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Refined blow-up results for nonlinear fourth order differential equations

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Abstract

We study a class of nonlinear fourth order differential equations which arise as models of suspension bridges. When it comes to power-like nonlinearities, it is known that solutions may blow up in finite time, if the initial data satisfy some positivity assumption. We extend this result to more general nonlinearities allowing exponential growth and to a wider class of initial data. We also give some hints on how to prevent blow-up.

Mathematics Subject Classification: 34A12, 65L05, 34C10, 35B05.

1 Introduction

In this paper we are interested in finite space blow up of solutions of the ordinary differential equation

$$w''''(x) - Tw''(x) + f(w(x)) = 0 \qquad (x \in \mathbb{R}), \tag{1}$$

where $T \in \mathbb{R}$ and f is a locally Lipschitz function. This equation arises in several contexts. We refer to the monograph by Peletier-Troy [15] and to [1, 3, 4, 7, 11, 16] and references therein for several different applications.

In this paper we investigate further an ideal model obeying to (1) which was introduced in [6]. Consider an infinite beam subject to the restoring forces of a large number of nonlinear two-sided springs as in Figure 1. If the beam had finite length, it could model the roadway of a suspension bridge and the springs could model

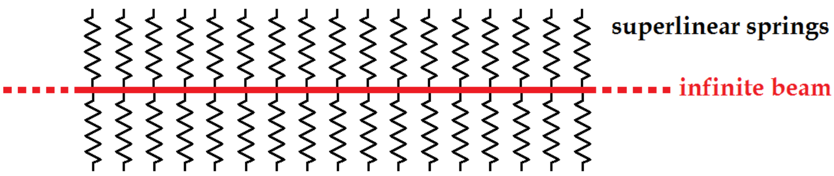

Figure 1: Beam subject to two-sided restoring springs.

the hangers. In recent years, the nonlinear structural behavior of suspension bridges has been uncovered, see e.g. [5, 6, 14, 17], and this suggests to consider semilinear versions of (1). Let $u = u(x)$ denote the vertical displacement of the beam in position x. Assume that, besides the nonlinear restoring force $g = g(u)$ due to the springs, there is a uniform downwards load $p(x) \equiv p > 0$ acting on the beam, for instance, its weight per unit length. The equation modeling an infinite beam having flexural rigidity EI, constant tension $T \geq 0$, and subject to both a downwards load p (its weight) and to the restoring action $g = g(u)$ due to some elastic springs reads

$$EI\, u''''(x) - Tu''(x) = p - g(u(x)) \qquad (x \in \mathbb{R})\,.$$

For physical reasons, one has that $u \mapsto g(u)$ is increasing. Let $u_p > 0$ be the unique solution of $g(u_p) = p$. Put $f(w) := g(w + u_p) - p$ so that f is also increasing and $f(0) = 0$. Put $w(x) = u(x) - u_p$; then w solves the equation

$$EI\, w''''(x) - Tw''(x) + f(w(x)) = 0 \qquad (x \in \mathbb{R})$$

which is precisely (1).

The first purpose of the present paper is to refine previous blow-up results for the solutions of (1) when the nonlinearity f satisfies

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})\,, \quad f(w)\, w > 0 \quad \text{for every } w \in \mathbb{R} \setminus \{0\}. \tag{2}$$

To start, let us recall the following statements proved in [2]:

Proposition 1. *Let $T \in \mathbb{R}$ and assume that f satisfies* (2)*.*
(i) *If a local solution w of* (1) *blows up at some finite $R \in \mathbb{R}$, then*

$$\liminf_{x\to R} w(x) = -\infty \qquad \text{and} \qquad \limsup_{x\to R} w(x) = +\infty\,. \tag{3}$$

(ii) *If f also satisfies*

$$\limsup_{w\to+\infty} \frac{f(w)}{w} < +\infty \qquad \text{or} \qquad \limsup_{w\to-\infty} \frac{f(w)}{w} < +\infty, \tag{4}$$

then any local solution of (1) *exists for all $x \in \mathbb{R}$.*

If *both* the conditions in (4) are satisfied then global existence follows from classical theory of ODE's; but (4) merely requires that f is "one-sided at most linear" so that statement (ii) is far from being trivial and, as shown in [8], it does not hold for differential equations of order at most 3. On the other hand, Proposition 1 (i) states that, under the sole assumption (2), the only way that finite space blow up can occur is with "wide and thinning oscillations" of the solution w; again, in [8] we showed that this kind of blow up is a phenomenon typical of (at least) fourth order problems such as (1) since it does not occur in related lower order equations. Note that assumption (4) includes, in particular, the cases where f is either concave or convex.

Subsequently, it was proved in [10] (see also [8, 9] for related work) that finite space blow up indeed occurs when f has a superlinear power-like behavior:

Proposition 2. *Let $T \geq 0$ and assume that*

$$\exists p > q \geq 1,\ \exists \mu \geq 0, \qquad f(w) = \mu|w|^{q-1}w + |w|^{p-1}w\,. \tag{5}$$

If $w = w(x)$ is a local solution of (1) *in a neighborhood of $x = 0$ which satisfies*

$$w'(0)w''(0) - w(0)w'''(0) + Tw(0)w'(0) > 0\,, \tag{6}$$

then w blows up in finite space for $x > 0$, that is, there exists $R < +\infty$ such that (3) *holds.*

A detailed description of the oscillations of the solution w was also given in [10] and slightly more general nonlinearities f were considered: one could replace (5) with the assumption that f is bounded from above and below by the same power. But the problem whether blow up occurs under more general assumptions on f was left open. In the present paper we prove a stronger version of Proposition 2, for more general superlinearities f satisfying (2) and for initial data possibly violating (6). Moreover, we give some hints on how to prevent blow up and to obtain global solutions.

2 Main results

We assume that f satisfies the regularity conditions

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\}) \tag{7}$$

and the growth conditions

$$\exists c, \delta > 0,\ \tau > 0, \quad \text{s.t.} \quad wf(w) \geq c|w|^{2+\delta} \quad \forall w \in \mathbb{R}\ , \qquad wf(w) \geq cF(w) \quad \forall |w| \geq \tau\ , \tag{8}$$

where $F(w) := \int_0^w f(s)\, ds$ is an antiderivative of f. We also assume that f is increasing and that

$$\exists \lambda \in (0,1),\ \exists \alpha > 0, \quad \text{s.t.} \quad \liminf_{w \to \pm\infty} \frac{F(\lambda w)}{F(w)^\alpha} > 0\ . \tag{9}$$

A typical function f that satisfies all these assumptions is the function f which appears in (5). A further example is obtained by defining $f(w)$ by (5) for all $w \leq 0$ and by $f(w) = \beta(e^{\gamma w} - 1)$ for all $w \geq 0$, where $\beta, \gamma > 0$ are arbitrary. In particular, $f(w)$ is allowed to grow exponentially as $w \to +\infty$.

