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WEIGHTED FRACTIONAL POROUS MEDIA EQUATIONS: EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS WITH MEASURE DATA

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ABSTRACT. We shall prove existence and uniqueness of solutions to a class of porous media equations driven by weighted fractional Laplacians when the initial data are positive finite measures on the Euclidean space $\mathbb{R}^d$. In particular, Barenblatt-type solutions exist and are unique for the evolutions considered. The weight can be singular at the origin, and must have a sufficiently slow decay at infinity (power-like). Such kind of evolutions seems to have not been treated before even as concerns their linear, non-fractional analogues.

CONTENTS

1. INTRODUCTION

The main goal of this note is to prove existence and uniqueness of solutions to the following problem:

$$\begin{cases} |x|^{-\gamma} u_t + (-\Delta)^s (u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+ , \\ |x|^{-\gamma} u = \mu & \text{on } \mathbb{R}^d \times \{0\} , \end{cases} \tag{1.1}$$

where we assume that $s \in (0,1)$, $d > 2s$, $\gamma \in (0, 2s)$, $m > 1$ and that μ is a positive finite measure on $\mathbb{R}^d$ (so that $u \geq 0$). The unweighted case (namely when $\gamma = 0$) is known as *fractional porous media equation* and has been thoroughly analysed in [16] and [17] for initial data in $L^1(\mathbb{R}^d)$. Here we study some of its possible weighted variants whose model is (1.1), and want to extend the set of initial data to positive finite measures. For greater readability we shall consider explicitly only the

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case of such a model problem, that is when the singular weight is exactly $|x|^{-\gamma}$. However, notice that all of our main results also apply to the more general problem

$$\begin{cases} \rho(x)u_t + (-\Delta)^s (u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+ \, , \\ \rho(x)u = \mu & \text{on } \mathbb{R}^d \times \{0\} \, , \end{cases} \tag{1.2}$$

where we the weight ρ is assumed to satisfy

$$c \leq \rho(x) \leq C|x|^{-\gamma} \quad \forall x \in B_1(0) \quad \text{and} \quad c|x|^{-\gamma} \leq \rho(x) \leq C|x|^{-\gamma} \quad \forall x \in B_1^c(0) \tag{1.3}$$

for some positive constants $c < C$. This is recalled in Section 3.

Above we listed the ranges in which the parameters d, s, γ and m are allowed to vary. Actually, the methods of proof we exploit will require a further restriction on d, s and γ, which we shall clarify later (see the hypotheses of Theorems 3.2 and 3.3).

As a particular case, our results entail the existence and uniqueness of Barenblatt solutions for the equations considered, which extends to the cases considered here recent deep results of Vázquez [41]. We shall show in [24] that such Barenblatt solutions determine the asymptotics of solutions corresponding to integrable data, and shall consider there also other weighted fractional porous media equations with rapidly decaying weights, for which the asymptotics of solutions is radically different.

The analysis of the evolutions considered here poses significant difficulties, as can be guessed even when considering their linear analogues. In fact, the first issue we have to deal with is the essential self-adjointness of the operator formally defined as $|x|^{\gamma}(-\Delta)^s$ on test functions and the validity of the Markov property for the associated linear evolution. This will be crucial in the uniqueness part, and seemed not to be known so far. We just sketch in Appendix B the long and technical proof of these properties, which takes into account the fact that γ is sufficiently close to zero. For larger γ one expects that conditions at zero and/or at infinity should be needed to get self-adjointness. Notice that the study of weighted linear differential operator of second order has a long story, see for example [12, Sect. 4.7], or [31]. Recently, the study of the spectral properties of operators which are modeled on the critical operator formally given by $|x|^2\Delta$ has been performed in [13].

As for nonlinear evolutions, the study of porous media and fast diffusion equations with measure data can be tracked back to the pioneering, fundamental papers [3, 7, 33, 10]. See [42, Sect. 13] for details and additional references. The fast diffusion case, which will not be dealt with here, is investigated in [8, 9]: notice that in such case Dirac delta may not be smoothed into regular solutions, so that different approaches must be used, see the recent paper [34] for a general approach. In the breakthrough papers [16, 17], the fractional porous media and fast diffusion equations were introduced and thoroughly studied when the data are integrable functions. The construction of Barenblatt solutions and the study of their role as asymptotic attractors for general integrable data is performed in [41]. Existence and uniqueness of solutions in the fractional, weighted case, is studied in [36, 37]: there, the weights are regular and data cannot be measures. Notice that fractional porous media equations are being used as a model in several applied situations, see [5, Appendix B] and reference quoted for details. We also remark that the terminology "measure data" is sometimes used in different contexts in which a measure appears as source term in certain evolution equations: see e.g. [29] and references quoted.

There is a huge literature on the weighted, non fractional porous media equation: with no claim of generality we quote [14, 15, 19, 20, 22, 23, 25, 26, 27, 35, 38, 39, 40] and references quoted therein. It should be noticed explicitly that the possible singularity of the weight, and the fact that we want to consider measure data as well, makes our problem significantly different both from the unweighted, fractional case, and from the weighted, non-fractional case: straightforward modifications of the strategies valid in such cases are then not applicable here.

The paper is organized as follows. Section 2 briefly collects some preliminary tools on measure theory, fractional Laplacians and fractional Sobolev spaces. In Section 3 we prove our main result on existence, whereas in Section 4 uniqueness is addressed. Appendix A includes some technical results used in the approximating procedures developed in the paper. Finally, in Appendix B we state the main properties of the linear operator formally given by $|x|^\gamma(-\Delta)^s$ in the appropriate range of γ, and give a sketch of the corresponding proofs.

2. Preliminary tools

In this section we outline some basic notation, definitions and properties that we shall make us of later concerning weighted Lebesgue spaces, measures, fractional Laplacian, fractional Sobolev spaces and Riesz potentials of measures.

Weighted Lebesgue spaces. For a given measurable function $\rho : \mathbb{R}^d \to \mathbb{R}^+$ (that is, a weight), we denote as $L^p_\rho(\mathbb{R}^d)$ (let $p \in [1,\infty)$) the Banach space constituted by all (classes of equivalence of) measurable functions $v : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|v\|_{p,\rho}^p = \int_{\mathbb{R}^d} |v(x)|^p \, \rho(x) \mathrm{d}x < \infty \, .$$

In the special case $\rho(x) = |x|^\alpha$ (let $\alpha \in \mathbb{R}$) we simplify notation and replace $L^p_\rho(\mathbb{R}^d)$ by $L^p_\alpha(\mathbb{R}^d)$ and $\|v\|_{p,\rho}$ by $\|v\|_{p,\alpha}$. For the usual unweighted Lebesgue spaces we keep the traditional notation $L^p(\mathbb{R}^d)$, denoting the corresponding norms as $\|v\|_p$. Later on we might also use the more detailed notation $\|v\|_{L^p(\mathbb{R}^N)}$ (let N be d or $d+1$) in order to avoid ambiguity.

Positive finite measures on $\mathbb{R}^d$. Since in (1.1) we deal with positive, finite measures μ on $\mathbb{R}^d$, it is convenient to recall some basic properties enjoyed by the set of such measures, which we denote as $\mathcal{M}(\mathbb{R}^d)$ (with a slight abuse of notation: this is the usual symbol for the space of *signed* measures on $\mathbb{R}^d$). To begin with, consider a sequence $\{\mu_n\} \subset \mathcal{M}(\mathbb{R}^d)$. Following the notation of [33], we say that $\{\mu_n\}$ converges to $\mu \in \mathcal{M}(\mathbb{R}^d)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ if there holds

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \phi \, \mathrm{d}\mu_n = \int_{\mathbb{R}^d} \phi \, \mathrm{d}\mu \quad \forall \phi \in C_c(\mathbb{R}^d) \, , \tag{2.1}$$

where $C_c(\mathbb{R}^d)$ is the space of continuous, compactly supported functions on $\mathbb{R}^d$. This is usually referred to as *local weak* convergence* (see [2, Def. 1.58]). A classical theorem in measure theory asserts that if

$$\sup_n \mu_n(\mathbb{R}^d) < \infty \tag{2.2}$$

then there exists $\mu \in \mathcal{M}(\mathbb{R}^d)$ such that $\{\mu_n\}$ converges to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ up to subsequences (see [2, Th. 1.59]). A stronger notion of convergence is the following. A sequence $\{\mu_n\} \subset \mathcal{M}(\mathbb{R}^d)$ is said to converge to $\mu \in \mathcal{M}(\mathbb{R}^d)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ if

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \phi \, \mathrm{d}\mu_n = \int_{\mathbb{R}^d} \phi \, \mathrm{d}\mu \quad \forall \phi \in C_b(\mathbb{R}^d) \, , \tag{2.3}$$

where $C_b(\mathbb{R}^d)$ is the space of continuous, bounded functions on $\mathbb{R}^d$. Trivially, (2.3) implies (2.1). The opposite holds true under a further hypothesis. That is, if $\{\mu_n\}$ converges to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ and

$$\lim_{n\to\infty} \mu_n(\mathbb{R}^d) = \mu(\mathbb{R}^d)$$

then $\{\mu_n\}$ converges to μ also in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ (see [2, Prop. 1.80]). Notice that if $\{\mu_n\}$ converges to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ and (2.2) holds, a priori one only has a weak* lower semi-continuity property:

$$\mu(\mathbb{R}^d) \le \liminf_{n\to\infty} \mu_n(\mathbb{R}^d)$$

(see again [2, Th. 1.59]).

