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Identification problem for a hyperbolic equation with Robin condition

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Abstract

We discuss an identification problem for the one-dimensional wave equation with the Robin condition on an unknown part of the boundary. We prove that it is possible to identify both the unknown boundary and the Robin coefficient by two pairs of additional measurements.

1 Introduction

Let Ω be a bounded simply connected domain in $\mathbb{R}^n$, $n \geq 1$; suppose $\partial\Omega = \overline{\Gamma^a} \cup \overline{\Gamma^i}$, where Γ^i and Γ^a are two open connected disjoint portions of $\partial\Omega$.
We consider the following mixed boundary value problem

$$\begin{cases} u_{tt} = \triangle u & \text{in } \Omega \times (0,T) \\ u(x,0) = \varphi(x) & x \in \Omega \\ u_t(x,0) = \psi(x) & x \in \Omega \\ u(x,t) = g(t) & \text{on } \Gamma^a \times (0,T) \\ \frac{\partial u}{\partial \nu}(x,t) + \gamma u(x,t) = 0 & \text{on } \Gamma^i \times (0,T) \end{cases} \tag{1.1}$$

where ν is the exterior unit normal to $\partial\Omega$, φ, ψ, g are assigned functions, $\gamma > 0$.
Assuming $\partial\Omega$ of class C^2, it is well known [1], [2] that the initial boundary value problem (1.1) has a unique solution $u \in H^1\big(\Omega \times (0,T)\big)$ provided g, φ and ψ are smooth enough.
In the case $n = 1$, the Robin condition can be interpreted as the end Γ^i of a vibrating string being attached to a spring. In fact, since the vertical component of the string tension is proportional to u_x, the condition states that such component at Γ^i is proportional to the opposite of the vertical displacement of the end. For $n = 3$, the Robin condition plays an interesting role in the context of the acustic waves: considering the wave equation $\phi_{tt} = \triangle\phi$ for the *velocity potential* ϕ and assuming that each point of the surface $\partial\Omega$ acts like a spring in response to the excess pressure ϕ_t, then the condition $\frac{\partial \phi_t}{\partial \nu} + \gamma\phi_t = 0$ holds on $\partial\Omega \times (0,T)$ [3].
We will assume that Γ^i is unknown and inaccessible, while Γ^a is known and accessible for input and output measurements. Then, we deal with the inverse problem of determine Γ^i and γ, provided additional measurements $\displaystyle\frac{\partial u}{\partial \nu}\Big|_{\Sigma\times(0,T)}$ are known, where $\Sigma \subset \Gamma^a$ is part of the accessible boundary.
This problem was considered by Isakov [4] in the case of *vanishing initial data*. Assuming that Γ^i is a closed polygonal surface, it is proved that the additional measurement $\displaystyle\frac{\partial u}{\partial \nu}\,|_{\Sigma\times(0,T)}$ uniquely determines Γ^i and γ; if $T = \infty$, one can uniquely identify general smooth Γ^i and γ.
In this note we consider the problem (1.1) with non zero initial data φ, ψ, in the case $n = 1$ and we show that Γ^i and γ are uniquely determined by two pairs $(g, \displaystyle\frac{\partial u}{\partial \nu}\,|_{\Sigma\times(0,T)})$, $(\widetilde{g}, \displaystyle\frac{\partial \widetilde{u}}{\partial \nu}\,|_{\Sigma\times(0,T)})$, provided T is sufficiently large and the boundary data g, $\widetilde{g}$ are suitably chosen.

We also prove that in the case of non vanishing initial data a single additional measurement $\frac{\partial u}{\partial \nu}\big|_{\Sigma\times\mathbb{R}_+}$ is *not* sufficient to identify the unkwnown boundary Γ^i.
Inverse problems involving Robin condition was previously considered for the elliptic equations in [5], [6] where it is shown that uniqueness of Γ^i and of γ can be achieved by two couples of measurements

$$\Big(g_1, \frac{\partial u_1}{\partial \nu}\Big|_{\Sigma}\Big) \qquad \Big(g_2, \frac{\partial u_2}{\partial \nu}\Big|_{\Sigma}\Big)$$

In this context, a counterexample was given in [7] showing that a single additional measurement is not sufficient to determine Γ^i, even if γ is known.
Finally, a parabolic equation with the same boundary conditions as in (1.1) was considered in [8]; the authors prove that two couples of measurements guarantee as well uniqueness and stability of Γ^i and of γ.
The present paper is organized as follows: in §2, we consider the direct problem and we give conditions on the datum g in order to assure that the solution u is strictly positive for t large enough.
Relying on this property, we state in §3 the main result, that is a uniqueness theorem for the inverse problem and provide a counterexample showing that a unique measurement is not sufficient to determine the unknown boundary Γ^i.