In order to study equation (1), we shall first introduce the *energy function*

$$\mathcal{E}(x) := \frac{1}{2}\, w''(x)^2 + \frac{T}{2}\, w'(x)^2 - w'(x)w'''(x) - F(w(x))\ . \tag{10}$$

Then, if w solves (1), there holds

$$\mathcal{E}'(x) = 0 \quad \Longrightarrow \quad \mathcal{E}(x) = C \tag{11}$$

for some $C \in \mathbb{R}$. Besides the energy (10), we also introduce the auxiliary function

$$G(x) := w'(x)^2 + \frac{T}{2}\, w(x)^2 - w(x)w''(x) \tag{12}$$

so that

$$\begin{aligned} H(x) := G'(x) &= w'(x)w''(x) + Tw(x)w'(x) - w(x)w'''(x), \\ H'(x) = G''(x) &= w''(x)^2 + Tw'(x)^2 + w(x)f(w(x)). \end{aligned} \tag{13}$$

If $T \geq 0$ and (2) holds, by (13) we infer that

$$G''(x) = H'(x) \geq 0 \qquad \text{so that} \qquad H \text{ is increasing and } G \text{ is convex.} \tag{14}$$

Note that (7)-(8) strengthen (2), while (6) reduces to $H(0) > 0$.

Theorem 3. *Let $T > 0$. Assume that f is increasing and satisfies* (7)-(8)-(9)*. Assume that $w = w(x)$ is a local solution of* (1) *in a neighborhood of $x = 0$ which satisfies*

$$\textit{either} \ \ H(0) > 0 \ \ \textit{or} \ \ \mathcal{E}(0) \neq 0. \tag{15}$$

Then, w blows up in finite space for $x > 0$, that is, there exists $R < +\infty$ such that (3) *holds.*

Theorem 3 can easily be extended to the case $T = 0$, if one also assumes that $wf(w) \geq Cw^2$. Under this additional assumption, one may replace our Lemma 6 below by [8, Lemma 9] and then the rest of our proof applies verbatim. However, we shall merely focus on the case $T > 0$ for simplicity.

If assumption (15) is violated, then Theorem 3 becomes false and the trivial solution $w(x) \equiv 0$ serves as a counterexample. One is thus led to investigate what happens when $H(0) \leq 0 = \mathcal{E}(0)$ but $w \not\equiv 0$. In this respect, we list a few simple conditions which are satisfied by all nontrivial solutions.

Theorem 4. *Let $T > 0$. Assume that f is increasing and satisfies* (7)-(8)-(9). *Assume that $w = w(x)$ is a local nontrivial solution of* (1) *in a neighborhood of $x = 0$ which may be continued on $[0, +\infty)$. Then*
(i) $\mathcal{E}(0) = 0$,
(ii) $H(x)$ *is increasing with* $\lim_{x\to\infty} H(x) = 0$,
(iii) *the solution w satisfies* $\lim_{x\to\infty} w(x) = 0$.

The condition $\mathcal{E}(0) = 0$ defines a 3D manifold in the phase space $\mathbb{R}^4$ but since the stable manifold of $\{0\}$ is a 2D manifold, see [2, Proposition 20], one has probability 0 to find a global solution, even if $\mathcal{E}(0) = 0$. Condition (ii) suggests to seek initial data with "very negative" $H(0)$ and one is led to conjecture that the blow up place might be monotonically linked to $H(0)$. In Section 3 we numerically show that this is not the case and that different conjectures should be made. A problem left open in our paper is precisely to determine a global nontrivial solution of (1).

3 Numerical results for zero energy solutions

Take $T = 1$ and $f(w) = w + 4w^3$ so that $F(w) = \frac{w^2}{2} + w^4$ and (1) becomes

$$w''''(x) - w''(x) + w(x) + 4w(x)^3 = 0 \qquad (x \in \mathbb{R}) . \tag{16}$$

Let us mention that Plaut-Davis [17, Section 3.5] make the same choice and that this nonlinearity appears in several elasticity contexts, see e.g. [12, (1)]. Note also that the assumptions (7)-(8)-(9) are satisfied. Fix

$$w'''(0) = 0 \ , \ w(0) = 1 \tag{17}$$

in such a way that a local solution of (16) has zero energy; by (10), one has $\mathcal{E} \equiv 0$ if and only if

$$w'(0)^2 + w''(0)^2 = 3 \ . \tag{18}$$

For these data, the function H in (13) satisfies

$$H(0) = w'(0)\Big(w''(0) + 1\Big) \ . \tag{19}$$

We take as initial data (17) plus

$$w'(0) = \sqrt{3}\,\cos\theta \ , \ w''(0) = \sqrt{3}\,\sin\theta, \qquad \theta \in [0, 2\pi)$$

so that $\mathcal{E}(0) = 0$, see (18). Figure 2 displays the behavior of $H(0)$ in (19) under the constraint (18).

The four black dots correspond to the case where $H(0) = 0$, namely the points $(0, \sqrt{3})$, $(-\sqrt{2}, -1)$, $(0, -\sqrt{3})$, $(\sqrt{2}, -1)$. In turn, these points correspond, respectively, to $\theta = \pi/2$, $\theta = \pi + \arctan(1/\sqrt{2})$, $\theta = 3\pi/2$, $\theta = 2\pi - \arctan(1/\sqrt{2})$. As a function of $\theta \in [0, 2\pi]$, the value of $H(0)$ is given by $\sqrt{3}\cos(\theta)[1 + \sqrt{3}\sin(\theta)]$ and satisfies the following:

- it increases for $\theta \in [0, \arctan(1/\sqrt{2})]$ and attains its global maximum at $\theta = \arctan(1/\sqrt{2})$ where $H(0) = 2\sqrt{2}$;
- it decreases for $\theta \in [\arctan(1/\sqrt{2}), \pi - \arctan(1/\sqrt{2})]$ and attains its global minimum at $\theta = \pi - \arctan(1/\sqrt{2})$ where $H(0) = -2\sqrt{2}$;
- it increases for $\theta \in [\pi - \arctan(1/\sqrt{2}), 4\pi/3]$ and attains a local maximum at $\theta = 4\pi/3$ where $H(0) = \sqrt{3}/4$;
- it decreases for $\theta \in [4\pi/3, 5\pi/3]$ and attains a local minimum at $\theta = 5\pi/3$ where $H(0) = -\sqrt{3}/4$;
- it increases for $\theta \in [5\pi/3, 2\pi]$.

In Figure 3 we display the behavior of the blow up place R as a function of θ, $R = R(\theta)$.

The maximum for R is found for $\theta = \theta_1 \approx 1.1475$ and $\theta = \theta_2 \approx 4.2891$ and, in both cases, $R \approx 3.857836$; note that $\theta_2 - \theta_1 \approx \pi$. The points corresponding to θ_1 and θ_2 satisfying the constraint (18) are denoted by M in Figure 2. They both lie in the region where $H(0) > 0$.

