The steady two-dimensional flow over a rectangular obstacle lying on the bottom

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Abstract

We study a plane problem with mixed boundary conditions for a harmonic function in an unbounded Lipschitz domain contained in a strip. The problem is obtained by linearizing the hydrodynamic equations which describe the steady flow of a heavy ideal fluid over an obstacle lying on the flat bottom of a channel. In the case of obstacles of rectangular shape we prove unique solvability for all velocities of the (unperturbed) flow above a critical value depending on the obstacle depth. We also discuss regularity and asymptotic properties of the solutions.

Key words: Linear water waves; Polygonal boundary

1 Introduction

A well known problem in hydrodynamics is the determination of the steady flow of a heavy ideal fluid, in a channel of finite depth, over localized perturbations of a horizontal bottom. Assuming the usual hypotheses, i.e., irrotational and divergence-free flow, non viscous fluid and negligible surface tension, we get a problem for the Laplace equation in an unbounded domain, with a non linear condition (the Bernoulli condition) on a free boundary (the free surface). Such a problem has been widely studied by analytical and numerical methods [1]-[3]; however, little is known about its solvability from a rigorous point of view, due to the difficulties related to the free boundary. Thus, the mathematical approach to this problem, even in the two-dimensional case, deals with a linearized version (in a domain with a fixed boundary) called

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the Neumann-Kelvin problem; by a suitable choice of the reference frame, the same problem also describes the (linear) *ship waves* generated in the fluid by the uniform horizontal motion of a submerged body [6]. The linear theory often gives results in good accordance with experimental data in situations of practical interest. Besides, from a mathematical point of view, the solution of the linear problem may represent a crucial step in the proof of the existence of solutions to the nonlinear, free-boundary problem [7]. For these reasons, it is a relevant question to determine whether the linear problem for a given obstacle in a current is uniquely solvable for all values of the flux velocity. It is a known fact that the answer depends on the geometry of the obstacle; for example, there exists a sufficient condition on the body profile [4] for the uniqueness of a solution with a finite Dirichlet integral. However, such condition seems to be applicable only in special cases [5] since the solutions of the plane problem can not be assumed to have finite energy for every value of the velocity. In fact, the (a priori) asymptotic properties of these solutions depend critically on the value of the Froude number F_r, defined by

$$F_r = \frac{c^2}{gH},$$

where c is the velocity of the fluid at infinity upstream, g the acceleration of gravity and H the channel depth. If $F_r > 1$ (supercritical regime) every solution is exponentially decreasing at infinity; if $F_r < 1$ (subcritical regime), on the contrary, non vanishing oscillations at infinity downstream may occur, preventing the solution from having finite energy. Correspondingly, with a supercritical flow there is unique solvability of the Neumann-Kelvin problem for an arbitrary number of obstacles of generic shape, totally or partially immersed [8]. In the subcritical regime, instead, existence and uniqueness (for every subcritical value of the velocity) have been proved for a submerged cylinder [5] and for a surface-piercing obstacle with symmetric, non bulbous profile [9].

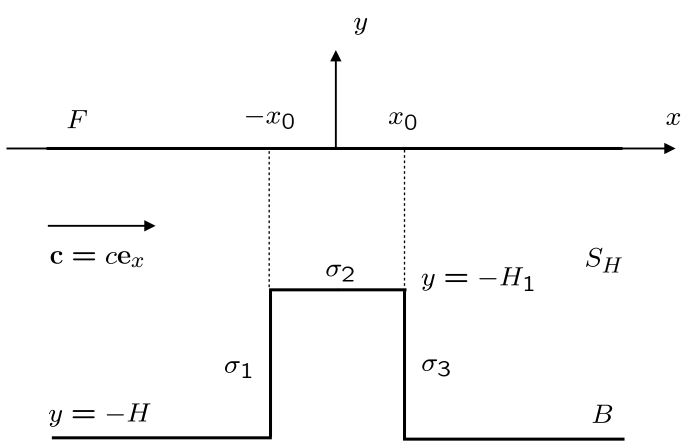
The first approach historically introduced to study the Neumann-Kelvin problem uses a suitable Green function to transform it in an integral equation [6], but in the present work we will follow a variational technique [9], which seems more suitable for the kind of obstacle considered. In Section 2 we introduce the plane Neumann-Kelvin problem for a rectangular obstacle lying on the bottom and describe a variational formulation in terms of a *perturbed stream function*. According to the previous discussion, if the flow is subcritical at infinity upstream the weak formulation of the problem in the usual Sobolev space H^1 presents some difficulties; in fact, it turns out that the associated bilinear form is not coercive. However, by assuming that the flow is *supercritical in the region of fluid above the obstacle*, we are able to find a subspace where coercivity holds by exploiting some a priori properties of the solutions. In Section 3 we discuss the main properties of the variational solution, including a careful analysis of its regularity; in particular, we show that the (distributional) laplacian of a weak solution has singularities, so that a regularization is required in order to obtain a harmonic solution. To this aim we introduce, for any considered value of the (unperturbed) velocity, two special variational solutions with Dirichlet data equal to the traces on the obstacle boundary of two independent periodic solutions of the free problem (no obstacle in the channel). The regularization procedure is completed in section 4 and allow us to prove unique solvability for every value of the flow velocity above a critical threshold, depending on the depth of the obstacle. The solvability of the problem for all the velocities remains an open question; we discuss some conjectures in the last section.

2 Strong and variational formulation of the problem

Let us consider the two-dimensional flow in a channel of constant depth H when a rectangular obstacle of height $H - H_1$ and width $2x_0$ lies on the bottom. If the perturbations of the free surface with respect to the horizontal plane are small, it is reasonable to approximate the region occupied by the fluid with the domain proper of calm water and the velocity field with its first order expansion

$$\mathbf{U} = \mathbf{c} + \nabla\phi,$$

where $\mathbf{c}$ is the velocity at infinity upstream.



If we choose a cartesian frame as depicted in the figure and define

$$\begin{aligned}
&F = \mathbb{R} \times \{0\}, \qquad \sigma_1 = \{-x_0\} \times (-H, -H_1), \\
&\sigma_2 = (-x_0, x_0) \times \{-H_1\}, \qquad \sigma_3 = \{x_0\} \times (-H, -H_1), \\
&B = \{(-\infty, -x_0) \cup (x_0, +\infty)\} \times \{-H\}, \\
&S_H = \{\mathbb{R} \times (-H, 0)\} \setminus \overline{\{\sigma_2 \times (-H, -H_1)\}},
\end{aligned}$$

it may be shown that the perturbed potential ϕ satisfies the system:

$$\begin{aligned}
&\Delta\phi = 0 \qquad \text{in } S_H, \\
&\phi_{xx} + \frac{g}{c^2}\phi_y = 0 \qquad \text{on } F, \\
&\frac{\partial\phi}{\partial n} = -c\mathbf{n}_i \cdot \mathbf{e}_x \qquad \text{on } \sigma_i,\ i = 1,2,3, \\
&\phi_y = 0 \qquad \text{on } B,
\end{aligned}$$

together with the asymptotic conditions

$$\begin{aligned}
&\sup_{S_H \setminus A} |\nabla\phi| < +\infty, \\
&\lim_{x\to-\infty} |\nabla\phi\,(x,y)| = 0,
\end{aligned}$$

where A is any neighborhood of the obstacle. Here $\mathbf{n}_i$ is the unit outward normal on σ_i and $c = |\mathbf{c}|$. From now on we will set $\nu = g/c^2$. We will study the subcritical regime ($\nu H > 1$) as the solvability of the problem for $\nu H < 1$ is known (see the discussion in the introduction). Since the domain S_H is simply connected, we can formulate the problem in a more convenient way in terms of the *perturbed stream function* ψ, which is a harmonic conjugate of ϕ vanishing for $x \to -\infty$. Then we obtain [6]

$$\begin{aligned}
&\Delta\psi = 0 \qquad \text{in } S_H, \\
&\psi_y - \nu\psi = 0 \qquad \text{on } F, \\
&\psi = c\,(y + H) \qquad \text{on } \sigma_1 \text{ and } \sigma_3, \\
&\psi = c\,(H - H_1) \qquad \text{on } \sigma_2, \\
&\psi = 0 \qquad \text{on } B, \\
&\sup_{S_H} |\psi| < +\infty, \\
&\lim_{x\to-\infty} \psi\,(x,y) = 0.
\end{aligned}$$

This problem is a particular case of the following:

