DIPARTIMENTO DI MATEMATICA
"Francesco Brioschi"
POLITECNICO DI MILANO

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Collezione dei *Quaderni di Dipartimento*, numero **QDD 178**
Inserito negli *Archivi Digitali di Dipartimento* in data 5-5-2014

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We present a fast and accurate pricing technique based on the Spitzer identity and the Wiener-Hopf factorization. We apply it to barrier and lookback options when the monitoring is discrete and the underlying evolves according to an exponential Lévy process. The numerical implementation exploits the fast Fourier transform and the Euler summation. The computational cost is independent of the number of monitoring dates; the error decays exponentially with the number of grid points, except for double-barrier options.

Key words: barrier options, lookback options, discrete monitoring, Hilbert transform, Fourier transform, FFT, Lévy process, sinc functions, Spitzer identity, Wiener-Hopf factorization, z-transform, Euler summation.

1. Introduction

The application of transform techniques to price derivative contracts is rather recent. The first and most important contributions are probably the articles by Heston [1] and Carr and Madan [2], where the authors show how to price European options with non-Gaussian models exploiting the Fourier transform. Similar techniques were developed later for path-dependent derivatives [3, 4]. Our paper provides a unified framework for pricing barrier and lookback (or hindsight) options when the underlying asset evolves as an exponential Lévy process. The monitoring condition, e.g., the event that the underlying asset value falls below a given barrier for a down-and-out barrier option, is assumed to be controlled at discrete time intervals.

The proposed methodology is based on the Spitzer identity [5], an important result about the distribution of the discrete extrema for a process with independent and identically distributed increments. Spitzer provided a closed formula for the z-transform (or moment generating function) of the characteristic function of the discrete extrema. Up to now its application has been difficult because it requires the Wiener-Hopf (WH) factorization of a function defined in the complex plane. Unfortunately, this factorization cannot be achieved analytically except in few cases, or its computation turns out to be very demanding requiring a multidimensional integral. In addition, with regard to a general Lévy process, very little is known for the two barriers problem. In this case the more difficult problem of a matrix factorization arises. Possible solutions have been suggested by approximating the Lévy process by a Lévy process with hyper-exponential jumps, when this is possible, and subsequently exploiting the availability of an analytic Wiener-Hopf factorization for the latter [6].

The key contributions of our paper are the following. First of all, we provide a constructive procedure for performing the Wiener-Hopf factorization. More precisely, we express the Wiener-Hopf factors arising in the Spitzer identity in terms of the Plemelj-Sokhotsky relations and then we compute them exploiting the Hilbert transform. Even if the Spitzer identity has already been used in option pricing [4, 7, 8, 9] and the present paper is mainly focused on this kind of applications, our method goes well beyond option pricing and opens up the way to a more extensive use of the Spitzer identity and the Wiener-Hopf factorization in other non-financial applications. In this regard we would like to mention the applicability to queuing theory due to the strict connection between random walks and queues, see Lindley [10] for pioneering contributions and Cohen [11], Prabhu [12], and Asmussen [13, 14]. Further applications include insurance [15] and sequential testing [16]. Finally, the Wiener-Hopf factorization arises in almost all branches of engineering, mathematical physics and applied mathematics. This is testified by the thousands of papers published on the subject since its conception. A review of the different applications is given by Lawrie and Abrahams [17].

Our methodology can deal with both a single and a double barrier. The solution in the second case is of interest in itself and our procedure solves a long-standing problem related to an efficient computation of the Wiener-Hopf factors in the presence of two barriers. The double-barrier case did not admit a simple feasible solution up to now, except under few special assumptions on the structure of the Lévy process. Indeed, it is related to the difficult problem of a matrix Wiener-Hopf factorization, where one has to solve two coupled Wiener-Hopf equations casted in matrix form. A general solution for the appropriate factorization of these matrices has not been found yet. Here, we propose a constructive fixed-point algorithm based on an extension of the single barrier case that achieves a fast convergence.

We would also like to stress that, when the proposed methodology is applied to pricing discretely monitored barrier and lookback options, its computational cost is independent of the number of monitoring dates thanks to the Euler acceleration, which bounds from above the number of Wiener-Hopf factorizations to be computed. Moreover, at least with regard to single-barrier and lookback options, the method provides exponential order of convergence due to the fact that the factorization is performed remaining in the complex plane. The existing methods are based on the backward recursive formula, see for example Refs. [18, 19, 20, 21, 22], and on exploiting the convolution structure of the transition density of the Lévy process by performing the computations efficiently and fast using the FFT, which leads to a CPU time that grows as $\mathcal{O}(M \log M)$, where M is the number of discretization points. However, all the above cited methods are characterized by a polynomial decay of the error with M. This order of accuracy is related to the fact that the backward procedure for barrier options involves a convolution, that can be computed in the complex plane, and a projection, which is applied in the real plane, to take into account the presence of the barrier. A noticeable exception was presented by Feng and Linetsky [3, 23], who reformulated the backward procedure for barrier and lookback options in terms of the Hilbert transform, so that all steps are performed in the complex plane. Computing the Hilbert transform with a sinc function expansion, they achieved an exponential decay of the error. However, the computational cost of all these methods, including the one by Feng and Linetsky, increases linearly with the number of monitoring dates.

Finally, the factorization procedure introduced here is quite general and can also be applied, without any additional complication, to continuously-monitored contracts. Even the best available method listed above, i.e., that by Feng and Linetsky, does not have this feature.

The structure of the paper is the following. Section 2 introduces the Spitzer identity and its relationship with the Wiener-Hopf factorization, proposing a fast numerical method to compute the distributions of the minimum and the maximum of a Lévy process, as well as the joint distributions of the process at maturity and of its minimum or maximum over the whole time interval. Section 3 shows how the proposed general methodology can be implemented efficiently and accurately using the Hilbert transform via a sinc expansion to compute the Wiener-Hopf factorization; we also discuss the inversion of the z-transform and its acceleration through the Euler summation rule to make the computational cost independent of the number of monitoring dates. Section 4 deals with the pricing problem for all the derivatives, describing how the Spitzer identity can be exploited to obtain our fast and accurate pricing methods. Section 5 reviews other pricing methods presented in the literature and related to the proposed methodology, while Section 6 validates numerically the pricing procedures, taking into consideration both the accuracy and the computational cost. Finally, Section 7 addresses the continuous monitoring case.

2. Spitzer identity and Wiener-Hopf factorization

We consider a Lévy process $X(t)$, i.e., a process with $X(0)=0$ and independent and identically distributed increments. The Lévy-Khincine formula states that the characteristic function of the process is given by $\Psi(\xi,t)=\mathbb{E}[e^{i\xi X(t)}]=e^{\psi(\xi)t}$, where ψ is the characteristic exponent of the process,

$$\psi(\xi)=ia\xi-\frac{1}{2}\sigma^2\xi^2+\int_{\mathbb{R}}\left(e^{i\xi\eta}-1-i\xi\eta\mathbf{1}_{|\eta|<1}\right)\nu(d\eta); \tag{1}$$

the parameters (a,σ,ν) are the Lévy-Khincine triplet which fully defines the Lévy process $X(t)$.

In several applications in queueing theory, insurance and financial mathematics, the key point is the determination of the law of the extrema of the Lévy process observed on an equally-spaced grid $X_n=X(n\Delta),\ n=0,\dots,N$, where $\Delta>0$ is the time step, i.e., the distance between two consecutive monitoring dates, which is assumed constant. We define the processes of the maximum M_N and of the minimum m_N up to the Nth monitoring date as

$$M_N=\max_{n=0,\dots,N}X_n \quad \text{and} \quad m_N=\min_{n=0,\dots,N}X_n. \tag{2}$$

To distinguish the present case, where the above processes, albeit evolving in continuous time, are recorded only at discrete times, the terminology discrete versus continuous monitoring is used.

In particular, besides the distribution $P_X(x,N)$ of the Lévy process at maturity $T=N\Delta$, we will need the distributions $P_m(x,N)$ of the minimum and $P_M(x,N)$ of the maximum over the whole set $\{n=0,\dots,N\}$, as well as the joint distributions $P_{X,m}(x,N)$ or $P_{X,M}(x,N)$ of the process at maturity and of its minimum or maximum over the interval with respect to a lower (upper) barrier l (u), and the joint distribution of the triplet (X_N,m_N,M_N), $P_{X,m,M}(x,N)$. These distributions are defined as

$$dP_X(x,N)=p_X(x,N)dx=\mathbb{P}[X_N\in[x,x+dx)] \tag{3}$$

$$dP_m(x,N)=p_m(x,N)dx=\mathbb{P}[m_N\in[x,x+dx)] \tag{4}$$

$$dP_M(x,N)=p_M(x,N)dx=\mathbb{P}[M_N\in[x,x+dx)] \tag{5}$$

$$dP_{X,m}(x,N)=p_{X,m}(x,N)dx=\mathbb{P}[X_N\in[x,x+dx)\cap m_N>l] \tag{6}$$

$$dP_{X,M}(x,N)=p_{X,M}(x,N)dx=\mathbb{P}[X_N\in[x,x+dx)\cap M_N<u] \tag{7}$$

$$dP_{X,m,M}(x,N)=p_{X,m,M}(x,N)dx=\mathbb{P}[X_N\in[x,x+dx)\cap m_N>l\cap M_N<u]. \tag{8}$$

We define the Fourier transform of a function $g(x)$ as

$$\widehat{g}(\xi)=\mathcal{F}_{x\to\xi}[g(x)](\xi):=\int_{-\infty}^{+\infty}g(x)e^{i\xi x}dx \tag{9}$$

and its inverse with

$$g(x)=\mathcal{F}^{-1}_{\xi\to x}[\widehat{g}(\xi)](x):=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\widehat{g}(\xi)e^{-ix\xi}d\xi. \tag{10}$$

When no misunderstanding about which variable is being Fourier-transformed is possible, notably when the argument function depends on a single variable, we will drop the subscript to the operators $\mathcal{F}$ and $\mathcal{F}^{-1}$. In some cases, for compatibility with previous literature we use an upper-case letter instead of a lower-case letter with a hat, i.e., $G(\xi)$ instead of $\widehat{g}(\xi)$. As an exception to these notations, the above defined characteristic function Ψ of the Lévy process is the Fourier transform of the probability density function f of the Lévy process,

$$\Psi(\xi,\Delta) = \mathcal{F}_{x\to\xi}[f(x,\Delta)](\xi,\Delta), \tag{11}$$

where the transition probability that $X(t+\Delta)=x$ when $X(t)=x'$ has density $f(x-x',\Delta)$ for any $t>0$. The convolution form of the density function is due to the assumption of independent increments. We would like to stress that $f(x,\Delta)$ is the forward-in-time density of the process, while we indicate with $f_{\mathrm{b}}(x,\Delta) := f(-x,\Delta)$, $x\in\mathbb{R}$, the backward-in-time density. As an example, the general pricing recursion to compute the price (or cost) c of a plain vanilla derivative at time t given its value at time $t+\Delta$ can be computed from its price at time $t+\Delta$ using the backward-in-time density

$$c(x,t) = e^{-r\Delta}\int_{-\infty}^{+\infty} f_{\mathrm{b}}(x-x',\Delta)c(x',t+\Delta)dx'. \tag{12}$$

The Fourier transform of the backward-in-time transition density is the conjugate $\Psi^*(\xi,\Delta)$ of the characteristic function. Therefore the above equation in Fourier space is

$$\widehat{c}(\xi,t) = e^{-r\Delta}\Psi^*(\xi)\widehat{c}(\xi,t+\Delta). \tag{13}$$

We define the z-transform (or generating function) of a discrete set of functions $v(x,n)$, $n\in\mathbb{N}_0$, as

$$\widetilde{v}(x,q) = \mathcal{Z}_{n\to q}[v(x,n)](x,q) := \sum_{n=0}^{\infty} q^n v(x,n), \tag{14}$$

with $q\in\mathbb{C}$ (in the more common definition, z^{-1} is used in place of q). It is a discrete version of the Laplace transform. The original function $v(x,n)$ can be recovered through the complex integral

$$v(x,n) = \mathcal{Z}^{-1}_{q\to n}[\widetilde{v}(x,q)](x,n) = \frac{1}{2\pi\rho^n}\int_0^{2\pi} \widetilde{v}(x,\rho e^{iu})e^{-inu}du, \tag{15}$$

where ρ must be within the radius of convergence [24].