In the whole, our numerical results suggest the following

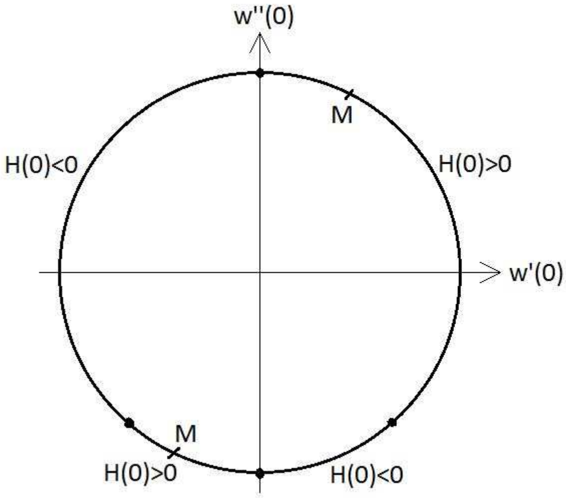

Figure 2: Behavior of $H(0)$ under the constraint (18).

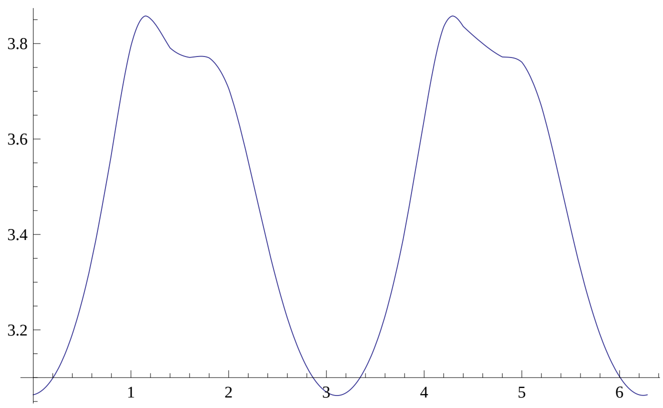

Figure 3: Dependence of the blow up place with respect to the initial data.

Conjecture 5. *Consider the equation*

$$w''''(x) - w''(x) + w(x) + 4w(x)^3 = 0 \qquad (x \geq 0)$$

with the initial conditions

$$w'''(0) = 0\ ,\ w''(0) = \sqrt{3}\,\sin\theta\ ,\ w'(0) = \sqrt{3}\,\cos\theta\ ,\ w(0) = 1, \qquad \theta \in [0, 2\pi].$$

Although $\mathcal{E}(0) = 0$ for all θ, for no values of θ may the solution be extended to $[0, +\infty)$. The blow up place $R(\theta)$ does not depend monotonically on the value of $H(0)$. For any θ we have $3.064 \leq R(\theta) \leq 3.857836$.

4 Preliminary lemmas

In this section we prove some lemmas which will enable us to reach the proofs of our main results. We point out that in some of the following statements the function f is not required to satisfy assumptions (7)-(8)-(9) but only weaker assumptions. However, all the results hold under assumptions (7)-(8)-(9).

We first prove some qualitative properties of the solution of (1).

Lemma 6. *Let $T > 0$ and assume that $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ satisfies* (8)*. Let w be a solution of* (1) *defined on some maximal interval $[0, R)$. Then, for the function H defined in* (13)*, the following alternative holds:*
(i) If $H(x)$ is bounded as $x \nearrow R$, then $R = +\infty$, $H(x) \leq 0$ for all x and

$$\lim_{x\to+\infty} H(x) = \lim_{x\to+\infty} w(x) = 0.$$

(ii) If $H(0) > 0$, then

$$\lim_{x\to R} H(x) = +\infty\,, \quad \lim_{x\to R} G(x) = +\infty\,,$$

and $w(x)$ is unbounded as $x \to R$.

Proof. If $R < +\infty$, Proposition 1 states that there exists a sequence of local maxima m_j such that $m_j \nearrow R$ and $w(m_j) \to \infty$ as $j \to \infty$. Hence, (11) shows that

$$\lim_{j\to\infty} \frac{w''(m_j)^2}{2} = \lim_{j\to\infty} [F(w(m_j)) + C] = +\infty\,.$$

Since $w''(m_j) < 0$, we infer that $G(m_j) = -w(m_j)w''(m_j) + \frac{T}{2} w(m_j)^2 \to \infty$ and, subsequently, $H(m_j) = G'(m_j) \to \infty$ in view of (14). We have so proved that if $H(x)$ remains bounded, then $R = +\infty$.

For the remaining statements, we refer to [2, Theorem 8]. In the case where $R < \infty$ (statement (ii)) one should invoke once more Proposition 1. □

Next, we turn our attention to geometric properties of the solution, such as monotonicity and concavity. The next two statements are also obtained by exploiting the features of the energy functions. In particular, the next result may not hold if the assumption $H(m) > 0$ is violated, see [10].

Lemma 7. *Let $T \geq 0$ and assume that f satisfies* (2)*. Assume that a solution $w = w(x)$ of* (1) *admits a local maximum at some m such that $w(m) > 0$ and $H(m) > 0$. Then w is strictly concave in some maximal interval (m, ξ). In particular, in such an interval the solution w is strictly decreasing. Moreover:*
– if $\xi = +\infty$, then $\lim_{x\to\infty} w(x) = -\infty$;
– if $\xi < +\infty$, then $w(\xi) < 0$ and $F(w(m)) < F(w(\xi))$.
Therefore, the solution w vanishes exactly once in (m, ξ).

Proof. The assumptions $w(m) > 0$ and $H(m) > 0$ imply that $w'''(m) < 0$. Hence, $w'''(x) < 0$ in some maximal right neighborhood (m, σ) of m. Since $w''(m) \leq 0$, we also have that $w''(x) < 0$ in some maximal interval (m, ξ) with $\xi \geq \sigma$ (equality holds only in the case where $\sigma = +\infty$).

If $\xi = +\infty$, then $\lim_{x\to\infty} w(x) = -\infty$ (recall that w is strictly decreasing).

