varepsilon\,\sharp(S_{\dot{w}}) \ + \ \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| & \text{if } w \in \mathcal{K} \\ +\infty \qquad \text{otherwise in } BH(\mathbb{R}) \end{cases} \tag{2.15}$$

where $\sharp$ denotes the counting measure.

Theorem 2.4. *Assume that $f \in L^\infty(\mathbb{R})$ satisfies (2.5) then $\mathcal{F}^\varepsilon$ achieves a finite minimum.*

Proof - By Lemma 2.2 we get, for any $w \in SBH$ such that $\operatorname{spt} w \subset [0,L]$,

$$\gamma\,\|v''\|_T - \int f v dx \ \ge\ \gamma\|v''\|_T - \|f\|_{L^\infty}\|v\|_{L^1} \ >\ \left(\gamma - \|f\|_{L^\infty}\frac{L^2}{16}\right)\|v''\|_T\ . \tag{2.16}$$

Then we apply the direct method as like as in [16] (Lemma 3.2) and we get the thesis. The only difference with respect to [16] is the use of $L^1{-}BH$ Poincaré inequality instead of $L^1{-}BH$ Poincaré inequality to get inequality (2.16). ■

Remark 2.5. *The safe load constant $16\gamma/L^2$ in (2.5) is sharp for load in L^∞.*
In fact, for any $\delta > 0$, there are examples with $\|f\|_{L^\infty} = 16\gamma/L^2 + \delta$ and $\inf \mathcal{F} = \inf \mathcal{F}^\varepsilon = -\infty$: e.g. the choices

$$f \;\equiv\; 16\,\frac{\gamma}{L^2} \,+\, \delta\,, \qquad z_t(x) \;=\; t\,\left(\frac{L}{2} - \left| x - \frac{L}{2} \right|\right)^+$$

entail

$$S_{\dot z_t} \;=\; \{\,0\,,\,L/2\,,\,L\,\}\,, \quad \ddot z_t \;\equiv\; 0, \quad \gamma \sum_{S_{\dot z_t}} |[\dot z_t]| = 4\gamma t \qquad \int_0^L f\,z_t\,dx \;=\; t\left(4\gamma + L^2\delta/4\right)$$

$$\mathcal{F}^\varepsilon(z_t) \;=\; 3\,\varepsilon \,-\, \delta\,\frac{L^2}{4}\,t \;\to\; -\infty \qquad \text{as } t \to +\infty,\ \forall \varepsilon \geq 0\,.$$

Remark 2.6. *The L^∞ safe load condition (2.5) in Theorem 2.1 is weaker (for bounded load) than the safe load for measure load (3.3) of* [16] *or (3.2) of* [15]*: say $|f|_T < 8\gamma/L$.*

The following result has been proven in Theorem 4.1 of [15] and provides a bound on hinges number for minimizers of $\mathcal{F}_\varepsilon$ which is independent of ε.

Theorem 2.7. *Assume (2.5) and let $v \in \operatorname{argmin} \mathcal{F}^\varepsilon$. Then $\sharp(S_{v'}) \leq 2$.*

We define $\mathcal{F}^* : BH(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$

$$\mathcal{F}^*(w) = \begin{cases} \displaystyle\int_{\mathbb{R}} (\varphi^{**}(\ddot w) - f w)\,dx \,+\, \gamma \sum_{S_{\dot w}} |[\dot w]| \,+\, \gamma \|(w'')^c\|_T & \text{if } w \in \mathcal{K}^* \\ +\infty \qquad \text{otherwise in } BH(\mathbb{R}) \end{cases} \tag{2.17}$$

where φ^{**} is the convex envelope of

$$\varphi(s) \;=\; \min\left\{\frac{s^2}{2}, \gamma|s|\right\} \tag{2.18}$$

that is

$$\varphi^{**}(s) = \begin{cases} \dfrac{s^2}{2} & \text{if } |s| \leq \gamma \\ \gamma|s| - \dfrac{\gamma^2}{2} & \text{otherwise.} \end{cases} \tag{2.19}$$

Theorem 2.8.

$$\Gamma\,(w^*\,BH)\,\lim_{\varepsilon\to 0} \mathcal{F}^\varepsilon(w) \;=\; \mathcal{F}^*(w)\,. \tag{2.20}$$

Proof - (*Lower bound*) We claim that for any $\varepsilon_n \to 0$, $w \in \mathcal{K}^*$, $w_n \in \mathcal{K}$, $w_n \overset{w^*\,BH}{\longrightarrow} w$

$$\liminf_n \ \mathcal{F}^{\varepsilon_n}(w_n) \;\geq\; \mathcal{F}^*(w)\,. \tag{2.21}$$

It is not restrictive assuming $\mathcal{F}^{\varepsilon_n}(w_n) < +\infty$, hence w_{ε_n}, w belong to SBH and have support in $[0, L]$.

Since $\varepsilon > 0$ and $\varphi \geq \varphi^{**}$, we get

$$\mathcal{F}^{\varepsilon_n}(w_n) \geq \mathcal{F}(w_n) \geq \int_0^L \varphi^{**}(\ddot{w}_n)\,dx + \gamma \sum_{S_{\dot{w}_n}} |[\dot{w}_n]| - \int_0^L f w_n\,dx = \mathcal{F}^*(w_n). \quad (2.22)$$

Since γ is the recession value at infinity for φ^{**} then $\mathcal{F}^*$ is seq. l.s.c. ([1], Th.5.2). By taking the liminf in (2.22) we obtain (2.21).

(*Recovery sequence*) Select $w \in \mathcal{K}^*$ such that $(\dot{w})_a$ is a continuous piecewise affine function. Recall that $\ddot{w} = (\dot{w})'_a(x)$; so that $|(\dot{w})'_a(x)| > \gamma$ if and only if x belongs to $\in \cup_{i=1}^p (a_i, b_i)$ where $(a_i, b_i) \subset [0, L]$. Set $l_i = b_i - a_i$, $I = [0, L] \setminus \cup_{i=1}^p (a_i, b_i)$.
If the intervals (a_i, b_i) are empty then the following approximation procedure is not necessary; otherwise choose $h_\varepsilon \to +\infty$ such that $\varepsilon h_\varepsilon \to 0$ as $\varepsilon \to 0_+$, fix an arbitrary choice of $s_{ij} \in (a_i + h_\varepsilon^{-1} j l_i, a_i + h_\varepsilon^{-1}(j+1) l_i)$ and $t_j \in [h_\varepsilon^{-1} j\, L, h_\varepsilon^{-1}(j+1)\, L)$. Then, by labelling $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ else (for any $E \subset \mathbb{R}$), we define, for a.e. $x \in [0, L]$,

$$\begin{aligned}
\theta_\varepsilon(x) &= (\dot{w})_a(x)\mathbf{1}_I(x) + \\
&+ \sum_{i=1}^p \left(\sum_{j=0}^{h_\varepsilon - 1} (w')_a(s_{ij}) \mathbf{1}_{(a_i + l_i h_\varepsilon^{-1} j,\, a_i + l_i h_\varepsilon^{-1}(j+1))}(x) \right) \mathbf{1}_{(a_i, b_i)}(x) + \\
&+ (\dot{w})_j(x) + \sum_{j=0}^{h_\varepsilon - 1} (w')_c(t_j) \mathbf{1}_{[L h_\varepsilon^{-1} j,\, L h_\varepsilon^{-1}(j+1))}(x)
\end{aligned}$$

so that $|(\vartheta_\varepsilon)'_a| \leq \gamma$ a.e. Moreover define $w_\varepsilon \in \mathcal{K}$ by $w_\varepsilon(0) = 0$ and

$$w'_\varepsilon(x) = \theta_\varepsilon(x) - \frac{1}{L}\int_0^L \theta_\varepsilon(x)\,dx \ \text{ if } x \in [0, L]\,, \qquad w'_\varepsilon(x) = 0 \ \text{ if } x \notin [0, L].$$

Then $|\ddot{w}_\varepsilon| \leq \gamma$ a.e. and

$$\varepsilon \sharp(S_{\dot{w}_\varepsilon}) \leq \varepsilon \sharp(S_{\dot{w}}) + \varepsilon(h_\varepsilon + 1)(1 + p) \to 0 \quad (2.23)$$

$$\begin{aligned}
&\sum_{S_{\dot{w}_\varepsilon}} |[\dot{w}_\varepsilon]| = \\
&= \sum_{S_{\dot{w}}} |[\dot{w}]| + \sum_{j=0}^{h_\varepsilon - 1} |(w')_c(t_j) - (w')_c(t_{j-1})| + \\
&\qquad + \sum_{i=1}^p \sum_{j=0}^{h_\varepsilon - 1} |(w')_a(s_{ij}) - (w')_a(s_{i,j-1})| \leq \qquad (2.24) \\
&\leq \sum_{S_{\dot{w}}} |[\dot{w}]| + \|(w'')^c\|_T + \int_{\cup_{i=1}^p (a_i, b_i)} |(\dot{w}_a)'(x)|\,dx \leq \\
&\leq \sum_{S_{\dot{w}}} |[\dot{w}]| + \|(w'')^c\|_T + \int_{\cup_{i=1}^p (a_i, b_i)} |\ddot{w}(x)|\,dx \ \leq \ \|w''\|_T\,.
\end{aligned}$$

Then

$$w_\varepsilon \rightharpoonup w \ \text{ in } w^* BH\,. \quad (2.25)$$

Moreover, since $\ddot{w}_\varepsilon = \ddot{w}$ a.e. in I and $\ddot{w}_\varepsilon = 0$ in $\cup_i (a_i, b_i)$, we obtain

$$\begin{aligned}
\mathcal{F}^\varepsilon(w_\varepsilon) \;&\leq\; \int_I \frac{1}{2}\,|\ddot{w}|^2\; dx \;+\; \int_{\cup_{i=1}^p (a_i,b_i)} \gamma |\ddot{w}|\, dx \;-\; \int_0^L f w\, dx \,+ \\
&\quad + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| \;+\; \gamma\, \|(w'')^c\|_T \;+\; \varepsilon\, \sharp(S_{\dot{w}_\varepsilon}) \;-\; \int_0^L f(w_\varepsilon - w)\, dx \\
&= \int_0^L (\varphi(\ddot{w}) - f w)\; dx \,+ \\
&\quad + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| \;+\; \gamma\, \|(w'')^c\|_T \;+\; \varepsilon\, \sharp(S_{\dot{w}_\varepsilon}) \;-\; \int_0^L f(w_\varepsilon - w)\, dx
\end{aligned}$$

and by taking the lim sup of both sides, by (2.23) and (2.25) we get

$$\limsup \mathcal{F}^\varepsilon(w_\varepsilon) \leq \int_0^L \varphi(\ddot{w})\, dx - \int_0^L f w\, dx + \gamma \|(w'')^c\|_T + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]| \;. \tag{2.26}$$