Using combinatorial arguments, Spitzer derived a formula for the z-transforms of the characteristic functions of the distributions defined in Equations (3)–(8), the celebrated Spitzer identities [5]. We recall them here. Let $\Phi_\pm$ be two functions which are analytic in the overlap of two half planes including the real line such that

$$\Phi(\xi,q) := 1 - q\mathbb{E}[e^{i\xi X(\Delta)}] = 1 - q\Psi(\xi,\Delta) = \Phi_+(\xi,q)\Phi_-(\xi,q). \tag{16}$$

$\Phi_\pm(\xi,q)$ are the positive and negative Wiener-Hopf factors of $1-q\Psi(\xi,\Delta)$. The Spitzer identities enable to express the desired characteristic functions through the inversion of a moment-generating function involving Φ, Φ_+ and Φ_-. We have

$$\widetilde{\widehat{p}}_X(\xi,q) = \sum_{n=0}^{\infty} q^n \widehat{p}_X(\xi,n) = \sum_{n=0}^{\infty} q^n \mathbb{E}\big(e^{i\xi X_n}\big) = \frac{1}{\Phi(\xi,q)} \tag{17}$$

$$\widetilde{\widehat{p}}_m(\xi,q) = \sum_{n=0}^{\infty} q^n \widehat{p}_m(\xi,n) = \sum_{n=0}^{\infty} q^n \mathbb{E}\big(e^{i\xi m_n}\big) = \frac{1}{\Phi_+(0,q)\Phi_-(\xi,q)} \tag{18}$$

$$\widetilde{\widehat{p}}_M(\xi,q) = \sum_{n=0}^{\infty} q^n \widehat{p}_M(\xi,n) = \sum_{n=0}^{\infty} q^n \mathbb{E}\big(e^{i\xi M_n}\big) = \frac{1}{\Phi_+(\xi,q)\Phi_-(0,q)} \tag{19}$$

$$\widetilde{\widehat{p}}_{X,m}(\xi,q) = \sum_{n=0}^{\infty} q^n \widehat{p}_{X,m}(\xi,n) = \frac{1}{\Phi(\xi,q)} - e^{il\xi}\frac{P_-(\xi,q)}{\Phi_+(\xi,q)} = e^{il\xi}\frac{P_+(\xi,q)}{\Phi_+(\xi,q)} \tag{20}$$

$$\widetilde{\widehat{p}}_{X,M}(\xi,q) = \sum_{n=0}^{\infty} q^n \widehat{p}_{X,M}(\xi,n) = \frac{1}{\Phi(\xi,q)} - e^{iu\xi}\frac{Q_+(\xi,q)}{\Phi_-(\xi,q)} = e^{iu\xi}\frac{Q_-(\xi,q)}{\Phi_-(\xi,q)} \tag{21}$$

$$\widetilde{\widehat{p}}_{X,m,M}(\xi,q) = \sum_{n=0}^{\infty} q^n \widehat{p}_{X,m,M}(\xi,n) = \frac{1}{\Phi(\xi,q)} - e^{il\xi}\frac{J_-(\xi,q)}{\Phi(\xi,q)} - e^{iu\xi}\frac{J_+(\xi,q)}{\Phi(\xi,q)}, \tag{22}$$

where

$$P(\xi,q) := \frac{e^{-il\xi}}{\Phi_-(\xi,q)} = P_+(\xi,q) + P_-(\xi,q) \tag{23}$$

$$Q(\xi,q) := \frac{e^{-iu\xi}}{\Phi_+(\xi,q)} = Q_+(\xi,q) + Q_-(\xi,q). \tag{24}$$

Notice that the joint probabilities in Equations (20)–(22) are given by the probability of the process at maturity, Equation (17), minus the probability to hit a barrier; the latter vanishes if the barrier moves to $\pm\infty$. Similar identities exist for the continuous-monitoring case too, where the quantity to be factorized is simply $1-q\psi(\xi)$.

The double-barrier problem, which is more difficult than the others, was not examined by Spitzer himself, but by Kemperman [25]. Unfortunately he did not present a constructive procedure for the determination of the quantities $J_+(\xi,q)$ and $J_-(\xi,q)$ in Equation (22). The problem was later solved in the Gaussian case by Green, Fusai and Abrahams [4, Section 2.4]. Here we generalize the latter construction to Lévy processes. In particular, Green, Fusai and Abrahams proved that $J_+(\xi,q)$ and $J_-(\xi,q)$ are the solution of the coupled integral equations

$$\frac{J_-(\xi,q)}{\Phi_-(\xi,q)} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{i(u-l)\xi'}J_+(\xi',q)}{(\xi-\xi')\Phi_-(\xi',q)}d\xi' = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-il\xi'}}{(\xi-\xi')\Phi_-(\xi',q)}d\xi',\ \operatorname{Im}\xi' > \operatorname{Im}\xi, \tag{25}$$

$$\frac{J_+(\xi,q)}{\Phi_+(\xi,q)} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{i(l-u)\xi'}J_-(\xi',q)}{(\xi'-\xi)\Phi_+(\xi',q)}d\xi' = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-iu\xi'}}{(\xi'-\xi)\Phi_+(\xi',q)}d\xi',\ \operatorname{Im}\xi' < \operatorname{Im}\xi, \tag{26}$$

where Im is the imaginary part.

As proved by Krein [26], the decomposition of a complex function $\widehat{f}(\xi)=\widehat{f_+}(\xi)+\widehat{f_-}(\xi)$ can be computed as

$$\widehat{f_+}(\xi)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\widehat{f}(\xi')}{\xi'-\xi}d\xi', \quad \operatorname{Im}\xi'<\operatorname{Im}\xi, \tag{27}$$

$$\widehat{f_-}(\xi)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\widehat{f}(\xi')}{\xi-\xi'}d\xi', \quad \operatorname{Im}\xi'>\operatorname{Im}\xi. \tag{28}$$

Therefore Equations (25)–(26) can be rewritten as

$$\frac{J_-(\xi,q)}{\Phi_-(\xi,q)}+\left[\frac{e^{i(u-l)\xi}J_+(\xi,q)}{\Phi_-(\xi,q)}\right]_-=\left[\frac{e^{-il\xi}}{\Phi_-(\xi,q)}\right]_- \tag{29}$$

$$\frac{J_+(\xi,q)}{\Phi_+(\xi,q)}+\left[\frac{e^{i(l-u)\xi}J_-(\xi,q)}{\Phi_+(\xi,q)}\right]_+=\left[\frac{e^{-iu\xi}}{\Phi_+(\xi,q)}\right]_+ \tag{30}$$

or

$$\frac{J_-(\xi,q)}{\Phi_-(\xi,q)}=\left[\frac{e^{-il\xi}-e^{i(u-l)\xi}J_+(\xi,q)}{\Phi_-(\xi,q)}\right]_-, \tag{31}$$

$$\frac{J_+(\xi,q)}{\Phi_+(\xi,q)}=\left[\frac{e^{-iu\xi}-e^{i(l-u)\xi}J_-(\xi,q)}{\Phi_+(\xi,q)}\right]_+. \tag{32}$$

To make the above expressions usable, we need to factorize (or decompose) a complex function, defined in a strip containing the real axis, into a product (or sum) of two functions which are analytic in the overlap of two half planes, including the real line, where they are defined. Once this has been done and the relevant quantities in Equations (17)–(22) have been obtained, we must compute numerically an inverse z-transform, followed by an inverse Fourier transform. The latter is done in a standard way using the fast Fourier transform (FFT). The inversion of the z-transform is rather easy too. It has been discussed in detail by Abate and Whitt [24], who showed that it can be well approximated by

$$v(x,n)=\mathcal{Z}^{-1}_{q\to n}\widetilde{v}(x,q)\approx\frac{\widetilde{v}(x,\rho)+2\sum_{j=1}^{n-1}(-1)^j\widetilde{v}\left(x,\rho e^{ij\pi/n}\right)+(-1)^n\widetilde{v}(x,-\rho)}{2n\rho^n}. \tag{33}$$

The more challenging part is the factorization of Φ in Equation (16), as well as the decomposition of P and Q. In general, this problem can be described as follows. Given a smooth enough function $\widehat{f}(\xi)$, analytic in a strip around the real axis, we need to compute $\widehat{f_\pm}(\xi)$ such that

$$\widehat{f}(\xi)=\widehat{f_+}(\xi)\widehat{f_-}(\xi); \tag{34}$$

$\widehat{f_+}(\xi)$ is such that its inverse Fourier transform $f_+(x)=0$ for $x<0$, while $\widehat{f_-}(\xi)$ is such that $f_-(x)=0$ for $x>0$. Taking logarithms, this can be accomplished by the decomposition

$$\log\widehat{f}(\xi)=\log\widehat{f_+}(\xi)+\log\widehat{f_-}(\xi). \tag{35}$$

The conditions under which the above factorization or decomposition gives proper results have been given by Krein [26]; the most important requirement is that $\widehat{f}(\xi)$ is not zero anywhere.

In general neither the factorization nor the decomposition can be done analytically. With continuous monitoring, where the quantity to be factorized is $1-q\psi(\xi)$ instead of $1-q\Psi(\xi,\Delta)$, an analytical treatment becomes possible for a Brownian motion or if we impose strong restrictions on the structure of the considered Lévy process [27], such as the assumption that it is spectrally one-sided, i.e., jumps are either always up or always down. Another assumption that makes the factorization feasible is if the jumps are of phase type [28], which includes the Kou double exponential jump model as a special case. In these cases the Wiener-Hopf factorization is tractable because $1-q\psi(\xi)$ is a rational function and its decomposition in upper/lower factors is quite immediate. For example, Jeannin and Pistorius [6] approximate different Lévy models by the class of generalized hyper-exponential models, which have a tractable Wiener-Hopf factorization. A similar idea is pursued by Asmussen, Madan and Pistorius [29].

Unfortunately, with discrete monitoring even under the above assumptions the factorization is not doable analytically, because $1-q\Psi(\xi,\Delta)$ is no more a rational function. In addition, all the above mentioned methods consider only the single-barrier case. An exception was given by Boyarchenko and Levendorskii [30], who obtained exact analytical pricing formulae in terms of Wiener-Hopf factors, and, under additional conditions on the process, derived simpler approximate formulae. For the general difficulty in computing the factors, with reference to the important financial engineering problem of pricing barrier options, Carr and Crosby [31] state: *"Pricing barrier options for arbitrary Lévy processes is far from trivial. There are, in principle, some results ... based on Wiener-Hopf analysis although they involve inversion of triple Laplace transforms and it is open to debate as to whether this could be done efficiently enough for use in a trading environment."* Similarly, Cont and Tankov [32], a popular reference text for applications of Lévy processes in finance, state: *"The Wiener-Hopf technique is too computationally expensive and we recommend Monte Carlo simulation or numerical solution of partial integro-differential equations."* These remarks are based on the representation of the Wiener-Hopf factors for the continuous monitoring case as double integrals. Indeed we have $\varphi(\xi) := 1-q\psi(\xi) = \varphi_+(\xi)\varphi_-(\xi)$ with

$$\log\varphi_+(\xi) = \int_0^{+\infty} t^{-1}e^{-t/q}dt \int_0^{+\infty}(1-e^{i\xi x})P_X(dx), \tag{36}$$

and

$$\log\varphi_-(\xi) = \int_0^{+\infty} t^{-1}e^{-t/q}dt \int_{-\infty}^{0}(1-e^{i\xi x})P_X(dx), \tag{37}$$

where $P_X(\cdot)$ is the distribution of the Lévy process X [32, Chapter 11.3]. With reference to financial applications, some attempts to compute the Wiener-Hopf factors have been done by Boyarchenko and Levendorskii [30], Kuznetsov et al. [33], among the others.