Fractional Laplacian and fractional Sobolev spaces. The fractional s-Laplacian operator which appears in (1.1) is defined, at least for any $\phi \in \mathcal{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$, as

$$(-\Delta)^s(\phi)(x) = p.v. \; C_{d,s} \int_{\mathbb{R}^d} \frac{\phi(x) - \phi(y)}{|x-y|^{d+2s}} \, \mathrm{d}y \quad \forall x \in \mathbb{R}^d \, , \tag{2.4}$$

where $C_{d,s}$ is a suitable positive constant depending only on d and s. However, since a priori we have no clue about the regularity of solutions to (1.1), it is necessary to reformulate the problem in a suitable weak sense, see Definition 3.1 below. Before doing it, we need to introduce some fractional Sobolev spaces. Here we shall mainly deal with $\dot{H}^s(\mathbb{R}^d)$, that is the closure of $\mathcal{D}(\mathbb{R}^d)$ w.r.t. the norm

$$\|\phi\|_{\dot{H}^s}^2 = C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d) .$$

Notice that the space usually denoted as $H^s(\mathbb{R}^d)$ is just $L^2(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$. For definitions and properties of the general fractional Sobolev spaces $W^{r,p}(\mathbb{R}^d)$ (let $r > 0$ and $p \in [1, \infty)$) we refer the reader to the survey paper [18].

At first glance the link between the fractional s-Laplacian and the space $\dot{H}^s(\mathbb{R}^d)$ might not be very clear. In order to make it apparent, one can first start from the validity of the identities

$$\begin{aligned} C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y &= \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(\phi)(x) \, (-\Delta)^{\frac{s}{2}}(\psi)(x) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \phi(x) (-\Delta)^s(\psi)(x) \, \mathrm{d}x \end{aligned} \tag{2.5}$$

for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$, namely a sort of "integration by parts" formula. For a rigorous proof of (2.5), which exploits Fourier transform methods, see [18, Sect. 3]. In particular, letting $\phi = \psi$, one gets the equality

$$\|\phi\|_{\dot{H}^s}^2 = \left\| (-\Delta)^{\frac{s}{2}}(\phi) \right\|_{L^2}^2 \quad \forall \phi \in \mathcal{D}(\mathbb{R}^d) . \tag{2.6}$$

Now fix $v \in \dot{H}^s(\mathbb{R}^d)$ and pick a sequence $\{\phi_n\} \subset \mathcal{D}(\mathbb{R}^d)$ converging to v in $\dot{H}^s(\mathbb{R}^d)$. Thanks to fractional Sobolev embeddings (see [18, Sect. 6] or Lemma 4.5 below) the sequence $\{\phi_n\}$ converges to v also in $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$. This is enough to pass to the limit on the r.h.s. of the second identity in (2.5), since $(-\Delta)^s(\psi)(x)$ is a regular function decaying at least like $|x|^{-d-2s}$ as $|x| \to \infty$ (see Lemma A.1 of Appendix A). On the l.h.s. of the first identity in (2.5) we can also pass to the limit because by definition of $\|\phi\|_{\dot{H}^s}$ the sequence

$$\frac{\phi_n(x) - \phi_n(y)}{|x-y|^{\frac{d}{2}+s}}$$

converges in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$, and by the convergence of $\{\phi_n\}$ to v in $L^{\frac{2d}{d-2s}}(\mathbb{R}^d)$ such limit must necessarily coincide a.e. with

$$\frac{v(x) - v(y)}{|x-y|^{\frac{d}{2}+s}} .$$

Arguing similarly and passing to the limit on the r.h.s. of the first identity in (2.5), one finds that there exists a function $h \in L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} h(x) \, (-\Delta)^{\frac{s}{2}}(\psi)(x) \, \mathrm{d}x = \int_{\mathbb{R}^d} v(x) (-\Delta)^s(\psi)(x) \, \mathrm{d}x \quad \forall \psi \in \mathcal{D}(\mathbb{R}^d) . \tag{2.7}$$

Formula (2.7) is nothing but the definition of $(-\Delta)^{\frac{s}{2}}(v) = h$ in the sense of distributions. Gathering all this information, one finally obtains the identities

$$\begin{aligned} C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))(\psi(x) - \psi(y))}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y &= \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(v)(x) \, (-\Delta)^{\frac{s}{2}}(\psi)(x) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^d} v(x) (-\Delta)^s(\psi)(x) \, \mathrm{d}x \end{aligned} \tag{2.8}$$

for all $v \in \dot{H}^s(\mathbb{R}^d)$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$, which clearly illustrate the link between the fractional s-Laplacian and the space $\dot{H}^s(\mathbb{R}^d)$. Moreover, letting ψ tend to $w \in \dot{H}^s(\mathbb{R}^d)$ and passing to the limit in the first identity of (2.8) yields

$$C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x) - v(y))(w(x) - w(y))}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(v)(x) \, (-\Delta)^{\frac{s}{2}}(w)(x) \, \mathrm{d}x \quad \forall v, w \in \dot{H}^s(\mathbb{R}^d) . \tag{2.9}$$

If we set $v = w$ in (2.9) we deduce that (2.6) also holds in $\dot{H}^s(\mathbb{R}^d)$. In Sections 4 and 5 (and in Appendix B) we shall focus on functions which belong to $\dot{H}^s(\mathbb{R}^d)$ and to weighted Lebesgue spaces.

Riesz potentials. Another mathematical object deeply linked with the fractional s-Laplacian is its Riesz kernel, namely the function

$$I_{2s}(x) = \frac{k_{d,s}}{|x|^{d-2s}} \, ,$$

where $k_{d,s}$ is a positive constant depending only on d and s. For a given positive finite measure ν, one can show that the convolution

$$U^\nu = I_{2s} * \nu$$

produces an $L^1_{loc}(\mathbb{R}^d)$ function referred to as the *Riesz potential* of ν, which formally satisfies

$$(-\Delta)^s(U^\nu) = \nu \, .$$

That is, still at a formal level, the convolution against I_{2s} coincides with the operator $(-\Delta)^{-s}$. One of the most important and classical references for Riesz potentials is the monograph [28] by N. S. Landkof. In the proof of Theorem 3.2 and throughout Section 5 we shall exploit some crucial properties of Riesz potentials collected in [28], along with their connections with the fractional s-Laplacian.

3. Statements of the main results

Having introduced all the basic mathematical tools we need, we can provide a suitable notion of weak solution to (1.1) (and (1.2)), in the spirit of [17] and [36]. Before going on note that, in the present and in the next sections, by the symbol $u(t)$ we shall mean the whole of the function $u(x,t)$ evaluated at time $t \geq 0$.

Definition 3.1. *Given a finite positive measure μ, by a weak solution to problem* (1.1) *we mean a nonnegative function u such that*

$$u \in L^\infty((0,\infty); L^1_{-\gamma}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (\tau,\infty)) \quad \forall \tau > 0 \, , \tag{3.1}$$

$$u \in L^2_{loc}((0,\infty); \dot{H}^s(\mathbb{R}^d)) \, , \tag{3.2}$$

$$-\int_0^\infty \int_{\mathbb{R}^d} u(x,t) \varphi_t(x,t) \, |x|^{-\gamma} \mathrm{d}x \, \mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x,t) \, (-\Delta)^{\frac{s}{2}}(\varphi)(x,t) \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{3.3}$$
$$\forall \varphi \in C_c^\infty(\mathbb{R}^d \times (0,\infty))$$

and

$$\operatorname*{ess\,lim}_{t\to 0} |x|^{-\gamma} u(t) = \mu \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d)) \, . \tag{3.4}$$

For problem (1.2) *weak solutions are understood analogously, provided one replaces $|x|^{-\gamma}$ with ρ accordingly.*

Our first main result concerns existence.

Theorem 3.2. *Let $d > 2s$ and $\gamma \in (0, 2s \wedge (d-2s))$. Let μ be a positive finite measure. Then there exists a weak solution u to* (1.1) *in the sense of Definition 3.1. It satisfies the smoothing effect*

$$\|u(t)\|_\infty \leq K \, t^{-\alpha} \, \mu(\mathbb{R}^d)^\beta \quad \forall t > 0 \tag{3.5}$$

where K is a suitable positive constant depending only on m, γ, s, d and

$$\alpha = \frac{d-\gamma}{(m-1)(d-\gamma) + (2s-\gamma)p_0} \, , \quad \beta = \frac{(2s-\gamma)p_0}{(m-1)(d-\gamma) + (2s-\gamma)p_0} \, .$$

In particular $u(t) \in L^p_{-\gamma}(\mathbb{R}^d)$ for all $t > 0$ and $p \in [1,\infty]$. The solution satisfies the energy estimates

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}} \left(u^m\right)(x,t)\right|^2 \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}^d} u^{m+1}(x,t_2) \, |x|^{-\gamma} \mathrm{d}x = \int_{\mathbb{R}^d} u^{m+1}(x,t_1) \, |x|^{-\gamma} \mathrm{d}x$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |z_t(x,t)|^2 \, |x|^{-\gamma} \mathrm{d}x \, \mathrm{d}t \leq C$$

for all $t_2 > t_1 > 0$*, where* $z = u^{\frac{m+1}{2}}$ *and* C *is a positive constant that depends only on* t_1*,* t_2 *and on*

$$\int_{\mathbb{R}^d} u^{m+1}(x, t_*) \, |x|^{-\gamma} \mathrm{d}x$$

for some $t_* \in (0, t_1)$*.*

The same results hold for weak solutions to (1.2)*, provided* $|x|^{-\gamma}$ *is replaced by any* ρ *satisfying conditions* (1.3)*.*

As for uniqueness, we have the following result.

Theorem 3.3. *Let* $d > 2s$ *and* $\gamma \in (0, 2s) \cap (0, d - 2s]$*. Let* u_1, u_2 *be two weak solutions to* (1.1) *in the sense of Definition 3.1. Suppose that they assume as initial datum the same finite positive measure* μ *in the sense of* (3.4)*. Then* $u_1 = u_2$*.*

The same result holds true for weak solutions to (1.2)*, provided* $|x|^{-\gamma}$ *is replaced by any* ρ *satisfying conditions* (1.3)*.*

Remark 3.4. Notice that, if $d \geq 4s$, then the assumptions on γ in the above theorems reduce to $\gamma \in (0, 2s)$.

Remark 3.5. As a consequence of our method of proof, uniqueness of the initial trace for weak solutions to the equations considered, in the spirit of [5, Sect. 7] and [4], can be proved. In fact, given a function u satisfying (3.1), (3.2), (3.3), it can be shown easily, thanks to the monotonicity in time of the associated potential (see the proof of Theorem 3.2 in this connection), that there exists a unique positive finite measure μ which is the initial trace of u in the sense that (3.4) holds true.