Problem $\mathcal{P}$ Given the positive numbers x_0, ν, H, H_1 and the functions $h_i \in H^{3/2}(\sigma_i)$ $(i = 1,2,3)$ such that

$$h_1(-x_0, -H) = 0, \tag{2.1}$$
$$h_1(-x_0, -H_1) = h_2(-x_0, -H_1), \tag{2.2}$$
$$h_2(x_0, -H_1) = h_3(x_0, -H_1), \tag{2.3}$$
$$h_3(x_0, -H) = 0, \tag{2.4}$$

find $\psi \in H^1_{loc}(S_H)$ satisfying

$$\Delta\psi = 0 \qquad \text{in } S_H, \tag{2.5}$$
$$\psi_y - \nu\psi = 0 \qquad \text{on } F, \tag{2.6}$$
$$\psi = h_i \qquad \text{on } \sigma_i,\ i = 1, 2, 3, \tag{2.7}$$
$$\psi = 0 \qquad \text{on } B, \tag{2.8}$$
$$\sup_{S_H} |\psi| < +\infty, \tag{2.9}$$
$$\lim_{x\to-\infty} \psi(x, y) = 0. \tag{2.10}$$

The symbol $\mathcal{P}^*$ will denote problem $\mathcal{P}$ without condition (2.10). It is worth to say that the equalities (2.1), (2.2), (2.3) and (2.4) are verified by the data of the physical problem and are necessary for the existence of a weak solution, because the trace on the obstacle's boundary of functions in $H^1(S_H)$ is not onto the space $\prod_{i=1}^{3} H^{1/2}(\sigma_i)$. The presence of compatibility conditions for Dirichlet data is a typical feature of polygonal boundaries [12].

The variational form of problem $\mathcal{P}$ can now be stated in the subspace of the functions of $H^1(S_H)$ vanishing on B equipped with the Dirichlet norm (which is equivalent to the H^1 norm by the Poincaré inequality). By standard arguments [9], we get:

Find $\psi \in H^1(S_H)$ satisfying (2.7), (2.8) and such that

$$a(\psi, v) = \int_{S_H} \nabla\psi \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y - \nu \int_{-\infty}^{+\infty} \psi(x, 0)\, v(x, 0)\,\mathrm{d}x = 0 \tag{2.11}$$

for every $v \in H^1_*(S_H)$, where

$$H^1_*(S_H) = \left\{ f \in H^1(S_H) : \ f = 0 \text{ on } B,\ f = 0 \text{ on } \sigma_i,\ i = 1, 2, 3 \right\}.$$

For $\nu > 1/H$, the continuous bilinear form a is not coercive in the subspace $H^1_*(S_H)$ (endowed with the Dirichlet norm); however, the following a priori property of the solutions gives us a hint of a subspace where coercivity may hold:

Lemma 2.1 *Let $f \in H^1(B \times (-H, 0))$ be a harmonic function satisfying (2.6) and (2.8). Then we have*

$$\int_{-H}^{0} \sinh(\nu_0 (y + H))\, \psi(x, y)\,\mathrm{d}y = 0 \qquad |x| > x_0,$$

where $\nu_0 > 0$ is the unique positive solution of the equation

$$\tanh(\nu_0 H) = \frac{\nu_0}{\nu}. \tag{2.12}$$

The proof is the same as in [9], section 2.1. □

Lemma 2.1 suggests to define a closed subspace of $H^1(S_H)$ as follows:

$$V_* = \Big\{ f \in H^1(S_H) : \ f = 0 \text{ on } B, \ f = 0 \text{ on } \sigma_i, \ i = 1,2,3, \\ \int_{-H}^{0} \sinh(\nu_0 (y+H)) f(x,y) \, \mathrm{d}y = 0 \qquad |x| > x_0 \Big\}.$$

Let us now assume that the following condition holds:

$$\nu H_1 < 1. \tag{2.13}$$

This amounts to consider flow velocities *above* the critical threshold $\sqrt{gH_1}$, depending on the height of the obstacle. In that case, the subspace V_* is the correct set of test functions for the variational formulation; in fact

Proposition 2.2 *The form a is coercive in V_*.*

Proof. For $f \in V_*$, integrating by parts the orthogonality relation

$$\int_{-H}^{0} \sinh(\nu_0 (y+H)) f(x,y) \, \mathrm{d}y = 0$$

we have

$$f(x,0) = \frac{1}{\cosh(\nu_0 H)} \int_{-H}^{0} \cosh(\nu_0 (y+H)) f_y(x,y) \, \mathrm{d}y$$

with $|x| > x_0$ and then

$$\nu \int_{\mathbb{R}\setminus[-x_0,x_0]} |f(x,0)|^2 \, \mathrm{d}x \le \alpha \int_{S_H \setminus \overline{R}_0} |\nabla f|^2 \, \mathrm{d}x \, \mathrm{d}y,$$

where $R_0 = (-x_0, x_0) \times (-H_1, 0)$ and

$$\alpha = \frac{1}{2} \left(1 + \frac{2\nu_0 H}{\sinh(2\nu_0 H)} \right).$$

Moreover, if $-x_0 < x < x_0$, by applying the Hölder inequality to the identity $f(x,0) = \int_{-H_1}^{0} f_y(x,t) \, \mathrm{d}t$, we get

$$\nu \int_{-x_0}^{x_0} |f(x,0)|^2 \, \mathrm{d}x \le \nu H_1 \int_{R_0} |\nabla f|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

Now, since $\alpha < 1$ and we assumed (2.13), coercivity holds by the estimate:

$$a(f,f) \ge \min\{1 - \nu H_1, 1 - \alpha\} \, \|\nabla f\|^2_{L^2(S_H)}.$$

□

Now we readily get

Proposition 2.3 *Let us define*

$$W = \Big\{ f \in H^1(S_H) : \ f = 0 \text{ on } B, \ f = h_i \text{ on } \sigma_i, \ i = 1, 2, 3,$$
$$\int_{-H}^{0} \sinh(\nu_0 (y+H)) f(x,y) \, \mathrm{d}y = 0 \qquad |x| > x_0 \Big\},$$

and let a be the bilinear form in (2.11). Then, there exists only one $\psi \in W$ such that

$$a(\psi, v) = 0 \qquad \forall\, v \in V_*. \tag{2.14}$$

Proof. First of all we notice that a function $f \in H^1(S_H)$ which verifies the Dirichlet conditions of problem $\mathcal{P}$ exists thanks to (2.1), (2.2), (2.3) and (2.4). Furthermore the map

$$\begin{aligned} g : \mathbb{R} \setminus [-x_0, x_0] &\longrightarrow \mathbb{R} \\ x &\longmapsto g(x) = \int_{-H}^{0} \sinh(\nu_0 (y+H)) f(x,y) \, \mathrm{d}y \end{aligned}$$

is in $H^1(\mathbb{R} \setminus [-x_0, x_0])$ and has an extension $w \in H^1(\mathbb{R})$. Then, chosen $\chi \in D(-H, 0)$ with support contained in $(-H_1, 0)$ and such that

$$\int_{-H}^{0} \sinh(\nu_0 (y+H)) \chi(y) \, \mathrm{d}y = 1,$$

the function

$$\begin{aligned} z : \quad S_H &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto z(x,y) = f(x,y) - w(x)\chi(y) \end{aligned}$$

clearly belongs to W and so $W \neq \emptyset$. A simple application of the Lax-Milgram lemma completes the proof. □

We will call variational solution of problem $\mathcal{P}$ the map uniquely determined by Proposition 2.3.

3 Properties of the variational solution

Remembering that a function which satisfies problem $\mathcal{P}$ has not finite energy in general, we expect the application given by Proposition 2.3 to be the true solution only in very special cases.

Theorem 3.1 *Given h_i $(i = 1, 2, 3)$ verifying (2.1), (2.2), (2.3) and (2.4), let $F_1 = (-\infty, -x_0) \times \{0\}$, $F_2 = (-x_0, x_0) \times \{0\}$, $F_3 = (x_0, +\infty) \times \{0\}$. Then* *the variational solution ψ of problem $\mathcal{P}$ is the only function in $H^1(S_H)$ such that*

$$\begin{aligned}
&\Delta\psi = (\lambda_+\delta(x-x_0) + \lambda_-\delta(x+x_0))\sinh(\nu_0(y+H)), && (3.1)\\
&\psi_y - \nu\psi = 0 \qquad \text{on } F_i,\ i=1,2,3, && (3.2)\\
&\psi = h_i \qquad \text{on } \sigma_i,\ i=1,2,3, && (3.3)\\
&\psi = 0 \qquad \text{on } B. && (3.4)
\end{aligned}$$

Here δ is the Dirac delta distribution and λ_+ and λ_- are real constants.