A more convenient representation of the Wiener-Hopf factors can be found using the Hilbert transform and the Plemelj-Sokhotsky relations. The Hilbert transform [34] of a function $\widehat{f}(\xi)$ is defined as

$$\mathcal{H}_\xi \widehat{f}(\xi) = \text{p.v.}\, \frac{1}{\pi\xi} * \widehat{f}(\xi) = \text{p.v.}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\widehat{f}(\xi')}{\xi' - \xi} d\xi', \tag{38}$$

where $*$ denotes convolution and p.v. the Cauchy principal value,

$$\text{p.v.}\, \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi' = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \left(\int_{-1/\epsilon}^{-\epsilon} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi' + \int_{\epsilon}^{1/\epsilon} \frac{\widehat{f}(\xi')}{\xi - \xi'} d\xi' \right); \tag{39}$$

the latter assigns a value to an improper integral which would otherwise result in the indefinite form $+\infty - \infty$. The convolution theorem

$$\widehat{f}(\xi) * \widehat{g}(\xi) = \mathcal{F}[f(x)g(x)], \tag{40}$$

which maps a convolution to a product via a Fourier transform, together with the inverse Fourier transform

$$\text{p.v.}\, \mathcal{F}^{-1} \frac{1}{\pi\xi} = -i \operatorname{sgn} x, \tag{41}$$

enables to express the Hilbert transform through an inverse Fourier transform (from $\widehat{f}(\xi)$ to $f(x)$) and a direct Fourier transform

$$i\mathcal{H}\widehat{f}(\xi) = \mathcal{F}\big[\operatorname{sgn} x\, f(x)\big]; \tag{42}$$

Thus a fast method to compute the Hilbert transform numerically consists simply in evaluating Eq. (42) through an inverse and a direct FFT. In Section 3.1 we will see a more sophisticated numerical method based on a sinc function expansion.

Define the projections of a function $f(x)$ on the positive or the negative half-axis through the multiplication with the indicator function of that set,

$$\mathcal{P}_x^+ f(x) := \mathbf{1}_{x>0} f(x) = f_+(x) \tag{43}$$

$$\mathcal{P}_x^- f(x) := \mathbf{1}_{x<0} f(x) = f_-(x). \tag{44}$$

Now substitute

$$\operatorname{sgn} x\, f(x) = (\mathbf{1}_{x>0} - \mathbf{1}_{x<0}) f(x) = f_+(x) - f_-(x) \tag{45}$$

into Equation (42), obtaining the remarkable property

$$\widehat{f_+}(\xi) - \widehat{f_-}(\xi) = i\mathcal{H}\widehat{f}(\xi). \tag{46}$$

Together with the identity

$$\widehat{f_+}(\xi) + \widehat{f_-}(\xi) = \widehat{f}(\xi), \tag{47}$$

this allows to achieve a decomposition of a function $\widehat{f}(\xi)$, and thus a factorization of $\exp\widehat{f}(\xi)$, via its Hilbert transform. To this end, Equations (46) and (47) are conveniently rearranged to the Plemelj-Sokhotsky relations

$$\widehat{f_+}(\xi) = \frac{1}{2}\big[\widehat{f}(\xi) + i\mathcal{H}\widehat{f}(\xi)\big] \tag{48}$$

$$\widehat{f_-}(\xi) = \frac{1}{2}\big[\widehat{f}(\xi) - i\mathcal{H}\widehat{f}(\xi)\big]. \tag{49}$$

Obtaining the Wiener-Hopf factors of $\exp\widehat{f}(\xi)$ through Equations (48)–(49) with the Hilbert transform computed in a straightforward way by Equation (42) corresponds to performing in sequence an inverse Fourier transform, a projection on the positive or negative half axis and a Fourier transform,

$$\widehat{f_+}(\xi) = \mathcal{F}_{x\to\xi}\big[\mathcal{P}^+_x\,\mathcal{F}^{-1}_{\xi\to x}\widehat{f}(\xi)\big](\xi) \tag{50}$$

$$\widehat{f_-}(\xi) = \mathcal{F}_{x\to\xi}\big[\mathcal{P}^-_x\,\mathcal{F}^{-1}_{\xi\to x}\widehat{f}(\xi)\big](\xi), \tag{51}$$

corresponding to the scheme

$$\begin{array}{ccc} & \overset{\mathcal{P}^+}{\nearrow} & \cdot \xrightarrow{\mathcal{F}} \widehat{f_+} \\ \widehat{f} \xrightarrow{\mathcal{F}^{-1}} \cdot & & \\ & \underset{\mathcal{P}^-}{\searrow} & \cdot \xrightarrow{\mathcal{F}} \widehat{f_-}. \end{array} \tag{52}$$

This factorization turns out to be very fast, because it can be accomplished numerically with two fast Fourier transforms (FFTs) and one projection [35, 36]. On the other hand, switching back and forth between the complex and the real planes, the application of the projection causes a loss of accuracy; at the end this procedure turns out to have only quadratic accuracy.

A numerically more accurate approach consists in the computation of the Hilbert transform, and thus of the Plemelj-Sokhotsky relations, using a sinc expansion approximation to analytic functions. This approach uses two FFTs too to multiply Toeplitz matrices with vectors and thus has a computational cost of $\mathcal{O}(M\log M)$, but it does not leave Fourier space and its discretization error decreases exponentially with respect to M; see Section 3.1 for details.

We stress here the similarities and differences with the approach followed by Feng and Linetsky [3, 23]. The analogy is due to the fact that in the mentioned papers the Hilbert transform is applied in the backward-in-time pricing procedure. In practice, the projection step is performed in the complex plane using the Hilbert transform; greater details on how this is possible will be given in Section 5.1. This transform is computed at a high degree of accuracy via sinc expansion. No direct relationship of their procedure with the Wiener-Hopf factorization can be devised. The analogy is that we are able to express the Wiener-Hopf factors via the Hilbert transform and then we can exploit their idea of performing this transform with a sinc expansion. At the end, we are able to achieve the same accuracy as their method, but with a significant saving of computational time, because our procedure turns out to have a cost independent of the number of monitoring dates N, whilst all existing methods, including [3, 23], this cost increases linearly with N.

For the sake of truth, an advantage of the Feng and Linetsky method with respect to our procedure is that it can easily deal with non-equally spaced monitoring dates. On the other side, our methodology can cope with the continuous monitoring case, as shown in Section 7, whilst Feng and Linetsky approach, and other Fourier methods, cannot.

The new approach proposed in the present paper is therefore summarized in performing the Wiener-Hopf factorization through the Plemelj-Sokhotsky relations (48)–(49), and computing the Hilbert transform in Fourier space using sinc functions. The detailed procedure is described in Section 4, considering different financial applications.

3. Discrete approximation error and efficient implementation

The implementation of the proposed procedure to estimate the distributions in the Equations (3)–(8) contains mainly two sources of error: the computation of the Wiener-Hopf factorizations using M discretization points, and the inversion of the z-transform. In the following, we detail how to efficiently implement both the factorization, exploiting sinc functions, and the z-transform inversion, via the Euler acceleration technique.

3.1. Hilbert transform with sinc functions

The Hilbert transform can be efficiently computed using the sinc expansion approximation of analytic functions. The use of sinc functions,

$$S_k(z,h) = \frac{\sin(\pi(z-kh)/h)}{\pi(z-kh)/h}, \quad k \in \mathbb{Z}, \tag{53}$$

has been deeply studied by Stenger [37, 38], who showed that a function $f(z)$ analytic on the whole complex plane and of exponential type with parameter π/h, i.e.,

$$|f(z)| \le C e^{\pi|z|/h}, \tag{54}$$

can be reconstructed exactly from the knowledge of its values on an equispaced grid of step h, as $f(z)$ admits the sinc expansion [37, Theorem 1.10.1]

$$f(z) = \sum_{k=-\infty}^{+\infty} f(kh) S_k(z,h). \tag{55}$$

Now, since

$$\mathcal{F}_{z\to\zeta} S_k(z,h) = h e^{ikh\zeta} \tag{56}$$

```
% Fast Hilbert transform: Hilbert transform through fast Fourier transforms.
function iHF = ifht(F)
% Setup
[M N] = size(F); % Dimension parameters: number of equations and of grid points
P = N; % Number of zero padding elements
Q = N+P; % Number of grid points after zero padding
% Define the auxiliary vector
t = (1-(-1).^(-Q/2:Q/2-1))./(pi*(-Q/2:Q/2-1));
t(Q/2+1) = 0;
vec = repmat(imag(fft(ifftshift(t))),M,1);
% Compute the Hilbert transform times the imaginary unit
f = ifft(F,Q,2); % Optional padding with P trailing zeros to length Q = N + P
iHF = fft(vec.*f,[],2);
iHF = iHF(:,1:N);
```

Figure 1 Matlab code to compute the Hilbert transform via sinc function expansion.

```
% Factorise L = 1-H
lL = log(1-H);
iHlL = ifht(lL); % imaginary unit times the fast Hilbert transform of L
lLp = (lL+iHlL)/2; % Plemelj-Sokhotsky
lLm = (lL-iHlL)/2; % Plemelj-Sokhotsky
Lp = exp(lLp);
Lm = exp(lLm);
```

Figure 2 Matlab code to compute the Wiener-Hopf factorization via the Hilbert transform.

and [3, Corollary 6.1]

$$\mathcal{H}_z S_k(z,h) = \frac{1-\cos(\pi(z-kh)/h)}{\pi(z-kh)/h}, \tag{57}$$

also the Fourier and Hilbert transforms of $f(z)$ admit the sinc expansions

$$\widehat{f}(\zeta) = h \sum_{k=-\infty}^{+\infty} f(kh)e^{ikh\zeta} \quad \text{if } |\zeta| < \pi/h, \tag{58}$$

$\widehat{f}(\zeta) = 0$ if $|\zeta| \geq \pi/h$, since functions analytic on the whole plain and of exponential type have Fourier transforms that vanish outside of the finite interval $(-\pi/h, \pi/h)$ [37, Theorem 1.10.1], and

$$\mathcal{H}f(z) = \sum_{k=-\infty}^{+\infty} f(kh)\frac{1-\cos(\pi(z-kh)/h)}{\pi(z-kh)/h}. \tag{59}$$

The integrals of f and $|f|^2$ can be written as sinc expansions too,

$$\int_{\mathbb{R}} f(x)dx = h\sum_{k=-\infty}^{+\infty} f(kh), \qquad \int_{\mathbb{R}} |f(x)|^2 dx = h\sum_{k=-\infty}^{+\infty} |f(kh)|^2. \tag{60}$$

The above results show in particular that the trapezoidal quadrature rule with step size h is exact.

This holds true for a function $f(z)$ that is analytic in the whole complex plane. However, this can be used also to approximate a function that is analytic only in a strip including the real axis, which is the case considered in this article. More precisely, Stenger shows [37, Theorems 3.1.3, 3.1.4 and 3.2.1] that in this case the trapezoidal approximation has an error that decays exponentially with respect to h. The computation of the Hilbert transform via a sinc expansion can be performed using the FFT [3, Section 6.5]. The idea is that to compute a discrete Hilbert transform it is necessary to do matrix-vector multiplications involving Toeplitz matrices. As is well known, this kind of multiplications can be performed exploiting the FFT, once those matrices are embedded in a circulant matrix [3, Appendix B][19]; see also Section 5.2. In particular, Feng and Linetsky, with respect to the computation of the Hilbert transform [23, Theorem 3.3] and of the whole Plemelj-Sokhotsky formulas (48)–(49) [3, Theorem 6.5] [23, Theorem 3.4] with sinc functions, prove the following convergence result: if a function is analytic in a suitable strip around the real axis, then the discretization error of its numerical factorization or decomposition decays exponentially with the number of discretization points M. We will show the connection of Feng and Linetsky's procedure with the Plemelj-Sokhotsky formula in Section 5.1. Further details can be found in the cited references [3, Section 6] [23, Section 3.4], while Matlab code to compute the Hilbert transform via sinc functions and the WH factorization via the Hilbert transform is reported in Figures 1 and 2.