If $\xi < +\infty$, then $\sigma < +\infty$ and

$$x \mapsto w''(x)^2 \qquad \text{is strictly increasing in } [m, \sigma]\,. \tag{20}$$

Note that $\sigma > m$ is the first stationary point of $w''(x)^2$ and $w'''(\sigma) = 0$ so that, by (11),

$$\frac{w''(m)^2}{2} - F(w(m)) = \mathcal{E}(m) = \mathcal{E}(\sigma) = \frac{w''(\sigma)^2}{2} + \frac{T}{2}\, w'(\sigma)^2 - F(w(\sigma))\,.$$

Since $w''(\sigma)^2 > w''(m)^2$ by (20) and since $T \geq 0$, we then have

$$F(w(\sigma)) - F(w(m)) = \frac{w''(\sigma)^2}{2} - \frac{w''(m)^2}{2} + \frac{T}{2}\, w'(\sigma)^2 > 0\,. \tag{21}$$

Since $w(\sigma) < w(m)$ and since (2) implies that $w \mapsto F(w)$ is increasing for $w \geq 0$, we necessarily have $w(\sigma) < 0$. Finally, since $\xi > \sigma$ and w is strictly decreasing in (σ, ξ), we have $w(\xi) < w(\sigma) < 0$ and, by (2) and (21), $F(w(m)) < F(w(\sigma)) < F(w(\xi))$. □

Let $w = w(x)$ be a local solution of (1) and assume that there exists an interval $[z_1, z_2] \subset (0, +\infty)$ such that

$$w(z_1) = w(z_2) = 0 \qquad \text{and} \qquad w(x) > 0 \quad \forall x \in (z_1, z_2) \,. \tag{22}$$

We now prove very precise geometric properties of w in these intervals; what follows can be extended to intervals where w is negative.

Lemma 8. *Let $T \geq 0$ and assume that f satisfies* (2)*. Let w be a solution of* (1) *defined on $[0, +\infty)$ and satisfying $H(0) > 0$ and $G(0) \geq 0$. Assume that there exists an interval $[z_1, z_2] \subset (0, +\infty)$ such that* (22) *holds. Then the following facts occur:*
(i) $0 < w'(z_1) < -w'(z_2)$ and there exists a unique $m \in (z_1, z_2)$ such that $w'(m) = 0$;
(ii) $w''(z_2) < 0 < w''(z_1)$, there exists a unique $r \in (z_1, z_2)$ where w'' changes sign, moreover $r \leq m$.

Proof. Since $H(0) > 0$ and $G(0) \geq 0$, by (13) and (14), we know that $0 \leq G(0) < w'(z_1)^2 = G(z_1) < G(z_2) = w'(z_2)^2$. Hence, $0 < w'(z_1) < -w'(z_2)$. Moreover, w cannot admit two critical points in view of Lemma 7. This proves Item (i).

By (14) we infer that $0 < H(0) < w'(z_1)w''(z_1) = H(z_1) < H(z_2) = w'(z_2)w''(z_2)$ which, together with the just proved Item (i), shows that $w''(z_2) < 0 < w''(z_1)$. This implies the existence of a first point $r \in (z_1, z_2)$ such that $w''(r) = 0$ and w'' changes sign at r. Since Lemma 7 ensures that $w'' < 0$ on (m, z_2), we have $r \in (z_1, m]$ and we need only show that r is unique. If not, there exists a second point $\sigma \in (r, m]$ such that $w''(\sigma) = 0$ and w'' changes sign at σ. Since w'' changes from positive to negative at the point r, we must have $w'''(r) \leq 0$. Similarly, we have $w'''(\sigma) \geq 0$. Hence,

$$w'(r)w'''(r) \leq 0 \leq w'(\sigma)w'''(\sigma) \,. \tag{23}$$

Moreover, since $w''(x) < 0$ for $x \in (r, \sigma)$, we have $0 < w'(\sigma) < w'(r)$ and, in turn,

$$0 \geq -\frac{T}{2}\, w'(\sigma)^2 \;\geq\; -\frac{T}{2}\, w'(r)^2 \tag{24}$$

with strict inequalities if $T > 0$. Finally, recalling that (2) implies the monotonicity of F in $[0, \infty)$, since $w'(x) > 0$ for $x \in (r, \sigma)$, we have $F(w(r)) < F(w(\sigma))$. Combined with (23) and (24), this gives

$$\mathcal{E}(r) = \frac{T}{2}\, w'(r)^2 - w'(r)w'''(r) - F(w(r)) > \frac{T}{2}\, w'(\sigma)^2 - w'(\sigma)w'''(\sigma) - F(w(\sigma)) = \mathcal{E}(\sigma)$$

in contradiction with (11). □

We now introduce two further energy functions. Let $w = w(x)$ be a local solution of (1) and let

$$\Phi(x) := \frac{w''(x)^2}{2} + F(w(x)) \,, \qquad \Psi(x) := w''(x)^2 + \frac{T}{2} w'(x)^2 - w'(x)w'''(x) \,. \tag{25}$$

Lemma 9. *Assume f is increasing with $f(0) = 0$ and satisfies* (7)*. Let $T \geq 0$ and let $w = w(x)$ be a nontrivial local solution of* (1)*. Then Φ and Ψ are strictly convex functions. Moreover, if w admits a local maximum at some m such that $w(m) > 0$ and $H(m) > 0$, then Φ and Ψ are strictly increasing for $x \geq m$.*

Proof. Since f is increasing, our assumption (7) implies that

$$f'(w) \geq 0 \qquad \forall w \neq 0 \,. \tag{26}$$

By differentiating and by using (1) we obtain

$$\Phi'(x) = w''(x)w'''(x) + f(w(x))w'(x) \,, \qquad \Phi''(x) = w'''(x)^2 + Tw''(x)^2 + f'(w(x))w'(x)^2 \,.$$

By (26) and recalling that $T \geq 0$, we obviously have $\Phi''(x) > 0$ for almost all x, except for at most some isolated x where $\Phi''(x) = 0$ or where Φ'' is not defined (when $w(x) = 0$). If the local maximum m exists, the assumptions $w(m) > 0$ and $H(m) > 0$ imply that $w'''(m) < 0$, that is, $\Phi'(m) \geq 0$ and hence $\Phi'(x) > 0$ for all $x > m$. This proves the statements for Φ. Since $\mathcal{E}(x) = \Psi(x) - \Phi(x)$, by (11) we obtain $\Psi'(x) = \Phi'(x)$ and $\Psi''(x) = \Phi''(x)$, which prove the statements also for Ψ. □

Finally, we prove a crucial result regarding solutions which are eventually of one sign. If $T \leq 0$, we recall from [2, Theorem 4] that no such solutions exist, if one assumes (2) and

$$\liminf_{w \to \pm\infty} |f(w)| > 0 \,. \tag{27}$$

Proposition 10. *Let $T \leq 0$ and let f satisfy* (2) *and* (27)*. If w is a nontrivial global solution of* (1)*, then $w(x)$ changes sign infinitely many times as $x \to +\infty$ and as $x \to -\infty$.*

It is also known [2] that, under the sole assumption (2), this phenomenon may not occur when $T > 0$. In our next result, we show that solutions which are eventually of one sign must necessarily decay to zero. To see that such solutions may actually exist, we note that $w = \alpha e^{-x}$ satisfies (1) with $T = 2$ and $f(w) = w$ for any $\alpha \neq 0$, while $f(w) = w$ satisfies (7) and (8) near $w = 0$.