4. Existence of weak solutions

We stress again the fact that we shall prove our results only for weak solutions to (1.1). The modifications required to deal with (1.2), provided ρ is any weight complying with (1.3), are straightforward.

Before proceeding with the proof of Theorem 3.2 (and associated preliminary lemmas), we shall show a first, direct consequence of Definition 3.1, namely the conservation in time of the quantity $\int_{\mathbb{R}^d} u(x,t)\,|x|^{-\gamma}\mathrm{d}x$, that is the $L^1_{-\gamma}(\mathbb{R}^d)$ norm of $u(t)$ since we consider nonnegative solutions.

Proposition 4.1. Let $\gamma \in (0, 2s)$ and u be the weak solution to (1.1) according to Definition 3.1. Then there holds

$$\|u(t)\|_{1,-\gamma} = \int_{\mathbb{R}^d} u(x,t)\,|x|^{-\gamma}\mathrm{d}x = \mu(\mathbb{R}^d) \quad \text{for a.e. } t > 0\,, \tag{4.1}$$

namely the *conservation of mass*.

Proof. In order to prove (4.1) we plug into (3.3) the following test function:

$$\varphi_R(x,t) = \vartheta(t)\xi_R(x)\,,$$

where ξ_R is a cut-off function as in Lemma A.3 of Appendix A and ϑ is a suitable positive, regular and compactly supported approximation of the function $\chi_{[t_1,t_2]}$ (let $t_2 > t_1 > 0$). As already mentioned, $(-\Delta)^s(\xi)(x)$ is a regular function which decays at least like $|x|^{-d-2s}$ as $|x| \to \infty$ (see Lemma A.1). Moreover, by the scaling properties recalled in Lemma A.3, there holds

$$(-\Delta)^s(\xi_R)(x) = \frac{1}{R^{2s}}(-\Delta)^s(\xi)\left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^d\,.$$

Thanks to these properties, we have:

$$\left||x|^\gamma(-\Delta)^s(\xi_R)(x)\right| = \left|\frac{1}{R^{2s-\gamma}}\left|\frac{x}{R}\right|^\gamma (-\Delta)^s(\xi)\left(\frac{x}{R}\right)\right| \leq \frac{1}{R^{2s-\gamma}}\left\||x|^\gamma(-\Delta)^s(\xi)\right\|_\infty \quad \forall x \in \mathbb{R}^d\,. \tag{4.2}$$

Now we let $R \to \infty$ in (3.3). Clearly,

$$\lim_{R\to\infty} \int_0^\infty \int_{\mathbb{R}^d} u(x,t)(\varphi_R)_t(x,t)\,|x|^{-\gamma} \mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^d} u(x,t)\vartheta'(t)\,|x|^{-\gamma} \mathrm{d}x\,\mathrm{d}t\,. \tag{4.3}$$

As for the second integral on the l.h.s. of (3.3), note that

$$\int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x,t)\,(-\Delta)^{\frac{s}{2}}(\varphi_R)(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^d} u^m(x,t)\,(-\Delta)^s(\varphi_R)(x,t)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{4.4}$$

where integration by parts is justified since φ_R is regular and compactly supported and $u^m(t) \in \dot{H}^s(\mathbb{R}^d)$ (recall (2.8)). Estimate (4.2) ensures that

$$\left| \int_0^\infty \int_{\mathbb{R}^d} u^m(x,t)\,(-\Delta)^s(\varphi_R)(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| \le \frac{\|\vartheta\|_\infty\, \||x|^\gamma(-\Delta)^s(\xi)\|_\infty}{R^{2s-\gamma}} \left| \int_{t_*}^{t^*} \int_{\mathbb{R}^d} u^m(x,t)\,|x|^{-\gamma} \mathrm{d}x\,\mathrm{d}t \right|, \tag{4.5}$$

where $t^* > t_* > 0$ are chosen so that $\operatorname{supp}\vartheta \subset [t_*, t^*]$. The integral on the r.h.s. of (4.5) is finite thanks to (3.1). Hence, letting $R \to \infty$, we infer that the integrals in (4.4) converge to zero (recall that $\gamma < 2s$), which together with (4.3) yields

$$\int_0^\infty \int_{\mathbb{R}^d} u(x,t)\vartheta'(t)\,|x|^{-\gamma} \mathrm{d}x\,\mathrm{d}t = 0\,. \tag{4.6}$$

Letting $\vartheta \to \chi_{[t_1,t_2]}$ in (4.6) and using Lebesgue differentiation Theorem we infer that $\|u(t_2)\|_{1,-\gamma} = \|u(t_1)\|_{1,-\gamma}$ for a.e. $t_2 > t_1$. This property and (3.4) finally yield (4.1). □

The proof of existence of weak solutions to (1.1) (Theorem 3.2) is based on an approximation procedure. That is, the idea is to approximate the measure μ with data $u_0 \in L^1_{-\gamma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. This calls first for an existence result of solutions to the following problem:

$$\begin{cases} |x|^{-\gamma} u_t + (-\Delta)^s\,(u^m) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+\,, \\ u = u_0 & \text{on } \mathbb{R}^d \times \{0\}\,. \end{cases} \tag{4.7}$$

In order to obtain it, we need in turn to approximate problem (4.7), just by regularizing the weight $|x|^{-\gamma}$ in a neighbourhood of $x = 0$. This is thoroughly described in the proof of Lemma 4.3 below. For the success of such a procedure, the following elementary lemma turns out to be crucial.

Lemma 4.2. *Let $\gamma \in (0, d+2s]$ and ρ be a weight that complies with* (1.3)*. Consider a function $v \in L^2_{loc}((0,\infty); \dot{H}^s(\mathbb{R}^d))$ such that, for all $t_2 > t_1 > 0$,*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |v(x,t)|^2\, \rho(x) \mathrm{d}x\,\mathrm{d}t \le C\,, \tag{4.8}$$

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}}\,(v)\,(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \le C \tag{4.9}$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |v_t(x,t)|^2\, \rho(x) \mathrm{d}x\,\mathrm{d}t \le C\,, \tag{4.10}$$

where C is a positive constant depending only on t_1 and t_2. Take any cut-off functions $\xi_1 \in C_c^\infty(\mathbb{R}^d)$, $\xi_2 \in C_c^\infty((0,\infty))$ and define $v_c : \mathbb{R}^d \to \mathbb{R}$ as follows:

$$v_c(x,t) = \xi_1(x)\xi_2(t)v(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}\,,$$

where we implicitly assume ξ_2 and v to be zero for $t < 0$. Then there holds

$$\|v_c\|^2_{H^s(\mathbb{R}^{d+1})} = \|v_c\|^2_{L^2(\mathbb{R}^{d+1})} + \|v_c\|^2_{\dot{H}^s(\mathbb{R}^{d+1})} \le C' \tag{4.11}$$

for a positive constant C' that depends only on ξ_1 and ξ_2 (also through C).

Proof. The validity of

$$\|v_c\|^2_{L^2(\mathbb{R}^{d+1})} \le C' \tag{4.12}$$

is an immediate consequence of (4.8) and of the fact that ρ is bounded away from zero on compact sets (from now on C' will be a constant as in the statement of the lemma, which we shall not relabel throughout the proof). Moreover, since

$$(v_c)_t = \xi_1 \xi_2' v + \xi_1 \xi_2 v_t \,,$$

by (4.8), (4.10) and again the fact that ρ is bounded away from zero on compact sets we deduce that

$$\|(v_c)_t\|^2_{L^2(\mathbb{R}^{d+1})} \le C' \,. \tag{4.13}$$

Now we have to handle the spatial regularity of v_c. First it is convenient to recall the identity

$$\int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} (v_c)(x,t) \right|^2 \mathrm{d}x = C_{d,s} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(v_c(x,t)-v_c(y,t))^2}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y = \|v_c(t)\|^2_{\dot{H}^s(\mathbb{R}^d)} \,.$$

Straightforward computations show that

$$\begin{aligned} \|v_c(t)\|^2_{\dot{H}^s(\mathbb{R}^d)} = & C_{d,s}\, \xi_2^2(t) \int_{\mathbb{R}^d} \xi_1^2(x) \left(\int_{\mathbb{R}^d} \frac{(v(x,t)-v(y,t))^2}{|x-y|^{d+2s}} \, \mathrm{d}y \right) \mathrm{d}x \\ & + C_{d,s}\, \xi_2^2(t) \int_{\mathbb{R}^d} |v(y,t)|^2 \left(\int_{\mathbb{R}^d} \frac{(\xi_1(x)-\xi_1(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}x \right) \mathrm{d}y \\ & + 2\, C_{d,s}\, \xi_2^2(t) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \xi_1(x) v(y,t) \frac{(v(x,t)-v(y,t))\,(\xi_1(x)-\xi_1(y))}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y \,. \end{aligned} \tag{4.14}$$

An immediate application of the Cauchy-Schwarz inequality entails that the third integral on the r.h.s. of (4.14) is controlled by the first two integrals. As concerns the first one, we have:

$$C_{d,s}\, \xi_2^2(t) \int_{\mathbb{R}^d} \xi_1^2(x) \left(\int_{\mathbb{R}^d} \frac{(v(x,t)-v(y,t))^2}{|x-y|^{d+2s}} \, \mathrm{d}y \right) \mathrm{d}x \le \chi_{\operatorname{supp} \xi_2}(t) \, \|\xi_2\|^2_\infty \, \|\xi_1\|^2_\infty \, \|v(t)\|^2_{\dot{H}^s(\mathbb{R}^d)} \,. \tag{4.15}$$