2 The direct problem

In the case $n = 1$, problem (1.1) takes the form

$$\begin{cases} u_{xx} = u_{tt} & 0 < x < b,\ 0 < t < T \\ u(x,0) = \varphi(x) & 0 \le x \le b, \\ u_t(x,0) = \psi(x) & 0 \le x \le b, \\ u(0,t) = g(t) & 0 < t < T \\ u_x(b,t) + \gamma u(b,t) = 0 & 0 < t < T \end{cases} \tag{2.1}$$

In this section, we solve problem (2.1) for $T = +\infty$ and show that one can always choose a *boundary datum* g in such a way that the solution is strictly positive for large enough t. To this aim, we need two lemmas.

Lemma 2.1. *Let $u(x,t)$ be the solution of the problem*

$$\begin{cases} u_{xx} = u_{tt} & 0 < x < b,\ t > 0 \\ u(x,0) = 0 & 0 \le x \le b, \\ u_t(x,0) = 0 & 0 \le x \le b, \\ u(0,t) = g(t) & t \ge 0 \\ u_x(b,t) + \gamma u(b,t) = 0 & t \ge 0, \end{cases} \tag{2.2}$$

where $b > 0$, $\gamma > 0$ and the boundary datum g satisfies the following assumptions

1. $g \in \mathcal{C}^2\big([0,+\infty)\big)$;
2. $g(t) \sim g'(0)\, t$ for $t \to 0^+$, *where* $g'(0) > 0$ *and* $\lim_{t\to+\infty} g(t) = M > 0$;
3. *if* $h(t) = g(t) - M\big(1 - e^{-\frac{g'(0)}{M}t}\big)$, *then* $\lim_{t\to+\infty} h'(t) = 0$ *and there are positive constants* N, ϵ, *such that*

$$|h''(t)| \le N\, e^{-2\epsilon t} \tag{2.3}$$

Then, if

4.
$$\frac{M}{g'(0)+\frac{5N}{2\epsilon}} > b\,(1+\gamma b)\,\big(1+\frac{4}{\pi^2}(\gamma b)^2\big)$$

there exists $t_0 > 0$ *such that*

$$u(x,t) > 0,\ \textit{for}\ 0 \le x \le b,\ \ t \ge t_0$$

Remark 2.2. *Roughly speaking, the assumptions of the Lemma are satisfied if the boundary datum* g *is increasing from zero at the origin and if* $g(t) = M + O(e^{-\varepsilon t})$ *for* $t \to +\infty$ *where* M *is large enough and* $0 < \varepsilon \le g'(0)/M$*. Note that the convergence to the limit* M *should not be too rapid. Actually, there are also examples of boundary data* g *for which the conclusion of the lemma holds even if* $g(t)$ *approaches a limit at infinity with a power rate.*

Proof. We will solve (2.2) by Laplace transform; let us define

$$\mathcal{L}\big[u(x,t)\big](s) = \int_0^{+\infty} u(x,t)e^{-st}dt \equiv \hat{u}(x,s); \qquad \hat{g}(s) = \mathcal{L}[g(t)](s)$$

Then, (2.2) becomes

$$\begin{cases} \hat{u}_{xx}(x,s) = s^2\hat{u}(x,s) & 0 \le x \le b, \quad s \in ð'(⊬) \\ \hat{u}(0,s) = \hat{g}(s) & \mathrm{Re}\, s > 0 \\ \hat{u}_x(b,s) + \gamma\hat{u}(b,s) = 0 & s \in ð'(⊬) \end{cases} \tag{2.4}$$