3.2. Acceleration of the inverse z-transform via Euler summation

The inverse z-transform $\mathcal{Z}^{-1}_{q\to n}$ is performed according to Equation (33), where $\rho \in (0,1)$ is a free parameter; setting $\rho = 10^{-6}$ yields a 10^{-12} accuracy of the option price [19, 24]. Moreover, we apply the Euler summation, which is a convergence-acceleration technique well suited to evaluate alternating series. The idea of the Euler summation is to approximate $\mathcal{Z}^{-1}_{q\to n}\widetilde{v}(\xi, q)$ by the binomial average, also called Euler transform, of its partial sums b_k from $k = n_{\mathrm{E}}$ to $k = n_{\mathrm{E}} + m_{\mathrm{E}}$, i.e.,

$$\mathcal{Z}^{-1}_{q\to n}\widetilde{v}(\xi, q) \approx \frac{1}{2^{m_{\mathrm{E}}} n\rho^n} \sum_{j=0}^{m_{\mathrm{E}}} \binom{m_{\mathrm{E}}}{j} b_{n_{\mathrm{E}}+j}(\xi), \tag{61}$$

where

$$b_k = \sum_{j=0}^{k} (-1)^j a_j \operatorname{Re} \widetilde{v}\big(\xi, \rho e^{ij\pi/n}\big), \tag{62}$$

with $a_0 = 0.5$, $a_j = 1,\ j = 1, \ldots, n_{\mathrm{E}} + m_{\mathrm{E}}$, and n_{E} and m_{E} are suitably chosen such that $n_{\mathrm{E}} + m_{\mathrm{E}} < n$. Thus the number of parameters $q = \rho e^{ij\pi/n}$ to be considered in Equation (33) drops from $n+1$ to $n_{\mathrm{E}} + m_{\mathrm{E}} + 1$. Numerical tests suggest to set $n_{\mathrm{E}} = 12$ and $m_{\mathrm{E}} = 20$. In Figure 3 we show graphically the convergence of the partial sums b_k to the inverse z-transform.

To conclude, the pricing algorithm proposed in Section 4 has a computational cost of

$$\mathcal{O}\left((\min\{n, n_{\mathrm{E}} + m_{\mathrm{E}}\} + 1) M \log M\right),$$

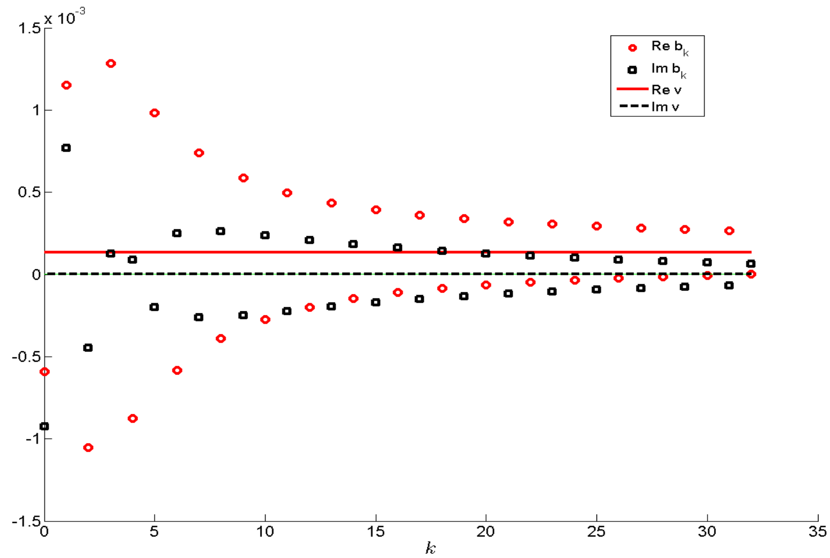

Figure 3 Convergence of the partial sum b_k to $\mathcal{Z}^{-1}_{q\to n}\widetilde{v}(\xi,q) \approx \frac{1}{2^{m_{\mathrm{E}}} n \rho^n} \sum_{j=0}^{m_{\mathrm{E}}} \binom{m_{\mathrm{E}}}{j} b_{n_{\mathrm{E}}+j}(\xi)$. The real (imaginary) part of b_k, $k=0,1,\ldots,n_E+m_E$, corresponds to the red circles (black squares), while the real (imaginary) part of the solution corresponds to the red (black dashed) line. The test case is related to the computation of $dP_{X,m}$ with $X(t)$ a double exponential Lévy process, a log-barrier $l=0.8$, $N=100$ monitoring dates and $M=2^{14}$ grid points.

and a discretization error which exponentially decays till it reaches an accuracy of about 10^{-12}. This is confirmed in the numerical experiments reported in Section 6 to price derivatives. The only exception is for double-barrier options, and therefore when we deal with the probability $dP_{X,m,M}$, where the decay of the error turns out to be polynomial. Likely, this is related to the fixed-point algorithm described in Section 4.3 and necessary to compute $J_\pm$ in Equations (25) and (26).

4. Applications to option pricing

In option pricing applications, we use the Lévy process $X(t)$ as driving source in describing the evolution of an asset price $S(t)$ according to

$$S(t) = S_0 e^{X(t)}, \tag{63}$$

$S_0 = S(0)$ being the initial spot price. The stock price dynamics is directly specified under the so-called risk-neutral measure, so that in Equation (1) $a = r - \delta - \frac{1}{2}\sigma^2 - \int_{\mathbb{R}} \left(e^\eta - 1 - \eta \mathbf{1}_{|\eta|<1}\right)\nu(d\eta)$, where r is the risk-free interest rate and δ the asset dividend rate.

In pricing path-dependent options such as barrier and lookback options, the relevant quantities are the maximum M_N and the minimum m_N up to time $N\Delta = T$. To price a fixed-strike lookback option we need the distribution $P_M(x,N)$ of the maximum or $P_m(x,N)$ of the minimum. For a single-barrier option we need the joint distribution $P_{X,M}(x,N)$ or $P_{X,m}(x,N)$ of the Lévy process at maturity $T=N\Delta$, Δ being the (constant) time interval between two subsequent monitoring dates, and of its maximum (up-and-out case) or minimum (down-and-out case) registered during the entire life $n=0,\dots,N$ of the option. For a double-barrier option we need the joint distribution $P_{X,m,M}(x,N)$ of the triplet (X_N,m_N,M_N).

In pricing the above mentioned contracts, we are interested in the truncated payoff

$$\phi(x)=e^{\alpha x}\left(S_0e^x-e^k\right)^+\mathbf{1}_{x\leq u} \tag{64}$$

for a call option and

$$\phi(x)=e^{\alpha x}\left(e^k-S_0e^x\right)^+\mathbf{1}_{x\geq l}, \tag{65}$$

for a put option, where $k=\log(K/S_0)$ is the rescaled log-strike of the option, and $l=\log(L/S_0)$ and $u=\log(U/S_0)$ are the rescaled lower and upper log-barriers. In the following we assume $l<k<u$. The damping factor $e^{\alpha x}$ with a suitable choice of the parameter α is introduced to make the Fourier transform of the above payoff well defined.

The option price can be obtained discounting the expectation value with respect to the appropriate distribution; this expectation can conveniently be computed through the Parseval relation [39] by a product in Fourier space and an inverse Fourier transform,

$$\mathbb{E}[\phi(x)]=\int_{-\infty}^{+\infty}\phi(x)p(x)dx=\int_{-\infty}^{+\infty}\hat{\phi}(\xi)\hat{p}^*(\xi)d\xi=\int_{-\infty}^{+\infty}\hat{\phi}^*(\xi)\hat{p}(\xi)d\xi=\mathcal{F}^{-1}\left[\hat{\phi}^*(\xi)\hat{p}(\xi)\right](0), \tag{66}$$

where $p(x)=p_M(x,N)$ or $p_m(x,N)$ for lookback options (to be synthetic, in the following we will consider only fixed-strike lookback options written on the minimum), $p(x)=p_{X,M}(x,N)$ or $p_{X,m}(x,N)$ for single-barrier options, and $p=p_{X,m,M}(x,N)$ for double-barrier options.

While it is known that the Fourier transform of the truncated payoff is

$$\widehat{\phi}(\xi)=Ke^{k(\alpha+i\xi)}\left(\frac{1-e^{b(\alpha+i\xi)}}{\alpha+i\xi}-\frac{1-e^{b(1+\alpha+i\xi)}}{1+\alpha+i\xi}\right) \tag{67}$$

with $b=\log(U/K)$ for a call option and $b=\log(L/K)$ for a put option in the barrier case [3], the main problem is the computation of the characteristic functions of the (joint) probability densities defined in Equations (5)–(8). Here we exploit the Spitzer identity and the factorization procedure previously described.

So let us assume for the moment that the quantities appearing on the right-hand side of Equations (18)–(22) are known; then if we take the z-transform of the undiscounted expectation value in Equation (66), we obtain

$$\sum_{n=0}^{\infty}q^n\mathbb{E}\left[\phi(x)\right]=\sum_{n=0}^{\infty}q^n\mathcal{F}^{-1}[\widehat{\phi}^*(\xi)\widehat{p}(\xi,n)](0), \tag{68}$$

and swapping the z-transform and the Fourier transform[1]

$$\mathcal{F}^{-1}_{\xi\to x}\left[\widehat{\phi}^*(\xi)\sum_{n=0}^{\infty}q^n\widehat{p}(\xi,n)\right](0)=\mathcal{F}^{-1}_{\xi\to x}\left[\widehat{\phi}^*(\xi)\widetilde{\widehat{p}}(\xi,q)\right](0). \tag{69}$$

Using the inverse z-transform defined in Equation (15), we obtain finally the option price through the double inverse transform

$$v(x,n)=e^{-rT}\mathcal{F}^{-1}_{\xi\to x}\left[\mathcal{Z}^{-1}_{q\to n}\left[\widehat{\phi}^*(\xi)\widetilde{\widehat{p}}(\xi,q)\right]\right], \tag{70}$$

evaluated for $(x,n)=(0,N)$. Later we will discuss a few little improvements to the above formula in order to enhance the numerical accuracy of the final result.

In Equation (70) the inverse z-transform is performed before the inverse Fourier transform to minimize the computational cost. The reason is that the inversion operator $\mathcal{Z}^{-1}_{q\to n}$ is well approximated by a sum of $n+1$ terms (or n_E+m_E+1 if the Euler acceleration is considered). Therefore, from a computational point of view it is advantageous to do a single inverse Fourier transform of the sum instead of a separate transform of each of the addends.

Since we have to deal with unbounded domains, we use a domain truncation based on a moments' bound with tolerance 10^{-8} [19]; thus the truncation error is constant, but, according to the reported numerical experiments [18, 19], it does not affect the first significant decimal digits.