Lemma 11. *Let $T > 0$ and assume that f satisfies* (7) *and* (8)*. If w is a solution of* (1) *on $[0, +\infty)$ and w is eventually of one sign, then its energy* (10) *is equal to zero and we have*

$$\lim_{x \to +\infty} H(x) = \lim_{x \to +\infty} w(x) = 0.$$

Proof. Assume that w is eventually positive, the other case being similar. We claim that w'' is eventually of one sign as well. If not, consider an interval (a, b) where $w(x) \geq 0$, $w''(x) < 0$ and $w''(a) = w''(b) = 0$. Multiplying (1) by w'' and integrating, we then obtain

$$-\int_a^b w'''(x)^2\,dx - T\int_a^b w''(x)^2\,dx = -\int_a^b w''(x) f(w(x))\,dx \geq 0.$$

Since the left hand side is the sum of non-positive terms, we get $w''' \equiv 0$ in (a, b); this makes w'' constant in (a, b) and hence $w'' \equiv 0$ in (a, b), being $w''(a) = w''(b) = 0$, a contradiction.

If the conclusions of Lemma 6(i) hold, then $w(x) \to 0$ as $x \to \infty$, so

$$\lim_{x \to \infty} w'''(x) = \lim_{x \to \infty} w''(x) = \lim_{x \to \infty} w'(x) = \lim_{x \to \infty} w(x) = 0 \tag{28}$$

by [2, Proposition 1] and the result follows easily. Otherwise, the conclusions of Lemma 6(ii) hold, instead. Since $w(x)$ attains a limit as $x \to \infty$, we must then have

$$\lim_{x \to \infty} H(x) = \lim_{x \to \infty} w(x) = +\infty. \tag{29}$$

Using our definition (13), we conclude that

$$\left(\frac{w''(x)}{w(x)} - T \log|w(x)| \right)' = -\frac{H(x)}{w(x)^2} < 0 \tag{30}$$

for large enough x, so the expression in brackets is bounded from above as $x \to \infty$. This implies

$$\begin{aligned} \frac{f(w(x)) - Tw''(x)}{w(x)} &= \left(\frac{f(w(x))}{w(x)} - T^2 \log|w(x)| \right) + T \left(T \log|w(x)| - \frac{w''(x)}{w(x)} \right) \\ &\geq \left(\frac{f(w(x))}{w(x)} - T^2 \log|w(x)| \right) + c_1 T \end{aligned}$$

as $x \to \infty$. Recalling (29) and also (8), we conclude that

$$\lim_{x\to\infty} [f(w(x)) - Tw''(x)] = +\infty. \tag{31}$$

In view of (1), this implies $w''''(x) \to -\infty$ as $x \to \infty$, which is contrary to (29). □

We conclude this section by remarking that the first part of (8) implies that $F(w) \geq c|w|^{2+\delta}$ for all $w \in \mathbb{R}$. Therefore, we have

$$\frac{1}{|w|} \geq \frac{c}{F(w)^{1/(2+\delta)}} \qquad \forall w \neq 0\ . \tag{32}$$

5 Proof of Theorem 3

Step 1. Organization of the proof.
Denote by $[0, R)$ the maximal interval of continuation of the local solution $w = w(x)$. In order to prove that $R < +\infty$, we need some delicate estimates, see Steps 2-3-4-5 below. Once these estimates are obtained, in Step 6 we prove that $R < +\infty$. However, before doing this, we need to remark some preliminary facts, regardless of whether R is finite or infinite.

Recall that we are assuming either $H(0) > 0$ or $\mathcal{E}(0) \neq 0$. In the former case, we have $G(x), H(x) \to \infty$ as $x \to R$ by Lemma 6(ii), so we may assume that $G(0), H(0) > 0$ by using a translation, if necessary. In the latter case, the conclusions of Lemma 6(i) cannot hold. If they did hold, then (28) would also hold by [2, Proposition 1] and the conserved energy (10) would be zero, contrary to assumption. In particular, the conclusions of Lemma 6(ii) hold, so we may still assume that $G(0), H(0) > 0$ by using a translation, if necessary. We may thus invoke Lemma 8 in what follows.

From Proposition 1 in the case $R < +\infty$ and from Lemma 11 in the case $R = +\infty$, we infer that there exists an increasing sequence $\{z_j\}_{j\in\mathbb{N}}$ such that:
(i) $z_j \nearrow R$ as $j \to \infty$;
(ii) $w(z_j) = 0$ and w has constant sign in (z_j, z_{j+1}) for all $j \in \mathbb{N}$.
Furthermore, in each interval (z_j, z_{j+1}) where $w(x) > 0$ the following facts occur:
(iii) $0 < w'(z_j) < -w'(z_{j+1})$ and there exists a unique $m_j \in (z_j, z_{j+1})$ such that $w'(m_j) = 0$;
(iv) $w''(z_{j+1}) < 0 < w''(z_j)$, there exists a unique $r_j \in (z_j, z_{j+1})$ where w'' changes sign, and $r_j \leq m_j$.
Similar facts as (iii)-(iv) (with obvious changes) occur in intervals (z_j, z_{j+1}) where $w(x) < 0$. So, for all j let $m_j \in (z_j, z_{j+1})$ be the point where $|w(x)|$ attains its maximum on $[z_j, z_{j+1}]$ and let $M_j = w(m_j)$. If $R < +\infty$, by Proposition 1 we infer that

$$\limsup_{j\to\infty} |M_j| = +\infty\ .$$

Therefore, there exists a subsequence $\{m_h\} \subset \{m_j\}$ such that

$$\lim_{h\to\infty} |M_h| = +\infty\ .$$

In view of (8) and (11) we then infer

$$\lim_{h\to\infty} w''(m_h)^2 = 2 \lim_{h\to\infty} [F(M_h) + C] = +\infty\ .$$

Hence, recalling the definition of G in (12) and noticing that $w(m_h)w''(m_h) < 0$, we get

$$\lim_{h\to\infty} G(m_h) = \lim_{h\to\infty} \left[\frac{T}{2} M_h^2 - M_h w''(m_h)\right] = +\infty\ .$$

By (14) we then deduce that $\lim_{x \nearrow R} G(x) = +\infty$ without extracting subsequences. In particular, we get that

$$\lim_{j\to\infty} G(m_j) = \lim_{j\to\infty} \left[\frac{T}{2}M_j^2 - M_j w''(m_j)\right] = +\infty \tag{33}$$

on the whole sequence $\{m_j\}$ of maxima of $|w(x)|$. Using again (11) we obtain that

$$|w''(m_j)| = \sqrt{2(F(M_j)+C)} \tag{34}$$

which, replaced into (33), proves that

$$\lim_{j\to+\infty} |M_j| = +\infty \tag{35}$$

whenever $R < +\infty$. If $R = +\infty$, in what follows we assume that the local solution $w = w(x)$ can be continued as $x \to +\infty$ so that w is defined (at least) on $[0, +\infty)$. By Lemma 11 we know that $w(x)$ changes sign infinitely many times as $x \to +\infty$. Note that by (12) and Lemma 6(ii), we have again (33), whereas by (11), we have again (34). Hence, we obtain (35) also in the case $R = +\infty$.