Proof. Chosen $\varphi \in D(S_H)$ with

$$\int_{-H_1}^{0} \sinh(\nu_0(y+H))\,\varphi(\pm x_0, y)\;\mathrm{d}y = 0, \tag{3.5}$$

let us define

$$g(x,y) = \begin{cases} 0 & (x,y) \in R_0, \\ -\dfrac{1}{C(\nu_0)}\alpha(x)\sinh(\nu_0(y+H)) & (x,y) \in S_H \setminus \overline{R_0}, \end{cases}$$

where $R_0 = (-x_0, x_0) \times (-H_1, 0)$,

$$C(\nu_0) = \int_{-H}^{0} \sinh^2(\nu_0(y+H))\;\mathrm{d}y \tag{3.6}$$

and

$$\alpha(x) = \int_{-H}^{0} \sinh(\nu_0(y+H))\,\varphi(x,y)\;\mathrm{d}y \qquad |x| > x_0.$$

Thanks to (3.5) it is simple to check that $w(x,y) = \varphi(x,y) + g(x,y) \in V_*$ and therefore, by Proposition 2.3,

$$\int_{S_H} \nabla\psi \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}y = \nu \int_{-\infty}^{+\infty} \psi(x,0)\,w(x,0)\;\mathrm{d}x. \tag{3.7}$$

Now, from $\psi \in H^1(S_H)$ and

$$\int_{-H}^{0} \sinh(\nu_0(y+H))\,\psi(x,y)\;\mathrm{d}y = 0 \qquad \text{a. e. } |x| > x_0 \tag{3.8}$$

we deduce also

$$\int_{-H}^{0} \sinh(\nu_0(y+H))\,\psi_x(x,y)\;\mathrm{d}y = 0 \qquad \text{a. e. } |x| > x_0. \tag{3.9}$$

Taking account of (3.8), (3.9), (2.12) and of the boundary conditions we get

$$\int_{S_H} \nabla\psi \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}y = \int_{S_H} \nabla\psi \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_{S_H} \nabla\psi \cdot \nabla g \,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{S_H} \nabla\psi \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y - \frac{\nu \sinh(\nu_0 H)}{C(\nu_0)} \int_{\mathbb{R}\setminus[-x_0,x_0]} \psi(x,0)\,\alpha(x)\;\mathrm{d}x. \tag{3.10}$$

On the other hand, it results

$$\nu \int_{-\infty}^{+\infty} \psi(x,0)\,w(x,0)\;\mathrm{d}x = -\frac{\nu \sinh(\nu_0 H)}{C(\nu_0)} \int_{\mathbb{R}\setminus[-x_0,x_0]} \psi(x,0)\,\alpha(x)\;\mathrm{d}x. \tag{3.11}$$

Comparing (3.7), (3.10) and (3.11) we have

$$\int_{S_H} \nabla\psi \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}y = 0 \tag{3.12}$$

and (3.1) follows by standard arguments. The equality (3.2) in F_2 follows easily from the variational equation, so we will focus to prove (3.2) in every subset $(a,b) \times \{0\}$ of F with $x_0 \le a < b$. For the sake of clarity, we indicate the trace and the trace of the normal derivative on $(a,b) \times \{0\}$ with γ and $\gamma\frac{\partial}{\partial y}$ respectively. For $u \in D((a,b) \times \{0\})$, there is $v \in D(\mathbb{R}^2)$ such that

$$v|_{(a,b)\times\{0\}} = u, \tag{3.13}$$
$$\operatorname{supp} v \subseteq \left\{(x,y) \in \mathbb{R}^2 :\, a < x < b,\, y > -H\right\}.$$

Then, taken a smooth function χ as in the proof of Proposition 2.3, if we define

$$m(x,y) = \begin{cases} \left(\int_{-H}^{0} \sinh(\nu_0(y+H))\,v(x,y)\;\mathrm{d}y\right)\chi(y) & (x,y) \in S_H :\, x > x_0, \\ 0 & \text{otherwise in } S_H \end{cases}$$

we have plainly $m \in D(S_H)$,

$$\int_{-H_1}^{0} \sinh(\nu_0(y+H))\,m(\pm x_0, y)\;\mathrm{d}y = 0$$

and $v - m \in V_*$. By Green's formula, Proposition 2.3, (3.12) and (3.13) we get

$$\left\langle \gamma\frac{\partial}{\partial y}(\psi), u \right\rangle = \left\langle \gamma\frac{\partial}{\partial y}(\psi), \gamma(v) \right\rangle = \int_{S_H} \nabla\psi \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{S_H} \nabla\psi \cdot \nabla(v-m)\;\mathrm{d}x\,\mathrm{d}y + \int_{S_H} \nabla\psi \cdot \nabla m \,\mathrm{d}x\,\mathrm{d}y$$
$$= \nu \int_{-\infty}^{+\infty} \psi(x,0)\,(v-m)(x,0)\;\mathrm{d}x = \nu \int_a^b \psi(x,0)\,v(x,0)\;\mathrm{d}x$$
$$= \langle \gamma(\psi), u \rangle.$$

Condition (3.2) follows for density. Moreover, it is apparent that ψ makes (3.3) and (3.4) true. It remains to show uniqueness; to this aim, we first state some regularity result. Let us indicate with θ the characteristic function of the interval $(0,+\infty)$ and introduce the map

$$
\begin{aligned}
s\left(x,y\right) &= \frac{\lambda_+}{\nu_0}\sin\left(\nu_0\left(x-x_0\right)\right)\sinh\left(\nu_0\left(y+H\right)\right)\theta\left(x-x_0\right)\\
&-\frac{\lambda_-}{\nu_0}\sin\left(\nu_0\left(x+x_0\right)\right)\sinh\left(\nu_0\left(y+H\right)\right)\theta\left(-x-x_0\right). \qquad (3.14)
\end{aligned}
$$

Clearly s belongs to $H^1_{loc}\left(S_H\right)$, vanishes on B and σ_i and has a laplacian as in (3.1) by direct computation. Thus, for every $z\in H^1\left(S_H\right)$ which satisfies (3.1) and (3.3), the difference $z-s$ is harmonic and verifies the same boundary conditions; thus, by known regularity results [12], in a neighborhood of $(x_0,-H_1)$ we can write

$$z=(z-s-\mathfrak{S})+s+\mathfrak{S}$$

with $(z-s-\mathfrak{S})\in H^2$ and

$$\mathfrak{S}\left(r,\varphi\right)=Mr^{\frac{2}{3}}\sin\left(\frac{2}{3}\varphi\right)$$

in polar coordinates with origin in $(x_0,-H_1)$ and such that $\varphi=0$ on σ_3 and $\varphi=3\pi/2$ on σ_2 (M is a suitable constant). An equivalent result holds near $(-x_0,-H_1)$. It follows that for a. e. $x\in(-x_0,x_0)$ and a. e. $y\in(-H,-H_1)$ there exist the traces

$$
\begin{aligned}
&z_y\left(\cdot,-H_1\right)\in L^2\left(-x_0,x_0\right), \qquad z_x\left(-x_0,\cdot\right)\in L^2\left(-H,-H_1\right),\\
&z_x\left(x_0,\cdot\right)\in L^2\left(-H,-H_1\right).
\end{aligned}
$$

Let now ψ_1, ψ_2 be two solutions of (3.1) (with possibly different coefficients $\lambda_\pm$) verifying the same conditions (3.3), (3.4) and define $g=\psi_1-\psi_2$. Furthermore, we set

$$
\begin{aligned}
&R_{1,\epsilon,l}=\left(-l,-x_0-\epsilon\right)\times\left(-H,0\right),\; R_{2,\epsilon,l}=\left(-x_0+\epsilon,x_0-\epsilon\right)\times\left(-H_1,0\right),\\
&R_{3,\epsilon,l}=\left(x_0+\epsilon,l\right)\times\left(-H,0\right)
\end{aligned}
$$

with $0<\epsilon<x_0/2$ and $l>x_0+\epsilon$. Then, by Lemma 2.1 and by the previous regularity results, we can calculate the limit for $l\to+\infty$ and $\epsilon\to0^+$ of the first member of the obvious equality

$$\sum_{i=1}^{3}\int_{R_{i,\epsilon,l}}g\Delta g\,\mathrm{d}x\,\mathrm{d}y=0$$

deducing

$$\int_{S_H} |\nabla g|^2 \, \mathrm{d}x \, \mathrm{d}y - \nu \int_{-\infty}^{+\infty} |g(x,0)|^2 \, \mathrm{d}x = 0.$$

Since $g \in V_*$, we get $g = 0$ by the coercivity of the bilinear form on V_*; hence $\psi_1 = \psi_2$. □

Now we investigate if it is possible to "regularize" the variational solution ψ. The first aim consists in removing the singularities of its laplacian.