So, by referring to notion of sequential Γ $\limsup$ of the family $\mathcal{F}^\varepsilon$

$$\mathcal{F}^*_+(w) \;=\; \inf\{\, \limsup \mathcal{F}^\varepsilon(w_\varepsilon) :\; w_\varepsilon \rightharpoonup w \;\, w^*BH \} \qquad \forall w \in \mathcal{K}^*\,,$$

we have proven that for every $w \in \mathcal{K}^*$ with continuous piecewise affine $(\dot{w})_a$

$$\mathcal{F}^*_+(w) \leq \int_0^L \varphi(\ddot{w})\, dx - \int_0^L f w\, dx + \gamma \|(w'')^c\|_T + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]|. \tag{2.27}$$

Choose $w \in \mathcal{K}^*$. Then there exists a sequence of continuous piecewise affine functions $\sigma_h \to (\dot{w})_a$ in $W^{1,1}(0,L)$. We set

$$z_h(x) \;=\; \sigma_h(x) \;+\; (\dot{w})_j(x) \;+\; (\dot{w})_c(x)$$

and we define $w_h \in \mathcal{K}^*$ by setting $w_h(0) = 0,\;\, w_h \equiv 0$ in $\mathbb{R} \setminus (0,L)$ and

$$\dot{w}_h(x) = z_h(x) - \frac{1}{L} \int_0^L z_h \qquad \text{a.e. } x \in [0,L]\,.$$

We have $w_h \to w$ strongly in BH. By recalling that $\mathcal{F}^*_+$ is sequentially l.s.c. in the w^*BH convergence ([8]), definition (2.16) together with $(w''_h)^c = (w'')^c$ and $(\dot{w}_h)_j = (\dot{w})_j$ yield

$$\begin{aligned}
&\mathcal{F}^*_+(w) \leq \liminf \mathcal{F}^*_+(w_h) \leq \\
&\leq \liminf \int_0^L \varphi(\ddot{w}_h)\, dx - \int_0^L f\, w_h\, dx + \gamma \|(w''_h)\|_T + \gamma \sum_{S_{\dot{w}_h}} |[\dot{w}_h]| = \\
&= \;\liminf \int_0^L \varphi(\ddot{w}_h)\, dx - \int_0^L f\, w_h\, dx + \gamma \|(w'')\|_T + \gamma \sum_{S_{\dot{w}_h}} |[\dot{w}]| = \\
&= \int_0^L \varphi(\ddot{w})\, dx - \int_0^L f w\, dx + \gamma \|(w'')^c\|_T + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]|.
\end{aligned} \tag{2.28}$$

By a classic relaxation result ([2] Th. 2.6.4, p.74) for every $w \in \mathcal{K}^*$ there exists a sequence $\zeta_h \in W^{1,1}(0,L)$ such that $\zeta_h \rightharpoonup \dot{w}_a$ weakly in $W^{1,1}$ and ,by $(\dot{w}_a)' = \ddot{w}$ a.e.,

$$\int_0^L \varphi(\dot{\zeta}_h)\,dx \to \int_0^L \varphi^{**}((\dot{w}_a)')\,dx = \int_0^L \varphi^{**}(\ddot{w})\,dx = \int_{\mathbb{R}} \varphi^{**}(\ddot{w})\,dx.$$

Now set $v_h(0) = 0,\ v_h \equiv 0$ in $\mathbb{R} \setminus (0,L)$ and for a.e. $x \in (0,L)$

$$v_h'(x) \ = \ \zeta_h(x) \ + \ (\dot{w})_j(x) \ + \ (\dot{w})_c(x) - \ \frac{1}{L}\int_0^L (\zeta_h + \dot{w}_j + \dot{w}_c)\,dx\,.$$

Then $v_h \in \mathcal{K}^*,\ v_h \rightharpoonup w$ in w^*BH. We exploit sequential lower semicontinuity of $\mathcal{F}_+^*$ once more by evaluating (2.28) at v_h, we get

$$\mathcal{F}_+^*(v_h) \ \leq \ \int_0^L \varphi(\ddot{w})\,dx - \int_0^L fw\,dx + \gamma\|(w'')^c\|_T + \gamma\sum_{S_{\dot{w}}}|[\dot{w}]|$$

$$\mathcal{F}_+^*(w) \ \leq \ \ \liminf \mathcal{F}_+^*(v_h) \ =$$

$$= \int_0^L \varphi^{**}(\ddot{w})\,dx \ - \ \int_0^L fw\,dx \ + \ \gamma\|(w'')^c\|_T \ + \ \gamma\sum_{S_{\dot{w}}}|[\dot{w}]| \ = \ \mathcal{F}^*(w).$$

The above inequality and (2.21) together entail (2.20). ■

Theorem 2.9. *(***Relaxed functional***)*

$$sc^-\mathcal{F}(w) \ = \ \Gamma(w^*\,BH)\lim_{\varepsilon\to 0}\ \mathcal{F}^\varepsilon(w) \ \ = \ \ \mathcal{F}^*(w) \tag{2.29}$$

Proof - The same proof of (2.8) apply to this case without any restriction in the choice of the sequence $h_{\varepsilon_n} \to +\infty$. ■

We define

$$\widetilde{\mathcal{F}}(w) = \begin{cases} \mathcal{F}(w) & \text{if } \sharp(S_{\dot{w}}) \leq 2 \\ \\ +\infty & \text{otherwise}\,. \end{cases} \tag{2.30}$$

$$\widetilde{\mathcal{F}^\varepsilon}(w) = \begin{cases} \mathcal{F}^\varepsilon(w) & \text{if } \sharp(S_{\dot{w}}) \leq 2 \\ \\ +\infty & \text{otherwise} \end{cases} \tag{2.31}$$

and show the following statement.

Theorem 2.10.

$$\Gamma(w^*BH)\lim \widetilde{\mathcal{F}^\varepsilon}(w) \ = \ \widetilde{\mathcal{F}}(w)$$

Proof - (*Lower bound*) Let $w_\varepsilon \in BH(\mathbb{R}),\ w_\varepsilon \rightharpoonup w$ in $w^* - BH$ and assume without restriction that $\widetilde{\mathcal{F}^\varepsilon}(w_\varepsilon) \leq C$. Since $w_\varepsilon \in SBH$ and $\sharp(S_{\dot{w}_\varepsilon}) \leq 2$, we get $\sharp(S_{\dot{w}}) \leq 2$. Then by applying semicontinuity Theorem 4.7 in [1] to the functional $w \to \widetilde{\mathcal{F}}(w) + \varepsilon\sharp(S_{\dot{w}})$, we get

$$\liminf \widetilde{\mathcal{F}^\varepsilon}(w_\varepsilon) \ \geq \ \liminf \widetilde{\mathcal{F}}(w_\varepsilon) \ \geq \ \widetilde{\mathcal{F}}(w) \ = \ \mathcal{F}(w)\,.$$

(*Upper bound*) For every $w \in BH$

$$\lim \widetilde{\mathcal{F}}^{\varepsilon}(w) = \widetilde{\mathcal{F}}(w)\,.$$

Then the proof is achieved. ■

Now we can prove the main result of this section.

Proof of Theorem 2.1 Let $v_\varepsilon \in \operatorname{argmin} \mathcal{F}^\varepsilon$, then $\mathcal{F}^\varepsilon(v_\varepsilon) \le \mathcal{F}^\varepsilon(0) = 0$, hence by Lemma 2.2

$$\begin{aligned} &\frac{1}{2}\int_{\mathbb{R}} |\ddot{v}_\varepsilon|^2\,dx + \varepsilon\sharp(S_{\dot{v}_\varepsilon}) + \gamma \sum_{S_{\dot{v}_\varepsilon}} |[\dot{v}_\varepsilon]| \;\le\; \int_{\mathbb{R}} f v_\varepsilon\,dx \;\le \\ &\le \|f\|_{L^\infty}\|v_\varepsilon\|_{L^1} \le \frac{L^2}{16}\|f\|_{L^\infty}\left\{\int_{\mathbb{R}} |\ddot{v}_\varepsilon|\,dx + \sum_{S_{\dot{v}_\varepsilon}} |[\dot{v}_\varepsilon]|\right\} \le \\ &\le\; \frac{L^5}{256}\,\|f\|^2_{L^\infty} + \frac{1}{4}\int_{\mathbb{R}} |\ddot{v}_\varepsilon|^2\,dx \,+\, \frac{L^2}{16}\,\|f\|_{L^\infty}\sum_{S_{\dot{v}_\varepsilon}} |[\dot{v}_\varepsilon]|. \end{aligned} \tag{2.32}$$

By (2.32) and (2.5) we get

$$\int_{\mathbb{R}} |\ddot{v}_\varepsilon|^2\,dx \le C; \qquad \sum_{S_{\dot{v}_\varepsilon}} |[\dot{v}_\varepsilon]| \le C \tag{2.33}$$

where $C > 0$ is independent of ε and, by Theorem 2.7, $\sharp(S_{\dot{v}_\varepsilon}) \le 2$.
Then by applying compactness Theorem 4.8 in [1] to the sequence $\dot{v}_\varepsilon$ in $SBV(\mathbb{R})$ we get, up to subsequences, that $v_\varepsilon \to v \in SBH(\mathbb{R})$ in w^*BH, $v \in \mathcal{K}$ and $\sharp(S_{\dot{v}}) \le 2$. Moreover $v \in \operatorname{argmin} \mathcal{F}^*$ and $\inf \mathcal{F} = \mathcal{F}^*(v)$ due to Theorems 2.8, 2.9
Theorem 2.10, $\sharp(S_{\dot{v}}) \le 2$ and $\min \mathcal{F}^\varepsilon = \min \widetilde{\mathcal{F}}^\varepsilon$ entail $\inf \mathcal{F} = \min \widetilde{\mathcal{F}} = \mathcal{F}(v) = \min \mathcal{F}$.
Hence $\mathcal{F}$ achieves a finite minimum and at least one among its minimizers fulfils $\sharp(S_{\dot{w}}) \le 2$.
By relaxation (Theorem 2.9)

$$\min \mathcal{F} = \min \mathcal{F}^*\,, \qquad \operatorname{argmin} \mathcal{F} \;\subset\; \operatorname{argmin} \mathcal{F}^*\,. \tag{2.34}$$