By elementary calculations, the solution can be written

$$\hat{u}(x,s) = \frac{A(x,s)}{B(s)}\hat{g}(s) \tag{2.5}$$

where
$$\frac{A(x,s)}{B(s)} = \frac{s\cosh s(b-x) + \gamma\sinh s(b-x)}{s\cosh bs + \gamma\sinh bs} \tag{2.6}$$

The function $B(s)$ has simple zeroes at the points s which solve

$$\tanh(bs) = -\frac{s}{\gamma}, \quad s \in \mathbb{C} \tag{2.7}$$

Any solution $s \neq 0$ is a (possible) simple pole of (2.6). We have $s = iy$, $y \in \mathbb{R}$, where y satisfies

$$\tan(by) = -\frac{1}{\gamma}y \tag{2.8}$$

The solutions of equation (2.8) are given by a sequence

$$\big\{\pm y_n\big\}_{n=0,1,2,...}$$

where
$$y_0 = 0 \quad \text{and} \quad (n-1/2)\pi < by_n < n\pi, \quad n = 1,2,...$$

Actually, we have $by_n \sim (n-1/2)\pi$ (see e.g. [9]).
In order to obtain the solution $u(x,t)$, we will now apply the inverse transformation. Let us write

$$g(t) = M(1-e^{-\delta t}) + h(t)$$

where

$$\delta = g'(0)/M$$

Then, we have

$$\hat{g}(s) = \frac{g'(0)}{s(s+\delta)} + \hat{h}(s) \tag{2.9}$$

where $\hat{h}$ denotes the Laplace transform of h.
Accordingly, the inverse transformation of (2.5) splits into two terms

$$u(x,t) = u_1(x,t) + u_2(x,t) \tag{2.10}$$

By the residue theorem and after some calculation one gets

$$\begin{aligned} u_1(x,t) &= M\Big[\frac{1+\gamma(b-x)}{1+\gamma b} - \frac{\delta\cosh\delta(b-x) + \gamma\sinh\delta(b-x)}{\delta\cosh\delta b + \gamma\sinh\delta b}\, e^{-t\delta}\Big] \\ &+ 2g'(0)\left[\sum_{n=1}^{+\infty} \frac{(y_n\cos y_n(b-x) + \gamma\sin y_n(b-x)}{(1+\gamma b)\cos by_n - by_n\sin by_n} \times \frac{\delta\cos ty_n - y_n\sin ty_n}{y_n(y_n^2+\delta^2)}\right] \end{aligned} \tag{2.11}$$

The series in the second term at the right hand side converges absolutely and uniformly in $[0,b]\times\mathbb{R}_+$. In fact, one has the estimate (see appendix)

$$\left|\frac{(y_n\cos y_n(b-x) + \gamma\sin y_n(b-x)}{(1+\gamma b)\cos by_n - by_n\sin by_n}\right| \le \frac{1}{b}\Big(1 + \frac{4}{\pi^2}(\gamma b)^2\Big) \tag{2.12}$$

Moreover, we have

$$\left|\frac{\delta\cos ty_n - y_n\sin ty_n}{y_n(y_n^2+\delta^2)}\right| \le \frac{1}{y_n^2}$$

Then, since $by_n \in \big((n-1/2)\pi, n\pi\big)$, the whole term between square brackets in (2.11) is uniformly bounded by

$$b\left(1+\frac{4}{\pi^2}(\gamma b)^2\right) \times \sum_{n=1}^{+\infty}\frac{1}{(n-1/2)^2\pi^2} = \frac{b}{2}\left(1+\frac{4}{\pi^2}(\gamma b)^2\right)$$

For any fixed $\sigma_0 > 0$, the second term in (2.10) is the inverse transform

$$u_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(\sigma_0+i\tau)t}\hat{h}(\sigma_0+i\tau)\frac{A(x,\sigma_0+i\tau)}{B(\sigma_0+i\tau)}\,d\tau \tag{2.13}$$

By assumptions 2 and 3 (and by observing that $h(0) = h'(0) = 0$) one can show that $\hat{h}$ is holomorphic for $\operatorname{Re} s > -2\epsilon$, with

$$|\hat{h}(s)| \le \frac{N}{4\epsilon^3}; \quad |s^2\hat{h}(s)| \le \frac{N}{\epsilon} \quad \text{for } \operatorname{Re} s \ge -\epsilon$$