Proposition 3.2 *Let ψ satisfy (3.1)-(3.4) and let s be given by (3.14). Then the map*

$$\hat{\psi}(x,y) = \psi(x,y) - s(x,y)$$

solves problem $\mathcal{P}^*$.

Proof. In the proof of Theorem 3.1 we demonstrated that $\psi - s$ is harmonic in S_H. Furthermore, since s makes (3.2) true and is null on B and σ_i, trivially $\psi - s$ satisfies the boundary conditions of problem $\mathcal{P}^*$ and is bounded too. □

We now introduce two particular variational solutions whose use will be apparent soon.

Proposition 3.3 *Let ψ^s and ψ^c be the variational solutions corresponding to the data*

$$\begin{aligned}
&h_1^s(-x_0,y) = \sin(\nu_0 x_0)\sinh(\nu_0(y+H)),\\
&h_2^s(x,-H_1) = -\sin(\nu_0 x)\sinh(\nu_0(-H_1+H)),\\
&h_3^s(x_0,y) = -\sin(\nu_0 x_0)\sinh(\nu_0(y+H))
\end{aligned}$$

and

$$\begin{aligned}
&h_1^c(-x_0,y) = -\cos(\nu_0 x_0)\sinh(\nu_0(y+H)),\\
&h_2^c(x,-H_1) = -\cos(\nu_0 x)\sinh(\nu_0(-H_1+H)),\\
&h_3^c(x_0,y) = -\cos(\nu_0 x_0)\sinh(\nu_0(y+H))
\end{aligned}$$

with $-x_0 < x < x_0$ and $-H < y < -H_1$. Then

$$\begin{aligned}
&\psi^s(x,y) = -\psi^s(-x,y) \qquad (x,y) \in S_H,\\
&\Delta\psi^s = \lambda^s(\delta(x-x_0) - \delta(x+x_0))\sinh(\nu_0(y+H)),\\
&\psi^c(x,y) = \psi^c(-x,y) \qquad (x,y) \in S_H,\\
&\Delta\psi^c = \lambda^c(\delta(x-x_0) + \delta(x+x_0))\sinh(\nu_0(y+H)).
\end{aligned}$$

Moreover, if $\nu_0 x_0 = k\pi$ ($k \in \mathbb{N}$), we have

$$\psi^s(x,y) = \begin{cases} 0 & |x| > x_0, \\ -\sin(\nu_0 x)\sinh(\nu_0(y+H)) & |x| < x_0, \end{cases} \tag{3.15}$$
$$\lambda^s = (-1)^k \nu_0.$$

Similarly, for $\nu_0 x_0 = (k - 1/2)\pi$,

$$\psi^c(x,y) = \begin{cases} 0 & |x| > x_0, \\ -\cos(\nu_0 x)\sinh(\nu_0(y+H)) & |x| < x_0, \end{cases}$$
$$\lambda^c = (-1)^k \nu_0.$$

Proof. As a consequence of the symmetry of h_i^s, the function $-\psi^s(-x,y)$ belongs to W and, by elementary changes of variables, it verifies (2.14). Then $\psi^s(x,y) = -\psi^s(-x,y)$ for Proposition 2.3 and so $\lambda_+^s = -\lambda_-^s = \lambda^s$. Finally, the map (3.15) is in $H^1(S_H)$ and by explicit computations satisfies (3.1), (3.2), (3.3) and (3.4) with $\lambda_+ = -\lambda_- = (-1)^k \nu_0$ and therefore by Theorem 3.1 coincides with the variational solution. The same argument applies to ψ^c. □

Remark 3.4 The data h_i^s and h_i^c (i=1,2,3) are the traces on the obstacle of the functions

$$\mathcal{S}(x,y) = -\sin(\nu_0 x)\sinh(\nu_0(y+H)),$$
$$\mathcal{C}(x,y) = -\cos(\nu_0 x)\sinh(\nu_0(y+H)),$$

which represent two linearly independent solutions of the "free problem"

$$\Delta\psi = 0 \qquad \text{in } S_H,$$
$$\psi_y - \nu\psi = 0 \qquad \text{on } F,$$
$$\psi = 0 \qquad \text{on } \mathbb{R}\times\{-H\},$$
$$\sup_{S_H} |\psi| < +\infty.$$

By means of ψ^s and ψ^c we can state a necessary and sufficient condition for the existence of a solution with finite energy for problem $\mathcal{P}$.

Theorem 3.5 *The following relations hold:*

$$
\begin{aligned}
&(\lambda_+ - \lambda_-) \sin(\nu_0 x_0)\, C_0(\nu_0) \\
&= \int_{-x_0}^{x_0} h_2(x, -H_1) \left[\psi_y^s(x, -H_1) + \nu_0 \sin(\nu_0 x) \cosh(\nu_0(-H_1 + H))\right] \mathrm{d}x \\
&+ \int_{-H}^{-H_1} (h_3(x_0, y) - h_1(-x_0, y)) \left[\psi_x^s(x_0, y)\right. \\
&\left.+ (\nu_0 \cos(\nu_0 x_0) - \lambda^s) \sinh(\nu_0(y + H))\right] \mathrm{d}y, \qquad (3.16) \\
&(\lambda_+ + \lambda_-) \cos(\nu_0 x_0)\, C_0(\nu_0) \\
&= \int_{-x_0}^{x_0} h_2(x, -H_1) \left[\psi_y^c(x, -H_1) + \nu_0 \cos(\nu_0 x) \cosh(\nu_0(-H_1 + H))\right] \mathrm{d}x \\
&+ \int_{-H}^{-H_1} (h_3(x_0, y) + h_1(-x_0, y)) \left[\psi_x^c(x_0, y)\right. \\
&\left.- (\lambda^c + \nu_0 \sin(\nu_0 x_0)) \sinh(\nu_0(y + H))\right] \mathrm{d}y, \qquad (3.17)
\end{aligned}
$$

where $C(\nu_0)$ is defined by (3.6). When $\nu_0 x_0 \neq k\pi/2$, problem $\mathcal{P}$ is solvable in $H^1(S_H)$ if and only if the quantities in the second member of (3.16) and (3.17) vanish for the Dirichlet data h_i ($i = 1, 2, 3$). Furthermore, if a solution with finite energy exists, it is unique and coincide with the variational solution.

Proof. Let us apply Green's formula to the weak solution ψ and to the harmonic function $-\mathcal{S}$ in the rectangle $R_\epsilon = (-x_0 + \epsilon, x_0 - \epsilon) \times (-H_1, 0)$ with $0 < \epsilon < x_0/2$. By recalling (3.8), (3.9) and the boundary conditions, for $\epsilon \to 0^+$ we get

$$
\begin{aligned}
&(\lambda_+ - \lambda_-) \sin(\nu_0 x_0)\, C_1(\nu_0) = \int_{-x_0}^{x_0} \sin(\nu_0 x) \left[-\psi_y(x, -H_1)\right. \\
&\left.\cdot \sinh(\nu_0(-H_1 + H)) + \nu_0 \cosh(\nu_0(-H_1 + H))\, h_2(x, -H_1)\right] \mathrm{d}x \\
&+ \int_{-H}^{-H_1} \sinh(\nu_0(y + H)) \left[-\sin(\nu_0 x_0)\, \psi_x(-x_0, y)\right. \\
&\left.-\nu_0 \cos(\nu_0 x_0)\, h_1(-x_0, y)\right] \mathrm{d}y + \int_{-H}^{-H_1} \sinh(\nu_0(y + H)) \\
&\cdot \left[-\sin(\nu_0 x_0)\, \psi_x(x_0, y) + \nu_0 \cos(\nu_0 x_0)\, h_3(x_0, y)\right] \mathrm{d}y \qquad (3.18)
\end{aligned}
$$

with

$$
C_1(\nu_0) = \int_{-H_1}^{0} \sinh^2(\nu_0(y + H))\, \mathrm{d}y. \qquad (3.19)
$$