Then $(w'')^c \equiv 0$ for any $w \in \operatorname{argmin} \mathcal{F}$.
From $\varphi^{**}(\ddot{w}) = (\ddot{w})^2/2$ in the set $\{|\ddot{w}| \le \gamma\}$ and $\mathcal{F}^*(w) = \mathcal{F}(w)$ for any $w \in \operatorname{argmin}\mathcal{F}$, we get

$$\int_{[0,L]\cap\{|\ddot{w}|>\gamma\}} \varphi^{**}(\ddot{w})\,dx \;=\; \int_{[0,L]\cap\{|\ddot{w}|>\gamma\}} \frac{1}{2}|\ddot{w}|^2\,dx \tag{2.35}$$

by substitution of (2.19)

$$0 \;=\; \int_{[0,L]\cap\{|\ddot{w}|>\gamma\}} \left(-\frac{1}{2}|\ddot{w}|^2 + \gamma|\ddot{w}| - \frac{\gamma^2}{2}\right) \;=\; -\int_{[0,L]\cap\{|\ddot{w}|>\gamma\}} \frac{1}{2}\left(|\ddot{w}|^2 - \gamma\right)^2 \tag{2.36}$$

say

$$|\,\{x :\; |\ddot{w}(x)| > \gamma\}\,| \;=\; 0\,. \tag{2.37}$$

Then (2.6) is proven. ■

Remark 2.11. *About Theorem 2.1 we notice that in order to achieve only the existence of a minimizer for $\mathcal{F}$ (without additional information) a shorter proof is the following one.*

Let $v_\varepsilon \in \operatorname{argmin} \mathcal{F}^\varepsilon$, then $\sharp(S_{\dot v_\varepsilon}) \le 2$ by Theorem 2.6. By reasoning as in the deduction of (2.32),(2.33) and exploiting Theorems 4.7 and 4.8 of [1] we find a subsequence s.t. (without relabelling) $v_\varepsilon \to v$ in $w^*\, BH$, with $v \in SBH(\mathbb{R})$ and $\sharp(S_{\dot v_\varepsilon}) \le 2$; moreover

$$\liminf_{\varepsilon\to 0} \int_0^L |\ddot v_\varepsilon|^2\,dx \;\ge\; \int_0^L |\ddot v|^2\,dx$$

$$\liminf_{\varepsilon\to 0} \sum_{S_{\dot v_\varepsilon}} |[\dot v_\varepsilon]| \;\ge\; \sum_{S_{\dot v}} |[\dot v]|$$

and by $\varepsilon\,\sharp(S_{\dot v_\varepsilon}) \to 0$ we get

$$\liminf_{\varepsilon\to 0} \; \mathcal{F}^\varepsilon(v_\varepsilon) \;\ge\; \mathcal{F}(v)\,.$$

If w is any other admissible function in $\mathcal{K}$ we get

$$\mathcal{F}(w) \;=\; \liminf_{\varepsilon\to 0}\; \mathcal{F}^\varepsilon(w) \;\ge\; \liminf_{\varepsilon\to 0}\; \mathcal{F}^\varepsilon(v_\varepsilon)$$

hence $\mathcal{F}(v) \le \mathcal{F}(w)\;\; \forall w \in \mathcal{K}$.

3. Clamped elastic-plastic beam: structure of minimizer

In this section we deduce sharp regularity conditions and structure properties for regular and non regular minimizers of $\mathcal{F}$ by suitable estimates based on Green function.
We start by deducing a complete set of Euler equations: a differential relationship in $(0, L)$ and Weierstrass-Erdmann type corner conditions at singular set, as shown by the following statement.

Theorem 3.1. *($\mathcal{F}$ **Euler-Lagrange equations**) Assume (2.5) and $w \in \operatorname{argmin}\mathcal{F}$. Then*

$$(\ddot w)'' \;=\; f \qquad \textit{in } (0,L) \tag{3.1}$$

$$\ddot w_-(x) \;=\; \gamma\,\operatorname{sign}([\dot w])(x) \qquad \textit{in } S_{\dot w}\cap(0,L] \tag{3.2}$$

$$\ddot w_+(x) \;=\; \gamma\,\operatorname{sign}([\dot w])(x) \qquad \textit{in } S_{\dot w}\cap[0,L) \tag{3.3}$$

In particular $\ddot w \in H^2(0,L)$, hence $\ddot w$ and $\dddot w = (\ddot w)'$ are continuous in $[0, L]$.

Proof - Let $\varphi \in C^2_0(\mathbb{R}),\ \operatorname{spt}\varphi \subset [0, L]$: the first variation of $\mathcal{F}$ yields

$$\int_0^L \{\ddot w\varphi'' - f\varphi\}\,dx = 0$$

thus proving (3.1) after integrating by parts twice; hence $\ddot w$ belongs to $H^2(0, L)$.
For any point $\overline{x} \in S_{\dot w}$ and any $\varphi \in BH(\mathbb{R}) \cap C^2(\mathbb{R}\setminus\{\overline{x}\}),\ \operatorname{spt}\varphi \subset [0, L]$, by recalling that $w \in SBH$, the first variation of $\mathcal{F}$ yields

$$0 = \int_0^{\overline{x}} \{\ddot w\varphi'' - f\varphi\}\,dx + \int_{\overline{x}}^L \{\ddot w\varphi'' - f\varphi\}\,dx + \gamma\,\operatorname{sign}([\dot w])(\overline{x})\,[\dot\varphi](\overline{x}) =$$

$$= -\int_0^L \{\ddot{w}\dot{\varphi} + f\varphi\}\, dx - \ddot{w}(\overline{x})[\dot{\varphi}](\overline{x}) + \gamma\,\mathrm{sign}([\dot{w}])(\overline{x})\,[\dot{\varphi}](\overline{x}) =$$

$$= + \int_0^L \{(\ddot{w})''\varphi - f\varphi\}\, dx + [\dddot{w}](\overline{x})\,\varphi(\overline{x}) - \ddot{w}(\overline{x})\,[\dot{\varphi}](\overline{x}) + \gamma\,\mathrm{sign}([\dot{w}])(\overline{x})\,[\dot{\varphi}](\overline{x})$$

and (3.2), (3.3) follow by (3.1).■

Theorem 3.2. *($\mathcal{F}$* **Compliance identity***) Assume (2.5). Then the following identity holds true for any w in $SBH(\mathbb{R})$ satisfying Euler-Lagrange equations (3.1)-(3.3) and* $\mathrm{spt}\, w \subset [0, L]$ *:*

$$\mathcal{F}(w) \;=\; -\frac{1}{2}\int_0^L |\ddot{w}|^2\, dx\,. \tag{3.4}$$

In particular any $w \in \operatorname{argmin} \mathcal{F}$ fulfills (3.4).

Proof - By (3.1) we have $(\ddot{w})'' = f$ in $\mathcal{D}'(0, L)$.
Then by taking into account $w(0) = w(L) = 0$ and

$$w'' = \ddot{w} + \sum_{S_{\dot{w}}\cap(0,L)} [\dot{w}]\, d\,\sharp \llcorner S_{\dot{w}} \qquad \text{in } \mathcal{D}'(0, L)$$

we get

$$\int_{\mathbb{R}} fw\, dx \;=\; \int_0^L fw\, dx \;=\; \int_0^L (\ddot{w})''w\, dx \;=\; -\int_0^L (\ddot{w})'w'\, dx \;=$$

$$= \int_0^L \ddot{w}\, d(w'') - \; \ddot{w}_-(L)\,\dot{w}_-(L) \;+\; \ddot{w}_+(0)\,\dot{w}_+(0) \;=$$

$$= \int_0^L |\ddot{w}|^2\, dx + \sum_{S_{\dot{w}}\cap(0,L)} \ddot{w}[\dot{w}] \;+\; \ddot{w}_-(L)\,[\dot{w}](L) \;+\; \ddot{w}_+(0)\,[\dot{w}](0)\,.$$

By substitution of (3.2),(3.3):

$$\int_0^L fw\, dx = \int_0^L |\ddot{w}|^2 + \gamma \sum_{S_{\dot{w}}} |[\dot{w}]|$$

and thesis follows by the definition of $\mathcal{F}$. ■

A straightforward consequence of compliance identity is the following remark about the relevant structure of minimizers.

Lemma 3.3. *Any $w \in \operatorname{argmin} \mathcal{F}$ can be uniquely decomposed (thank to (3.1)) as follows*

$$w \;=\; u + v\,, \tag{3.5}$$

where u is the solution of (3.9) and

$$v \in SBH(\mathbb{R})\,, \quad \mathrm{spt}\, v \subset [0, L]\,, \quad (\ddot{v})'' = 0 \text{ in } (0, L)\ . \tag{3.6}$$

Hence, for suitable a, $b \in \mathbb{R}$,

$$\ddot{v}(x) \;=\; (ax + b)\,\mathbf{1}_{(0,L)}(x) \tag{3.7}$$

and, by $\int_0^L u''\,\ddot v = \int_0^L u\,(\ddot v)'' = 0$ *and the compliance identity,*

$$\begin{aligned}\mathcal{F}(w) &= -\frac{1}{2}\int_0^L |\ddot w|^2 = -\frac{1}{2}\int_0^L |u''|^2 - \int_0^L u''\,\ddot v - \frac{1}{2}\int_0^L |\ddot v|^2 = \\ &= -\frac{1}{2}\int_0^L |u''|^2 - \frac{1}{2}\int_0^L |\ddot v|^2 = -\frac{1}{2}\int_0^L |u''|^2 - \left(\frac{a^2}{6}L^3 + \frac{ab}{2}L^2 + \frac{b^2}{2}L\right).\end{aligned} \tag{3.8}$$

∎

We proceed by recalling some regularity results concerning minimizers of $\mathcal{F}$ which are in the same spirit of those proven in ([15]) and ([16]).

Theorem 3.4. *(*L^∞ **bending moment regularity condition for clamped beam***)*
Let u be the unique solution of

$$\begin{cases} u \in H^2(\mathbb{R}) \\ u'''' = f \qquad \textit{in } (0,L) \\ \operatorname{spt} u \subset [0,L]\,. \end{cases} \tag{3.9}$$

If

$$\|u''\|_{L^\infty(0,L)} \le \gamma \tag{3.10}$$

then u is a minimizer of $\mathcal{F}$.
If the inequality in (3.10) is strict then the minimizer is unique.