Hence, we have the estimate

$$|\hat{h}(s)| \le \frac{5N}{4\epsilon}\,\frac{1}{|s|^2+\epsilon^2} \quad \text{for } \operatorname{Re} s \ge -\epsilon \tag{2.14}$$

In order to evaluate the integral in (2.13), let us choose a sequence of rectangular paths $\partial Q_{\epsilon,m}$, where

$$Q_{\epsilon,m} = [-\epsilon, \sigma_0] \times [-R_m, R_m]$$

and the R_m are such that $\lim_{m\to\infty} R_m = +\infty$ and $\partial Q_{\epsilon,m}$ contains no poles. Let us now calculate

$$\int_{\partial Q_{\epsilon,m}} e^{st}\frac{A(x,s)}{B(s)}\hat{h}(s)\,ds$$

with the residue theorem and take the limit for $m \to \infty$. Then, by the bound (2.14), by using again (2.12) and by observing that the ratio $A(x,s)/B(s)$ is uniformly bounded if s remains at a positive distance from the set $\{iy_n\}_{n=1,2,\ldots}$, we obtain after some calculation

$$\begin{aligned} |u_2(x,t)| &\leq \frac{5N}{4\epsilon}\left[\frac{1}{b}\Big(1+\frac{4}{\pi^2}(\gamma b)^2\Big)\sum_{n=1}^{\infty}\frac{2}{y_n^2+\epsilon^2} + e^{-\epsilon t}\int_{-\infty}^{+\infty}\frac{K_\epsilon}{|\tau|^2+2\epsilon^2}\,d\tau\right] \\ &\leq \frac{5N}{4\epsilon}\left[K + \frac{\pi}{\epsilon\sqrt{2}}\,K_\epsilon e^{-\epsilon t}\right] \end{aligned} \tag{2.15}$$

where

$$K(b,\gamma) = b\,\big(1+\frac{4}{\pi^2}(\gamma b)^2\big) \tag{2.16}$$

and K_ϵ depends on γ, b and ϵ.
Now, by taking t large enough we may achieve

$$\frac{\pi}{\epsilon\sqrt{2}}\,K_\epsilon e^{-\epsilon t} \leq K$$

so that

$$|u_2(x,t)| \leq \frac{5N}{2\epsilon}\,K \tag{2.17}$$

By the previous estimates, we finally get

$$\begin{aligned} u(x,t) &= u_1(x,t)+u_2(x,t) \geq u_1(x,t) - |u_2(x,t)| \\ &\geq M\left[\frac{1+\gamma(b-x)}{1+\gamma b} - \frac{\delta\cosh\delta(b-x)+\gamma\sinh\delta(b-x)}{\delta\cosh\delta b+\gamma\sinh\delta b}\,e^{-t\delta}\right] - K\Big(g'(0)+\frac{5N}{2\epsilon}\Big) \\ &\geq M\Big(\frac{1}{1+\gamma b} - e^{-t\delta}\Big) - K\Big(g'(0)+\frac{5N}{2\epsilon}\Big) \end{aligned} \tag{2.18}$$

Now, the Lemma follows by the identity

$$(1+\gamma b)\,K = b\,(1+\gamma b)\,\Big(1+\frac{4}{\pi^2}(\gamma b)^2\Big) \tag{2.19}$$

□

Corollary 2.3. *Let us choose the boundary datum $g(t) = M(1-e^{-\delta t})$ in problem (2.2). Then, if the condition $\delta\,b\,(1+\gamma b)\,\big(1+\frac{4}{\pi^2}(\gamma b)^2\big) < 1$ holds, the time t_0 can be taken*

$$t_0 = \frac{1}{\delta}\log\frac{2(1+\gamma b)}{1-\delta b(1+\gamma b)(1+\frac{4}{\pi^2}(\gamma b)^2)} \tag{2.20}$$

Proof. Since $h = 0$ in the previous theorem, the bound (2.18) holds with $N = 0$; by recalling that $g'(0) = \delta M$ we get

$$u(x,t) \geq M\Big(\frac{1}{1+\gamma b} - e^{-t\delta} - \delta\,K\Big)$$

Hence, u is strictly positive if

$$e^{-t\delta} \leq \frac{1}{2}\Big(\frac{1}{1+\gamma b} - \delta\, K\Big)$$

that is for

$$t \geq t_0 = \frac{1}{\delta}\log\frac{2(1+\gamma b)}{1-\delta\,(1+\gamma b)\,K}$$