On the other hand, using the Green formula to ψ and to ψ^s in the same domains $R_{i,\epsilon,l}$ ($i = 1, 2, 3$) as in the proof of Theorem 3.1, from

$$
\sum_{i=1}^{3} \int_{R_{i,\epsilon,l}} (\psi^s \Delta\psi - \psi \Delta\psi^s)\, \mathrm{d}x\, \mathrm{d}y = 0
$$

if we suppose $l \to +\infty$ and $\epsilon \to 0^+$ we obtain

$$
\begin{aligned}
&-\int_{-x_0}^{x_0} \psi_y(x,-H_1)\sin(\nu_0 x)\sinh(\nu_0(-H_1+H))\,\mathrm{d}x \\
&-\int_{-H}^{-H_1} \psi_x(-x_0,y)\sin(\nu_0 x_0)\sinh(\nu_0(y+H))\,\mathrm{d}y \\
&-\int_{-H}^{-H_1} \psi_x(x_0,y)\sin(\nu_0 x_0)\sinh(\nu_0(y+H))\,\mathrm{d}y \\
&=\int_{-x_0}^{x_0} \psi_y^s(x,-H_1)\,h_2(x,-H_1)\,\mathrm{d}x \\
&+\int_{-H}^{-H_1} \left(\psi_x^s(x_0,y)\,h_3(x_0,y)-\psi_x^s(-x_0,y)\,h_1(-x_0,y)\right)\mathrm{d}y \\
&+\int_{-H}^{-H_1} (\lambda_- - \lambda_+)\sin(\nu_0 x_0)\sinh^2(\nu_0(y+H))\,\mathrm{d}y \\
&+\int_{-H}^{-H_1} \lambda^s \sinh(\nu_0(y+H))\left(h_1(-x_0,y)-h_3(x_0,y)\right)\mathrm{d}y. \qquad (3.20)
\end{aligned}
$$

The substitution of (3.20) into (3.18) yields (3.16). Similarly, when we consider $-\mathcal{C}$ and ψ^c we can write (3.17). Now let $\nu_0 x_0 \neq k\pi/2$, which means that $\sin(\nu_0 x_0) \neq 0$ and $\cos(\nu_0 x_0) \neq 0$. If the second members of (3.16) and (3.17) are null, we deduce $\lambda_+ = \lambda_- = 0$ and then ψ is harmonic. From Proposition 3.2 we have $\hat{\psi} = \psi$, thus ψ solves problem $\mathcal{P}^*$. Moreover condition (2.10) is satisfied because $\psi \in H^1(S_H)$ and hence ψ is a solution of problem $\mathcal{P}$. Viceversa, if a solution in $H^1(S_H)$ exists, thanks to Theorem 3.1 it is unique and coincide with ψ, so we deduce $\lambda_+ = \lambda_- = 0$ and the second members of (3.16) and (3.17) necessarily vanish. □

4 Regularization and unique solvability

The function given by Proposition 3.2 is harmonic but unfortunately it does not generally vanish for $x \to -\infty$ owing to the oscillations introduced by the term s. Here we attempt to modify the solution $\hat{\psi}$ of problem $\mathcal{P}^*$ in order satisfy condition (2.10). The maps ψ^s and ψ^c of Proposition 3.3 give us a help again.

Proposition 4.1 *Let $\hat{\psi}^s$, $\hat{\psi}^c$ be defined as in Proposition 3.3 and $\mathcal{S}$, $\mathcal{C}$ as in Remark 3.4. Then the functions*

$$
\begin{aligned}
\zeta^s(x,y) &= \hat{\psi}^s(x,y) - \mathcal{S}(x,y), \\
\zeta^c(x,y) &= \hat{\psi}^c(x,y) - \mathcal{C}(x,y)
\end{aligned}
$$

solve problem $\mathcal{P}^*$ with homogeneous boundary conditions.

Proof. By Proposition 3.2 $\hat{\psi}^s$ is a solution of problem $\mathcal{P}^*$ with the Dirichlet data h_i^s which are the traces on σ_i of $\mathcal{S}$ thanks to Remark 3.4. The same holds for ζ^c. □

Exploiting Proposition 4.1 we deduce that

$$\omega(x,y) = \hat{\psi}(x,y) + q^s \zeta^s(x,y) + q^c \zeta^c(x,y) \tag{4.1}$$

solves problem $\mathcal{P}^*$ with data h_i for every scalars q^s and q^c. In order to investigate if condition (2.10) can be verified by a particular choice of the coefficients of the linear combination of ζ^s and ζ^c, it is useful to introduce an asymptotic expression for each solution of problem $\mathcal{P}^*$.

Lemma 4.2 *Let $z \in H^1_{loc}(S_H)$ satisfy (2.5), (2.6), (2.8) and (2.9) with $\nu H > 1$. Then*

$$\begin{aligned} z(x,y) &= \sum_{n=1}^{+\infty} a_n e^{-\mu_n x} \sin(\mu_n (y+H)) \\ &+ \Big(A^+ \sin(\nu_0 x) + B^+ \cos(\nu_0 x)\Big) \sinh(\nu_0 (y+H)) \end{aligned}$$

for $(x,y) \in (x_0, +\infty) \times (-H, 0)$, where $\nu_0 > 0$ and $\mu_n > 0$ are the solutions of (2.12) and of

$$\tan(\mu_n H) = \frac{\mu_n}{\nu}.$$

When $\nu H < 1$ equation (2.12) has no solution and we have

$$z(x,y) = \sum_{n=1}^{+\infty} a_n e^{-\mu_n x} \sin(\mu_n (y+H)).$$

Analogous expansions hold for $(x,y) \in (-\infty, -x_0) \times (-H, 0)$.

Proof. The argument is the same as in [10]. □

Now we can state

Proposition 4.3 *If $\nu H_1 < 1$ and*

$$\lambda^s \cos(\nu_0 x_0) - \lambda^c \sin(\nu_0 x_0) \neq \nu_0 \tag{4.2}$$

problem $\mathcal{P}$ admits one solution only.

Proof. By (4.1) and by Lemma 4.2, the function ω in the region $(-\infty, -x_0) \times (-H, 0)$ has the expression

$$\omega(x,y) = O(e^{-\mu_1 |x|}) - A^- \mathcal{S}(x,y) - B^- \mathcal{C}(x,y)$$

where

$$A^- = \left(1 - \frac{\lambda^s}{\nu_0}\cos(\nu_0 x_0)\right) q^s + \frac{\lambda^c}{\nu_0}\cos(\nu_0 x_0)\, q^c + \frac{\lambda^-}{\nu_0}\cos(\nu_0 x_0)\,,$$
$$B^- = -\frac{\lambda^s}{\nu_0}\sin(\nu_0 x_0)\, q^s + \left(1 + \frac{\lambda^c}{\nu_0}\sin(\nu_0 x_0)\right) q^c + \frac{\lambda^-}{\nu_0}\sin(\nu_0 x_0)\,.$$

Thanks to (4.2), there exists an unique choice of q^s and q^c which makes the coefficients A^- and B^- both vanishing, namely $q^s = \lambda^- p^s$ and $q^c = \lambda^- p^c$ with

$$p^s = -\frac{\cos(\nu_0 x_0)}{\lambda^c \sin(\nu_0 x_0) - \lambda^s \cos(\nu_0 x_0) + \nu_0}, \tag{4.3}$$
$$p^c = -\frac{\sin(\nu_0 x_0)}{\lambda^c \sin(\nu_0 x_0) - \lambda^s \cos(\nu_0 x_0) + \nu_0}. \tag{4.4}$$

Uniqueness of the solution found by the above described procedure can be proved following the same lines as in [10]. □

When $\nu_0 x_0 = k\pi/2$, from Proposition 3.3 we know the analytic expressions of λ^s or λ^c and it results

$$\lambda^s \cos(\nu_0 x_0) - \lambda^c \sin(\nu_0 x_0) = \nu_0.$$

We wonder whether other values of $\nu_0 x_0$ satisfy this equality. The answer is negative, in fact

Proposition 4.4 *For every $\nu_0 > 0$ the following relation holds:*

$$\lambda^s \cos(\nu_0 x_0) - \lambda^c \sin(\nu_0 x_0) = \nu_0 - K(\nu_0)\sin(\nu_0 x_0)\cos(\nu_0 x_0)$$

with $K(\nu_0) > 0$.