Proof - The proof could be deduced by the same argument (here $\beta = 0$) in the proofs of Theorem 3.6 of [16] and related excess estimate (Lemma 3.5 of [16]), where β strictly positive does not paly any role. For reader convenience we make it explicit in the present simpler situation as follows.
Let u solves (3.9) and choose $v \in \mathcal{K}$. Then

$$\ddot v = v'' - [\dot v]\, d\sharp \llcorner S_{\dot v}\,, \qquad u'''' = f \text{ in } (0,L)$$

and by exploiting convexity, $u'' \in C^0([0,L])$, $u(0) = u(L) = v(0) = v(L) = 0$, we get

$$\begin{aligned}\mathcal{F}(v) \;&\ge\; \mathcal{F}(u) + \int_0^L u''(\ddot v - u'')dx - \int_0^L f(v-u)dx + \gamma \sum_{S_{\dot v}} |[\dot v]| = \\ &=\; \mathcal{F}(u) + \int_0^L u''(v'' - u'')dx - \int_0^L f(v-u)dx + \sum_{S_{\dot v}} (\gamma|[\dot v]| - u''[\dot v]) = \\ &= \mathcal{F}(u) + \textstyle\sum_{S_{\dot v}} (\gamma|[v]| - u''[v])\end{aligned} \tag{3.11}$$

Inequalities (3.10) and (3.11) entail

$$\mathcal{F}(v) \;\ge\; \mathcal{F}(u) + \sum_{S_{\dot v}} (\gamma|[\dot v]| - u''[\dot v]) \tag{3.12}$$

so that $\|u''\|_{L^\infty(\mathbb{R})} \le \gamma$ entails u is a minimizer and if in addition $\|u''\|_{L^\infty(\mathbb{R})} < \gamma$ then no minimizer can have creases. ∎

From now on u will always denote the function introduced by (3.9) in Theorem 3.4.
The following result provides a sharp condition on the external load f which in turn implies (3.10) and hence regularity of minimizers.

Theorem 3.5. *(**L^∞ load regularity condition for clamped beam**) Assume*

$$\|f\|_{L^\infty(0,L)} \leq \frac{12\gamma}{L^2} \tag{3.13}$$

then $\mathcal{F}$ achieves its finite minimum when evaluated at the solution u of (3.9) (which has no crease: $S_{\dot u} = \emptyset$).
Moreover (3.13) entails that u is the unique minimizer of $\mathcal{F}$.

Before proving Theorem 3.5 we state a useful representation formula and an estimate for u'' in the following Lemma.

Lemma 3.6. *Let u be the unique solution of problem (3.9). Then*

$$u''(x) \ = \ \int_0^L K(x,y)\, f(y)\ dy \tag{3.14}$$

where for $x, y \in [0, L]$

$$K(x,y) = \frac{1}{2\,L^3}(2\,x - L)\ y^2\,(3L - 2y)\ -\ \frac{1}{2\,L}\,y^2\ +\ (y - x)^+\,. \tag{3.15}$$

Moreover

$$\max_{x\in[0,L]} \int_0^L |K(x,y)|\,dy \ = \ \frac{L^2}{12}\,, \tag{3.16}$$

$$\|u''\|_{L^\infty} \ \leq \ \frac{L^2}{12}\|f\|_{L^\infty}\,. \tag{3.17}$$

Proof - Let u be the unique solution of problem (3.9). Then by direct computation (see [16] Lemma 3.10) via Green function $\mathcal{G}$ for the operator d^4/dy^4 in $(0, L)$ with homogeneous boundary conditions ($\mathcal{G}_{yy}(x, y) = \delta(x - y),\ \mathcal{G}(x, 0) = \mathcal{G}(x, L) = \mathcal{G}_y(x, 0) = \mathcal{G}_y(x, L)$) we get (3.14),(3.15) hence, by substitution,

$$K(x,y) \ = \ K(L-x, L-y) \qquad \forall\, x, y \ \in \ [0, L]\,. \tag{3.18}$$

Since $K(x, L) = 0$ and

$$K_y(x,y) \ = \ (L-y)\,\left(\frac{3y\,(2x-L)}{L^3} \ + 1\right) \ \geq \ 0 \qquad \text{for } L/2 \ \leq \ x \ \leq \ y \ \leq \ L$$

we get

$$K(x,y) \ \leq \ 0 \qquad \text{for } L/2 \ \leq \ x \ \leq \ y \ \leq \ L\,. \tag{3.19}$$

Moreover, if $x \geq L/2,\ 0 \leq y \leq x$, then

$$K(x,y) \ = \ \frac{y^2}{2\,L^3}\left((2x-L)(3L-2y) - L^2\right) \ \leq \ \frac{y^2}{2\,L^3}\,2\,(3Lx - 2L^2)$$

hence

$$K(x,y) \ \leq \ 0 \qquad \text{for } L/2 \ \leq \ x \ \leq \ 2L/3\,,\ 0 \ \leq \ y \ \leq \ x\,. \tag{3.20}$$

Eventually

$$\begin{array}{c} K(x,y) \;=\; \dfrac{y^2}{2\,L^3}\,\Big(2\,(3Lx-2L^2)\;-\;2y\,(2x-L)\Big)\leq 0 \\ \text{for} \;\; x\geq 2L/3\,, \quad \dfrac{3Lx-2L^2}{2x-L}\leq y\leq x\,, \end{array} \tag{3.21}$$

while

$$\begin{array}{c} K(x,y) \;=\; \dfrac{y^2}{2\,L^3}\,\Big(2\,(3Lx-2L^2)\;-\;2y\,(2x-L)\Big)\;\geq\;0 \\ \text{for} \;\; x\;\geq\;2L/3\,, \;\; 0\;\leq\;y\;\leq\;\dfrac{3Lx-2L^2}{2x-L}\;\leq\;x\,. \end{array} \tag{3.22}$$

Thanks to (3.18),(3.19),(3.20),(3.21),(3.22) we have

$$\int_0^L |K(x,y)|\,dy \;=\; \int_0^L -K(x,y)\,dy \;=\; \frac{1}{12}\Big(6Lx-6x^2-L^2\Big) \qquad \text{if } \frac{L}{2}\leq x\leq\frac{2L}{3}\,,$$

$$\int_0^{\frac{3Lx-2L^2}{2x-L}} K(x,y)\,dy + \int_{\frac{3Lx-2L^2}{2x-L}}^1 -K(x,y)\,dy \;=\; \frac{L\,(3x-2L)^4}{6\,(2x-L)^3}+\frac{1}{12}(6Lx-6x^2-L^2)$$

$$\text{if } \frac{2L}{3}\leq x\leq L.$$

By (3.18) we have

$$\int_0^L |K(x,y)|\,dy = \int_0^L |K(L-x,L-y)|\,dy = \int_0^L |K(L-x,y)|\,dy$$

hence

$$\max_{x\in[0,L]}\int_0^L |K(x,y)|\,dy \;=\; \max_{x\in[L/2,L]}\int_0^L |K(x,y)|\,dy \;=\; \frac{L^2}{12}$$

say (3.16). Then (3.14),(3.16) together entail (3.17):

$$\|u''\|_{L^\infty} \;\leq\; \|f\|_{L^\infty}\,\max_{x\in[0,L]}\int_0^L |K(x,y)|\,dy \;=\; \frac{L^2}{12}\|f\|_{L^\infty}\,. \blacksquare \tag{3.23}$$

Proof of Theorem 3.5 - By (3.13),(3.17) and Theorem 3.4 we get the first part of thesis: that is u is a minimizer.

Now assume w is any minimizer of $\mathcal{F}$. Then, by referring to structural splitting (3.5) in Lemma 3.3 ($w=u+v$), we exploit compliance identity for the minimizer u and (3.8) to get

$$-\frac{1}{2}\int_0^L |u''|^2 \;=\; \mathcal{F}(u)=\mathcal{F}(w)= -\frac{1}{2}\int_0^L |u''|^2 - \;\frac{L}{6}\,(a^2L^2+3abL+3b^2)\,. \tag{3.24}$$

The positive definiteness of quadratic form in aL and b entails $a=b=0$.

Then $\ddot{v}\equiv 0$, $v''=[\dot{v}]d\sharp\llcorner S_{\dot{v}}$ and

$$\mathcal{F}(u)=\mathcal{F}(w)=\mathcal{F}(u)+\gamma\sum_{S_{\dot{v}}}|[\dot{v}]|-\int_0^L f v\,dx\,, \tag{3.25}$$

hence by (3.25),(3.13) and Poincaré inequality (Lemma 2.2)

$$\gamma \sum_{S_{\dot v}} |[\dot v]| = \int_0^L f v \, dx \le \|f\|_{L^\infty} \|v\|_{L^1} \le \frac{3\gamma}{4} \|v''\|_T = \frac{3\gamma}{4} \sum_{S_{\dot v}} |[\dot v]| \tag{3.26}$$

that is $\sum_{S_{\dot v}} |[\dot v]| = 0$. Hence (by $\operatorname{spt} v \subset [0,L]$) $v \equiv 0$ and u is the unique minimizer of $\mathcal{F}$. ■

Euler equations (3.1)-(3.3) together with strict sign of load entail a severe bound on the number of hinges allowed for a minimizer, as shown in the following statement.

Theorem 3.7. *Assume (2.5) and $f > 0$ a.e. in $[0,L]$ (or $f < 0$ a.e. in $[0,L]$). Then $\sharp(S_{\ddot w}) \le 2$ for any $w \in \operatorname{argmin} \mathcal{F}$. Moreover when $\sharp(S_{\ddot w}) = 2$ then at least one endpoint of $[0,L]$ belongs to $S_{\ddot w}$.*

Proof - By (3.1), (3.2), (3.3) we get $(\ddot w)'' = f$ in $(0,L)$ and $|\ddot w(x)| = \gamma$ for every $x \in S_{\ddot w}$. Inequality $f > 0$ entails that $\ddot w$ is strictly convex in $(0,L)$, then there exists unique $\bar x \in [0,L]$ such that $\ddot w(\bar x) = \min\{\ddot w(x) : x \in [0,L]\}$ and by estimate (2.6).

$$\|\ddot w\|_{L^\infty} \;=\; \max\{|\ddot w(0)|,\ |\ddot w(L)|,\ |\ddot w(\bar x)|\} \;\le\; \gamma\,. \tag{3.27}$$

Moreover by strict convexity $|\ddot w(x)| < \gamma$ for every $x \notin \{0, L, \bar x\}$. Suppose now by contradiction that $\sharp(S_{\ddot w}) > 2$: then $\bar x \in (0,L)$, $\sharp(S_{\ddot w}) = 3$, $S_{\ddot w} = \{0, L, \bar x\}$ and $\ddot w(0) = \ddot w(L) = \gamma = -\ddot w(\bar x)$.

Then $\eta(x) = \ddot w(x) - \gamma$ solves

$$\begin{cases} \eta'' = f \quad \text{in } (0,L) \\ \eta(0) = \eta(L) = 0 \end{cases} \tag{3.28}$$

If G is the Green function of d^2/dx^2 in $(0,L)$ with homogeneous boundary conditions ($G_{xx}(x,y) = \delta(x,y)$, $G(0,y) = G(L,y) = 0$), we have

$$G(x,y) \;=\; \frac{1}{L}\, y\,(x-L)\, \mathbf{1}_{\{y<x\}} \;+\; \frac{1}{L}\, x\,(y-L)\, \mathbf{1}_{\{y>x\}} \qquad x,y \in (0,L) \tag{3.29}$$

$$\eta(\bar x) \;=\; \ddot w(\bar x) - \gamma \;=\; \int_0^L G(\bar x, y) f(y)\, dy \tag{3.30}$$

and therefore by using the safe load condition (2.5)

$$|\,\ddot w(\bar x) \,-\, \gamma\,| \;<\; \frac{16\,\gamma}{L^2} \int_0^L |G(\bar x,y)|\, dy \;=\; \frac{16\,\gamma}{L^2}\, \frac{L}{2}\, \frac{\bar x}{L}\, (L - \bar x) \;\le\; 2\,\gamma \tag{3.31}$$

which contradicts $\ddot w(\bar x) = -\gamma$, so statement $\sharp(S_{\ddot w}) \le 2$ is proven.