We now prove some estimates related to the points found in Lemma 8. For the sake of simplicity, we denote by (z_j, z_{j+1}) an interval where $w(x) > 0$ and by (z_{j-1}, z_j) an interval where $w(x) < 0$; moreover, we put $M_j = w(m_j) > 0$ and $M_{j-1} = w(m_{j-1}) < 0$. Clearly the estimates below can be reversed on intervals where w has the opposite sign.

Step 2. We prove that

$$\lim_{j\to\infty} (r_j - z_j) = 0\,. \tag{36}$$

Assume for contradiction that the claim is false so that $\limsup_{j\to\infty}(r_j - z_j) > 0$. Then there exists $a > 0$ and a subsequence (still denoted in the same way) such that $(r_j - z_j) \geq a$ for all j. By Lemma 6 we know that $w'(z_j)^2 = G(z_j) \to +\infty$ as $j \to \infty$. In turn, by Lemma 8, we know that $w'(x) \to +\infty$ for all $x \in [z_j, r_j]$ as $j \to \infty$. Finally, this proves that

$$w(z_j + \sigma) \to +\infty \qquad \forall \sigma \in (0, r_j - z_j) \quad \text{as } j \to \infty\,. \tag{37}$$

Let $h(x) := (x - z_j)^3 (r_j - x)^4$. By (37) and assumption (8), we infer that

$$h(z_j+\sigma) f(w(z_j+\sigma)) - T h''(z_j+\sigma) w(z_j+\sigma) + h''''(z_j+\sigma) w(z_j+\sigma) \to +\infty \qquad \forall \sigma \in (0, r_j - z_j) \tag{38}$$

as $j \to \infty$. Multiply (1) by $h(x)$ and integrate over $[z_j, r_j]$. Since h, h', h'' vanish in $\{z_j, r_j\}$ and $h'''(r_j) = 0$, four integrations by parts yield

$$\int_{z_j}^{r_j} [h(x) f(w(x)) - T h''(x) w(x) + h''''(x) w(x)]\, dx = 0\,.$$

This contradicts (38) unless (36) holds. This proves the claim of Step 2.

Step 3. We prove that there exists $C_1 > 0$ such that if j is sufficiently large, then

$$r_j - z_j \leq \frac{C_1}{F(M_{j-1})^{\delta/4(4+\delta)}}\,. \tag{39}$$

In what follows c denotes a positive constant which depends on δ and which may vary from line to line, and also within the same formula. Let

$$h(x) := \sin^4\left(\pi\, \frac{x - z_j}{r_j - z_j}\right)$$

so that

$$h'(x)=\frac{4\pi}{r_j-z_j}\,\sin^3\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\,\cos\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\,,$$

$$h''(x)=\frac{4\pi^2}{(r_j-z_j)^2}\,\left[3\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)-4\sin^4\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]\,,$$

$$h'''(x)=\frac{8\pi^3}{(r_j-z_j)^3}\,\left[3\sin\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\,\cos\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)-8\sin^3\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\,\cos\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]\,,$$

$$h''''(x)=\frac{8\pi^4}{(r_j-z_j)^4}\,\left[3-30\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)+32\sin^4\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]\,.$$

Multiply (1) by $h(x)$ and integrate over $[z_j,r_j]$. Since h,h',h'',h''' vanish in $\{z_j,r_j\}$, four integrations by parts yield

$$\int_{z_j}^{r_j} h(x)f(w(x))\,dx=\int_{z_j}^{r_j}[Th''(x)-h''''(x)]w(x)\,dx\,. \tag{40}$$

We now estimate the terms in (40). Before doing this, we need some energy arguments. Since m_{j-1} is a minimum for w we have $w''(m_{j-1})>0$ so that, by (11), we have $w''(m_{j-1})=\sqrt{2(F(M_{j-1})+C)}$. Hence, by (35),

$$G(m_{j-1})=|M_{j-1}|w''(m_{j-1})+\frac{T}{2}M_{j-1}^2\geq|M_{j-1}|\sqrt{2(F(M_{j-1})+C)}\geq c|M_{j-1}|\sqrt{F(M_{j-1})}\,.$$

By taking into account (14), we then infer that

$$w'(z_j)^2=G(z_j)>G(m_{j-1})\geq c|M_{j-1}|\sqrt{F(M_{j-1})}\,.$$

Since $w(x)$ is convex in $[z_j,r_j]$, see Lemma 8, we then deduce

$$w(x)\geq c|M_{j-1}|^{1/2}F(M_{j-1})^{1/4}\,(x-z_j)\qquad\forall x\in[z_j,r_j]\,.$$

In particular, by (8) we also have

$$f(w(x))\geq cw(x)^{1+\delta}\geq c|M_{j-1}|^{\delta/2}F(M_{j-1})^{\delta/4}\,(x-z_j)^\delta\,w(x)\qquad\forall x\in[z_j,r_j]\,. \tag{41}$$

Next, we estimate

$$\begin{aligned}Th''(x)-h''''(x)\;&=\;\frac{4T\pi^2}{(r_j-z_j)^2}\,\left[3\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)-4\sin^4\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]\\&\quad-\frac{8\pi^4}{(r_j-z_j)^4}\,\left[3-30\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)+32\sin^4\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]\\&\leq\;\frac{12T\pi^2}{(r_j-z_j)^2}\,\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)+\frac{8\pi^4}{(r_j-z_j)^4}\,\left[-3+30\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]\\&\leq\;\frac{24\pi^4}{(r_j-z_j)^4}\,\left[-1+11\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]\end{aligned} \tag{42}$$

where the last inequality follows from (36), provided j is sufficiently large. By inserting (41) and (42) into (40) we obtain

$$\begin{aligned}&c|M_{j-1}|^{\delta/2}F(M_{j-1})^{\delta/4}\int_{z_j}^{r_j}\sin^4\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)(x-z_j)^\delta w(x)\,dx\\&\leq\frac{c}{(r_j-z_j)^4}\int_{z_j}^{r_j}\left[-1+11\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]\,w(x)\,dx\,.\end{aligned} \tag{43}$$

Let

$$\gamma := \frac{1}{\pi}\arcsin\frac{1}{\sqrt{11}} \simeq 0.0975$$

and notice that

$$11\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right) \le 1 \qquad \forall x\in[z_j, z_j+\gamma(r_j-z_j)]\cup[r_j-\gamma(r_j-z_j), r_j]\,.$$