Proof. If we take the limit for $l \to +\infty$ and $\epsilon \to 0^+$ of

$$\sum_{i=1}^{3} \int_{R_{i,\epsilon,l}} \psi^s \Delta \psi^s \, \mathrm{d}x \, \mathrm{d}y = 0,$$

where $R_{i,\epsilon,l}$ has the usual meaning, we get

$$\begin{aligned}
&-\sinh(\nu_0(-H_1 + H)) \int_{-x_0}^{x_0} \psi^s_y(x, -H_1)\sin(\nu_0 x)\, \mathrm{d}x \\
&-2\sin(\nu_0 x_0) \int_{-H}^{-H_1} \psi^s_x(x_0, y)\sinh(\nu_0(y + H))\, \mathrm{d}y \\
&= -\int_{S_H} |\nabla \psi^s| \, \mathrm{d}x \, \mathrm{d}y + \nu \int_{-\infty}^{+\infty} |\psi^s(x, 0)| \, \mathrm{d}x \\
&-2\lambda^s \sin(\nu_0 x_0) \int_{-H}^{-H_1} \sinh^2(\nu_0(y + H))\, \mathrm{d}y
\end{aligned}$$

and using this equality in (3.18) it results

$$
\begin{aligned}
&2\lambda^s \sin(\nu_0 x_0)\, C(\nu_0) = -\int_{S_H} |\nabla \psi^s| \,\mathrm{d}x\,\mathrm{d}y + \nu \int_{-\infty}^{+\infty} |\psi^s(x,0)| \,\mathrm{d}x \\
&-\nu_0 \sinh(\nu_0(-H_1+H)) \cosh(\nu_0(-H_1+H)) \int_{-x_0}^{x_0} \sin^2(\nu_0 x) \,\mathrm{d}x \\
&-2\nu_0 \sin(\nu_0 x_0) \cos(\nu_0 x_0) \int_{-H}^{-H_1} \sinh^2(\nu_0(y+H)) \,\mathrm{d}y. \qquad (4.5)
\end{aligned}
$$

We remember that the constant $C(\nu_0)$ is given by (3.6). The analogous relation for ψ^c is

$$
\begin{aligned}
&2\lambda^c \cos(\nu_0 x_0)\, C(\nu_0) = -\int_{S_H} |\nabla \psi^c| \,\mathrm{d}x\,\mathrm{d}y + \nu \int_{-\infty}^{+\infty} |\psi^c(x,0)| \,\mathrm{d}x \\
&-\nu_0 \sinh(\nu_0(-H_1+H)) \cosh(\nu_0(-H_1+H)) \int_{-x_0}^{x_0} \cos^2(\nu_0 x) \,\mathrm{d}x \\
&+2\nu_0 \sin(\nu_0 x_0) \cos(\nu_0 x_0) \int_{-H}^{-H_1} \sinh^2(\nu_0(y+H)) \,\mathrm{d}y. \qquad (4.6)
\end{aligned}
$$

For $\nu_0 x_0 \neq k\pi/2$ we introduce the quantities

$$
\begin{aligned}
v^s(x,y) &= \frac{1}{\sin(\nu_0 x_0)} \psi^s(x,y), \qquad & v^c(x,y) &= \frac{1}{\cos(\nu_0 x_0)} \psi^c(x,y), \\
z^s(x,y) &= \frac{\psi^s(x,y) - \mathcal{S}(x,y)}{\sin(\nu_0 x_0)}, \qquad & z^c(x,y) &= \frac{\psi^c(x,y) - \mathcal{C}(x,y)}{\cos(\nu_0 x_0)}.
\end{aligned}
$$

Then, from (4.5) and (4.6) we deduce

$$
\begin{aligned}
&\lambda^s \cos(\nu_0 x_0) - \lambda^c \sin(\nu_0 x_0) = \nu_0 - \frac{1}{2C(\nu_0)} \sin(\nu_0 x_0) \cos(\nu_0 x_0) \\
&\cdot \Bigg[\Bigg(2 \int_{R_H} |\nabla v^s|^2 \,\mathrm{d}x\,\mathrm{d}y - 2\nu \int_{x_0}^{+\infty} |v^s(x,0)|^2 \,\mathrm{d}x \\
&+ \int_{R_0} |\nabla z^s|^2 \,\mathrm{d}x\,\mathrm{d}y - \nu \int_{-x_0}^{x_0} |z^s(x,0)|^2 \,\mathrm{d}x \Bigg) \\
&- \Bigg(2 \int_{R_H} |\nabla v^c|^2 \,\mathrm{d}x\,\mathrm{d}y - 2\nu \int_{x_0}^{+\infty} |v^c(x,0)|^2 \,\mathrm{d}x \\
&+ \int_{R_0} |\nabla z^c|^2 \,\mathrm{d}x\,\mathrm{d}y - \nu \int_{-x_0}^{x_0} |z^c(x,0)|^2 \,\mathrm{d}x \Bigg) \Bigg] \qquad (4.7)
\end{aligned}
$$

with $R_0 = (-x_0, x_0) \times (-H_1, 0)$ and $R_H = (x_0, +\infty) \times (-H, 0)$. Let us consider the set

$$\Lambda = \Big\{ u \in H^1(R_H) : \ u(x,-H) = 0 \quad x > x_0,$$
$$u(x_0, y) = -\sinh(\nu_0 (y+H)) \quad -H < y < -H_1,$$
$$u(x_0, \cdot) \text{ Hölder continuous in } [-H_1, 0],$$
$$\int_{-H}^{0} \sinh(\nu_0 (y+H))\, u(x,y)\, \mathrm{d}y = 0 \text{ for a. e. } x > x_0 \Big\},$$

which is not empty because it contains $v^s|_{R_H}$ and $v^c|_{R_H}$; by the Lax-Milgram lemma it is immediate to show for every $u \in \Lambda$ the existence of a unique $Z^s(u) \in H^1(R_0)$ and a unique $Z^c(u) \in H^1(R_0)$ such that, respectively,

$$\begin{aligned}
&\Delta Z^s(u) = 0 \qquad \text{in } R_0,\\
&Z^s(u)_y - \nu Z^s(u) = 0 \qquad \text{on } (-x_0, x_0) \times \{0\},\\
&Z^s(u) = 0 \qquad \text{on } (-x_0, x_0) \times \{-H_1\},\\
&Z^s(u)(x_0, y) = u(x_0, y) + \sinh(\nu_0(y+H)) \qquad -H_1 < y < 0,\\
&Z^s(u)(-x_0, y) = -u(x_0, y) - \sinh(\nu_0(y+H)) \qquad -H_1 < y < 0
\end{aligned}$$

and

$$\begin{aligned}
&\Delta Z^c(u) = 0 \qquad \text{in } R_0,\\
&Z^c(u)_y - \nu Z^c(u) = 0 \qquad \text{on } (-x_0, x_0) \times \{0\},\\
&Z^c(u) = 0 \qquad \text{on } (-x_0, x_0) \times \{-H_1\},\\
&Z^c(u)(x_0, y) = u(x_0, y) + \sinh(\nu_0(y+H)) \qquad -H_1 < y < 0,\\
&Z^c(u)(-x_0, y) = u(x_0, y) + \sinh(\nu_0(y+H)) \qquad -H_1 < y < 0.
\end{aligned}$$

Moreover

$$\begin{aligned}
&Z^s(u)(x,y) = -Z^s(u)(-x,y), \qquad Z^c(u)(x,y) = Z^c(u)(-x,y), \qquad (4.8)\\
&Z^s\left(v^s|_{R_H}\right) = z^s, \qquad Z^c\left(v^c|_{R_H}\right) = z^c.
\end{aligned}$$

Thus, if we define the functionals

$$\begin{aligned}
J^s : \Lambda &\longrightarrow \mathbb{R}\\
u &\longmapsto J^s(u) = 2\left(\int_{R_H} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y - \nu \int_{x_0}^{+\infty} |u(x,0)|^2 \,\mathrm{d}x\right)\\
&\qquad + \left(\int_{R_0} |\nabla Z^s(u)|^2 \,\mathrm{d}x\,\mathrm{d}y - \nu \int_{-x_0}^{x_0} |Z^s(u)(x,0)|^2 \,\mathrm{d}x\right)
\end{aligned}$$

and

$$J^c : \Lambda \longrightarrow \mathbb{R}$$
$$u \longmapsto J^c(u) = 2\left(\int_{R_H} |\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y - \nu \int_{x_0}^{+\infty} |u(x,0)|^2 \,\mathrm{d}x\right) + \left(\int_{R_0} |\nabla Z^c(u)|^2 \,\mathrm{d}x\,\mathrm{d}y - \nu \int_{-x_0}^{x_0} |Z^c(u)(x,0)|^2 \,\mathrm{d}x\right),$$

equation (4.7) can be written

$$\lambda^s \cos(\nu_0 x_0) - \lambda^c \sin(\nu_0 x_0) = \nu_0 - \frac{1}{2C(\nu_0)}\left[J^s\left(v^s|_{R_H}\right) - J^c\left(v^c|_{R_H}\right)\right] \sin(\nu_0 x_0)\cos(\nu_0 x_0).$$