4.1. Lookback options

In the case of lookback options (we recall that, to be synthetic, we deal only with fixed-strike lookback on the minimum), assuming a number of monitoring dates $N>1$, it is convenient to modify the pricing formula (70) into

$$v(x,N)=e^{-rT}\mathcal{F}^{-1}_{\xi\to x}\left[\Psi(\xi)\widehat{\phi}^*(\xi)\mathcal{Z}^{-1}_{q\to N-1}\left[\frac{1}{\Phi_+(0,q)\Phi_-(\xi,q)}\right]\right], \tag{71}$$

evaluated for $x=0$, i.e., to apply the inverse z-transform to a number of monitoring dates reduced by 1, and to account for the extra date multiplying the conjugated Fourier transform of the payoff function by the characteristic function Ψ. This smooths the payoff function, giving it the required regularity to ensure an exponential decay of the error. From a financial point of view, this smoothing is equivalent to price a lookback option with $N-1$ monitoring dates and a payoff $v_1=v(x,1)$, where $v(x,n)$ is the value of the option at time $(N-n)\Delta$ and log-price x. Thus, lookback options are priced according to the algorithm

$$\phi\equiv v_0\xrightarrow{\mathcal{F}}\widehat{v}_0\xrightarrow{\Psi^*}\widehat{v}_1\xrightarrow{\mathcal{ZS}}\widehat{v}_N\xrightarrow{\mathcal{F}^{-1}}v_N, \tag{72}$$

[1] The interchange of integration and summation requires $\sum_{n=0}^{\infty}q^nf(z,n)$ to converge uniformly. The z-transform $\sum_{n=0}^{\infty}q^nf(z,n)$ is in fact a power series in q with coefficients $f(z,n)$ and radius of convergence given by ρ. A power series converges uniformly in a closed and bounded interval contained in the interval of convergence [40].

where the operator $\mathcal{ZS}$ is defined as

$$\mathcal{ZS}: F(\xi) \to F^*(\xi)\mathcal{Z}^{-1}_{q\to N-1}\left[\frac{1}{\Phi_+(0,q)\Phi_-(\xi,q)}\right]. \tag{73}$$

Recall that the conjugate operator applied to the generic function F is due to the Parseval relation. The full procedure consists of the following steps:

1. For each q necessary to invert the z-transform, factorize

$$\Phi(\xi,q) := 1 - q\Psi(\xi,q) = \Phi_+(\xi,q)\Phi_-(\xi,q) \tag{74}$$

and compute the Spitzer identity

$$\widetilde{\widehat{p}}_m(\xi,q) = \frac{1}{\Phi_+(0,q)\Phi_-(\xi,q)}. \tag{75}$$

2. Apply the inverse z-transform $\mathcal{Z}^{-1}_{q\to N-1}$ to $\widetilde{\widehat{p}}_m(\xi,q)$ and multiply the result by $\Psi(\xi)\widehat{\phi}^*(\xi)$, obtaining $\widehat{v}(\xi,N)$.

3. Apply the inverse FFT to $\widehat{v}(\xi,N)$ and pick the value for $x=0$, obtaining the option price.

A similar procedure is valid for fixed-strike lookback options written on the maximum, where $\widetilde{\widehat{p}}_M(\xi,q)$ is used in place of $\widetilde{\widehat{p}}_m(\xi,q)$.

4.2. Single-barrier options

Without loss of generality, let us consider the case of a down-and-out barrier option. Assuming again a number of monitoring dates $N>2$, we reduce this number by 1 and multiply the payoff function by the characteristic function, as we did for lookback options. Then our methodology is applied to the remaining $N-1$ monitoring dates. The full procedure can be summarized as follows:

$$\phi \equiv v_0 \xrightarrow{\mathcal{F}} \widehat{v}_0 \xrightarrow{\Psi^*} \widehat{v}_1 \xrightarrow{\mathcal{ZS}} \widehat{v}_N \xrightarrow{\mathcal{F}^{-1}} v_N \tag{76}$$

$$\delta \xrightarrow{\mathcal{F}} \cdot \xrightarrow{\Psi} \overset{\uparrow}{\cdot} \tag{77}$$

In this case we denote with $\mathcal{ZS}$ the operator

$$\mathcal{ZS}: F(\xi) \to F^*(\xi)\mathcal{Z}^{-1}_{q\to N-2}\left[e^{il\xi}\frac{\overline{P}_+(\xi,q)}{\Phi_+(\xi,q)}\right], \tag{78}$$

and

$$\overline{P}(\xi,q) := \Psi(\xi)\frac{e^{-il\xi}}{\Phi_-(\xi,q)} = \overline{P}_+(\xi,q) + \overline{P}_-(\xi,q). \tag{79}$$

If we compare the above operator with the definition of $\widetilde{\widehat{p}}_{X,m}$ in Equation (20), we notice that the only difference is the presence of $\overline{P}$, which differs from P only for the factor Ψ. This is necessary to smooth the function P, substituting it with $\overline{P}$, that has the regularity required to ensure an exponential decay of the error. The substitution is related to the procedure sketched in Equation (77):

in computing the distribution $P_{X,m}$ we do not start from time 0, but we move one step forward with a convolution procedure, which corresponds to multiplying with Ψ the Fourier transform of the Dirac delta function, i.e., the value of the probability at time 0, and then apply the Spitzer identity. Moreover, notice that the procedures given by Equation (76) and Equation (77) are performed backward and forward-in-time, respectively, since the first one is related to the price of the derivative (starting point: payoff at time T), while the second one to the probability distribution of the log-price (starting point: Dirac delta at time 0).

Therefore, for a down-and-out barrier option we perform the following steps:

1. For each q necessary to invert the z-transform, factorize

$$\Phi(\xi,q) := 1 - q\Psi(\xi,q) = \Phi_+(\xi,q)\Phi_-(\xi,q), \tag{80}$$

decompose

$$\overline{P}(\xi,q) := \Psi(\xi)\frac{e^{-il\xi}}{\Phi_-(\xi,q)} = \overline{P}_+(\xi,q) + \overline{P}_-(\xi,q), \tag{81}$$

and compute the Spitzer identity

$$R(\xi,q) := e^{il\xi}\frac{\overline{P}_+(\xi,q)}{\Phi_+(\xi,q)}. \tag{82}$$

The function $R(\xi,q)$ is related to $\widetilde{\widehat{p}}_{X,m}(\xi,q)$ in Equation (18): more precisely, $\mathcal{Z}^{-1}_{q\to N-1}R(\xi,q) = \mathcal{Z}^{-1}_{q\to N}\widetilde{\widehat{p}}_{X,m}(\xi,q) = \widehat{p}_{X,m}(\xi,N)$.

2. Apply the inverse z-transform $\mathcal{Z}^{-1}_{q\to N-2}$ and then the inverse Fourier transform, obtaining the option price from

$$v(x,N) = e^{-rT}\mathcal{F}^{-1}_{\xi\to x}\left[\Psi(\xi)\widehat{\phi}^*(\xi)\mathcal{Z}^{-1}_{q\to N-2}\left[e^{il\xi}\frac{\overline{P}_+(\xi,q)}{\Phi_+(\xi,q)}\right]\right] \tag{83}$$

evaluated for $x = 0$, where $\widehat{\phi}^*$ is the conjugated Fourier transform of the payoff, possibly with a damping factor to make its transform computable, Equation (67), and conjugated because of its use within the Parseval relation, Equation (66).

4.3. Double-barrier options

For the double-barrier option pricing problem the missing piece is the computation of the factors J_+ and J_- in Equation (22). This requires the solution of a system of two integral equations. We are not aware of any previous efficient computational procedure able to deal with this problem, and we propose here a new procedure via an iterative computation based on the fixed-point algorithm.

Starting from Equations (25)–(26), as for the single-barrier case we assume a number of monitoring dates $N>2$ and we move one step forward in the computation of the probability $dP_{X,m,M}$ via convolution. This corresponds to consider

$$\frac{J_-(\xi,q)}{\Phi_-(\xi,q)} = \left[\frac{e^{-il\xi}\Psi(\xi) - e^{i(u-l)\xi}J_+(\xi,q)}{\Phi_-(\xi,q)}\right]_-, \tag{84}$$

$$\frac{J_+(\xi,q)}{\Phi_+(\xi,q)} = \left[\frac{e^{-iu\xi}\Psi(\xi) - e^{i(l-u)\xi}J_-(\xi,q)}{\Phi_+(\xi,q)}\right]_+, \tag{85}$$

instead of Equations (31)–(32). To compute $J_\pm$ we propose the following iterative procedure: starting from a guess function $J_+^{(0)}(\xi,q)=0$, compute $J_\pm(\xi,q)$ and thus the solution with the following fixed-point algorithm: for $i=1,\dots$

1. Decompose

$$\overline{P}^{(i)}(\xi,q) := \frac{e^{-il\xi}\Psi(\xi)}{\Phi_-(\xi,q)} - \frac{e^{i(u-l)\xi}J_+^{(i-1)}(\xi,q)}{\Phi_-(\xi,q)} = \overline{P}_+^{(i)}(\xi,q) + \overline{P}_-^{(i)}(\xi,q) \tag{86}$$

and compute

$$J_-^{(i)}(\xi,q) = \overline{P}_-^{(i)}(\xi,q)\Phi_-(\xi,q). \tag{87}$$

2. Decompose

$$\overline{Q}^{(i)}(\xi,q) := \frac{e^{-iu\xi}\Psi(\xi)}{\Phi_+(\xi,q)} - \frac{e^{i(l-u)\xi}J_-^{(i-1)}(\xi,q)}{\Phi_+(\xi,q)} = \overline{Q}_+^{(i)}(\xi,q) + \overline{Q}_-^{(i)}(\xi,q) \tag{88}$$

and compute

$$J_+^{(i)}(\xi,q) = \overline{Q}_+^{(i)}(\xi,q)\Phi_+(\xi,q). \tag{89}$$

3. Compute

$$R^{(i)}(\xi,q) := \frac{\Psi}{\Phi(\xi,q)} - e^{il\xi}\frac{J_-^{(i)}(\xi,q)}{\Phi(\xi,q)} - e^{iu\xi}\frac{J_+^{(i)}(\xi,q)}{\Phi(\xi,q)}; \tag{90}$$

if the distance between the new and old functions $R^{(i)}$ and $R^{(i-1)}$ is greater than a given tolerance, increase i and return to Step (a), otherwise stop and set $R=R^{(i)}$, $J_-=J_-^{(i)}$, $J_+=J_+^{(i)}$. The function $R(\xi,q)$ is related to $\widetilde{\widehat{p}}_{X,m,M}(\xi,q)$ in Equation (22): more precisely, $\mathcal{Z}^{-1}_{q\to N-1}R(\xi,q) = \mathcal{Z}^{-1}_{q\to N}\widetilde{\widehat{p}}_{X,m,M}(\xi,q) = \widehat{p}_{X,m,M}(\xi,N)$.

Therefore the scheme for the computation of the option price is the following:

1. For each q necessary to invert the z-transform, factorize

$$\Phi(\xi,q) = 1 - q\Psi(\xi,q) = \Phi_+(\xi,q)\Phi_-(\xi,q), \tag{91}$$

and compute $R(\xi,q)$ via the iterative scheme.

2. Apply the inverse z-transform $\mathcal{Z}^{-1}_{q\to N-2}$ to $R(\xi,q)$ and then the inverse Fourier transform, obtaining the option price from

$$v(x,N) = e^{-rT}\mathcal{F}^{-1}_{\xi\to x}\left[\Psi(\xi)\widehat{\phi}^*(\xi)\mathcal{Z}^{-1}_{q\to N-2}\left[\frac{\Psi}{\Phi(\xi,q)} - e^{il\xi}\frac{J_-(\xi,q)}{\Phi(\xi,q)} - e^{iu\xi}\frac{J_+(\xi,q)}{\Phi(\xi,q)}\right]\right] \tag{92}$$

evaluated for $x = 0$. Thus, the methodology to price a double-barrier option is close to the one proposed for single-barrier contracts and consists of the same steps as sketched in Equations (76)–(77), with a modified $R(\xi, q)$ inside the operator $\mathcal{ZS}$, i.e.,

$$\mathcal{ZS} : F(\xi) \to F^*(\xi)\mathcal{Z}^{-1}_{q \to N-2}\left[R(\xi, q)\right], \tag{93}$$

replacing $R(\xi, q)$ defined in Equation (82) with the one computed, via the fixed-point algorithm, in Equation (90). Even if the factorization is performed with a Hilbert transform computed as proposed in Section 3.1, our numerical experiments show that this pricing algorithm provides a quadratic convergence of the error, instead of the exponential one of single-barrier (and look-back) options. In practice, the fixed-point algorithm causes a loss of accuracy with respect to the backward-in-time procedure used by Feng and Linetsky [3]. On the other hand, the above numerical scheme solves a long-standing problem related to an efficient computation of the Wiener-Hopf factors in the double-barrier case.