In order to bound the second integral, it is important to recall that the function $l_s(\xi_1)(y)$ (see Lemma A.3) is regular and decays at least like $|y|^{-d-2s}$ as $|y| \to \infty$ (for the definition and properties of $l_s(\cdot)$ see Lemma A.2). Hence, by the assumptions on ρ and γ, we infer that

$$\xi_2^2(t) \int_{\mathbb{R}^d} |v(y,t)|^2 \left(\int_{\mathbb{R}^d} \frac{(\xi_1(x)-\xi_1(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}x \right) \mathrm{d}y \le c' \chi_{\operatorname{supp} \xi_2}(t) \, \|\xi_2\|^2_\infty \int_{\mathbb{R}^d} |v(y,t)|^2 \, \rho(y) \mathrm{d}y \tag{4.16}$$

for a suitable positive constant c'. Integrating in time (4.14) and using (4.15), (4.16), (4.8) and (4.9) we then get

$$\left\| (-\Delta)^{\frac{s}{2}} (v_c) \right\|^2_{L^2(\mathbb{R}^{d+1})} \le C' \,. \tag{4.17}$$

By exploiting (4.12), (4.13) and (4.17) one deduces (4.11). This is easily justified by means of Fourier transforms. In fact, upon denoting $\mathcal{F}(f)(x',t')$ as the Fourier transform of a function $f(x,t)$, from (4.13) we obtain

$$\int_{\mathbb{R}^{d+1}} |t'|^2 \left| \mathcal{F}(v_c)(x',t') \right|^2 \mathrm{d}x' \mathrm{d}t' = \int_{\mathbb{R}^{d+1}} (v_c)_t^2(x,t) \, \mathrm{d}x \, \mathrm{d}t \le C' \,, \tag{4.18}$$

whereas (4.17) gives

$$\int_{\mathbb{R}^{d+1}} |x'|^{2s} \left| \mathcal{F}(v_c)(x',t') \right|^2 \mathrm{d}x' \mathrm{d}t' = \int_{\mathbb{R}^{d+1}} \left| (-\Delta)^{\frac{s}{2}} (v_c)(x,t) \right|^2 \mathrm{d}x \, \mathrm{d}t \le C' \,. \tag{4.19}$$

Thus, thanks to (4.12), (4.18) and (4.19) we finally get the estimate

$$\int_{\mathbb{R}^{d+1}} \left(1 + |x'|^2 + |t'|^2 \right)^s \left| \mathcal{F}(v_c)(x',t') \right|^2 \mathrm{d}x' \mathrm{d}t' \le C' \,,$$

which is equivalent to (4.11) (see e.g. [18, Sect. 3]). □

We are now able to prove existence of weak solutions to (4.7). Such solutions are understood in the sense of Definition 3.1, just by replacing μ with $|x|^\gamma u_0$.

Lemma 4.3. *Let $d > 2s$, $\gamma \in (0, 2s)$ and $u_0 \in L^1_{-\gamma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, with $u_0 \geq 0$. There exists a weak solution u to (4.7) which satisfies the following energy estimates:*

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}} (u^m)(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} u^{m+1}(x,t_2)\,|x|^{-\gamma}\mathrm{d}x = \int_{\mathbb{R}^d} u^{m+1}(x,t_1)\,|x|^{-\gamma}\mathrm{d}x \qquad \forall t_2 > t_1 \geq 0 \tag{4.20}$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |z_t(x,t)|^2\ |x|^{-\gamma}\mathrm{d}x\,\mathrm{d}t \leq C \quad \forall t_2 > t_1 > 0\,, \tag{4.21}$$

where $z = u^{\frac{m+1}{2}}$ and C is a positive constant that depends only on t_1, t_2 and on the initial datum u_0 through the integral

$$\int_{\mathbb{R}^d} u^{m+1}(x,t_*)\,|x|^{-\gamma}\mathrm{d}x \leq \int_{\mathbb{R}^d} u_0^{m+1}(x)\,|x|^{-\gamma}\mathrm{d}x\,, \tag{4.22}$$

for some $t_* \in (0, t_1)$.

Proof. First of all it is convenient to introduce the following approximation of problem (4.7):

$$\begin{cases} \rho_\eta(x)\,(u_\eta)_t + (-\Delta)^s\left(u_\eta^m\right) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+\,, \\ u_\eta = u_0 & \text{on } \mathbb{R}^d \times \{0\}\,, \end{cases}$$

where $\{\rho_\eta\} \subset C(\mathbb{R}^d)$ is a family of positive weights (depending on the positive parameter η) which behave like $|x|^{-\gamma}$ at infinity and approximate $|x|^{-\gamma}$ monotonically from below. For instance, one can pick

$$\rho_\eta(x) = \left(|x|^2 + \eta\right)^{-\frac{\gamma}{2}} \quad \forall x \in \mathbb{R}^d\,. \tag{4.23}$$

Notice that, thanks to the properties of ρ_η, one has that $u_0 \in L^1_{\rho_\eta}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Existence (and uniqueness) of weak solutions to (4.23) for such weights and initial data have already been established in [37, Th. 3.1]. Actually the solutions constructed there also belong to $C([0,\infty); L^1_{\rho_\eta}(\mathbb{R}^d))$ and satisfy the bound

$$\|u_\eta\|_{L^\infty(\mathbb{R}^d\times(0,\infty))} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}\,. \tag{4.24}$$

Exploiting these properties it is easy to show that each u_η satisfies a weak formulation which is slightly stronger than the one of Definition 3.1:

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^d} u_\eta(x,t)\varphi_t(x,t)\,\rho_\eta(x)\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u_\eta^m)(x,t)\,(-\Delta)^{\frac{s}{2}}(\varphi)(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ =&\int_{\mathbb{R}^d} u_0(x)\varphi(x,0)\,\rho_\eta(x)\mathrm{d}x \end{aligned} \tag{4.25}$$

for all $T > 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$ such that $\varphi(T) = 0$, where $u_\eta^m \in L^2((0,\infty); \dot{H}^s(\mathbb{R}^d))$. The latter property follows from the validity of the key energy identity

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}}\left(u_\eta^m\right)(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t + \frac{1}{m+1}\int_{\mathbb{R}^d} u_\eta^{m+1}(x,t_2)\rho_\eta(x)\mathrm{d}x \\ =&\frac{1}{m+1}\int_{\mathbb{R}^d} u_\eta^{m+1}(x,t_1)\rho_\eta(x)\mathrm{d}x \end{aligned} \tag{4.26}$$

for all $t_2 > t_1 \geq 0$. Formally, (4.26) can be proved by plugging the test function $\varphi(x,t) = \vartheta(t)u_\eta^m(x,t)$ into the weak formulation (4.25) and letting ϑ tend to $\chi_{[t_1,t_2]}$ as in the proof of Proposition 4.1. The problem is that, a priori, such a φ is not admissible as a test function. In order to justify (4.26) rigorously one must proceed as in Section 8 of [17]. A crucial point concerns the fact that solutions can be proved to be *strong*, that is

$$(u_\eta)_t \in L^\infty((\tau,\infty); L^1_{\rho_\eta}(\mathbb{R}^d)) \quad \forall \tau > 0 \,.$$

We refer the reader to Sections 4.1 and 4.2 below for the details. The discussion there is focussed, for simplicity, on the special case of the weight $|x|^{-\gamma}$, but as recalled in the Introduction it applies safely to any weight ρ complying with (1.3). Another fundamental energy estimate that we shall exploit in order to manage the passage to the limit in (4.25) as $\eta \to 0$ is the following:

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |(z_\eta)_t(x,t)|^2 \, \rho_\eta(x) \mathrm{d}x \, \mathrm{d}t \le C \quad \forall t_2 > t_1 > 0 \,,$$

where $z_\eta = u_\eta^{\frac{m+1}{2}}$ and C is a suitable positive constant that depends only on t_1, t_2 and on the initial datum u_0 through the integral

$$\int_{\mathbb{R}^d} u_\eta^{m+1}(x,t_*) \, \rho_\eta(x) \mathrm{d}x \le \int_{\mathbb{R}^d} u_0^{m+1}(x) \, \rho_\eta(x) \mathrm{d}x$$

for some $t_* \in (0,t_1)$. Again, (4.3) can be *formally* proved by picking the test function $\varphi(x,t) = \zeta(t)(u_\eta^m)_t(x,t)$ and integrating by parts in time, where ζ is any positive regular function with compact support in $(0,\infty)$ (t_* is the infimum of its support) such that $\zeta = 1$ on $[t_1,t_2]$. Actually the validity of (4.3) is one of the main tools that one uses to prove that solutions are strong, and its rigorous proof follows exactly as in [17, Lem. 8.1].

Now notice that, since

$$\left(u_\eta^m\right)_t = \frac{2m}{m+1} z_\eta^{\frac{m-1}{m+1}} \left(z_\eta\right)_t$$

and

$$\|z_\eta\|_{L^\infty(\mathbb{R}^d\times(0,\infty))} = \|u_\eta\|_{L^\infty(\mathbb{R}^d\times(0,\infty))}^{\frac{m+1}{2}} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}^{\frac{m+1}{2}}$$

(recall (4.24)), from (4.3) we deduce that

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left| \left(u_\eta^m\right)_t(x,t) \right|^2 \rho_\eta(x) \mathrm{d}x \, \mathrm{d}t \le \left(\frac{2m}{m+1}\right)^2 \|u_0\|_\infty^{m-1} C \quad \forall t_2 > t_1 > 0 \,. \tag{4.27}$$