We notice that the restrictions of $Z^s(u)$ and $Z^c(u)$ to $R_1 = (0, x_0) \times (-H_1, 0)$ satisfy the same boundary conditions on $(0, x_0) \times \{0\}$, $(0, x_0) \times \{-H_1\}$ and $\{x_0\} \times (-H_1, 0)$. On the other hand, on the segment $\{0\} \times (-H_1, 0)$, thanks to the symmetry properties (4.8), $Z^s(u)$ vanishes while $Z^c(u)$ verifies a homogeneous Neumann condition. Hence, by coercivity (recall that $\nu H_1 < 1$) and by the Dirichlet principle we get:

$$\int_{R_1} |\nabla Z^s(u)|^2 \,\mathrm{d}x\,\mathrm{d}y - \nu \int_0^{x_0} |Z^s(u)(x,0)|^2 \,\mathrm{d}x \geq \int_{R_1} |\nabla Z^c(u)|^2 \,\mathrm{d}x\,\mathrm{d}y - \nu \int_0^{x_0} |Z^c(u)(x,0)|^2 \,\mathrm{d}x \qquad \forall\, u \in \Lambda$$

and therefore

$$J^s(u) \geq J^c(u), \qquad \forall\, u \in \Lambda.$$

Now, it can be shown [11] that the minimum of J^s is attained at $v^s|_{R_H}$, the minimum of J^c is attained at $v^c|_{R_H}$ and that the strict inequality

$$J^c\left(v^s|_{R_H}\right) > J^c\left(v^c|_{R_H}\right)$$

holds by uniqueness. Thus, the proof is complete. □

By the discussion of the introduction and by Propositions 4.3 and 4.4 we obtain the unique solvability of problem $\mathcal{P}$ for every $\nu < 1/H$ and for every ν in the interval $1/H < \nu < 1/H_1$, provided the condition $\nu_0 x_0 \neq k\pi/2$ holds. We now discuss the extension of the result when $\nu_0 x_0 = k\pi/2$; in this case, the previous technique for constructing the solution must be reviewed because either ζ^s or ζ^c identically vanishes. However, it turns out that the quantities $p^s\zeta^s$ and $p^c\zeta^c$, where p^s and p^c are defined by (4.3) and (4.4), have well defined limits for $\nu_0 x_0 \to k\pi/2$, which represent two non trivial solutions of problem $\mathcal{P}^*$. As a consequence, we can still get a unique solution by suitable limit of the solutions defined for $\nu_0 x_0 \neq k\pi/2$. The proof can be found in the appendix. Summarizing the discussion, we can state

Theorem 4.5 *If the condition $\nu H_1 < 1$ holds, problem $\mathcal{P}$ is well posed for every choice of the data satisfying the compatibility conditions.*

Remark 4.6 By inspection of the arguments of the previous sections, it can be readily shown that Theorem 4.5 also holds in the case of an obstacle represented by a region the form

$$Q_f = \{(x, y) : -x_0 < x < x_0,\ -H < y < -H + f(x)\},$$

where f belongs to $C^{1,1}(-x_0, x_0)$ with $H - H_1 \leq f(x) < H$.

5 Conclusion and open problems

We considered the linearized problem of the flow of a heavy ideal fluid over a rectangular obstacle; when the flow is supercritical in the fluid region above the obstacle, we proved the unconditional solvability of the problem. This means that the possible "singular values" of the velocity are confined in the interval $0 < c \leq \sqrt{gH_1}$. Clearly, the first open problem is the study of a flow in this range of velocities: in this case, one needs two different conditions in either the regions $S_H \cap \{|x| < x_0\}$ and $S_H \cap \{|x| > x_0\}$ in order to achieve coercivity. As a consequence, the regularization procedure will become more delicate, as well as the proof of unique solvability. We will treat this problem in a forthcoming paper. A further interesting question is the applicability of the variational approach to obstacles of generic shape in the subcritical regime; again, the crucial point is the determination of a suitable a priori condition for the coercivity of the associated bilinear form.

Appendix

Here we prove Theorem 4.5 when $\nu_0 x_0 = k\pi/2$. For every $t > 0$, let $\psi^{s,t}$ and $\psi^{c,t}$ be the variational solutions of problem $\mathcal{P}$, with $x_0 = t/\nu_0$, having the same traces on the obstacle as $\mathcal{S}$ and $\mathcal{C}$. We indicate with $\lambda^s(t)$ and $\lambda^c(t)$ the constants in the expression of the their laplacian. From Lemma 4.2 we have the expansions

$$\psi^{s,t}(x, y) = \sum_{n=1}^{+\infty} a_n^s(t)\, e^{-\mu_n\left(x - \frac{t}{\nu_0}\right)} \sin(\mu_n(y + H)) \qquad x > \frac{t}{\nu_0},$$
$$\psi^{c,t}(x, y) = \sum_{n=1}^{+\infty} a_n^c(t)\, e^{-\mu_n\left(x - \frac{t}{\nu_0}\right)} \sin(\mu_n(y + H)) \qquad x > \frac{t}{\nu_0}.$$

Moreover, for $\zeta^{s,t}$ and $\zeta^{c,t}$ we can write

$$\zeta^{s,t}(x,y) = \sum_{n=1}^{+\infty} b_n^s(t)\sinh(\tilde{\mu}_n x)\sin(\tilde{\mu}_n(y+H_1)) \qquad -\frac{t}{\nu_0} < x < \frac{t}{\nu_0},$$
$$\zeta^{c,t}(x,y) = \sum_{n=1}^{+\infty} b_n^c(t)\cosh(\tilde{\mu}_n x)\sin(\tilde{\mu}_n(y+H_1)) \qquad -\frac{t}{\nu_0} < x < \frac{t}{\nu_0},$$

where the coefficients $\tilde{\mu}_n$ are the positive solutions of $\tan(\tilde{\mu}_n H_1) = \tilde{\mu}_n/\nu$. The functions $\lambda^s(t)$, $\lambda^c(t)$, $a_n^s(t)$, $a_n^c(t)$, $b_n^s(t)$, $b_n^c(t)$ are smooth (see [7], [9]) and from Proposition 3.3 we have for $k \in \mathbb{N}$

$$\lim_{t\to k\pi} \lambda^s(t) = (-1)^k \nu_0, \qquad \lim_{t\to\left(k-\frac{1}{2}\right)\pi} \lambda^c(t) = (-1)^k \nu_0,$$
$$\lim_{t\to k\pi} a_n^s(t) = 0, \qquad \lim_{t\to\left(k-\frac{1}{2}\right)\pi} a_n^c(t) = 0,$$
$$\lim_{t\to k\pi} b_n^s(t) = 0, \qquad \lim_{t\to\left(k-\frac{1}{2}\right)\pi} b_n^c(t) = 0.$$

Besides, if we introduce also $\Delta(t) = \lambda^c(t)\sin(t) - \lambda^s(t)\cos(t) + \nu_0$, with the help of Proposition 4.4 we get $\Delta(t) = \mathcal{K}(t)\sin(t)\cos(t)$ with $\mathcal{K}(t) > 0$ for all $t > 0$ and so

$$\lim_{t\to k\pi} \Delta(t) = 0, \qquad \lim_{t\to\left(k-\frac{1}{2}\right)\pi} \Delta(t) = 0,$$
$$\lim_{t\to k\pi} \Delta'(t) = \Delta'(k\pi) = \mathcal{K}(k\pi) > 0, \tag{5.1}$$
$$\lim_{t\to\left(k-\frac{1}{2}\right)\pi} \Delta'(t) = \Delta'\left(\left(k-\frac{1}{2}\right)\pi\right) = -\mathcal{K}\left(\left(k-\frac{1}{2}\right)\pi\right) < 0. \tag{5.2}$$