Therefore, from (43) we deduce

$$c|M_{j-1}|^{\delta/2}F(M_{j-1})^{\delta/4}\int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)}\sin^4\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)(x-z_j)^\delta w(x)\,dx$$

$$\le \frac{c}{(r_j-z_j)^4}\int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)}\left[-1+11\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\right]w(x)\,dx\,. \tag{44}$$

On the new interval of integration $[z_j+\gamma(r_j-z_j), r_j-\gamma(r_j-z_j)]$ we have uniform bounds such as

$$\sin^4\left(\pi\,\frac{x-z_j}{r_j-z_j}\right) \ge \frac{1}{121}\,, \qquad -1+11\sin^2\left(\pi\,\frac{x-z_j}{r_j-z_j}\right)\le 10\,.$$

Hence, from (44) we finally obtain

$$c|M_{j-1}|^{\delta/2}F(M_{j-1})^{\delta/4}(r_j-z_j)^\delta\int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)} w(x)\,dx$$

$$\le c|M_{j-1}|^{\delta/2}F(M_{j-1})^{\delta/4}\int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)}(x-z_j)^\delta w(x)\,dx \le \frac{c}{(r_j-z_j)^4}\int_{z_j+\gamma(r_j-z_j)}^{r_j-\gamma(r_j-z_j)} w(x)\,dx\,.$$

In view of (35), from the latter inequality we deduce (39) for some $C_1>0$.

Step 4. We prove that there exists $C_2>0$ such that if j is sufficiently large, then

$$z_{j+1}-m_j \le \frac{C_2}{F(M_j)^{\delta\alpha/4(2+\delta)}}\,. \tag{45}$$

Let $h(x) := (x-m_j)^2(z_{j+1}-x)^3$ and note that

$$h(m_j)=h'(m_j)=h(z_{j+1})=h'(z_{j+1})=h''(z_{j+1})=0\,. \tag{46}$$

For all $\ell\in\{0,1,2,3,4\}$ let $h^{(\ell)}$ denote the ℓ-th derivative of h. Clearly, $h^{(\ell)}$ is a linear combination of polynomials such as $(x-m_j)^a(z_{j+1}-x)^b$ with $a+b=5-\ell$. Therefore,

$$\forall\ell\in\{0,1,2,3,4\}\quad \exists c_\ell>0\quad \text{such that } |h^{(\ell)}(x)|\le c_\ell(z_{j+1}-m_j)^{5-\ell} \qquad \forall x\in[m_j,z_{j+1}]\,. \tag{47}$$

Recall that $w(m_j)=M_j$ and $w(z_{j+1})=w'(m_j)=0$; then, by (46), four integrations by parts yield

$$-\int_{m_j}^{z_{j+1}} w''''(x)h(x)\,dx = -\int_{m_j}^{z_{j+1}} w(x)h''''(x)\,dx - h'''(m_j)M_j$$

so that, by (47),

$$\begin{aligned}-\int_{m_j}^{z_{j+1}} w''''(x)h(x)\,dx &\le c_4(z_{j+1}-m_j)\int_{m_j}^{z_{j+1}} w(x)\,dx + c_3(z_{j+1}-m_j)^2M_j\\ &\le (c_3+c_4)(z_{j+1}-m_j)^2M_j\,. \end{aligned} \tag{48}$$

Similarly, two integrations by parts yield

$$T\int_{m_j}^{z_{j+1}} w''(x)h(x)\,dx = T\int_{m_j}^{z_{j+1}} w(x)h''(x)\,dx \le c_2 T(z_{j+1}-m_j)^4 M_j\ . \tag{49}$$

By Lemma 8 we know that w is concave over $[m_j, z_{j+1}]$ so that

$$w(x) \ge \frac{M_j(z_{j+1}-x)}{z_{j+1}-m_j} \qquad \forall x\in[m_j, z_{j+1}]\,. \tag{50}$$

In particular, due to (35), if $\lambda\in(0,1)$ is as in (9), we have

$$w(x)\ge \lambda M_j \to \infty \qquad \forall x\in[m_j,(1-\lambda)z_{j+1}+\lambda m_j]\,.$$

Therefore, provided j is sufficiently large in such a way that $\lambda M_j \ge \tau$, by (8) and (32) we get

$$f(w(x)) \ge c\,\frac{F(w(x))}{w(x)} \ge c\,F(w(x))^{(1+\delta)/(2+\delta)} \qquad \forall x\in[m_j,(1-\lambda)z_{j+1}+\lambda m_j]\,. \tag{51}$$

By (51), (50), and monotonicity of F we infer that

$$\begin{aligned} I &:= \int_{m_j}^{z_{j+1}} h(x)f(w(x))\,dx \ge \int_{m_j}^{(1-\lambda)z_{j+1}+\lambda m_j} h(x)f(w(x))\,dx \\ &\ge c\int_{m_j}^{(1-\lambda)z_{j+1}+\lambda m_j} h(x)F\left(\frac{M_j(z_{j+1}-x)}{z_{j+1}-m_j}\right)^{(1+\delta)/(2+\delta)} dx\ . \end{aligned}$$

By replacing h and by means of the change of variables $\sigma=(z_{j+1}-x)/(z_{j+1}-m_j)$, we then obtain

$$I \ge c(z_{j+1}-m_j)^6\int_\lambda^1 (1-\sigma)^2\sigma^3 F(M_j\sigma)^{(1+\delta)/(2+\delta)}\,d\sigma\ .$$

Moreover, by monotonicity of F and by using (8) and (9), we get

$$\begin{aligned} I &\ge c(z_{j+1}-m_j)^6\,F(\lambda M_j)^{(1+\delta)/(2+\delta)}\int_\lambda^1 (1-\sigma)^2\sigma^3\,d\sigma \ge c(z_{j+1}-m_j)^6\,F(\lambda M_j)^{\delta/(2+\delta)}M_j \\ &\ge c(z_{j+1}-m_j)^6\,F(M_j)^{\delta\alpha/(2+\delta)}M_j\ . \end{aligned}$$

Recalling also (48)-(49) we have so obtained that

$$c(z_{j+1}-m_j)^6\,F(M_j)^{\delta\alpha/(2+\delta)}M_j \le \int_{m_j}^{z_{j+1}} f(w(x))h(x)\,dx = -\int_{m_j}^{z_{j+1}} [w''''(x)-Tw''(x)]h(x)\,dx$$

$$\le c(z_{j+1}-m_j)^2M_j + cT(z_{j+1}-m_j)^4M_j\ ,$$

that is,

$$F(M_j)^{\delta\alpha/(2+\delta)}\,(z_{j+1}-m_j)^4 - cT(z_{j+1}-m_j)^2 - c \le 0\ .$$

By solving this biquadratic algebraic inequality we infer that (45) holds for some $C_2>0$.