Moreover, the validity of

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^d} \left|u_\eta^m(x,t)\right|^2 \rho_\eta(x) \mathrm{d}x \, \mathrm{d}t \le C' \quad \forall t_2 > t_1 \ge 0 \tag{4.28}$$

for another suitable positive constant C' that depends only on t_1, t_2 and u_0 is ensured by the conservation of mass (4.1) (with $|x|^{-\gamma}$ replaced by ρ_η), which yields

$$\|u_\eta(t)\|_{1,\rho_\eta} = \|u_0\|_{1,\rho_\eta} \le \|u_0\|_{1,\gamma} \quad \forall t > 0 \,,$$

and by the uniform boundedness of u_η given by (4.24). Thanks to (4.26), (4.27) and (4.28) we are in position to apply Lemma 4.2 with the choice $v = u_\eta^m$. In place of the weight ρ there, exploiting the monotonicity of $\{\rho_\eta\}$, we can pick for instance ρ_1. Hence, estimate (4.11) and the fact that $H^s(\mathbb{R}^{d+1})$ is compactly embedded in $L^2_{loc}(\mathbb{R}^{d+1})$ (see e.g. [18, Th. 7.1]) imply that, up to subsequences, $\{u_\eta\}$ converges at least pointwise to some limit function u as $\eta \to 0$. Furthermore, from (4.26) we deduce that $\{u_\eta^m\}$ admits (still up to subsequences) a weak limit w in $L^2((0,T); \dot{H}^s(\mathbb{R}^d))$ for all $T > 0$. The identification between w and u^m is just a consequence of the pointwise convergence of $\{u_\eta\}$ to u. We can therefore pass to the limit in the weak formulation (4.25) and obtain that such u satisfies

$$\begin{aligned} & -\int_0^T \int_{\mathbb{R}^d} u(x,t)\varphi_t(x,t) \, |x|^{-\gamma} \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x,t) \, (-\Delta)^{\frac{s}{2}}(\varphi)(x,t) \, \mathrm{d}x \, \mathrm{d}t \\ & = \int_{\mathbb{R}^d} u_0(x)\varphi(x,0) \, |x|^{-\gamma} \mathrm{d}x \end{aligned} \tag{4.29}$$

for all $T>0$ and $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$ such that $\varphi(T)=0$. In fact, the passage to the limit in the first integral on the l.h.s. of (4.25) is justified by dominated convergence (there holds (4.24) and the weight $|x|^{-\gamma}$ is locally integrable), while in the second integral one directly exploits the weak convergence of $\{u_\eta^m\}$ to u^m in $L^2((0,T);\dot{H}^s(\mathbb{R}^d))$. Finally, on the r.h.s. one uses again dominated convergence. As remarked in the beginning of the proof, it is not difficult to show that (4.29) implies that u is a weak solution also in the sense of Definition 3.1. Indeed, the only nontrivial point is to show that (3.4) holds true. In order to do it, one proceeds similarly to the proof of Proposition 4.1. We omit details, but recall that the idea is to plug into (4.29) the test function $\varphi(x,t)=\vartheta(t)\phi(x)$, where ϕ is either a function of $\mathcal{D}(\mathbb{R}^d)$ or an approximation of 1, while ϑ is a regular approximation of $\chi_{[0,t_2]}$. Then, one lets $t_2 \to 0$.
As concerns (4.20) and (4.21), they can be obtained reasoning exactly as we did for the proof of (4.26) and (4.3) (we use again the fact that solutions are strong, see Sections 4.1 and 4.2). □

Having at our disposal an existence result for problem (4.7), we can let $|x|^{-\gamma}u_0$ approximate μ. In order to show that the corresponding solutions converge to a solution of (1.1), we need first some technical Lemmas.

The next result is a slight modification (in the hypotheses) of the classical Stroock-Varopoulos inequality. A simple proof of the latter (with different assumptions on the function v below), which exploits the extension in the upper plane, can be found in [17, Sect. 5]. See also [12, formula (2.2.7)] for a similar inequality involving general Dirichlet forms.

Lemma 4.4. *Let $d>2s$. For all nonnegative $v \in L^\infty(\mathbb{R}^d)\cap \dot{H}^s(\mathbb{R}^d)$ such that $(-\Delta)^s(v) \in L^1(\mathbb{R}^d)$, the inequality*

$$\int_{\mathbb{R}^d} v^{q-1}(x)(-\Delta)^s(v)(x)\,\mathrm{d}x \geq \frac{4(q-1)}{q^2}\int_{\mathbb{R}^d}\left|(-\Delta)^{\frac{s}{2}}(v^{\frac{q}{2}})(x)\right|^2\mathrm{d}x \tag{4.30}$$

holds true for any $q>1$.

Proof. We shall assume, with no loss of generality, that v is a regular function. Indeed, by standard mollification arguments, one can always pick a sequence $\{v_n\}\subset C^\infty(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\cap \dot{H}^s(\mathbb{R}^d)$ such that $\{v_n\}$ converges pointwise to v, $\|v_n\|_\infty \leq \|v\|_\infty$ and $\{(-\Delta)^s(v_n)\}$ converges to $(-\Delta)^s(v)$ in $L^1(\mathbb{R}^d)$. This suffices to pass to the limit as $n\to\infty$ on the l.h.s. of (4.30), while on the r.h.s. one exploits the weak lower semi-continuity of the L^2 norm.
Consider now the following sequences of functions:

$$\psi_n(x)=\int_0^{x\wedge\frac{1}{n}} y^{\frac{4s}{d-2s}}\,\mathrm{d}y+(q-1)\int_{\frac{1}{n}}^{x\vee\frac{1}{n}} y^{q-2}\,\mathrm{d}y \quad \forall x\in\mathbb{R}^+,$$

$$\Psi_n(x)=\int_0^{x\wedge\frac{1}{n}} y^{\frac{2s}{d-2s}}\,\mathrm{d}y+(q-1)^{\frac{1}{2}}\int_{\frac{1}{n}}^{x\vee\frac{1}{n}} y^{\frac{q}{2}-1}\,\mathrm{d}y \quad \forall x\in\mathbb{R}^+.$$

It is plain that ψ_n and Ψ_n are absolutely continuous, monotone increasing functions such that

$$\psi_n'(x)=\left[\Psi_n'(x)\right]^2 \quad \forall x\in\mathbb{R}^+.$$

For any $R>0$, take a cut-off function ξ_R as in Lemma A.3 of Appendix A. To the function $\xi_R v$ one can apply Lemma 5.2 of [17] with the choices $\psi=\psi_n$ and $\Psi=\Psi_n$, which yields

$$\int_{\mathbb{R}^d}\psi_n(\xi_R v)(x)\,(-\Delta)^s(\xi_R v)(x)\,\mathrm{d}x \geq \int_{\mathbb{R}^d}\left|(-\Delta)^{\frac{s}{2}}(\Psi_n(\xi_R v))(x)\right|^2\mathrm{d}x\,. \tag{4.31}$$

Expanding the s-Laplacian of the product of two functions, we get that the l.h.s. of (4.31) equals

$$\begin{aligned}&\int_{\mathbb{R}^d}\psi_n(\xi_R v)(x)\,\xi_R(x)(-\Delta)^s(v)(x)\,\mathrm{d}x+\int_{\mathbb{R}^d}\psi_n(\xi_R v)(x)(-\Delta)^s(\xi_R)(x)v(x)\,\mathrm{d}x\\&+2\,C_{d,s}\int_{\mathbb{R}^d}\psi_n(\xi_R v)(x)\int_{\mathbb{R}^d}\frac{(\xi_R(x)-\xi_R(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned} \tag{4.32}$$

By dominated convergence,

$$\lim_{R\to\infty}\int_{\mathbb{R}^d}\psi_n(\xi_R v)(x)\,\xi_R(x)(-\Delta)^s(v)(x)\,\mathrm{d}x=\int_{\mathbb{R}^d}\psi_n(v)(x)(-\Delta)^s(v)(x)\,\mathrm{d}x\,.$$

Our aim is to show that the other two integrals in (4.32) go to zero as $R\to\infty$. We have:

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\psi_n(\xi_R v)(x)(-\Delta)^s(\xi_R)(x)v(x)\,\mathrm{d}x\right|\\&\le\|(-\Delta)^s(\xi_R)\|_\infty\left(\frac{d-2s}{d+2s}\int_{\{v\le\frac1n\}}v^{\frac{2d}{d-2s}}(x)\,\mathrm{d}x+\psi_n(\|v\|_\infty)\|v\|_\infty\int_{\{v>\frac1n\}}\mathrm{d}x\right)\end{aligned}\tag{4.33}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\psi_n(\xi_R v)(x)\int_{\mathbb{R}^d}\frac{(\xi_R(x)-\xi_R(y))(v(x)-v(y))}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x\right|\\&\le\|v\|_{\dot H^s}\left(\int_{\mathbb{R}^d}[\psi_n(\xi_R v)(x)]^2\int_{\mathbb{R}^d}\frac{(\xi_R(x)-\xi_R(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}y\,\mathrm{d}x\right)^{\frac12}\\&\le\|v\|_{\dot H^s}\,\|l(\xi_R)\|_\infty^{\frac12}\left(\left[\frac{d-2s}{d+2s}\right]^2\int_{\{v\le\frac1n\}}v^{2\frac{d+2s}{d-2s}}(x)\,\mathrm{d}x+[\psi_n(\|v\|_\infty)]^2\int_{\{v>\frac1n\}}\mathrm{d}x\right)^{\frac12},\end{aligned}\tag{4.34}$$

where $l(\cdot)$ is defined in Lemma A.2. Thanks to the scaling properties of both $(-\Delta)^s(\xi_R)$ and $l(\xi_R)$ (see again Lemma A.3), it is immediate to check that $\lim_{R\to\infty}\|(-\Delta)^s(\xi_R)\|_\infty=\lim_{R\to\infty}\|l(\xi_R)\|_\infty=0$. Moreover, notice that $v\in L^{\frac{2d}{d-2s}}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ (see [18, Sect. 6] or Lemma 4.5 below). In particular, v also belongs to $L^{2\frac{d+2s}{d-2s}}(\mathbb{R}^d)$. Thus, letting $R\to\infty$ in (4.33) and (4.34), we deduce that the last two integrals in (4.32) vanish, so that we can pass to the limit on the l.h.s. of (4.31). On the r.h.s. we just use the fact that $(-\Delta)^{\frac s2}(\Psi_n(\xi_R v))$ converges to $(-\Delta)^{\frac s2}(\Psi_n(v))$ weakly in $L^2(\mathbb{R}^d)$. This proves the validity of

$$\int_{\mathbb{R}^d}\psi_n(v)(x)(-\Delta)^s(v)(x)\,\mathrm{d}x\ge\int_{\mathbb{R}^d}\left|(-\Delta)^{\frac s2}(\Psi_n(v))(x)\right|^2\mathrm{d}x\,.\tag{4.35}$$