If the equality $\sharp(S_{\ddot w}) = 2$ holds true, then $|\ddot w(x)| = \gamma$ for every $x \in S_{\ddot w}$ and $|\ddot w(x)| < \gamma$ for every $x \notin \{0, L, \bar x\}$ together entail that either 0 or L belongs to $S_{\ddot w}$. ■

Theorem 3.8. *Assume that $f \in L^\infty(\mathbb{R})$ satisfies (2.5),*

$$f > 0 \;\; \textit{a.e. in } [0,L]\,, \tag{3.32}$$

$$f(x) = f(L-x) \;\; \textit{a.e in } [0,L]\,. \tag{3.33}$$

Then the minimizer of $\mathcal{F}$ is unique and is given by

$$z(x) \;=\; u(x) \;+\; \frac{1}{2}\, (u''(0) - \gamma)^+\, x\, (L - x) \tag{3.34}$$

where u is the unique solution of (3.9) and

$$\mathcal{F}(z) \;=\; -\,\frac{1}{2}\int_0^L |u''|^2\,dx \;-\; \frac{L}{2}\,\left(\,(u''(0)-\gamma)^+\right)^2\,. \tag{3.35}$$

In any case $u''(0)=u''(L)=\|u''\|_{L^\infty}>0$. In particular:
if $0<u''(0)\le\gamma$ then $w=u$ (solution without hinges of problem (3.9));
if $u''(0)>\gamma$ (or equivalently $\|u''\|_{L^\infty}>\gamma$) then $S_{\dot z}=\{0,L\}$ (hinges at the boundary).

The proof of Theorem 3.8 will proceed in several steps: by Theorem 3.7 we know that any minimizer w fulfils $\sharp(S_{\dot w})\le 2$; we examine separately cases $\sharp(S_{\dot w})=0,1,2$, and conclude by matching them together.

Lemma 3.9. *Assume (2.5) and $u\in\operatorname{argmin}\mathcal{F}$, where u solves (3.9). Then*

$$\mathcal{F}(u) \;=\; -\,\frac{1}{2}\int_0^L |u''|^2\,dx\,. \quad \blacksquare \tag{3.36}$$

Proof - Straightforward consequence of Theorems 3.1 and 3.2.

Lemma 3.10. *Assume: $f\in L^\infty(\mathbb{R})$ fulfils (2.5), (3.32) and (3.33), u solves (3.9).*
Then $-2\gamma/3<u''(L/2)=\min u''\le\max u''=u''(0)=u''(L)$.
If in addition $u''(0)\le\gamma$ then: $u\in\operatorname{argmin}\mathcal{F}$ and (3.36) holds true. $\blacksquare$

Proof - By (3.32),(3.33) and $u''''=f$ in $(0,1)$ we deduce the strict convexity and symmetry for u'', hence $u''(L/2)=\min u''$ and $u''(0)=u''(L)=\max u''$.
By exploiting assumptions (3.32), (3.33) in the Green representation of u'' (see [15], formulae (3.12),(3.13)) we deduce

$$u''\left(\frac{L}{2}\right) \;=-\frac{1}{2\,L}\int_0^{\frac{L}{2}} y^2 f(y)dy-\frac{1}{2\,L}\int_{\frac{L}{2}}^L (y-L)^2 f(y)dy=\;-\frac{1}{L}\int_0^{L/2} y^2\,f(y)\,dy \tag{3.37}$$

hence (2.5) gives $-2\gamma/3<u''(L/2)$ and first part of thesis follows.
Then $u''(0)\le\gamma$ gives $\|u''\|_{L^\infty}\le\gamma$ and Theorem 3.4 entails $u\in\operatorname{argmin}\mathcal{F}$.
Eventually we can apply Lemma 3.9 to get (3.36). $\blacksquare$

Lemma 3.11. *Assume $f\in L^\infty(\mathbb{R})$ fulfils (2.5), (3.32), (3.33) and there is $w\in\operatorname{argmin}\mathcal{F}$ such that $\sharp(S_{\dot w})=2$.*
Then $u''(0)=u''(L)>\gamma$, $S_{\dot w}=\{0,L\}$, $w\;=\;u\,+\,1/2\,(u''(0)-\gamma)\,x(L-x)$ and

$$\mathcal{F}(w) \;=\; -\,\frac{1}{2}\int_0^L |u''|^2\,dx \;-\; \frac{L}{2}\,(u''(0)-\gamma)^2 \tag{3.38}$$

Proof - Let $w\in\operatorname{argmin}\mathcal{F}$ such that $\sharp(S_{\dot w})=2$.
By Lemma 3.10 and (3.33) we deduce $u''(0)=u''(L)>\gamma$.
By using the decomposition $w=u+v$ given in (3.5),(3.6), by taking into account boundary

conditions ($\operatorname{spt} w \subset [0,L]$) and by setting $S_{\dot w} = \{t_1, t_2\}$ we get

$$\frac{a}{2}\, L^2 \,+\, b\, L \,+\, [\dot w](t_1) \,+\, [\dot w](t_2) \,=\, 0 \tag{3.39}$$

$$\frac{a}{6}\, L^3 \,+\, \frac{b}{2}\, L^2 \,+\, (L-t_1)\, [\dot w](t_1) \,+\, (L-t_2)\, [\dot w](t_2) \,=\, 0 \tag{3.40}$$

Then Theorem 3.7 shows that either $S_{\dot w} = \{0, L\}$ or $S_{\dot w} = \{0, \bar x\}$ or $S_{\dot w} = \{\bar x, L\}$ for a suitable $\bar x \in (0,L)$.

If $S_{\dot w} = \{0, \bar x\}$ then (2.6),(3.1)-(3.3), strict convexity of $\ddot w$ and $\dddot w_+ = \dddot w_- = (\ddot w)' = 0$ at $\bar x$ (minimum point for $\ddot w$) together yield (by the same argument in the proof of Theorem 3.7)

$$\ddot w(0) = \gamma = u''(0) + b\,, \qquad \ddot w(\bar x) = -\gamma = u''(\bar x) + a\bar x + b\,, \quad u'''(\bar x) + a = 0\,.$$

Hence

$$[\dot w](0) > 0\,, \qquad [\dot w](\bar x) < 0\,. \tag{3.41}$$

Then by (2.5), $u'''' = f$ and Taylor formula

$$2\gamma \,=\, u''(0) - u''\,(\bar x) + \bar x\, u'''\,(\bar x) \,=\, \int_0^{\bar x} y f(y)\, dy \,<\, 8\,\gamma\, \bar x^2 \,/ L^2\,. \tag{3.42}$$

hence $\bar x > \frac{L}{2}$. By u''' strictly monotone and u'' symmetric we get $-a = u'''(\bar x) > u'''(\frac{L}{2}) = 0$ and by subtracting (3.40) from L times (3.39) with $t_1 = 0$, $t_2 = \bar x$, we get the inequality $-bL^2/2 = aL^3/3 + \bar x\,[\dot w](\bar x) < 0$ thus contradicting $b = \gamma - u''(0) < 0$.

The case $S_{\dot w} = \{\bar x, L\}$ can be discarded in the same way. Precisely, (2.6),(3.1)-(3.3), strict convexity of $\ddot w$ and $\dddot w_+ = \dddot w_- = (\ddot w)' = 0$ at $\bar x$ (minimum point for $\ddot w$) together yield (by the same argument in the proof of Theorem 3.7)

$$\ddot w(\bar x) = -\gamma = u''(\bar x) + a\bar x + b\,, \qquad \ddot w(L) = \gamma = u''(L) + aL + b\,, \quad u'''(\bar x) + a = 0\,,$$

$$[\dot w](\bar x) < 0\,, \qquad [\dot w](L) > 0\,. \tag{3.43}$$

Then by (2.5), $u'''' = f$ and Taylor formula

$$2\gamma \,=\, u''(L) - u''\,(\bar x) + (\bar x - L)\, u'''\,(\bar x) \,=\, \int_{\bar x}^{L} (L-y) f(y)\, dy \,<\, 8\,\gamma\, \frac{(L-\bar x)^2}{L^2}\,, \tag{3.44}$$

hence the inequality $\bar x < \frac{L}{2}$. By u''' strictly monotone increasing and u'' symmetric we get $-a = u'''(\bar x) < u'''(\frac{L}{2}) = 0$ and by subtracting (3.40) from $L/2$ times (3.39) with $t_1 = \bar x$, $t_2 = L$, we get the equality $a\, L^3 \,/\, 12 \,+\, (\bar x - L/2)\, [\dot w](\bar x) \,+\, L\, [\dot w](L)\, /\, 2 \,=\, 0$, thus contradicting $a > 0$.

Then we must have $S_{\dot w} = \{0, L\}$. Now (3.39),(3.40) read

$$a\, L^2/2 \,+\, b\, L \,+\, [\dot w](0) \,+\, [\dot w](L) \,=\, 0 \tag{3.45}$$

$$a\, L^3/6 \,+ b\, L^2/2 \,+\, L\, [\dot w](0) \,=\, 0\,, \tag{3.46}$$

by subtracting (3.46) from L times (3.45):

$$[\dot w](L) \,=\, -\,a\, \frac{L^3}{3} \,-\, b\, \frac{L^2}{2}\,. \tag{3.47}$$

By the same argument of Theorem 3.7 and strict convexity of $\ddot{w}$ we get $|\ddot{w}(0)| = |\ddot{w}(L)| = \gamma$. So that only three subcases are allowed:

- (i) $\ddot{w}(0) \,=\, \ddot{w}(L) \,=\, \gamma\,,$
- (ii) $\ddot{w}(0) \,=\, \gamma\,, \quad \ddot{w}(L) \,=\, -\gamma\,,$
- (iii) $\ddot{w}(0) \,=\, -\gamma\,, \quad \ddot{w}(L) \,=\, \gamma\,.$

In subcase (ii) we deduce $u''(0) + b = \ddot{w}(0) = \gamma = -\ddot{w}(L) = -u''(L) - aL - b$. Hence

$$a \;=\; -\,L^{-1}\,(u''(0)+u''(L)+2b) \;=\; -\,2\,L^{-1}\,(u''(0)+b) \;=\; -\,2\,\gamma/L\,; \qquad (3.48)$$

by substituting (3.48) and $b = \gamma - u''(0)$ in (3.47) and taking into account $u''(0) > \gamma > 0$:

$$[\dot{w}](L) \;=\; \left(\frac{\gamma}{6} + \frac{u''(0)}{2}\right) L^2 \;>\; 0\,, \qquad (3.49)$$

say a contradiction with $\ddot{w}(L) < 0$ and (3.2), (3.3).