4.4. Defaultable bonds

It is straightforward to apply our fast Wiener-Hopf factorization to compute the survival probability of a firm and to price defaultable bonds in the context of a structural approach to credit risk. According to Black and Cox [41], the credit event is defined as the first time that the firm value falls below a predefined lower barrier. Consider the firm value process

$$V(t) = V_0 e^{X(t)}, \tag{94}$$

where $X(t)$ is a Lévy process. The firm will default when its value falls below a barrier L. Here, we assume that the default event is monitored at discrete dates and the default time is defined as the first hitting time of a level L,

$$\tau = \min_{j=0,\ldots,N} \left\{ j\Delta : V(j\Delta) \leq L \right\}. \tag{95}$$

The default probability $\mathbb{P}(\tau \leq j\Delta)$ is related to the distribution of the minimum value of the underlying asset P_m defined in Equation (4). Indeed the relationship between default time and minimum firm value is, for any $j = 0, \ldots, N$,

$$\mathbb{P}(\tau \leq j\Delta) = \mathbb{P}(m_j \leq \log L). \tag{96}$$

A defaultable zero-coupon bond issued by the firm V is a bond which at maturity T pays a unit notional if the firm does not default, or pays the recovery fraction $R < 1$ of the notional otherwise (R could also be equal to 0). Once the probability of default $p = \mathbb{P}(m_N \leq \log L)$, as well as its complement, the survival probability $1-p$, have been computed, and the recovery rate is assigned, the price $P_{\mathrm{d}}(T)$ of the defaultable zero-coupon with maturity T is

$$P_{\mathrm{d}}(T) = e^{-rT}\left(1-p+Rp\right). \tag{97}$$

In the above computations, the knowledge of the distribution of the minimum is fundamental, and it can be computed as

$$\mathcal{F}^{-1}_{\xi\to x}\mathcal{Z}^{-1}_{q\to N}\left[\frac{1}{\Phi_+(0,q)\Phi_-(\xi,q)}\right], \tag{98}$$

as shown in Equation (18). See Ref. [42] for further details.

5. Other Fourier-based transform methods

In this section we discuss other numerical methods presented in the literature which are based on Fourier and Hilbert transforms. We will not try to be exhaustive, but limit ourselves to those approaches that are most related to our own, and thus we will not cover e.g. the Cos method [43]. For ease of exposition, we will consider only a down-and-out barrier option and neglect the damping factor.

5.1. Convolution and Hilbert transform

First of all, we briefly describe the convolution method [21, 22], as well as the method based on the Hilbert transform due to Feng and Linetsky [3]. Both are based on observing that the option price can be obtained recursively via

$$v(x,j) = e^{-r\Delta}\int_l^{+\infty} f(z-x,\Delta)v(z,j-1)dz \tag{99}$$

Therefore, it holds

$$\begin{aligned} v(x,j) &= e^{-r\Delta}\mathcal{P}_\Omega\big(f(-x,\Delta)*v(x,j-1)\big)\\ &= e^{-r\Delta}\mathcal{P}_\Omega\mathcal{F}^{-1}_{\xi\to x}\big(\Psi^*(\xi)\widehat{v}(\xi,j-1)\big), \end{aligned} \tag{100}$$

where we recall that $*$ is the convolution operator and $\mathcal{P}_\Omega$ is the projector operator on $\Omega := (l,+\infty)$, i.e., $\mathcal{P}_\Omega f(x) = \mathbf{1}_{x\in\Omega}(x)f(x)$. The indicator function $\mathbf{1}_{x\in\Omega}$ can be replaced by the Heaviside step function centered on l: it is 1 if $x>l$ and 0 if $x<l$, while for $x=l$ it can be assigned the values 0 (left-continuous choice), 1 (right-continuous choice) or 1/2 (symmetric choice). The value for $x=l$ matters only from a numerical point of view, as the measure of this point is zero.

At each time step the convolution method proceeds by moving from the real to the Fourier space and backward trough the iteration

$$v_{j-1}\xrightarrow{\mathcal{F}}\widehat{v}_{j-1}\xrightarrow{*}\Psi^*\widehat{v}_{j-1}\xrightarrow{\mathcal{P}\mathcal{F}^{-1}}v_j,\quad j=1,\dots,N. \tag{101}$$

This method has been used, among the others, by Jackson et al. [21] and Lord et al. [22]. Lord et al. improved this numerical methods in order to have a monotonic convergence to zero of the discretization error.

The method of Feng and Linetsky [3] is based on the Hilbert transform (38). In fact, considering the Plemelj-Sokhotsky relation

$$\mathcal{F}\mathcal{P}_{\Omega}h = \frac{1}{2}\big[\mathcal{F}h + ie^{i\xi l}\mathcal{H}_{\xi}(e^{-i\xi l}\mathcal{F}h)\big], \tag{102}$$

the Fourier transform of Equation (100) yields

$$\widehat{v}(\xi,j) = \frac{1}{2}e^{-r\Delta}(\Psi^*(\xi)\widehat{v}(\xi,j-1) + ie^{i\xi l}\mathcal{H}_{\xi}(e^{-i\xi l}\Psi^*(\xi)\widehat{v}(\xi,j-1))). \tag{103}$$

Thus all the computations are in Fourier space:

$$v_0 \xrightarrow{\mathcal{F}} \widehat{v}_0 \longrightarrow \ldots \longrightarrow \widehat{v}_{j-1} \xrightarrow{*} \Psi^*\widehat{v}_{j-1} \xrightarrow{\mathcal{H}} \widehat{v}_j \longrightarrow \ldots \longrightarrow \widehat{v}_N \xrightarrow{\mathcal{F}^{-1}} v_N. \tag{104}$$

The Hilbert transform is computed in the Fourier space via sinc functions and thanks to this procedure the pricing error decays exponentially, as stated in Section 3.1. Therefore the Hilbert method has to be preferred to the convolution approach. The computational cost of both methods is $\mathcal{O}(NM\log M)$.

5.2. Quadrature methods

Fusai et al. [18, 20] solved the recursion given by Equation (99) using quadrature. If the domain is truncated as in [19], the quadrature nodes are denoted with $x_i,\ i=1,\ldots,M$, $\mathbf{K}$ is the $M\times M$ square matrix with elements $K_{ij} = e^{-r\Delta}f(x_j - x_i, \Delta)$, $\mathbf{D}$ is the $M\times M$ diagonal matrix which contains the quadrature weights, and $(\mathbf{v}_j)_i = v(x_i, j),\ i=1,\ldots,M,\ j=0,\ldots,N$, then Equation (99) becomes

$$\mathbf{v}_j = \mathbf{K}\mathbf{D}\mathbf{v}_{j-1} \tag{105}$$

for $j=1,\ldots,N$. Thus, in order to compute the option price, one only has to perform N matrix-vector multiplications.

This approach can be efficiently implemented using the FFT, provided Newton-Cotes quadrature rules are considered. In fact, if the quadrature formula is characterized by equidistant nodes, the matrix $\mathbf{K}$ is a Toeplitz matrix and the matrix-vector multiplication in Equation (105) can be performed using the FFT as follows.

We recall that an $M\times M$ Toeplitz matrix $\mathbf{T}$ can be embedded in a $2M\times 2M$ circulant matrix[2] $\mathbf{C}$. Thus, given an $M\times 1$ vector $\mathbf{x}$, we can compute the component i of $\mathbf{T}\mathbf{x}$, $i=1,\ldots,M$, as

$$(\mathbf{T}\mathbf{x})_i = \left(\mathrm{FFT}^{-1}\left(\mathrm{FFT}(\mathbf{c})\,\mathrm{FFT}(\mathbf{x}^*)\right)\right)_i, \tag{106}$$

[2] A circulant matrix is a special kind of Toeplitz matrix where each row vector is rotated one element to the right relative to the preceding row vector.

$\mathbf{c}$ being the first column of the circulant matrix $\mathbf{C}$ and $\mathbf{x}^*$ being the extension of the vector $\mathbf{x}$ obtained appending M zeros to $\mathbf{x}$. Thus, in our case, Equation (105) becomes

$$(\mathbf{v}_j)_i = \left(\mathrm{FFT}^{-1}\left(\mathrm{FFT}(\mathbf{c})\,\mathrm{FFT}\left((\mathbf{D}\mathbf{v}_{j-1})^*\right)\right)\right)_i, \tag{107}$$

$i = 1, \dots, M$, $\mathbf{c}$ being the first column of the circulant matrix embedding $\mathbf{K}$. Since $(\mathbf{K})_{i,j} = e^{-r\Delta} f(x_j - x_i, \Delta) = e^{-r\Delta} f((j-i)h, \Delta)$, h being the distance between the quadrature nodes, and f is computed with an inverse Fourier transform of the characteristic function Ψ, it follows that $\widehat{\mathbf{c}} := \mathrm{FFT}(\mathbf{c})$ can be computed directly by using Ψ, avoiding one FFT. At the end the computational cost of this pricing procedure becomes $2NM\log M$, since for each iteration of the pricing recursion we have to compute one FFT and one inverse FFT. Notice that we also have to compute the matrix-vector multiplication $\mathbf{D}\mathbf{v}_{j-1}$, however, being $\mathbf{D}$ a diagonal matrix, the computational cost consists of M operations. Thus the quadrature-FFT based approach is implemented through the following procedure:

$$\mathbf{v}_{j-1} \longrightarrow \mathbf{D}\mathbf{v}_{j-1} \xrightarrow{\mathcal{F}} \mathcal{F}[\mathbf{D}\mathbf{v}_{j-1}] \xrightarrow{*} \widehat{\mathbf{c}}\mathcal{F}[\mathbf{D}\mathbf{v}_{j-1}] \xrightarrow{\mathcal{F}^{-1}} \mathbf{v}_j. \tag{108}$$

5.3. The Z-WH algorithm

Another approach consists in relating the pricing problem to the solution of an integral equation. This approach is presented in [44, 19], where, after applying the z-transform to Equation (99) and by defining

$$w(x) := w(x, q) = \sum_{n=0}^{+\infty} q^n v(x, n), \tag{109}$$

it is shown that $w(x)$ solves the Wiener-Hopf integral equation

$$w(x) = q e^{-r\Delta} \int_l^{+\infty} f(z - x, \Delta) w(z) dz + \phi(x) \quad \text{for } x \geq l. \tag{110}$$

Two solutions strategies are possible to solve the integral equation. The first one, considered in [44, 19], consists to apply a quadrature scheme to Equation (110) and therefore it reduces the problem to the solution of the linear system

$$(\mathbf{I} - q\mathbf{K}\mathbf{D})\mathbf{w} = \mathbf{g}, \tag{111}$$

with parameter q, $\mathbf{I}$ being the $M \times M$ identity matrix, before inverting the z-transform to obtain the option price. This approach was introduced in Fusai et al. [19]: the authors presented a numerical scheme, based on a preconditioning technique, to speed up the solution of these linear systems: in fact, considering Newton-Cotes quadrature rules and an iterative linear system solution method, like the generalized minimal residual (GMRes) method, the authors provide an FFT based method which computational cost is $\mathcal{O}(\min\{N, m_{\mathrm{E}} + n_{\mathrm{E}}\} IM \log M)$, M being the number of quadrature nodes and I denoting the average number of GMRes iterations necessary to solve a linear system. The authors showed that the scheme provides a great accuracy at a low computational cost if the matrix $\mathbf{D}$ is nearly equal to $c\mathbf{I}$, for any constant c: in fact only in this case I is independent on the number of monitoring dates. This is true for the trapezoidal rule ($\mathrm{diag}\,(\mathbf{D}) = h[0.5, 1, 1, \ldots, 1, 0.5]$), but not, for example, for the Simpson rule.