Then, (2.20) follows by the definition (2.16). □

Lemma 2.4. *Let $u(x,t)$ be the solution of the problem*

$$\begin{cases} u_{xx} = u_{tt} & 0 < x < b,\ t > 0 \\ u(x,0) = \varphi(x) & 0 \leq x \leq b, \\ u_t(x,0) = \psi(x) & 0 \leq x \leq b, \\ u(0,t) = 0 & t \geq 0 \\ u_x(b,t) + \gamma u(b,t) = 0 & t \geq 0, \end{cases} \tag{2.21}$$

where $b > 0$, $\gamma > 0$, $\varphi \in \mathcal{C}^2([0,b])$, $\psi \in \mathcal{C}^1([0,b])$. Then, if $\varphi(0) = 0$ the following estimate holds

$$|u(x,t)| \leq \gamma b\,\|\varphi\|_{L^\infty} + b\left(\|\varphi'\|_{L^\infty} + 2\,\|\psi\|_{L^\infty}\right) + \frac{b^2}{\sqrt{2}}\left(\|\varphi''\|_{L^\infty} + \|\psi'\|_{L^\infty}\right) \tag{2.22}$$

Moreover, if φ also satisfy $\varphi'(b) + \gamma\varphi(b) = 0$, the estimate reduces to

$$|u(x,t)| \leq 2\,b\,\|\psi\|_{L^\infty} + \frac{b^2}{\sqrt{2}}\left(\|\varphi''\|_{L^\infty} + \|\psi'\|_{L^\infty}\right) \tag{2.23}$$

which is independent of γ.

Proof. The solution can be represented by a Fourier series

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos(y_n t) + b_n\sin(y_n t)\right]\sin(y_n x) \tag{2.24}$$

where the y_n solve (2.8). Then,

$$u(x,0) = \sum_{n=1}^{\infty} a_n\sin(y_n x) = \varphi(x); \qquad u_t(x,0) = \sum_{n=1}^{\infty} y_n b_n\sin(y_n x) = \psi(x)$$

Let us multiply both the equations by $\sin(y_k x)$, $k = 1, 2, ...$ and integrate on the interval $[0,b]$. Taking account of the orthogonality relations and by defining

$$\sigma_k = \int_0^b \sin^2(y_k x) > \frac{b}{2}$$

we get

$$\sigma_k\, a_k = \int_0^b \varphi(x)\sin(y_k x); \qquad \sigma_k\, y_k\, b_k = \int_0^b \psi(x)\sin(y_k x) \tag{2.25}$$

Integrating by parts, we obtain

$$\int_0^b \varphi(x)\sin(y_k x)dx = -\frac{1}{y_k}\varphi(b)\cos(y_k b) + \frac{1}{y_k}\int_0^b \varphi'(x)\cos(y_k x)dx$$

$$= \quad -\frac{1}{y_k}\varphi(b)\cos(y_k b) + \frac{1}{y_k^2}\varphi'(b)\sin(y_k b) - \frac{1}{y_k^2}\int_0^b \varphi''(x)\sin(y_k x)dx \tag{2.26}$$

$$\int_0^b \psi(x)\sin(y_k x)dx = \frac{1}{y_k}\big[\psi(0) - \psi(b)\cos(y_k b)\big] + \frac{1}{y_k}\int_0^b \psi'(x)\cos(y_k x)dx \tag{2.27}$$