In subcase (iii) we deduce $u''(0) + b = \ddot{w}(0) = \,-\,\gamma \,=\, -\,\ddot{w}(L) \,= -u''(L) - aL - b$. Hence

$$a \;=\; -\,L^{-1}\,(u''(0)+u''(L)+2b) \;=\; -\,2\,L^{-1}\,(u''(0)+b) \;=\; 2\,\gamma/L\,; \qquad (3.50)$$

by substituting (3.50) and $b = -\gamma - u''(0)$ in (3.46) and taking into account $u''(0) > \gamma > 0$:

$$[\dot{w}](0) \;=\; \left(\frac{\gamma}{6} + \frac{u''(0)}{2}\right) L \;>\; 0\,, \qquad (3.51)$$

say a contradiction with $\ddot{w}(0) < 0$ and (3.2), (3.3).

Then the first subcase (i) is the only admissible one. In this case (2.6),(3.2),(3.3),(3.5)-(3.7),(3.33) and strict convexity of $\ddot{w}$ yield

$$u''(0) + b = \ddot{w}(0) = \gamma = \ddot{w}(L) = u''(L) + aL + b = u''(0) + a + b\,.$$

Hence $a = 0$, $b = \gamma - u''(0) = \gamma - u''(L)$, $\ddot{w}(0) = \ddot{w}(L) = \gamma$ and, by (3.2)(3.3), $\mathrm{sign}([\dot{w}(0)]) = \mathrm{sign}([\dot{w}(L)]) > 0$.
By subtracting (3.40) from (3.39) times L and then evaluating at $t_1 = 0$, $t_2 = L$ we get $0 \,<\, [\dot{w}](L) \,=\, bL^2/2 \,=\, (\gamma - u''(0))\,L^2/2\,.$
By (3.5)-(3.8) we get $w = u + \,1/2\,(u''(0) - \gamma)\,x(L-x)$ and

$$\mathcal{F}(w) = -\frac{1}{2}\int_0^1 |u''|^2 - \frac{b^2}{2} = -\frac{1}{2}\int_0^1 |u''|^2 - \frac{L}{2}(u''(0)-\gamma)^2 \qquad (3.52)$$

and the Lemma is proven. ■

Lemma 3.12. *Assume* $f \in L^\infty(\mathbb{R})$ *fulfils (2.5), (3.32), (3.33) and there is* $w \in \operatorname{argmin} \mathcal{F}$ *such that* $\sharp(S_{\dot{w}}) = 1$.
Then: $u''(0) = u''(L) > \gamma\,,$

$$\mathcal{F}(w) \;=\; -\,\frac{1}{2}\int_0^L |u''|^2\,dx \;-\; \frac{L}{8}(u''(0)-\gamma)^2\,. \qquad (3.53)$$

and either $S_{\dot{w}} = \{0\}$ *or* $S_{\dot{w}} = \{L\}$,

Proof - Let $w \in \operatorname{argmin} \mathcal{F}$ such that $\sharp(S_{\dot w}) = 1$.
By Lemma 3.10 and (3.33) we deduce $u''(0) = u''(L) > \gamma$.
By the same procedure in the proof of Theorem 3.7 we have: either $S_{v'} = \{0\}$ or $S_{\dot w} = \{L\}$ or $S_{\dot w} = \{\bar x\},\ \bar x \in (0, L)$.
Assume $S_{\dot w} = \{\bar x\},\ \bar x \in (0, L)$: then by taking into account $\operatorname{spt} w \subset [0, L]$ we get

$$\frac{a}{2} L^2 + b\,L + [\dot w](\bar x) = 0 \tag{3.54}$$

$$\frac{a}{6} L^3 + \frac{b}{2} L^2 + \bar x\,[\dot w](\bar x) = 0 \tag{3.55}$$

and by Euler-Lagrange equations (3.1)-(3.3)

$$\ddot w(\bar x) = \gamma \operatorname{sign}([\dot w](\bar x)) = u''(\bar x) + \bar x\, a + b. \tag{3.56}$$

Since by (3.32) and (3.33) $\ddot w$ is strictly convex, then by (2.6) and $\bar x$ minimum point for $\ddot w$ we get $\ddot w(\bar x) = -\gamma,\ [\dot w](\bar x) < 0$. By (3.5)-(3.7) and $\bar x$ minimium point of $\ddot w$, we get $(\ddot w)'(\bar x) = u'''(\bar x) + a = 0$.
By subtracting (3.55) from (3.54) times $L/2$ we get $-a = u'''(\bar x) = 6\,L^{-3}\,(L - 2\bar x)\,[\dot w](\bar x)$ which is a contradiction unless $S_{\dot w} = \bar x = L/2$ since u''' is strictly increasing and $u'''(L/2) = 0$ by (3.33).
So in the case we are facing we must have $\bar x = L/2 = S_{\dot w}$ and $a = 0$. Then (3.54)(3.55)(3.56) become

$$b\,L + [\dot w](L/2) = 0 \tag{3.57}$$

$$\frac{b}{2} L^2 + \frac{L}{2}\,[\dot w](L/2) = 0 \tag{3.58}$$

$$\ddot w(L/2) = \gamma \operatorname{sign}([\dot w](L/2)) = u''(L/2) + b. \tag{3.59}$$

By reading the previous analysis with $\bar x = L/2$ we find: $\ddot w(L/2) = -\gamma,\ \ [\dot w](L/2)/L < 0$. Moreover, from (3.57), we obtain $b = -[\dot w](L/2)\,/L > 0$ and, by (3.59),

$$u''\left(\frac{L}{2}\right) = -\gamma - b < -\gamma\,.$$

On the other hand by recalling (3.37) we have

$$u''\left(\frac{L}{2}\right) = -L^{-1} \int_0^{\frac{L}{2}} y^2 f(y)\,dy\ .$$

Then by estimating the above representation of $u''(L/2)$ with (2.5) we find the contradiction

$$-\frac{2}{3}\gamma < u''\left(\frac{L}{2}\right) = -\gamma$$

Then the only possibilities are: either $S_{\dot w} = \{0\}$ or $S_{\dot w} = \{L\}$.
Assume $S_{\dot w} = \{0\}$. We set $s = [\dot w](0)$ and evaluate (3.54),(3.55),(3.56) at $\bar x = 0$:

$$\begin{cases} aL^2/2 + bL + s = 0 \\ aL^3/3 + bL^2/2 = 0 \\ u''(0) + b - \gamma \operatorname{sign}(s) = 0\,, \end{cases} \tag{3.60}$$

hence $a = 6s$, $b = -4s$ and

$$u''(0) = \gamma \operatorname{sign}(s) + 4s \;=\; \operatorname{sign}(s)\,(\,\gamma + 4|s|\,).$$

By (3.14), (3.15)

$$0 \;<\; L^{-2}\int_0^L y(y-L)^2 f(y)\,dy \;= u''(0)$$

then $s > 0$, and

$$a = -3bL/2 = 6s = -\frac{3}{2}(\gamma - u''(0)) > 0\,, \qquad b = -4sL = \gamma - u''(0) < 0. \quad (3.61)$$

By summarizing together with (3.5), if $S_{\dot w} = \{0\}$ then $\gamma - u''(0) < 0$ and

$$\mathcal{F}(w) = -\frac{1}{2}\int_0^L |u''|^2\,dy \;-\; \frac{L}{8}\,(u''(0)-\gamma)^2. \quad (3.62)$$

The case $S_{\dot w} = \{L\}$ can be examined exactly in the same way obtaining (3.62) with $u''(L)$ replacing $u''(0)$. Since $u''(0) = u''(L)$, the energy is still given by (3.62). ■

By comparison of the possibilities analyzed in the previous Lemmas we can now prove the main theorem of this section.

Proof of Theorem 3.8 - Let $w \in \operatorname{argmin}\mathcal{F}$. Recall that the set $\operatorname{argmin}\mathcal{F}$ is not empty due to Theorem 2.1, while Theorem 3.7 tells that $\sharp(S_{\dot w}) \le 2$ for any $w \in \operatorname{argmin}\mathcal{F}$.
If $u''(0) = u''(L) \le \gamma$, then we get $w \equiv u$, $\sharp(S_{\dot w}) = 0$ and (3.36) (by Lemma 3.10).
If $u''(0) = u''(L) > \gamma$, then by comparison of (3.38) and (3.53) we see that case ($\sharp(S_{\dot w}) = 1$) studied in Lemma 3.12 never happens; while case $\sharp(S_{\dot w}) = 0$ (e.g. $u \in \operatorname{argmin}\mathcal{F}$) cannot take place since Lemma 3.9 and comparison of (3.36) and (3.38) lead to a contradiction; then case $\sharp(S_{\dot w}) = 2$ is the only possibility: so Lemma 3.11 tells that w is the unique minimizer and fulfils $S_{\dot w} = \{0, L\}$, $w \;=\; u \,+\, 1/2\;(u''(0)-\gamma)\,x(L-x)$ and (3.38).
Then we have uniqueness of minimizer in any case; moreover by taking into account the explicit values (3.36) and (3.38) of energy in the two cases we get, for any case,

$$\mathcal{F}(w) = -\,\frac{1}{2}\int_0^L |u''|^2\,dy \;-\; \frac{L}{2}\,\left[(u''(0)-\gamma)^+\right]^2. \quad (3.63)$$

Eventually by $\ddot w(0) = \gamma$ and decomposition $w = u + v$ (introduced by Lemma 3.3) we get $\ddot v = -(u''(0) - \gamma)^+$. This relationship together with $v(0) = v(L) = 0$ yields

$$v(x) = \frac{1}{2}\,\left(u''(0)-\gamma)\right)^+ x(1-x)\,.$$

This completes the proof of the representation formula (3.34). ■

The above analysis can be repeated for symmetric strictly negative f up to analogous conclusions, leading to the following statement.