Another possibility consists in relating the Spitzer-Wiener-Hopf factorization to the solution of the integral equations. Indeed, we remark that the well known methodology to solving a WH integral equation also requires the knowledge of the WH factors. More precisely, the main steps for solving the Wiener-Hopf integral equation (110) are the following:

1. Factorize the function $L(\xi, q) := 1 - qe^{-r\Delta}\Psi^*(\xi, \Delta)$ into

$$L(\xi, q) = L_+(\xi, q) L_-(\xi, q). \tag{112}$$

2. Given the payoff function $\phi(x)$, define $P(\xi, q) := e^{-il\xi}\widehat{\phi}(\xi)/L_-(\xi, q)$ and decompose it into additive components that are analytic in the appropriate complex half planes,

$$P(\xi, q) = P_+(\xi, q) + P_-(\xi, q). \tag{113}$$

3. The Fourier transform of the solution of the integral equation (110) is now given by

$$W(\xi, q) = P_+\left(\xi, q\right) / L_+\left(\xi, q\right). \tag{114}$$

Therefore, the following pricing methodology can be considered, assuming that the number of monitoring dates is greater than 2.

1. Compute the value of $\widehat{v}(\xi, 1)$ by convolution, i.e.,

$$\widehat{v}(\xi, 1) = \Psi^*(\xi)\widehat{\phi}(\xi). \tag{115}$$

2. Compute $\widehat{v}(\xi, N-1)$, i.e., consider an option with $N-2$ monitoring dates and payoff $\phi(x) = \mathcal{F}^{-1}_{\xi\to x}\widehat{v}(\xi, 1)$, whose price is $v(\cdot, N-1)$, with

$$\widehat{v}(\xi, N-1) = \mathcal{Z}^{-1}_{q\to N-2}\left[\widetilde{v}(\xi, q)\right], \tag{116}$$

solving the Wiener-Hopf integral equations using the Wiener-Hopf factorization to obtain $\widetilde{v}(\xi, q)$ for the different values of q necessary to invert the z-transform.

3. Compute the value of $\widehat{v}(x, N)$ by convolution, as in Equation (115).

4. Apply an inverse Fourier transform to obtain the option price $v(x, N)$.

All the computations are performed in Fourier space:

$$v_0 \xrightarrow{\mathcal{F}} \widehat{v}_0 \xrightarrow{\Psi^*} \widehat{v}_1 \xrightarrow{\mathcal{ZWH}} \widehat{v}_{N-1} \xrightarrow{\Psi^*} \widehat{v}_N \xrightarrow{\mathcal{F}^{-1}} v_N, \tag{117}$$

where $\mathcal{ZWH}$ stands for the second step of the above algorithm. As in the algorithm described in Section 4.2, Steps 1 and 3 are necessary in order to smooth the tails of the payoff and of the inverse of the z-transform in Fourier space, $\widehat{v}(x, N-1)$, before applying the z-transform (Step 2) and the inverse Fourier transform (Step 4), respectively, and thus to obtain an exponential convergence considering the Wiener-Hopf factorization described in Section 3.1. We would like to stress that $\phi(x) = \mathcal{F}^{-1}_{\xi\to x}\widehat{v}(x,1)$ differs from $v(x,1)$ because of a projection, i.e., $v(x,1) = \mathcal{P}_{\{x>l\}}\mathcal{F}^{-1}_{\xi\to x}\widehat{v}(\xi,1)$.

In the case of double-barrier options, the WH equation becomes a Fredholm equation of the second type with a convolution kernel. This problem is very old but up to now no efficient and accurate procedure has been devised for its solution. So the scheme here presented deserves some interest on its own. More precisely, the pricing equation becomes

$$w(x) = qe^{-r\Delta}\int_l^u f(z-x,\Delta)w(z)dz + \phi(x), \tag{118}$$

and this can be solved using the fixed-point algorithm similar to the one presented in Section 4.3, being Equations (84)–(85) replaced by

$$\frac{J_-(\xi,q)}{L_-(\xi,q)} = \left[\frac{e^{-il\xi}\Psi^*(\xi)\widehat{\phi}(\xi) - e^{i(u-l)\xi}J_+(\xi,q)}{L_-(\xi,q)}\right]_-, \tag{119}$$

$$\frac{J_+(\xi,q)}{L_+(\xi,q)} = \left[\frac{e^{-iu\xi}\Psi^*(\xi)\widehat{\phi}(\xi) - e^{i(l-u)\xi}J_-(\xi,q)}{L_+(\xi,q)}\right]_+. \tag{120}$$

Once $J_\pm$ are computed via the fixed-point algorithm described in Section 4.3, the option price is given by

$$v(x,N) = \mathcal{F}^{-1}_{\xi\to x}\left[\Psi^*(\xi)\mathcal{Z}^{-1}_{q\to N-2}\left[\frac{\widehat{\phi}(\xi)\Psi^*(\xi)}{L(\xi,q)} - e^{il\xi}\frac{J_-(\xi,q)}{L(\xi,q)} - e^{iu\xi}\frac{J_+(\xi,q)}{L(\xi,q)}\right]\right]. \tag{121}$$

Thus the pricing algorithm consists of the following steps:

$$\phi \equiv v_0 \xrightarrow{\mathcal{F}} \widehat{v}_0 \xrightarrow{\Psi^*} \widehat{v}_1 \xrightarrow{\mathcal{ZWH}} \widehat{v}_{N-1} \xrightarrow{\Psi^*} \widehat{v}_N \xrightarrow{\mathcal{F}^{-1}} v_N, \tag{122}$$

where here we denote with $\mathcal{ZWH}$ the operator

$$\mathcal{ZWH}: F(\xi) \to \mathcal{Z}^{-1}_{q\to N-2}\left[\frac{F(\xi) - e^{il\xi}J_-(\xi,q) - e^{iu\xi}J_+(\xi,q)}{L(\xi,q)}\right]. \tag{123}$$

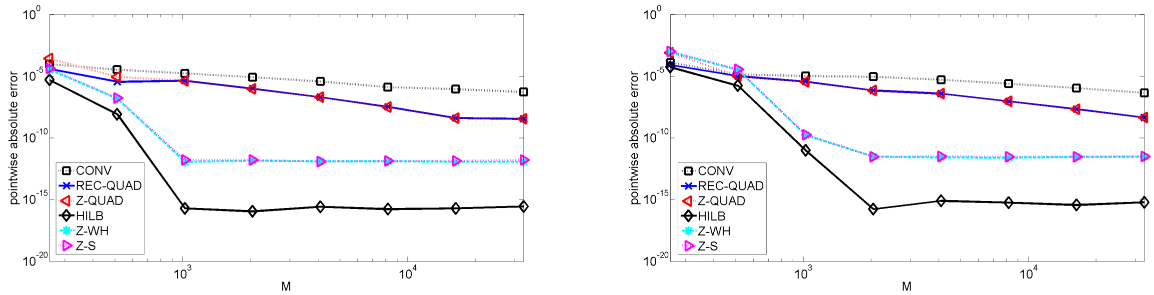

Figure 4 Down-and-out barrier call option: pointwise absolute errors for $N = 100$ (left) and $N = 252$ (right).

6. Numerical experiments

In this section we compare the proposed pricing techniques with others presented in the literature. We consider:

- Z-S and Z-WH, i.e., the new fast methods presented in this article, Z-S in Section 4 and Z-WH in Section 5.3. Both methods exploit the Wiener-Hopf factorization via the Hilbert transform.
- CONV, i.e., the convolution method of Lord et al. [22] described in Section 5.1.
- HILB, i.e., the recursive method of Feng and Linetsky [3] based on the Hilbert transform and described in Section 5.1.
- REC-QUAD, i.e., the recursive method based on the trapezoidal quadrature rule and described in Section 5.2.
- Z-QUAD, i.e., the method of Fusai et al. [19] based on the z-transform and the trapezoidal quadrature rule, described in Section 5.3.

All the numerical experiments have been performed with Matlab R2011b running under Windows 7 on a personal computer equipped with an Intel Core i7 Q720 1600 MHz processor and 6 GB of RAM. We would like to stress that with lookback and single-barrier options and with all Fourier-based methods we have unbounded domains. Therefore, as already stated in Section 4, we truncate the domain with a Chernoff bound computed according to the first ten moments, as suggested in Fusai et al. [19].

First of all, we consider a down-and-out call barrier option assuming that the underlying asset evolves according to a Merton jump diffusion process with the same parameters as in Feng and Linetsky [3], including the procedure to choose the damping parameter α. The lower barrier is $L = 0.80$, the initial spot price S_0 and the strike price K are both set to 1, and the time to maturity is $T = 1$. The underlying asset has a dividend rate $\delta = 0.02$ and the risk-free interest rate is $r = 0.05$.

In Figure 4 we consider the case with $N = 100$ and $N = 252$ monitoring dates: we report in double logarithmic scale both the pointwise error, computed at the spot price $S_0 = 1$, taking for reference as the exact solution the price computed with the HILB method and a grid of 2^{16} points.

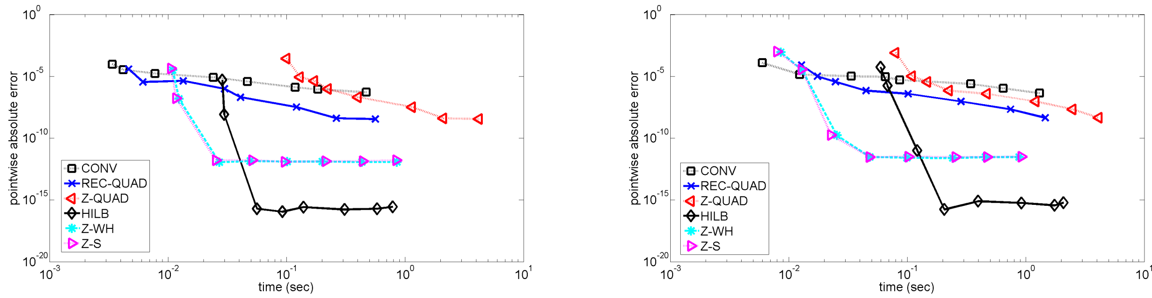

Figure 5 Down-and-out barrier call option: pointwise absolute errors, for $N = 100$ (left) and $N = 252$ (right), as a function of CPU time.

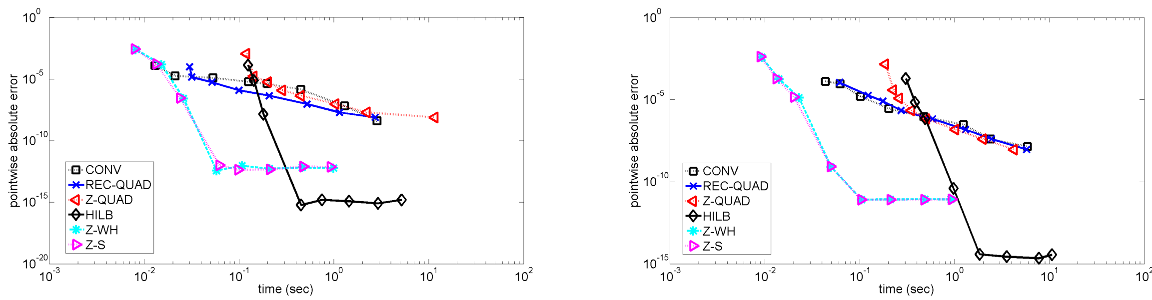

Figure 6 Down-and-out barrier call option: pointwise absolute errors, for $N = 504$ (left) and $N = 1260$ (right), as a function of CPU time.