Now, by the inequalities

$$|\cos(y_k b)| = \frac{\gamma}{\sqrt{\gamma^2 + y_k^2}} \le \frac{\gamma}{y_k} \le \frac{\gamma b}{(k-1/2)\pi}$$

$$\Big|\int_0^b \varphi''(x)\sin(y_k x)dx\Big| \le \|\varphi''\|_{L^2(0.b)}\sqrt{\sigma_k} \le \sqrt{b\,\sigma_k}\|\varphi''\|_{L^\infty} \le \sqrt{2}\sigma_k\|\varphi''\|_{L^\infty}$$

$$\Big|\int_0^b \psi'(x)\cos(y_k x)dx\Big| \le \|\psi'\|_{L^2(0.b)}\sqrt{b-\sigma_k} \le \sqrt{b(b-\sigma_k)}\|\psi'\|_{L^\infty} \le \frac{b}{\sqrt{2}}\|\psi'\|_{L^\infty}$$

we easily get from (2.25)-(2.27) the following estimates

$$|a_k| \le \Big(2\,\gamma b\,\|\varphi\|_{L^\infty} + 2\,b\,\|\varphi'\|_{L^\infty} + \sqrt{2}\,b^2\|\varphi''\|_{L^\infty}\Big) \times \frac{1}{(k-1/2)^2\pi^2} \tag{2.28}$$

$$|b_k| \le \Big(4\,b\,\|\psi\|_{L^\infty} + \sqrt{2}\,b^2\|\psi'\|_{L^\infty}\Big) \times \frac{1}{(k-1/2)^2\pi^2} \tag{2.29}$$

Then,

$$\begin{aligned} |u(x,t)| &\le \sum_{k=1}^{\infty}(|a_k| + |b_k|) \qquad (2.30)\\ &\le \gamma b\,\|\varphi\|_{L^\infty} + b\,\|\varphi'\|_{L^\infty} + \frac{1}{\sqrt{2}}\,b^2\|\varphi''\|_{L^\infty} + 2\,b\,\|\psi\|_{L^\infty} + \frac{1}{\sqrt{2}}\,b^2\|\psi'\|_{L^\infty} \end{aligned}$$

which is the estimate (2.22). Finally, if the initial datum φ satisfies the boundary condition at $x = b$, the first two terms at the right hand side of (2.26) disappear and we get the bound (2.23). □

We can now state the main result of this section; preliminarily, let us define

$$K^*(b,\gamma;\varphi,\psi) = \gamma b\,\|\varphi\|_{L^\infty} + b\,\big(\|\varphi'\|_{L^\infty} + 2\,\|\psi\|_{L^\infty}\big) + \frac{b^2}{\sqrt{2}}\big(\|\varphi''\|_{L^\infty} + \|\psi'\|_{L^\infty}\big) \tag{2.31}$$

Notice that the quantity K^* is adimensional.

Theorem 2.5. *Let $u(x,t)$ be the solution of problem (2.1) with $T = \infty$, where the boundary datum g satisfies the assumptions 1-4 of lemma 2.1 and the initial data φ, ψ satisfy the assumptions of lemma 2.4. Then, if $M = \lim_{t\to+\infty} g(t)$ is large enough, there is $\bar{t} > 0$ such that,*

$$u(x,t) > 0, \ \textit{for } 0 \le x \le b, \ t \ge \bar{t}$$

Proof. By linearity, the solution is the sum of the solution to problem (2.21) with the solution of (2.2); then, by lemmas 2.1, 2.4, we get

$$u(x,t) \ge M\Big(\frac{1}{1+\gamma b} - e^{-t\delta}\Big) - K\Big(g'(0) + \frac{5N}{2\epsilon}\Big) - K^* \tag{2.32}$$

where K is given by (2.16) and K^* by (2.31). Then, the theorem holds provided

$$\frac{M}{1+\gamma b} > K\Big(g'(0) + \frac{5N}{2\epsilon}\Big) + K^* \tag{2.33}$$

□

Remark 2.6. *As in corollary 2.3 by choosing* $g(t) = M(1 - e^{-\delta t})$*, where* M *satisfies* (2.33) *with* $N = 0$*,* $g'(0) = \delta M$*, one readily shows that the time* $\bar{t}$ *can be taken*

$$\bar{t} = -\frac{1}{\delta} \log \Big[\frac{1}{2(1+\gamma b)} - \frac{\delta b}{2} \big(1 + \frac{4}{\pi^2}(\gamma b)^2\big) - \frac{K^*}{2M}\Big] \tag{2.34}$$

3 The inverse problem

We can now state the main result of the paper, namely that two pair of measurements on a sufficiently large interval $(0, T)$ allow to uniquely identify the unknown boundary and Robin coefficient.