The final step is to let $n\to\infty$ in (4.35). It is clear that the sequence $\{\psi_n(x)\}$ converges locally uniformly to the function x^{q-1}, while $\{\Psi_n(x)\}$ converges locally uniformly to $2(q-1)^{\frac12}x^{\frac q2}/q$. Hence, $\{\psi_n(v)\}$ and $\{\Psi_n(v)\}$ converge in $L^\infty(\mathbb{R}^d)$ to v^{q-1} and $2(q-1)^{\frac12}v^{\frac q2}/q$, respectively. This is enough in order to pass to the limit in (4.35) and obtain (4.30). □

Lemma 4.5. *Let $d>2s$ and $\gamma\in[0,2s)$. There exists a positive constant $C_{CKN}=C_{CKN}(\gamma,s,d)$ such that the Caffarelli-Kohn-Nirenberg type inequalities*

$$\|v\|_{q,-\gamma}\le C_{CKN}\left\|(-\Delta)^{\frac s2}(v)\right\|_2^{\frac{1}{\alpha+1}}\|v\|_{p,-\gamma}^{\frac{\alpha}{\alpha+1}}\quad\forall v\in L^p_{-\gamma}(\mathbb{R}^d)\cap\dot H^s(\mathbb{R}^d)\tag{4.36}$$

hold true for any $\alpha\ge0$, $p\ge1$ and

$$q=2\frac{(d-\gamma)(\alpha+1)}{(d-\gamma)\frac{\alpha}{p}+d-2s}\,.$$

For $\alpha=0$ one recovers the fractional Sobolev inequalities

$$\|v\|_{2\frac{d-\gamma}{d-2s},-\gamma}\le C_S\left\|(-\Delta)^{\frac s2}(v)\right\|_2\quad\forall v\in\dot H^s(\mathbb{R}^d)\,.\tag{4.37}$$

Proof. Inequality (4.36) is just a particular case of [11, Th. 1.8]. Alternatively, one can prove it by interpolating between the fractional Sobolev inequality (4.37) in the case $\gamma=0$ (see [18, Th. 6.5]) and the fractional Hardy inequality (see e.g. [21] and references quoted therein)

$$\|v\|_{2,-2s}\le C_H\left\|(-\Delta)^{\frac s2}(v)\right\|_2\quad\forall v\in\dot H^s(\mathbb{R}^d)\,.$$

□

Lemmas 4.4 and 4.5 provide us with functional inequalities which are key in order to prove the following smoothing effect for solutions to (4.7), a result which is in turn crucial for the rest of this Section. The strategy of proof is standard and we stress the main points only.

Proposition 4.6. *Let $d > 2s$ and $\gamma \in (0, 2s)$. There exists a constant $K > 0$ depending only on m, γ, s, d such that, for all nonnegative initial datum $u_0 \in L^1_{-\gamma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and the corresponding weak solution u to (4.7) constructed in Lemma 4.3, the following $L^{p_0}_{-\gamma}$-L^∞ smoothing estimate holds true for any $p_0 \in [1, \infty)$:*

$$\|u(t)\|_\infty \le K\, t^{-\alpha}\, \|u_0\|_{p_0,-\gamma}^{\beta} \qquad \forall t > 0\,, \tag{4.38}$$

where

$$\alpha = \frac{d-\gamma}{(m-1)(d-\gamma) + (2s-\gamma)p_0}\,, \quad \beta = \frac{(2s-\gamma)p_0}{(m-1)(d-\gamma) + (2s-\gamma)p_0}\,. \tag{4.39}$$

Proof. We proceed exactly as in [17, Th. 8.2], i.e. by means of a standard parabolic Moser iteration, so we stress just the main steps for the convenience of the reader. First of all, let us fix any $t > 0$ and consider the time sequence $t_k := (1 - 2^{-k})\, t$. Let us also denote as $\{p_k\} \subset (1, \infty)$ another numerical sequence to be chosen later. By multiplying the differential equation in (4.7) by $u^{p_k - 1}(x, t)$, integrating over $\mathbb{R}^d \times [t_k, t_{k+1}]$, applying Lemma 4.4 to the function $v = u^m$ (with the choice $q = (p_k + m - 1)/m$) and exploiting the fact that the $L^p_{-\gamma}$ norms do not increase along the evolution (see Section 4.2), we get:

$$\|u(t_k)\|_{p_k,-\gamma}^{p_k} \ge \frac{c_k}{\|u(t_k)\|_{p_k,-\gamma}^{p_k}} \int_{t_k}^{t_{k+1}} \left\| (-\Delta)^{\frac{s}{2}} \left(u^{\frac{p_k+m-1}{2}}(\tau) \right) \right\|_2^2 \|u(\tau)\|_{p_k,-\gamma}^{p_k} \, \mathrm{d}\tau\,, \tag{4.40}$$

where $c_k = 4mp_k(p_k - 1)/(p_k + m - 1)^2$. The above computations are justified since, as we recall in Section 4.1, our solutions are strong. In particular both sides of the differential equation in (4.7) belong to $L^1(\mathbb{R}^d)$.
Now note that, using (4.36) with the choices $p = 2p_k/(p_k + m - 1)$ and $\alpha = p_k/(p_k + m - 1)$, we obtain:

$$\left\| (-\Delta)^{\frac{s}{2}} \left(u^{\frac{p_k+m-1}{2}}(\tau) \right) \right\|_2^2 \|u(\tau)\|_{p_k,-\gamma}^{p_k} \ge C_{CKN}^{-2\frac{2p_k+m-1}{p_k+m-1}} \|u(\tau)\|_{\frac{(d-\gamma)(2p_k+m-1)}{2d-\gamma-2s},-\gamma}^{2p_k+m-1}\,. \tag{4.41}$$

Thanks to (4.41) and again the fact that the $L^p_{-\gamma}$ norms of $u(\tau)$ do not grow, we have:

$$\int_{t_k}^{t_{k+1}} \left\| (-\Delta)^{\frac{s}{2}} \left(u^{\frac{p_k+m-1}{2}}(\tau) \right) \right\|_2^2 \|u(\tau)\|_{p_k,-\gamma}^{p_k} \, \mathrm{d}\tau \ge C_{CKN}^{-2\frac{2p_k+m-1}{p_k+m-1}} 2^{-(k+1)} t \, \|u(t_{k+1})\|_{\frac{(d-\gamma)(2p_k+m-1)}{2d-\gamma-2s},-\gamma}^{2p_k+m-1}\,. \tag{4.42}$$

Gathering (4.40) and (4.42) we get the recursive inequality

$$\|u(t_{k+1})\|_{p_{k+1},-\gamma} \le \left(\frac{2^{k+1}\, C_{CKN}^{2\frac{2p_k+m-1}{p_k+m-1}}}{c_k\, t} \right)^{\frac{\sigma}{2p_{k+1}}} \|u(t_k)\|_{p_k,-\gamma}^{\sigma \frac{p_k}{p_{k+1}}}\,,$$

where

$$p_{k+1} = \frac{\sigma}{2}\,(2p_k + m - 1)\,, \quad \sigma = \frac{2(d-\gamma)}{2d - \gamma - 2s}\,.$$

Observe that, since $0 < \gamma < 2s$, $\sigma > 1$. Furthermore, if we take $p_0 > 1$, it is easy to check that

$$p_k = A(\sigma^k - 1) + p_0\,, \quad A = p_0 + \frac{(d-\gamma)(m-1)}{2s - \gamma} > 0\,,$$

whence $p_{k+1} > p_k$ and $\lim_{k\to\infty} p_k = \infty$. Thus, upon setting $U_k := \|u(t_k)\|_{p_k,-\gamma}$, one can find a constant $c_0 = c_0(p_0, \gamma, m, s, d) > 0$ (in particular, independent of k) such that

$$U_{k+1} \le c_0^{\frac{k}{p_{k+1}}}\, t^{-\frac{\sigma}{2p_{k+1}}}\, U_k^{\sigma \frac{p_k}{p_{k+1}}}\,. \tag{4.43}$$

Iterating (4.43) yields

$$U_k \leq c_0^{\frac{1}{p_k}\sum_{j=1}^{k-1}(k-j)\sigma^j}\, t^{-\frac{1}{2p_k}\sum_{j=1}^{k}\sigma^j}\, U_0^{\sigma^k \frac{p_0}{p_k}} . \tag{4.44}$$

Letting $k \to \infty$ in (4.44) one infers the validity of

$$\|u(t)\|_\infty \leq K'\, t^{-\frac{d-\gamma}{(m-1)(d-\gamma)+(2s-\gamma)p_0}}\, \|u_0\|_{p_0,-\gamma}^{\frac{(2s-\gamma)p_0}{(m-1)(d-\gamma)+(2s-\gamma)p_0}} \tag{4.45}$$

for some positive constant $K' = K'(p_0,\gamma,m,s,d) > 0$. However, estimate (4.45) only holds for $p_0 > 1$, and to go down to $p_0 = 1$ one must proceed as in the proof of [17, Cor. 8.1]. □

Before proving the existence Theorem 3.2, we still need two technical lemmas concerning Riesz potentials.