Remark 3.13. *Assume that $f \in L^\infty(\mathbb{R})$ satisfies (2.5),*

$$f > 0 \;\; \textit{a.e. in } [0, L] \qquad \textit{or} \qquad f < 0 \;\; \textit{a.e. in } [0, L]\,, \quad (3.64)$$

$$f(x) = f(L-x) \;\; \textit{a.e in } [0, L] \quad (3.65)$$

then the minimizer of $\mathcal{F}$ *is unique and is given explicitly by*

$$z(x) \;=\; u(x) \,+\, \frac{1}{2}\,\mathrm{sign}(f)\,(|u''(0)|-\gamma)^{+}\,x\,(L-x) \tag{3.66}$$

where u *is the unique solution of (3.9) and* $\mathrm{sign}(u''(0)) = \mathrm{sign}(f)$. *In particular:*
if $|u''(0)| \le \gamma$ *then* $z \equiv u \in H^2(\mathbb{R})$ *and* $S_{\dot z} = \emptyset$;
if $|u''(0)| > \gamma$ *then* $S_{\dot z} = \{0, L\}$.

We emphasize that a similar analysis holds true also for penalized functionals $\mathcal{F}^{\varepsilon}$ (defined by (2.15)) as it is sketched below.

Remark 3.14. *When* $w^{\varepsilon} \in \mathrm{argmin}\,\mathcal{F}^{\varepsilon}$ *and (2.5) is satisfied, then analogous formulation of statements 3.1, 3.7, 3.9, 3.10, 3.11, 3.12, 3.13 and the following modified compliance identity (see also Lemma 3.8* [16]*) hold true:*

$$\mathcal{F}^{\varepsilon}(w_{\varepsilon}) \;=\; -\frac{1}{2}\int_0^L |\ddot w_{\varepsilon}|^2\,dx \;+\; \varepsilon\,\sharp(S_{\dot w_{\varepsilon}}) \qquad \forall w_{\varepsilon} \in \mathrm{argmin}\,\mathcal{F}^{\varepsilon}\,. \tag{3.67}$$

Moreover, if (3.64),(3.65) are satistfied and w_{ε} *is the unique element in* $\mathrm{argmin}\,\mathcal{F}^{\varepsilon}$, *then, by comparison of (3.36),(3.38),(3.53), we get*

$$w \;=\; w_{\varepsilon} \qquad \forall \varepsilon \in \left(0\;,\;\frac{L}{4}\,(u''(0)-\gamma)\right) \tag{3.68}$$

where $w \in \mathrm{argmin}\,\mathcal{F}$, *hence without extracting subsequences,* $w_{\varepsilon} \rightharpoonup w\ w^{*}BH$. ∎

Theorem 3.15. *Assume (2.5),(3.64),(3.65). Then:*
if $u''(0) \le \gamma$ *then minimizer* w_{ε} *of* $\mathcal{F}^{\varepsilon}$ *is unique and* $w_{\varepsilon} = u \in \mathrm{argmin}\,\mathcal{F}\ \ \forall \varepsilon > 0$.
if $u''(0) > \gamma$ *and* $0 < \varepsilon < \frac{L}{4}(u''(0)-\gamma)^2$ *then* $w_{\varepsilon} = z$ *where* z *is given by (3.66).*
In any case, from (3.67) we get

$$\mathcal{F}^{\varepsilon}(w^{\varepsilon}) \;= -\frac{1}{2}\int_0^L |u''|^2\,dx \,-\, \frac{L}{2}\,\Big[(u''(0)-\gamma)^{+}\Big]^2 + 2\varepsilon. \tag{3.69}$$

If $u''(0) > \gamma$ *and* $\varepsilon = \frac{L}{4}(u''(0)-\gamma)^2$, z *given by (3.66) and* u *(solution of (3.9)) have the same energy level.*

Proof - With the same proof of Theorem 3.8 (and Remark 3.13) we can prove the thesis up to (3.69).
The last statement follows by straightforward computations: we emphasize the fact that by denoting z the 2-hinges displacement in (3.66) and z_1 anyone among the 1-hinge displacements faced in Lemma 3.12, we have $\mathcal{F}^{\varepsilon}(z) = \mathcal{F}(z)+2\varepsilon$, $\mathcal{F}^{\varepsilon}(z_1) = \mathcal{F}(z_1)+\varepsilon$, $\mathcal{F}^{\varepsilon}(u) = \mathcal{F}(u)$.
Then, due to (3.38)(3.53),

$$\mathcal{F}^{\varepsilon}(z_1) \;>\; (\,\mathcal{F}^{\varepsilon}(u) \,\wedge\, \mathcal{F}^{\varepsilon}(z)\,) \qquad \forall \varepsilon > 0\,. \ \blacksquare$$

In case of constant load we can show the following explicit example of non uniqueness of minimizer for functional $\mathcal{F}^{\varepsilon}$.

Example 3.16 - The constant load $f \equiv 12(\gamma + 2\sqrt{\varepsilon/L})/L^2$ satisfies the safe load condition (2.5) iff $\varepsilon < \frac{\gamma^2}{36}\,L$ and entails $u''(0) = \frac{L^2}{12} f = \gamma + 2\sqrt{\varepsilon/L} > \gamma$ and $\varepsilon = \frac{L}{4}(u''(0)-\gamma)^2$. This means that for every $\gamma > 0$ and for every $\varepsilon < \frac{\gamma^2}{36} L$ there exists a constant admissible load $f \equiv 12(\gamma + 2\sqrt{\varepsilon/L})/L^2$ such that the corresponding energy functional $\mathcal{F}^\varepsilon$ has two different minimum points: one is regular and the other one has two hinges at the endpoints.

Remark 3.17 *Assume (2.5). Then for any* $v \in \operatorname{argmin} \mathcal{F}$ *s.t.* $\sharp(S_{\dot v}) < \infty$ *it is possible to define* $w \in \operatorname{argmin} \mathcal{F}$ *such that* $\sharp(S_{\dot w}) \leq 2$.
This w *can be constructed explicitly by adding a suitable piece-wise affine continuous function to* v *: the construction developed in the proof of Theorem 4.1 of* [15] *to reduce hinges number and strictly decrease the value of* $\mathcal{F}^\varepsilon$ *can be replayed in the present case with the effect to reduce hinges number and keep unchanged the value of* $\mathcal{F}$.

Remark 3.18 *A straightforward consequence of Theorem 3.7 and remark 3.12 is that inequality (3.13) is sharp as a regularity condition based on* L^∞ *estimate of load (see Theorem 3.5). Indeed, if load is constant:* $f \equiv c$, *then by (3.14) and (3.15) we get* $u''(0) = c\,L^2/12$, *say* $|u''(0)| > \gamma$ *iff* $\|f\|_{L^\infty} = |c| > 12\,\gamma/L^2$, *hence* $u \in H^2(\mathbb{R})$ *(given by* $u(x) = c\,x^2(x-L)^2/24$, *if* $x \in [0, L]$, $u(x) = 0$ *else) is the unique minimizer of* $\mathcal{F}$ *if and only if* $\|f\|_{L^\infty} = |c| \leq 12\,\gamma/L^2$.

4. The hinged-hinged elastic-plastic beam

In this section we study the functional

$$\Lambda(w) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^L |\ddot w|^2 - \int_0^L f w \, dx + \gamma \sum_{S_{\dot w}} |[\dot w]| & \text{if } w \in \mathcal{S} \\ +\infty & \text{otherwise in } BH(0,L) \end{cases} \tag{4.1}$$

where $\gamma > 0$ is a given constant, $f \in L^\infty(0, L)$ and

$$\mathcal{S} \;=\; \{w \in SBH(0,L) : w(0) = w(L) = 0\}\,.$$

We emphasize that by definition $S_{\dot w} \subset (0, L)$ for any $w \in \mathcal{S}$ and hence wherever in this section, while in the previous sections 0 and/or L could belong to $S_{\dot w}$ since any $w \in \mathcal{K}$ is defined in whole $\mathbb{R}$. The interval $[0, L]$ represents the reference configuration of an elastic plastic beam which is hinged at both the endpoints, f is the vertical dead load acting on the beam, w is the vertical displacement, $\gamma > 0$ is a constant depending on the material and the functional Λ describes the total energy related to deformation the hinged-hinged elastic-plastic beam with unitary flexural rigidity.
We introduce the penalized functionals

$$\Lambda^\varepsilon(w) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^L |\ddot w|^2 - \int_0^L f w \, dx + \gamma \sum_{S_{\dot w}} |[\dot w]| + \varepsilon\, \sharp(S_{\dot w}) & \text{if } w \in \mathcal{S}\,, \\ +\infty & \text{otherwise in } BH(0,L)\,. \end{cases} \tag{4.2}$$

The main result of this section is the following Theorem which provides the sharp value $8\gamma/L^2$ for both safe load condition and regularity load condition of hinged-hinged elastic-plastic beam.

Theorem 4.1. *Assume that $f \in L^\infty(0,L)$ satisfies*

$$\|f\|_{L^\infty(0,L)} \ < \ \frac{8\,\gamma}{L^2} \tag{4.3}$$

then all the functionals Λ, Λ^ε achieve the same finite minimum, uniqueness holds true for minimizer of Λ and of Λ^ε for every $\varepsilon > 0$ and all these minimizers coincide with the unique solution ω of hinged-hinged purely elastic beam:

$$\begin{cases} \omega \in H^2(0,L) \cap H^1_0(0,L) \\ \omega'''' = f \ \text{in} \ (0,L) \\ \omega(0) = \omega(L) = \omega''(0) = \omega''(L) = 0\,. \end{cases} \tag{4.4}$$

The proof of Theorem 4.1 is postponed at the end of this section.

Lemma 4.2. *(L^1–BH* **Poincarè Inequality for hinged-hinged beam***)*
Let $v \in BH(0,L)$, $v(0) = v(L) = 0$ then

$$\|v\|_{L^1} \ \le \ \frac{L^2}{8}\,\|v''\|_{T((0,L))}\ . \tag{4.5}$$

The equality in (4.5) holds true iff $v = r_s$:

$$r_s(x) \ = \ s\left(\frac{L}{2} - \left|x - \frac{L}{2}\right|\right)^+ \qquad s \in \mathbb{R} \tag{4.6}$$

Proof - The proof is identical to the one of Lemma 2.2 besides the fact that now the relevant quotient is $\|v''\|_{T((0,L))}\,/\,\|v\|_{L^1}$ for every $v \in \mathcal{S}$. So that, with the same choice for v, $\widetilde{v}$ and $\check{v}$ it is enough modifying only the deduction of (2.14) at the very end:
if $v \in BH(0,L)$, $v(0) = v(L) = 0$, v convex in $[0,L]$ and $v \not\equiv 0$, then $-\infty < v'_+(0) < 0$, $0 < v'_-(L) < +\infty$ and we can define

$$\check{v}(x) \ = \ (v'_+(0)x) \vee \big((v'_-(L)(x-L)\big) \qquad x \in [0,L]\ .$$

Then $\check{v} \le v$ and $\|\check{v}\,''\|_{T((0,L))} = (v_-(L) - v'_+(0)) = \|v\,''\|_{T((0,L))}$ (the difference with respect to Lemma 2.2 is the lack of coefficient 2, since hinges at boundary have no cost). So

$$\inf\ \Big\{\ \|v''\|_{T((0,L))}\,/\,\|v\|_{L^1}\ : v \in \mathcal{S},\ v \text{ convex in } [0,L]\,\Big\} \ge$$

$$\ge \inf\ \Big\{\ \|v''\|_{T((0,L))}\,/\,\|v\|_{L^1}\ : v(x) = (-ax) \vee (b(x-L)),\ a > 0,\ b > 0\,\Big\} =$$

$$\inf\ \big\{2(a+b)^2/(abL^2) : a > 0,\ b > 0\,\big\} = 8/L^2.$$

Actually the infimum is a minimum and is achieved iff $a = b$ say iff v is a roof function. ■

The following existence statement follows by Lemma 4.2 by the same argument in the proof of Theorem 2.4.