Theorem 3.1. *Let* $\varphi \in \mathcal{C}^2([0, +\infty))$*,* $\psi \in \mathcal{C}^1([0, +\infty))$ *and define* $\varphi_i = \varphi|_{[0,b_i]}$*,* $\psi_i = \psi|_{[0,b_i]}$*,* $i = 1, 2$*,* $0 < b_1 \le b_2$*. Let* $u_i(x, t), i = 1, 2$ *be solution to problem* (2.1) *with* $b = b_i$*,* $\gamma = \gamma_i$*, initial data* φ_i*,*ψ_i *and boundary datum* $g|_{(0,T)}$*, such that* φ_i*,* ψ_i *and* g *satisfy the assumptions of Lemmas 2.1, 2.4. Moreover, let* $\tilde{u}_i(x, t)$*,* $i = 1, 2$ *be solution to problem* (2.1) *with the same initial data, but with* $\tilde{u}(0, t) = \tilde{g}$*, where*

$$\begin{cases} \tilde{g}(t) = g(t), & 0 \le t \le \bar{t} \\ \tilde{g}(t) \ne g(t), & t > \bar{t} \end{cases} \tag{3.1}$$

$\bar{t}$ *being the same as in theorem 2.5 (we may assume* $\bar{t} \ge b_1$*). Let* $T \ge 5\bar{t}$ *and assume further that*

$$\begin{aligned} u_{1x}(0, t) &= u_{2x}(0, t), \quad 0 \le t \le T \\ \tilde{u}_{1x}(0, t) &= \tilde{u}_{2x}(0, t), \quad 0 \le t \le T \end{aligned} \tag{3.2}$$

Then,

$$b_2 = b_1, \ \gamma_2 = \gamma_1.$$

Proof. Suppose by contradiction that $b_2 > b_1$. Since $u_2(x, t)$ and $\tilde{u}_2(x, t)$ solve (2.1) in the domain $0 \le x \le b_2$, $0 \le t \le \bar{t}$ with the same data, they coincide on $[0, b_2] \times [0, \bar{t}]$. In particular

$$u_2(x, \bar{t}) = \tilde{u}_2(x, \bar{t}), \qquad 0 \le x \le b_2 \tag{3.3}$$

By the assumptions $u_1(0, t) = u_2(0, t) = g(t)$ for $t \ge 0$ and $u_{1x}(0, t) = u_{2x}(0, t)$, for $0 \le t \le T$; now, by considering the *domain of dependence* at (x, t) of a solution u of the wave equation with 'initial data' $u(0, t)$, $u_x(0, t)$, we also have

$$u_1(x, t) = u_2(x, t), \text{ for } 0 \le x \le b_1, \ x \le t \le T - x.$$

In particular, $u_1(b_1, t) = u_2(b_1, t)$ and $u_{1x}(b_1, t) = u_{2x}(b_1, t)$ for $b_1 \le t \le T - b_1$ and such interval has width larger that $2b_1$, being $b_1 \le \bar{t} \le 4\bar{t} \le T - b_1$.
By the previous identity, we readily get

$$u_{2x}(b_1, t) + \gamma_1 u_2(b_1, t) = 0, \text{ for } \bar{t} \le t \le 4\bar{t} \tag{3.4}$$

Obviously, an analogous relation also holds for $\widetilde{u_2}(x, t)$. Let us now consider the function

$$\lambda(x, t) = \frac{\widetilde{u_2}(x, t)}{u_2(x, t)} - 1, \qquad \text{for } \bar{t} \le t \le 4\bar{t}, \ b_1 \le x \le b_2$$

By theorem 2.5 and by the uniqueness of the solution of problem 2.1, we have $u_2(x, t) > 0$ for $\bar{t} \le t \le T$, $0 \le x \le b_2$; hence, the above function is well defined. Moreover, λ solves the homogeneous problem

$$\begin{cases} \lambda_{xx} - \lambda_{tt} = -\frac{2}{u_2}(u_{2x}\lambda_x - u_{2t}\lambda_t) & \bar{t} \le t \le 4\bar{t}, \ b_1 \le x \le b_2, \\ \lambda(x, \bar{t}) = 0 & b_1 \le x \le b_2, \\ \lambda_t(x, \bar{t}) = 0 & b_1 \le x \le b_2, \\ u_2^2(b_1, t) \ \lambda_x(b_1, t) = 0 & \bar{t} \le t \le 4\bar{t}, \\ u_2^2(b_2, t) \ \lambda_x(b_2, t) = 0 & \bar{t} \le t \le 4\bar{t}. \end{cases} \tag{3.5}$$