Now we suppose $\nu_0 x_0 = k\pi$. Thanks to (5.1) we can calculate by the De Hôpital rule the limits

$$
\begin{aligned}
\alpha_n^s &= \lim_{t\to k\pi} -\frac{\cos(t)}{\Delta(t)} a_n^s(t) = -\frac{a_n^{s'}(k\pi)}{\lambda^c(k\pi)-\lambda^{s'}(k\pi)},\\
\beta_n^s &= \lim_{t\to k\pi} -\frac{\cos(t)}{\Delta(t)} b_n^s(t) = -\frac{b_n^{s'}(k\pi)}{\lambda^c(k\pi)-\lambda^{s'}(k\pi)},\\
A^s &= \lim_{t\to k\pi} -\frac{\cos(t)}{\Delta(t)} \left(-\frac{\lambda^s(t)}{\nu_0}\cos(t)+1\right) = \frac{(-1)^k \lambda^{s'}(k\pi)}{\nu_0\left(\lambda^c(k\pi)-\lambda^{s'}(k\pi)\right)},\\
B^s &= \lim_{t\to k\pi} -\frac{\cos(t)}{\Delta(t)} \frac{\lambda^s(t)\sin(t)}{\nu_0} = -\frac{1}{\lambda^c(k\pi)-\lambda^{s'}(k\pi)},\\
\alpha_n^c &= \lim_{t\to k\pi} -\frac{\sin(t)}{\Delta(t)} a_n^c(t) = -\frac{a_n^{c}(k\pi)}{\lambda^c(k\pi)-\lambda^{s'}(k\pi)},\\
\beta_n^c &= \lim_{t\to k\pi} -\frac{\sin(t)}{\Delta(t)} b_n^c(t) = -\frac{b_n^{c}(k\pi)}{\lambda^c(k\pi)-\lambda^{s'}(k\pi)},\\
A^c &= \lim_{t\to k\pi} -\frac{\sin(t)}{\Delta(t)} \left(-\frac{\lambda^c(t)\cos(t)}{\nu_0}\right) = \frac{(-1)^k \lambda^{c}(k\pi)}{\nu_0\left(\lambda^c(k\pi)-\lambda^{s'}(k\pi)\right)},\\
B^c &= \lim_{t\to k\pi} -\frac{\sin(t)}{\Delta(t)} \left(\frac{\lambda^c(t)}{\nu_0}\sin(t)+1\right) = -\frac{1}{\lambda^c(k\pi)-\lambda^{s'}(k\pi)}.
\end{aligned}
$$

By means of (4.3) and (4.4) we have

$$
p^{s,t} = -\frac{\cos(t)}{\Delta(t)}, \qquad p^{c,t} = -\frac{\sin(t)}{\Delta(t)}
$$

when $t \neq k\pi/2$ and so from the analytic expressions of $\zeta^{s,t}$ and $\zeta^{c,t}$ it is easy to check that the functions

$$
u^s(x,y) = \begin{cases}
-\sum_{n=1}^{+\infty} \alpha_n^s e^{\mu_n(x+x_0)} \sin(\mu_n(y+H)) + (A^s \sin(\nu_0 x) - B^s \cos(\nu_0 x)) \\
\cdot \sinh(\nu_0(y+H)) \quad (x,y) \in (-\infty,-x_0)\times(-H,0), \\
\sum_{n=1}^{+\infty} \beta_n^s \sinh(\tilde{\mu}_n x) \sin(\tilde{\mu}_n(y+H_1)) \quad (x,y)\in(-x_0,x_0)\times(-H_1,0), \\
\sum_{n=1}^{+\infty} \alpha_n^s e^{-\mu_n(x-x_0)} \sin(\mu_n(y+H)) + (A^s \sin(\nu_0 x) + B^s \cos(\nu_0 x)) \\
\cdot \sinh(\nu_0(y+H)) \quad (x,y)\in(x_0,+\infty)\times(-H,0)
\end{cases}
$$

and

$$u^c(x,y)=\begin{cases}\sum_{n=1}^{+\infty}\alpha_n^c e^{\mu_n(x+x_0)}\sin\left(\mu_n\left(y+H\right)\right)+\left(-A^c\sin\left(\nu_0 x\right)+B^c\cos\left(\nu_0 x\right)\right)\\ \cdot\sinh\left(\nu_0\left(y+H\right)\right)\quad (x,y)\in(-\infty,-x_0)\times(-H,0),\\ \sum_{n=1}^{+\infty}\beta_n^c\cosh\left(\tilde{\mu}_n x\right)\sin\left(\tilde{\mu}_n\left(y+H_1\right)\right)\quad (x,y)\in(-x_0,x_0)\times(-H_1,0),\\ \sum_{n=1}^{+\infty}\alpha_n^c e^{-\mu_n(x-x_0)}\sin\left(\mu_n\left(y+H\right)\right)+\left(A^c\sin\left(\nu_0 x\right)+B^c\cos\left(\nu_0 x\right)\right)\\ \cdot\sinh\left(\nu_0\left(y+H\right)\right)\quad (x,y)\in(x_0,+\infty)\times(-H,0)\end{cases}$$

represent the limits of $p^{s,t}\zeta^{s,t}$ and $p^{c,t}\zeta^{c,t}$ for $t\to k\pi$. Furthermore, they are two non trivial solutions of the homogeneous problem $\mathcal{P}^*$. Now we define the map

$$\omega(x,y)=\hat{\psi}(x,y)+\lambda_-u^s(x,y)+\lambda_-u^c(x,y)$$

and make use of Lemma 4.2 again. By direct calculation, the coefficients $\mathfrak{A}^-$ and $\mathfrak{B}^-$ of

$$\sin\left(\nu_0 x\right)\sinh\left(\nu_0\left(y+H\right)\right),\qquad \cos\left(\nu_0 x\right)\sinh\left(\nu_0\left(y+H\right)\right)$$

in the series expansion of ω in the region $(-\infty,-x_0)\times(-H,0)$ are

$$\mathfrak{A}^-=\lambda_-\left(\frac{(-1)^k}{\nu_0}+A^s-A^c\right)=0,\qquad \mathfrak{B}^-=\lambda_-\left(B^c-B^s\right)=0,$$

then (2.10) is verified and we have a solution of problem $\mathcal{P}$. Let ω' and ω'' be two solutions. If $\mathfrak{A}^+$ and $\mathfrak{B}^+$ are the coefficients of the oscillatory part of $z=\omega'-\omega''$ in $(x_0,+\infty)\times(-H,0)$, we deduce [7], [10]

$$\begin{aligned}&B^s\mathfrak{A}^+-A^s\mathfrak{B}^+=0,\\ &B^c\mathfrak{A}^+-A^c\mathfrak{B}^+=0.\end{aligned}$$

Since

$$-B^sA^c+A^sB^c=\frac{(-1)^k}{\nu_0\left(\lambda^c(k\pi)-\lambda^{s'}(k\pi)\right)}\neq 0,$$

it results $\mathfrak{A}^+=\mathfrak{B}^+=0$, which means $z\in H^1(S_H)$. Thus it must be $\omega'=\omega''$ for Theorem 3.1. When $\nu_0x_0=t^*=(k-1/2)\pi$, from (5.2) we can prove unique solvability using in the definitions of u^s and u^c the new coefficients

$$\alpha_n^s = \lim_{t \to \left(k - \frac{1}{2}\right)\pi} -\frac{\cos(t)}{\Delta(t)} a_n^s(t) = \frac{a_n^s(t^*)}{\lambda^s(t^*) + \lambda^{c'}(t^*)},$$

$$\beta_n^s = \lim_{t \to \left(k - \frac{1}{2}\right)\pi} -\frac{\cos(t)}{\Delta(t)} b_n^s(t) = \frac{b_n^s(t^*)}{\lambda^s(t^*) + \lambda^{c'}(t^*)},$$

$$A^s = \lim_{t \to \left(k - \frac{1}{2}\right)\pi} -\frac{\cos(t)}{\Delta(t)} \left(-\frac{\lambda^s(t)}{\nu_0} \cos(t) + 1 \right) = \frac{1}{\lambda^s(t^*) + \lambda^{c'}(t^*)},$$

$$B^s = \lim_{t \to \left(k - \frac{1}{2}\right)\pi} -\frac{\cos(t)}{\Delta(t)} \frac{\lambda^s(t)\sin(t)}{\nu_0} = \frac{(-1)^{k+1} \lambda^s(t^*)}{\nu_0 \left(\lambda^s(t^*) + \lambda^{c'}(t^*)\right)},$$

$$\alpha_n^c = \lim_{t \to \left(k - \frac{1}{2}\right)\pi} -\frac{\sin(t)}{\Delta(t)} a_n^c(t) = -\frac{a_n^{c'}(t^*)}{\lambda^s(t^*) + \lambda^{c'}(t^*)},$$

$$\beta_n^c = \lim_{t \to \left(k - \frac{1}{2}\right)\pi} -\frac{\sin(t)}{\Delta(t)} b_n^c(t) = -\frac{b_n^{c'}(t^*)}{\lambda^s(t^*) + \lambda^{c'}(t^*)},$$

$$A^c = \lim_{t \to \left(k - \frac{1}{2}\right)\pi} -\frac{\sin(t)}{\Delta(t)} \left(-\frac{\lambda^c(t)\cos(t)}{\nu_0} \right) = \frac{1}{\lambda^s(t^*) + \lambda^{c'}(t^*)},$$

$$B^c = \lim_{t \to \left(k - \frac{1}{2}\right)\pi} -\frac{\sin(t)}{\Delta(t)} \left(\frac{\lambda^c(t)}{\nu_0} \sin(t) + 1 \right) = \frac{(-1)^k \lambda^{c'}(t^*)}{\nu_0 \left(\lambda^s(t^*) + \lambda^{c'}(t^*)\right)}.$$

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