Theorem 4.3. *(L^∞* **safe load condition for** *Λ^ε) Assume*

$$\|f\|_{L^\infty(0,L)} \ < \ \frac{8\,\gamma}{L^2} \tag{4.7}$$

then Λ^ε achieves a finite minimum.

Remark 4.4. *The safe load constant* $8\,\gamma/L^2$ *in (4.7) is sharp for load in* L^∞, *as shown (analogously to Remark 2.5) by the choices* $f \equiv 8\,\gamma/L^2+\delta,\ \delta>0,\ r_s(x),\ s>0,$ *entailing* $S_{\dot r_s(x)}=\{L/2\}\,,\ \gamma\sum_{S_{\dot r_s}}|[\dot r_s]|=2\gamma s,\ \mathcal{F}^\varepsilon(r_s(x))=\varepsilon-\delta L^2/4\,s\ \to\ -\infty$ *as* $s\to+\infty$.

Inequality (4.7) is a very stringent condition: actually we show that it prevents formation of creases, leading to uniqueness and regularity of minimizers.
By using exactly the same proof of Theorems 3.1 and 3.2 we obtain Euler Lagrange equations and a compliance identity for hinged-hinged beam as stated below.

Theorem 4.5. *(***Euler-Lagrange equations for functionals** $\Lambda,\ \Lambda^\varepsilon$*)*
If w *belongs to* $\operatorname{argmin}\Lambda^\varepsilon$ *or to* $\operatorname{argmin}\Lambda$ *then*

$$(\ddot w)''\ =f\quad in\ (0,L) \tag{4.8}$$
$$\ddot w_\pm\ =\ \gamma\,\mathrm{sign}([\dot w])\quad in\ S_{\dot w} \tag{4.9}$$
$$\ddot w_+(0)=\ddot w_-(L)=0\,. \tag{4.10}$$

In particular $\ddot w\in H^2(0,L)$, *hence* $\ddot w$ *and* $\dddot w=(\ddot w)'$ *are continuous in* $[0,L]$.

Theorem 4.6. *(***Compliance identity***)*
Assume that w *belongs to* $\mathcal{S}$ *and* w *fulfils conditions (4.8),(4.9),(4.10). Then*

$$\Lambda^\varepsilon(w)\ =\ -\frac{1}{2}\int_0^L|\ddot w|^2\,dx\,+\,\varepsilon\,\sharp\,(S_{\dot w}),$$
$$\Lambda(w)\ =\ -\frac{1}{2}\int_0^L|\ddot w|^2\,dx\,.$$

Theorems 4.5, 4.6 entail that minimizers of Λ^ε have a very simple structure, as stated precisely by the following Theorem.

Theorem 4.7. *If* $w_\varepsilon\in\operatorname{argmin}\Lambda^\varepsilon$ *then* $S_{\dot w_\varepsilon}=\emptyset$, *so that* $w_\varepsilon=\omega$ *(unique solution of (4.4)) and* $\Lambda(w_\varepsilon)=\Lambda^\varepsilon(w_\varepsilon)$.

Proof - By contradiction assume that $w\in\operatorname{argmin}\Lambda^\varepsilon$ and $x_1\in(0,L)$ belongs to $S_{\dot w}$. Then w fulfills Euler-Lagrange equations (4.8)-(4.10). We can modify w by eliminating the crease at x_1 and strictly reducing the energy at the same time.
We define a function $w_*\in SBH(0,L)$ such that $w_*(0)=0$ and

$$\dot w_*\ =\ \dot w\,+\,(1-x_1/L)\,[\dot w](x_1)\ \chi_{[0,x_1)}\,-\,(x_1/L)\,[\dot w](x_1)\ \chi_{[x_1,L]}, \tag{4.11}$$

hence $\int_0^L\dot w_*=\int_0^L\dot w=0,\ w_*(0)=w_*(L)=\ 0$ and $S_{\dot w_*}=\emptyset$.
Since $\ddot w_*=\ddot w$ we deduce that also w_* satisfies Euler-Lagrange equations (4.8)-(4.10), then compliance identity for Λ^ε yield the contradiction since $\Lambda^\varepsilon(w_*)=\Lambda(w)-\varepsilon$ hence first part of thesis. Obviously $\Lambda^\varepsilon(\omega)=\Lambda(\omega)$. ■

Theorem 4.8. *(*L^∞ **bending moment regularity condition for hinged-hinged beam***)*
Let ω *be the unique solution of (4.4) If*

$$\|\omega''\|_{L^\infty(0,L)}\ \le\ \gamma \tag{4.12}$$

then ω *is a minimizer of* Λ, *which is unique if the inequality in(4.12) is strict.*

Proof - Identical to the one of Theorem 3.4.

Proof of Theorem 4.1 - Referring to solution ω of (4.4), we evaluate ω'' by Green function for d^2/dx^2 in $(0,L)$ with homogeneous boundary conditions: $\omega''(x) = \int_0^L G(x,y)f(y)dy$ (see (3.29)), hence

$$|\,\omega''(x)\,| \;<\; \|f\|_{L^\infty} \int_0^L |G(x,y)|\,dy \;=\; \frac{L^2}{8}\,\|f\|_{L^\infty} \qquad \forall x \in (0,L)\,. \tag{4.13}$$

Then by (4.3),(4.13) we get $\|\omega''\|_{L^\infty} \le L^2\,\|f\|_{L^\infty}/8 \;<\; \gamma\,.$
So Theorem 4.8 ensures that Λ achieves finite a minimum and that ω is the unique minimizer of Λ.
Then, for every $\varepsilon > 0$, Theorem 4.7 entails that $v \in \operatorname{argmin} \Lambda^\varepsilon$ if and only if v is a minimizer of Λ and if and only if v is the unique solution ω of (4.4), say the unique minimizer among $w \in H^2(0,L) \cap H_0^1(0,L)$ of the hinged-hinged purely elastic beam functional

$$\mathcal{E}(w) = \int_0^L \frac{1}{2}|\ddot{w}|^2\,dx - \int_0^L f w\,dx \tag{4.14}$$

since $\Lambda^\varepsilon(\omega) = \Lambda(\omega) = \mathcal{E}(\omega)$, and the proof of Theorem 4.1 is achieved. ■

We conclude by clarifying the slight changes to be performed when the flexural rigidity EJ is differen from 1.

Remark 4.9. *If the flexural rigidity EJ is strictly positive but $EJ \neq 1$ (i.e. by considering functional (1.1) instead of (2.4) or (1.8) instead of (4.1)) then Theorems 2.1, 3.5, 4.1 still hold true.*
In fact, by direct inspection of the proofs, there are differences only in technical intermediate steps (Theorem 3.4, Euler equations and compliance identity) which are listed below in detail:

substitution of φ^{**} by φ^{**}_{EJ} in definition (2.17) of functional $\mathcal{F}^*$ where φ^{**}_{EJ} is the convex envelope of $\varphi_{EJ}(s) \;=\; \min\left\{EJs^2/2, \gamma|s|\right\}$ that is

$$\varphi^{**}_{EJ}(s) = \begin{cases} EJ\,s^2/2 & \text{if } |s| \le \gamma/(EJ) \\ \gamma|s| - \gamma^2/2EJ & \text{otherwise.} \end{cases} \tag{4.15}$$

substitution of (2.6) by

$$\|\ddot{w}\|_{L^\infty} \;\le\; \gamma/(EJ) \qquad \forall w \in \operatorname{argmin} \mathcal{F}\,. \tag{4.16}$$

substitution of (3.9) by

$$\begin{cases} u_{EJ} \in H^2(\mathbb{R}) \\ u''''_{EJ} = f/(EJ) & \text{in } (0,L) \\ \operatorname{spt} u_{EJ} \subset [0,L]\,, \end{cases} \tag{4.17}$$

substitution of (3.10) by analogous relationship for the solution u_{EJ} of (4.17)

$$\|u''_{EJ}\|_{L^\infty(\mathbb{R})} \;\le\; \gamma/(EJ)\,, \tag{4.18}$$

substitution of kernel K by $K/(EJ)$ in (3.14),(3.15) and hence the substitution of (3.17) by

$$\|u''_{EJ}\|_{L^\infty} \;\le\; L^2\,\|f\|_{L^\infty}/(12\,E\,J)\,, \tag{4.19}$$

so that (4.17)(4.19) together produce cancellation of the term EJ in the load regularity condition (3.13) which holds true without any change.
Euler equations and compliance identity read as follows:
if (2.5) holds true and $w \in \operatorname{argmin} \mathcal{F}$ then

$$\begin{aligned} EJ\,(\ddot{w})'' &= f && \text{in } (0,L) && (4.20)\\ EJ\;\ddot{w}_-(x) &= \gamma\,\mathrm{sign}([\dot{w}])(x) && \text{in } S_{\dot{w}} \cap (0,L] && (4.21)\\ EJ\;\ddot{w}_+(x) &= \gamma\,\mathrm{sign}([\dot{w}])(x) && \text{in } S_{\dot{w}} \cap [0,L)\,, && (4.22)\end{aligned}$$

if (2.5) holds true, w in $SBH(\mathbb{R})$ fulfils (4.20)-(4.22) and $\operatorname{spt} w \subset [0,L]$ *then*

$$\mathcal{F}(w) \;=\; -\,\frac{EJ}{2}\int_0^L |\ddot{w}|^2\,dx\,. \tag{4.23}$$

Eventually Theorem 3.8 holds true provided (3.34),(3.35) are respectively substituted by

$$z_{EJ}(x) \;=\; u_{EJ}(x) + \frac{1}{2}\,(u''_{EJ}(0) - \gamma/(EJ))^+\,x\,(L-x) \tag{4.24}$$

$$\mathcal{F}(z_{EJ}) \;=\; -\,\frac{EJ}{2}\int_0^L |u''_{EJ}|^2\,dx \;-\; L\,\frac{EJ}{2}\,\left((u''_{EJ}(0) - \gamma/(EJ))^+\right)^2 \tag{4.25}$$

where u_{EJ} is the unique solution of (4.17).

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