Step 5. We prove that there exists $C_3>0$ such that if j is sufficiently large, then

$$m_j - r_j \le \frac{C_3}{F(M_j)^{\delta\alpha/4(2+\delta)}}\ . \tag{52}$$

We use a slightly different test function h and we proceed as in Step 4; therefore, we omit some details. Let $h(x) := (x - r_j)^4 (m_j - x)^2$ and note that

$$h(m_j) = h'(m_j) = h(r_j) = h'(r_j) = h''(r_j) = h'''(r_j) = 0 \,. \tag{53}$$

For all $\ell \in \{0, 1, 2, 3, 4\}$ let $h^{(\ell)}$ denote the ℓ-th derivative of h. Clearly, $h^{(\ell)}$ is a linear combination of polynomials such as $(x - r_j)^a (m_j - x)^b$ with $a + b = 6 - \ell$. Therefore,

$$\forall \ell \in \{0, 1, 2, 3, 4\} \quad \exists c_\ell > 0 \quad \text{such that } |h^{(\ell)}(x)| \le c_\ell (m_j - r_j)^{6-\ell} \qquad \forall x \in [r_j, m_j] \,. \tag{54}$$

Recall that $w(m_j) = M_j$ and $w'(m_j) = 0$; then, by (53), four integrations by parts yield

$$-\int_{r_j}^{m_j} w''''(x) h(x)\, dx = -\int_{r_j}^{m_j} w(x) h''''(x)\, dx + h'''(m_j) M_j$$

so that, by (54),

$$-\int_{r_j}^{m_j} w''''(x) h(x)\, dx \le c_4 (m_j - r_j)^2 \int_{r_j}^{m_j} w(x)\, dx + c_3 (m_j - r_j)^3 M_j \le (c_3 + c_4)(m_j - r_j)^3 M_j \,. \tag{55}$$

Similarly, two integrations by parts yield

$$T \int_{r_j}^{m_j} w''(x) h(x)\, dx = T \int_{r_j}^{m_j} w(x) h''(x)\, dx \le c_2 T (m_j - r_j)^5 M_j \,. \tag{56}$$

By Lemma 8 we know that w is concave over $[r_j, m_j]$ so that

$$w(x) \ge \frac{M_j (x - r_j)}{m_j - r_j} \qquad \forall x \in [r_j, m_j] \,.$$

In particular, if $\lambda \in (0, 1)$ is as in (9), we have

$$w(x) \ge \lambda M_j \to +\infty \qquad \forall x \in [(1 - \lambda) r_j + \lambda m_j, m_j] \,.$$

We now use the counterpart of (51), together with the monotonicity of F, to infer

$$\begin{aligned} I \;&:=\; \int_{r_j}^{m_j} h(x) f(w(x))\, dx \ge \int_{(1-\lambda) r_j + \lambda m_j}^{m_j} h(x) f(w(x))\, dx \\ &\ge\; c \int_{(1-\lambda) r_j + \lambda m_j}^{m_j} h(x) F\left(\frac{M_j (x - r_j)}{m_j - r_j} \right)^{(1+\delta)/(2+\delta)} dx \,. \end{aligned}$$

By means of the change of variables $\sigma = (x - r_j)/(m_j - r_j)$, we then obtain

$$I \ge c (m_j - r_j)^7 \int_\lambda^1 (1 - \sigma)^2 \sigma^4 F(M_j \sigma)^{(1+\delta)/(2+\delta)}\, d\sigma \,.$$

Moreover, by monotonicity of F and by using (8) and (9), we get

$$I \ge c (m_j - r_j)^7 F(\lambda M_j)^{(1+\delta)/(2+\delta)} \int_\lambda^1 (1 - \sigma)^2 \sigma^4\, d\sigma \ge c (m_j - r_j)^7\, F(M_j)^{\delta\alpha/(2+\delta)} M_j \,.$$

By combining this estimate with (55)-(56), we obtain

$$F(M_j)^{\delta\alpha/(2+\delta)}\, (m_j - r_j)^4 - c T (m_j - r_j)^2 - c \le 0 \,.$$

By solving this biquadratic algebraic inequality we infer that (52) holds for some $C_3 > 0$.

Step 6. We show that $R < +\infty$.

Using Lemma 9 and the monotonicity of F in $[0, \infty)$, we get

$$\begin{aligned} F(M_j) &= F(w(m_j)) > F(w(r_j)) = \Phi(r_j) \\ &> \Phi(m_{j-1}) = \frac{w''(m_{j-1})^2}{2} + F(w(m_{j-1})) = 2F(M_{j-1}) + C \end{aligned} \tag{57}$$

for all $j \in \mathbb{N}$, the last equality being a consequence of

$$\mathcal{E}(m_{j-1}) = \frac{w''(m_{j-1})^2}{2} - F(w(m_{j-1})) = C$$

which holds in view of (11). In particular, (57) shows that

$$j \mapsto F(M_j) \quad \text{is eventually increasing and} \quad \lim_{j\to\infty} F(M_j) = +\infty \, . \tag{58}$$

By iterating (57) we find $F(M_j) > 2^j[F(M_0) + C] - C$ for all $j \geq 1$. In turn, by (58) we may relabel the indices j (in such a way that $F(M_0) > -2C$) and obtain

$$F(M_j) > 2^{j-1}F(M_0) \qquad \forall j \in \mathbb{N} \, . \tag{59}$$

By combining (39), (45), (52) and (59), we readily obtain that

$$z_{j+1} - z_j \leq \frac{C_1}{F(M_{j-1})^{\delta/4(4+\delta)}} + \frac{C_2 + C_3}{F(M_j)^{\delta\alpha/4(2+\delta)}} \leq \frac{c}{F(M_{j-1})^{\eta}} \leq \frac{c}{(2^\eta)^j} \qquad \forall j \geq 1 \tag{60}$$

where $\eta = \min\{\frac{\delta\alpha}{4(2+\delta)}, \frac{\delta}{4(4+\delta)}\}$ and where we used the monotonicity of $j \mapsto F(M_j)$. Finally, we obtain

$$R - z_0 = \sum_{j=0}^{\infty}(z_{j+1} - z_j) \leq \sum_{j=0}^{\infty} \frac{c}{(2^\eta)^j} < +\infty$$

since the geometric series has ratio $\left(\frac{1}{2}\right)^\eta < 1$. Therefore, $R < +\infty$ and the solution blows up in finite space.

6 Proof of Theorem 4

If $\mathcal{E}(0) \neq 0$, then blow up occurs by Theorem 3. The same is true by translation, if $H(x) > 0$ for some $x \geq 0$. In particular, we have $\mathcal{E}(0) = 0$ and $H(x) \leq 0$ for all $x \geq 0$. Using Lemma 6(i), we conclude that both $w(x)$ and $H(x)$ approach zero as $x \to \infty$. Since $H(x)$ is increasing by (14), the result follows.

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