By the energy estimate, we conclude $\lambda(x,t) \equiv 0$, for $\bar{t} \le t \le 4\bar{t}$, $b_1 \le x \le b_2$ and therefore

$$u_2(x,t) = \widetilde{u_2}(x,t),\ \bar{t} \le t \le 4\bar{t},\ b_1 \le x \le b_2$$

In particular,

$$u_2(b_1,t) = \tilde{u}_2(b_1,t) \quad \text{and} \quad u_{2x}(b_1,t) = \tilde{u}_{2x}(b_1,t), \qquad \bar{t} \le t \le 4\bar{t} \tag{3.6}$$

But the points (b_1,t), $\bar{t} \le t \le 4\bar{t}$ lie in the *domain of influence* of a solution at $(0,t)$ for $2\bar{t} \le t \le 3\bar{t}$; hence

$$u_2(0,t) = \widetilde{u_2}(0,t),\ 2\bar{t} \le t \le 3\bar{t},$$

contradicting the assumption $\widetilde{g}(t) \neq g(t)$ for $t > \bar{t}$. Hence, $b_1 = b_2$; finally, by equation (3.4) one gets $\gamma_1 = \gamma_2$. □

We now exhibit a simple counterexample showing that a single measurement, even on an arbitrarily long interval of time, is not enough to identify the value b. Consider the initial boundary value problem:

$$\begin{cases} u_{xx} = u_{tt} & 0 \le x \le b,\ t \ge 0 \\ u(x,0) = 0 & 0 \le x \le b, \\ u_t(x,0) = \sin x & 0 \le x \le b, \\ u(0,t) = 0 & t \ge 0 \\ u_x(b,t) + u(b,t) = 0 & t \ge 0 \end{cases} \tag{3.7}$$

The problem has the solution

$$u(x,t) = \sin x\ \sin t;$$

provided that

$$\cos b \sin t + \sin b \sin t = 0,$$

i.e.

$$b = \frac{3}{4}\pi + k\pi, \qquad k = 1, 2, ...$$

Hence, the output measurement $u_x(0,t) = \sin t$, $t \ge 0$, does not allow to identify b.

4 Appendix

We prove the bound (2.12). To begin with, we have

$$\left| \frac{(y_n \cos y_n(b-x) + \gamma \sin y_n(b-x)}{(1+\gamma b)\cos by_n - by_n \sin by_n} \right| \le \frac{\sqrt{y_n^2 + \gamma^2}}{\left|(1+\gamma b)\cos by_n - by_n \sin by_n\right|} \tag{4.1}$$

Moreover, by recalling that

$$\cos(by_n) = -\frac{\gamma}{y_n} \sin(by_n)$$

we also have

$$\begin{aligned} \left|(1+\gamma b)\cos by_n - by_n \sin by_n\right| &= |\sin(by_n)| \left|(1+\gamma b)\frac{\gamma}{y_n} + by_n\right| \\ \ge by_n\,|\sin(by_n)| \ge by_n\ \sin(by_1) &= by_n \frac{y_1}{\sqrt{y_1^2+\gamma^2}} \ge \frac{by_n}{\sqrt{1+\frac{4}{\pi^2}(\gamma b)^2}} \end{aligned} \tag{4.2}$$

where the last inequality follows by $\pi/2 < by_1 < \pi$.

By inserting (4.3) in (4.1) we get

$$\left|\frac{(y_n \cos y_n(b-x)+\gamma \sin y_n(b-x)}{(1+\gamma b)\cos by_n - by_n \sin by_n}\right| \le \frac{\sqrt{y_n^2+\gamma^2}}{by_n}\sqrt{1+\frac{4}{\pi^2}(\gamma b)^2} \le \frac{1}{b}\Big(1+\frac{4}{\pi^2}(\gamma b)^2\Big) \tag{4.3}$$

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