The CONV, REC and Z-QUAD methods have a polynomial convergence; moreover the REC and the Z-QUAD algorithms show a similar polynomial accuracy. Our newly proposed methods, Z-S and Z-WH, and the HILB algorithm exhibit an exponential convergence due to the use of the sinc expansion and to the fact that all computations are performed in Fourier space, as already described in [3]. As expected, both the Z-S and Z-WH methods rapidly reach the maximum accuracy allowed by the approximation used to invert the z-transform, i.e., 10^{-12}.

In Figures 5 and 6 we report the pointwise absolute error against the CPU time necessary for the price computation for different numbers of monitoring dates. It is clear that the Z-S, the Z-WH and the HILB methods are the most accurate. Their exponential convergence enables them to be used with a limited number M of grid nodes. The Z-S and the Z-WH are able to compute option prices with an accuracy of 10^{-12} in less than a quarter of a second. Notice that increasing the number of monitoring dates from 252 to 504 or 1260, the computational costs of the z-transform based methods do not change, due to the Euler acceleration technique. From this experiments, it appears that, among the methods proposed in this paper, the Z-S and the Z-WH methods are preferable when the number of dates is large. However, if a greater accuracy is necessary and the number of monitoring dates is not too large, the HILB method by Feng and Linetsky [3] should also be considered.

Table 1 Down-and-out barrier call option: option price and CPU time in seconds; $M = 2^{14}$.

	Z-S		Z-WH		HILB	
N	Price	CPU time	Price	CPU time	Price	CPU time
50	0.04775954751	0.604597	0.04775954751	0.615977	0.04775954750	0.411529
100	0.04775180473	0.598856	0.04775180473	0.585755	0.04775180472	0.719666
252	0.04774580616	0.613833	0.04774580616	0.600996	0.04774580615	1.745266
504	0.04774337792	0.601078	0.04774337791	0.591950	0.04774337791	3.468807

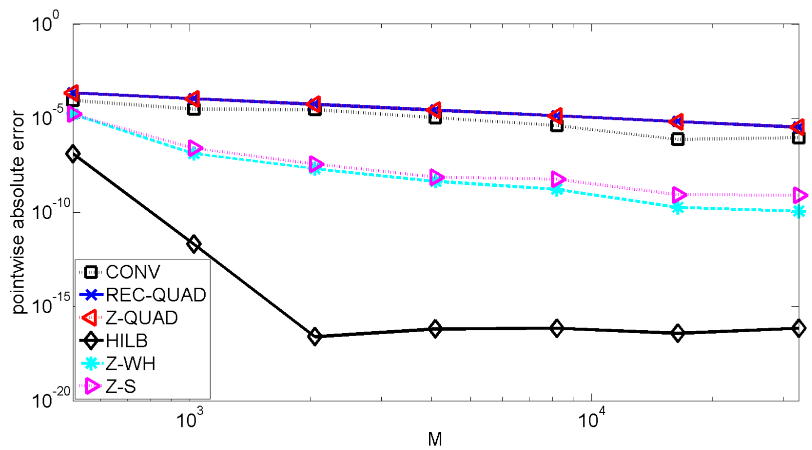

Figure 7 Knock-and-out barrier call option: pointwise absolute error with $N = 252$.

To complete the numerical tests on single-barrier options, Table 1 shows results for a down-and-out barrier call option, assuming that the underlying asset evolves according to a NIG process with the same parameters as in Feng and Linetsky [3]. All the other parameters are as before. These results confirm the good performance of the Z-S and Z-WH algorithms when the number of monitoring dates increases.

In Figure 7 we consider a double-barrier option and we plot the pointwise absolute error for the fixed-point algorithm presented in Section 5.3. We use a a double exponential model, again with the same parameters as in Feng and Linetsky [3]. The lower (upper) barrier is $L = 0.8$ ($U = 1.2$), the initial spot price is $S_0 = 1$ and the strike price is $K = 1.1$. A one year daily monitoring is assumed, i.e., $T = 1$ and $N = 252$. The error is again computed considering as exact the solution computed with the HILB method and $M = 2^{16}$ grid points. The numerical experiments show that the orders of convergence of the newly proposed algorithms, Z-WH and Z-S, are no more exponential as in the single-barrier case, but approximately quadratic. We would like to stress that the average number of fixed-point iterations necessary to reach a tolerance of 10^{-12} is as low as 3. Moreover, the newly proposed methods are still slightly more accurate than the CONV, REC and Z-QUAD ones.

Finally, in Table 2 we price a fixed-strike lookback put option written on the minimum, as well as a call on the maximum, with $N = 50$ monitoring dates. We set $S_0 = K = 1$. We assume that the underlying asset evolves as a geometric Brownian motion with the same parameters as in Feng

Table 2 Fixed-strike lookback call (on the maximum) and put (on the minimum) options: option price and CPU time in seconds.

	Call		Put	
M	Price	CPU time	Price	CPU time
2^8	0.183264603755	0.0097	0.117871584305	0.0087
2^9	0.183264598264	0.0169	0.117871585215	0.0114
2^{10}	0.183264598276	0.0214	0.117871585217	0.0175
2^{11}	0.183264598268	0.0361	0.117871585212	0.0371
2^{12}	0.183264598273	0.0722	0.117871585216	0.0964
2^{13}	0.183264598262	0.1933	0.117871585210	0.1753
2^{14}	0.183264598287	0.3211	0.117871585214	0.3052
2^{15}	0.183264598282	0.6192	0.117871585214	0.5601
2^{16}	0.183264598276	1.2780	0.117871585214	1.0442

and Linetsky [23], i.e., $\sigma = 0.3,\ r = 0.1,\ T = 0.5$. We report the option price and the computational cost of the Z-S approach for different numbers of grid points M. For the call option, compare the benchmark price 0.183264598300 provided by Feng and Linetsky [23, Table 1]. From this table we notice the same exponential convergence of the algorithm as in the single-barrier case.

7. Continuous versus discrete monitoring

As mentioned in Section 2, identities similar to Equations (17)–(22) exist for continuous monitoring too. The discrete minimum and maximum operators are replaced with their continuous versions,

$$M_t^{\mathrm{c}} = \sup_{s\in(0,t)} X(s) \quad \text{and} \quad m_t^{\mathrm{c}} = \sup_{s\in(0,t)} X(s). \tag{124}$$

In this case the quantities to be factorized are of the kind $\lambda_1 + \lambda_2\psi(\xi)$, for suitable parameters λ_1, λ_2, instead of $1 - q\Psi(\xi,\Delta)$. Moreover the z-transform is replaced by the Laplace transform. The following relation holds [4, Section 4.1.2]: if we define $\Phi^{\mathrm{c}}(\xi) := \lambda + \psi(z),\ \lambda > 0$, and set $q = e^{-\lambda\Delta}$, then

$$\lim_{\Delta\to 0} \frac{\Phi(\xi,q)}{\Delta} = \lim_{\Delta\to 0} \frac{1 - q\Psi(\xi,\Delta)}{\Delta} = \lim_{\Delta\to 0} \frac{1 - e^{-(\lambda+\psi(\xi))\Delta}}{\Delta} = \lambda + \psi(\xi) = \Phi^{\mathrm{c}}(\xi). \tag{125}$$

Similar limits hold for the WH factors of Φ and Φ^{c}:

$$\lim_{\Delta\to 0} \frac{\Phi_\pm(\xi,q)}{\sqrt{\Delta}} = \Phi^{\mathrm{c}}_\pm(\xi). \tag{126}$$

Remarkably, the WH factorization of Φ^{c} is not obtained through a passage to the limit of the WH factorization of Φ, but directly from Φ^{c} itself using the Hilbert transform like for the factorization of Φ. Therefore, our procedure can price contracts in the continuous monitoring case too, once the z-transform is replaced with the Laplace transform. To show that the WH factorization of both the discrete and continuous case can be computed with our procedure, in Figure 8 we consider a double exponential distribution, whose characteristic exponent is

$$\psi(\xi) = i\gamma\xi - \frac{1}{2}\sigma^2\xi^2 + \eta\left(p\frac{\eta_1}{\eta_1 + i\xi} + (1-p)\frac{\eta_2}{\eta_2 - i\xi} - 1\right). \tag{127}$$

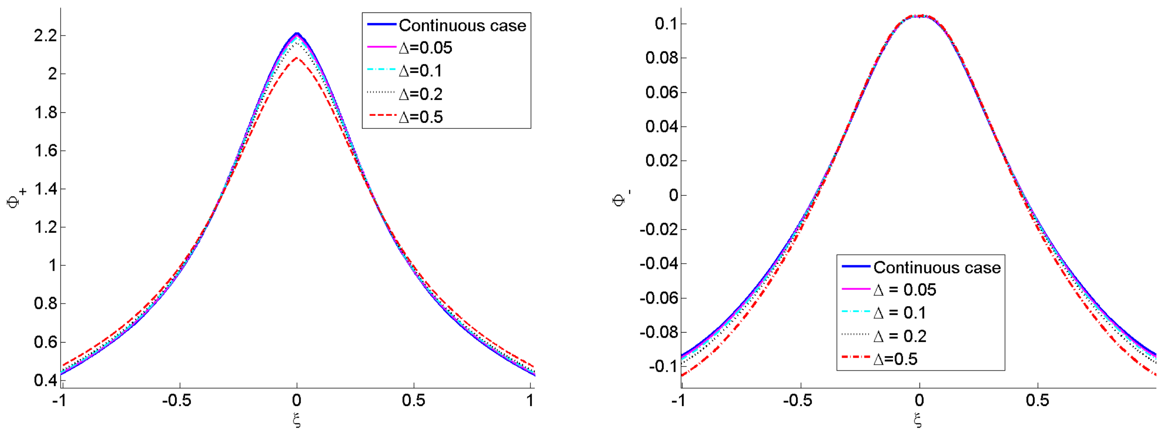

Figure 8 Convergence of $\Phi_+(\xi,q)/\sqrt{\Delta}$ to Φ_+^c (left) and of $\Phi_-(\xi,q)/\sqrt{\Delta}$ to Φ_-^c (right), $\xi\in[-1,1]$.

We set $\gamma=0.2$, $\sigma=0.2$, $\eta=0.5$, $p=0.5$, $\eta_1=0.4$, $\eta_2=0.4$ and $\lambda=0.23$, and we plot $\Phi_\pm^c$ as well as $\Phi_\pm(\xi,q)/\sqrt{\Delta}$ for different values of Δ, showing numerically the convergence of the latter to the former. The method by Feng and Linetsky, as well as all the other methods described in Section 5, can deal only with the discrete monitoring case. In these cases, the continuous monitoring value can be obtained only through a passage to the limit, but it is well known that the convergence is slow. This clarifies the importance of an efficient numerical method able to deal with both the discrete and continuous monitoring. The methodology proposed here factorizes directly $\lambda_1+\lambda_2\psi(\xi)$ and is exempt from the problem of the slow convergence from discrete to continuous monitoring.

8. Conclusions

In this article we present fast and accurate pricing methodologies based on the Spitzer identity and the Wiener-Hopf factorization. We apply them to barrier and lookback options, as well as defaultable zero-coupon bonds, when the monitoring is discrete and the underlying evolves according to an exponential Lévy process. First of all, in order to use the Spitzer identity, we provide a constructive procedure to perform the Wiener-Hopf factorization, and to employ it in pricing the above mentioned contracts. The numerical implementation exploits the fast Fourier transform and the Euler summation, and the computational cost is independent of the number of monitoring dates, while the error decays exponentially with the number of grid points. For double-barrier options we introduce an iterative algorithm based on the Wiener-Hopf factorization. Our procedure applies also to continuous monitoring. Extensions to other exotic derivatives like perpetual Bermudan, occupation time, quantile and step options are straightforward too combining our method with the Wendel-Port-Dassios identity [45].

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