Lemma 4.7. *Let $d > 2s$ and $\phi : \mathbb{R}^d \to \mathbb{R}$ be a continuous function which belongs to $L^1(\mathbb{R}^d)$ and decays at least like $|x|^{-d}$ as $|x| \to \infty$. Then, the convolution $I_{2s} * \phi$ (namely, the Riesz potential of ϕ) is also a continuous function, decaying at least like $|x|^{-d+2s}$ as $|x| \to \infty$.*

Proof. The fact that $(I_{2s} * \phi)$ is continuous easily follows from continuity and integrability properties of both I_{2s} and ϕ. In order to prove the claimed decay behaviour as $|x| \to \infty$ we have to work a bit more. To begin with, let us split the convolution in this way:

$$(I_{2s} * \phi)(x) = \int_{\mathbb{R}^d} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y = \underbrace{\int_{B_{2|x|}(0)} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y}_{F_1(x)} + \underbrace{\int_{B_{2|x|}^c(0)} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y}_{F_2(x)} .$$

As concerns F_2, we have:

$$|F_2(x)| = \left| \int_{B_{2|x|}^c(0)} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y \right| \leq 2^{d-2s}\, C\, k_{d,s} \int_{B_{2|x|}^c(0)} \frac{1}{|y|^{2d-2s}}\, \mathrm{d}y \leq \frac{C\, k_{d,s}\, d|B_1|}{(d-2s)|x|^{d-2s}} , \tag{4.46}$$

where we used the inequalities

$$|\phi(y)| \leq \frac{C}{|y|^d} \quad \forall y \in \mathbb{R}^d , \quad |x-y| \geq \frac{|y|}{2} \quad \forall y \in B_{2|x|}^c(0) ,$$

valid for some $C > 0$. On the other, hand F_1 can be handled as follows:

$$|F_1(x)| = \left| \int_{B_{2|x|}(0)} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y \right| \leq \left| \int_{B_{\frac{|x|}{2}}(x)} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y \right| + \left| \int_{B_{2|x|}(0)\setminus B_{\frac{|x|}{2}}(x)} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y \right| .$$

Since

$$\left| \int_{B_{\frac{|x|}{2}}(x)} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y \right| \leq \frac{2^d C\, k_{d,s}}{|x|^d} \left| \int_{B_{\frac{|x|}{2}}(x)} \frac{1}{|x-y|^{d-2s}}\, \mathrm{d}y \right| \leq \frac{2^{d-2s-1} C\, k_{d,s}\, d|B_1|}{s|x|^{d-2s}} \tag{4.47}$$

and

$$\left| \int_{B_{2|x|}(0)\setminus B_{\frac{|x|}{2}}(x)} \frac{k_{d,s}\,\phi(y)}{|x-y|^{d-2s}}\, \mathrm{d}y \right| \leq \frac{2^{d-2s} k_{d,s}}{|x|^{d-2s}}\, \|\phi\|_1 , \tag{4.48}$$

by gathering (4.46), (4.47) and (4.48) we finally deduce that $(I_{2s} * \phi)(x)$ decays at least like $|x|^{-d+2s}$ as $|x| \to \infty$. □

Lemma 4.8. *Let $d > 2s$, $\gamma \in (0, 2s)$, $v \in L^1_{-\gamma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and U_γ^v be the Riesz potential of $|x|^{-\gamma} v$, that is*

$$U_\gamma^v = I_{2s} * \left(|x|^{-\gamma} v\right) . \tag{4.49}$$

The following properties hold true:

- U_γ^v *belongs to* $C(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ *for all* p *such that*

$$p \in \left(\frac{d}{d-2s}, \infty\right]. \tag{4.50}$$

- *Under the additional condition*

$$\gamma < d-2s\,, \tag{4.51}$$

one has that $U_\gamma^v \in W^{r,p}(\mathbb{R}^d)$ *for any* $r\in(0,2s)$ *and* p *such that*

$$p \in \left(\frac{d}{d-2s}, \frac{d}{\gamma}\right). \tag{4.52}$$

In all of the above cases, the norms $\|U_\gamma^v\|_{L^p(\mathbb{R}^d)}$ *and* $\|U_\gamma^v\|_{W^{r,p}(\mathbb{R}^d)}$ *can be bounded from above by a constant that depends on* v *only through* $\|v\|_{1,-\gamma}$ *and* $\|v\|_\infty$*.*

Proof. In order to prove that U_γ^v belongs to $C(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ for all p complying with (4.50), it is convenient to split the convolution (4.49) as follows:

$$U_\gamma^v(x) = \underbrace{\int_{B_1(0)} |y|^{-\gamma} v(y)\, I_{2s}(x-y)\,\mathrm{d}y}_{U_{\gamma,1}^v(x)} + \underbrace{\int_{\mathbb{R}^d} \chi_{B_1^c(0)}(y)\,|y|^{-\gamma} v(y)\, I_{2s}(x-y)\,\mathrm{d}y}_{U_{\gamma,2}^v(x)}\,. \tag{4.53}$$

Exploiting the fact that $v\in L^\infty(\mathbb{R}^d)$ and $\gamma<2s$ (so that $|y|^{-d+2s-\gamma}$ is locally integrable), it is easily seen that $U_{\gamma,1}^v(x)$ is a continuous function which decays at least like $|x|^{-d+2s}$ as $|x|\to\infty$. In particular, it belongs to $L^p(\mathbb{R}^d)$ for all p satisfying (4.50). As concerns the second integral on the r.h.s. of (4.53), notice that since $v\in L^1_{-\gamma}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ we have that the function $\chi_{B_1^c(0)}|y|^{-\gamma}v$ belongs to $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Hence, thanks to the properties of the Riesz kernel I_{2s}, it is easy to check that even $U_{\gamma,2}^v$ is a continuous function. In order to prove that it belongs to $L^p(\mathbb{R}^d)$ for all p satisfying (4.50), let us first write it in this way:

$$U_{\gamma,2}^v = \left(\chi_{B_1(0)}\, I_{2s}\right) * \left(\chi_{B_1^c(0)}|y|^{-\gamma} v\right) + \left(\chi_{B_1^c(0)}\, I_{2s}\right) * \left(\chi_{B_1^c(0)}|y|^{-\gamma} v\right). \tag{4.54}$$

Since $\chi_{B_1(0)}\, I_{2s}\in L^1(\mathbb{R}^d)$ and we have just seen that $\chi_{B_1^c(0)}|y|^{-\gamma}v\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, the first convolution in (4.54) belongs to $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Using the fact that $\chi_{B_1^c(0)}\, I_{2s}\in L^p(\mathbb{R}^d)$ for all p satisfying (4.50) and in particular $\chi_{B_1^c(0)}|y|^{-\gamma}v\in L^1(\mathbb{R}^d)$, we infer that the second convolution in (4.54) belongs to $L^p(\mathbb{R}^d)$ for all p satisfying (4.50). The latter property is then inherited by $U_{\gamma,2}^v$.

Now we aim at proving the second part of the lemma. To begin with, we need to establish for which values of $p\geq 1$ the function $|x|^{-\gamma}v$ belongs to $L^p(\mathbb{R}^d)$. We have:

$$\begin{aligned}\left\||x|^{-\gamma}v\right\|_p^p &= \int_{\mathbb{R}^d}\left||x|^{-\gamma}v(x)\right|^p \mathrm{d}x = \int_{B_1(0)} |v(x)|^p\, |x|^{-\gamma p}\,\mathrm{d}x + \int_{B_1^c(0)} |v(x)|^p\, |x|^{-\gamma p}\,\mathrm{d}x \\ &\leq \|v\|_\infty^{p-1}\left(\|v\|_\infty \int_{B_1(0)} |x|^{-\gamma p}\,\mathrm{d}x + \int_{B_1^c(0)} |v(x)|\, |x|^{-\gamma}\,\mathrm{d}x\right).\end{aligned} \tag{4.55}$$

The last line of (4.55) is finite provided $|x|^{-\gamma p}$ is integrable in $B_1(0)$, namely for $p<d/\gamma$. From Proposition 3.1.7 and Theorem 1.1.1 of [1] it is immediate to deduce that, whenever a (nonnegative) function f and its potential $I_{2s}*f$ belong to the same $L^p(\mathbb{R}^d)$ space for some $p\in[1,\infty)$, then the potential actually belongs to $W^{r,p}(\mathbb{R}^d)$ for all $r\in(0,2s)$, with estimates on the corresponding $W^{r,p}$ norm that depend on f and $I_{2s}*f$ only through $\|f\|_p$ and $\|I_{2s}*f\|_p$. Thanks to the integrability properties of $|x|^{-\gamma}v$ and U_γ^v we proved above, it is clear that for any p complying with (4.52) both $|x|^{-\gamma}v$ and U_γ^v belong to $L^p(\mathbb{R}^d)$ and so $U_\gamma^v\in W^{r,p}(\mathbb{R}^d)$ for all $r\in(0,2s)$. Condition (4.51) is necessary and sufficient to prevent that the interval in (4.52) is empty.

Finally, the fact that in all of the cases the norms $\|U_\gamma^v\|_{L^p(\mathbb{R}^d)}$ and $\|U_\gamma^v\|_{W^{r,p}(\mathbb{R}^d)}$ can be bounded from above by constants depending on v only through $\|v\|_{-1,\gamma}$ and $\|v\|_\infty$ is just a consequence of the above computations. □

Proof of Theorem 3.2. We start dealing with the case of a measure μ which is also compactly supported. In order to construct solutions to problem (1.1) for such a μ, the idea is to exploit the existence result provided by Lemma 4.3. That is, we first consider the family of weak solutions that take on the regular initial data $\mu_\varepsilon = \psi_\varepsilon * \mu$ (let $\varepsilon > 0$), where

$$\psi_\varepsilon = \frac{1}{\varepsilon^d}\psi\left(\frac{x}{\varepsilon}\right)$$

and ψ is a nonnegative function of $\mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}\psi = 1$. Namely, $\{\psi_\varepsilon\}$ is a regular approximation of the Dirac δ and so $\{\mu_\varepsilon\}$ is a regular approximation (the mollification) of μ. It is straightforward to check that $\mu_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ (recall that μ is compactly supported) and

$$\|\mu_\varepsilon\|_1 = \int_{\mathbb{R}^d} \mu_\varepsilon(x)\,\mathrm{d}x = \mu(\mathbb{R}^d)\,. \tag{4.56}$$

We shall denote as u_ε the weak solution to (1.1) corresponding to the initial datum μ_ε, whose existence is ensured by Lemma 4.3 (with $u_0 = |x|^\gamma \mu_\varepsilon$). What follows in the proof aims at showing that, as $\varepsilon \to 0$, $\{u_\varepsilon\}$ suitably converges to a weak solution of (1.1) starting from μ.
The first problem we consider is the convergence of $\{u_\varepsilon\}$ (up to subsequences) to a certain function u which satisfies (3.1), (3.2) and (3.3). Afterwards we shall come to the initial condition (3.4). To this end, the idea is to exploit estimates (4.20) and (4.21) (with u replaced by u_ε) from Lemma 4.3. Combining the smoothing effect (4.38) with (4.56) and the conservation of mass (4.1), we obtain:

$$\begin{aligned}\int_{\mathbb{R}^d} u_\varepsilon^{m+1}(x,t)\,|x|^{-\gamma}\mathrm{d}x \le \|u_\varepsilon(t)\|_\infty^m \int_{\mathbb{R}^d} u_\varepsilon(x,t)\,|x|^{-\gamma}\mathrm{d}x = \|u_\varepsilon(t)\|_\infty^m\,\|\mu_\varepsilon\|_1 \\ \le K^m\,t^{-\alpha m}\,\mu(\mathbb{R}^d)^{1+\beta m}\end{aligned} \tag{4.57}$$

for all $t > 0$. Hence, using (4.20), (4.21) and (4.57) (evaluated at $t = t_1$ and $t = t_*$) we get the validity of the following energy estimates:

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left|(-\Delta)^{\frac{s}{2}}\left(u_\varepsilon^m\right)(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} u_\varepsilon^{m+1}(x,t_2)\,|x|^{-\gamma}\mathrm{d}x \le K^m\,t_1^{-\alpha m}\,\mu(\mathbb{R}^d)^{1+\beta m}\,, \tag{4.58}$$

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d} \left|(z_\varepsilon)_t\,(x,t)\right|^2\,|x|^{-\gamma}\mathrm{d}x\,\mathrm{d}t \le C \tag{4.59}$$

for all $t_2 > t_1 > 0$, where $z_\varepsilon = u_\varepsilon^{\frac{m+1}{2}}$ and C is a positive constant that depends on t_1, t_2 and $\mu(\mathbb{R}^d)$ but is independent of ε. Thanks to (4.58), (4.59) and the smoothing effect (which, in particular, bounds $\{u_\varepsilon\}$ in $L^\infty(\mathbb{R}^d\times(\tau,\infty))$ for all $\tau > 0$ independently of ε), we are allowed to proceed exactly as in the proof of Lemma 4.3. That is, we obtain that the pointwise limit u of $\{u_\varepsilon\}$ satisfies (3.1) (consequence of the smoothing effect and the conservation of mass), (3.2) (consequence of (4.58)) and, passing to the limit (up to subsequences) in the weak formulation (3.3) solved by u_ε,

$$\begin{gathered}-\int_0^\infty\int_{\mathbb{R}^d} u(x,t)\varphi_t(x,t)\,|x|^{-\gamma}\mathrm{d}x\,\mathrm{d}t + \int_0^\infty\int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}}(u^m)(x,t)\,(-\Delta)^{\frac{s}{2}}\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0 \\ \forall\varphi\in C_c^\infty(\mathbb{R}^d\times(0,\infty))\,.\end{gathered} \tag{4.60}$$

Notice that we cannot pass to the limit directly in the stronger weak formulation (4.29): the problem is that estimate (4.58) blows up as $t_1 \to 0$. Hence, we can only take test functions which are compactly supported in $\mathbb{R}^d\times(0,\infty)$. In particular, (4.60) does not provide any information over the initial datum assumed by $u(x,t)$. In order to prove that such initial datum is indeed μ (in the sense of (3.4)) we have to work more and exploit some results in potential theory, following [33] or [41].
To begin with, let us introduce the Riesz potential $U_\varepsilon(t)$ of $|x|^{-\gamma}u_\varepsilon(t)$, A first crucial point is to check the differential equation solved by U_ε. Note that, formally, there holds

$$|x|^{-\gamma}(u_\varepsilon)_t(x,t) = -\left(-\Delta\right)^s(u_\varepsilon^m)(x,t) \quad \forall(x,t)\in\mathbb{R}^d\times(0,\infty) \tag{4.61}$$

(a posteriori (4.61) is rigorous at least in $L^1(\mathbb{R}^d)$, see Section 4.1). Hence, still at a formal level, we can apply to both sides of (4.61) the operator $(-\Delta)^{-s}$, namely the convolution against the Riesz kernel I_{2s} (recall the discussion in Section 2), which yields

$$(U_\varepsilon)_t\,(x,t) = -u_\varepsilon^m(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times (0,\infty)\,. \tag{4.62}$$

Now we prove (4.62) rigorously. For any given $t_2 > t_1 > 0$ and for any given function $\phi \in \mathcal{D}(\mathbb{R}^d)$, let us plug into (3.3) (with $u = u_\varepsilon$) the test function $\varphi(x,t) = \vartheta(t)\phi(x)$, where ϑ is a smooth and compactly supported approximation of $\chi_{[t_1,t_2]}$ (we follow [33, proof of Lemma 2]). Integrating by parts (in space) and letting ϑ tend to $\chi_{[t_1,t_2]}$ (and also using Lebesgue differentiation Theorem for vector valued functions) we get

$$\int_{\mathbb{R}^d} u_\varepsilon(y,t_2)\phi(y)\,|y|^{-\gamma}\mathrm{d}y - \int_{\mathbb{R}^d} u_\varepsilon(y,t_1)\phi(y)\,|y|^{-\gamma}\mathrm{d}y = -\int_{\mathbb{R}^d}\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)(-\Delta)^s(\phi)(y)\,\mathrm{d}y\,. \tag{4.63}$$

For any fixed $x \in \mathbb{R}^d$ we can replace in (4.63) the function $\phi(y)$ by $\phi(y+x)$, thus obtaining

$$\begin{aligned}&\int_{\mathbb{R}^d} u_\varepsilon(y,t_2)\phi(y+x)\,|y|^{-\gamma}\mathrm{d}y - \int_{\mathbb{R}^d} u_\varepsilon(y,t_1)\phi(y+x)\,|y|^{-\gamma}\mathrm{d}y \\ &= -\int_{\mathbb{R}^d}\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)(-\Delta)^s(\phi)(y+x)\,\mathrm{d}y\,.\end{aligned} \tag{4.64}$$

Integrating (4.64) against the Riesz kernel $I_{2s}(x)$ gives

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^{-\gamma}u_\varepsilon(y,t_2)\phi(y+x)I_{2s}(x)\,\mathrm{d}y\,\mathrm{d}x - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^{-\gamma}u_\varepsilon(y,t_1)\phi(y+x)I_{2s}(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)(-\Delta)^s(\phi)(y+x)I_{2s}(x)\,\mathrm{d}y\,\mathrm{d}x\,,\end{aligned} \tag{4.65}$$

whence (let $z = y + x$)

$$\begin{aligned}&\int_{\mathbb{R}^d} U_\varepsilon(z,t_2)\phi(z)\,\mathrm{d}z - \int_{\mathbb{R}^d} U_\varepsilon(z,t_1)\phi(z)\,\mathrm{d}z \\ &= -\int_{\mathbb{R}^d}\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)\left(\int_{\mathbb{R}^d}(-\Delta)^s(\phi)(y+x)I_{2s}(x)\,\mathrm{d}x\right)\mathrm{d}y = -\int_{\mathbb{R}^d}\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)\phi(y)\,\mathrm{d}y\,.\end{aligned} \tag{4.66}$$

The exchange of order of integration between (4.65) and (4.66) (that is, the application of Fubini Theorem) is justified since

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left||y|^{-\gamma}u_\varepsilon(y,t)\phi(y+x)I_{2s}(x)\right|\mathrm{d}y\,\mathrm{d}x < \infty \quad t \in \{t_1,t_2\} \tag{4.67}$$

and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\left(\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\,\mathrm{d}t\right)(-\Delta)^s(\phi)(y+x)I_{2s}(x)\right|\mathrm{d}y\,\mathrm{d}x < \infty\,. \tag{4.68}$$

In fact, both the functions

$$y \to \int_{\mathbb{R}^d} |\phi(y+x)|I_{2s}(x)\,\mathrm{d}x \tag{4.69}$$

and

$$y \to \int_{\mathbb{R}^d} |(-\Delta)^s(\phi)(y+x)|I_{2s}(x)\,\mathrm{d}x \tag{4.70}$$

are continuous and decay at least like $|y|^{-d+2s}$ as $|y| \to \infty$. These are consequences of Lemma 4.7 (recall that $(-\Delta)^s(\phi)(y)$ is regular and decays at least like $|y|^{-d-2s}$ as $|y| \to \infty$, see Lemma A.1). In particular, (4.69) is bounded and in (4.67) is integrated against $|y|^{-\gamma}u_\varepsilon(t)$, an $L^1(\mathbb{R}^d)$ function. As concerns (4.70), we see that in (4.68) it is integrated against the function $\int_{t_1}^{t_2} u_\varepsilon^m(y,t)\mathrm{d}t$, which belongs to $L^1_{-\gamma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ thanks to (3.1). Hence, by the just remarked decay properties of (4.70), the integral in (4.68) is finite provided $d - 2s \geq \gamma$, which holds by hypothesis.

Clearly, the validity of (4.66) implies that $U_\varepsilon(x,t)$ has an absolutely continuous version (w.r.t. t) satisfying (4.62), from which we deduce that it is decreasing in time (a priori $U_\varepsilon(x,t_2) \le U_\varepsilon(x,t_1)$ for a.e. $x \in \mathbb{R}^d$, but Lemma 4.8 ensures that our potentials are continuous w.r.t. x, whence the inequality actually holds for *any* $x \in \mathbb{R}^d$). In particular, $U_\varepsilon(t)$ admits a pointwise limit as $t \to 0$. Since we also know that $|x|^{-\gamma} u_\varepsilon(t)$ converges to μ_ε in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ as $t \to 0$ (consequence of Lemma 4.3 and Definition 3.1), Theorem 3.8 of [28] guarantees the identification between such pointwise limit and the potential $U^{\mu_\varepsilon} = I_{2s} * \mu_\varepsilon$ of μ_ε:

$$\lim_{t\to 0} U_\varepsilon(x,t) = U^{\mu_\varepsilon}(x) \quad \text{for a.e. } x \in \mathbb{R}^d \, . \tag{4.71}$$

Now we need to deal with the convergence of $\{U_\varepsilon\}$ as $\varepsilon \to 0$. We have already mentioned the fact that, at any fixed $t > 0$, $U_\varepsilon(x,t)$ is a continuous function of x. However, we can exploit Lemma 4.8 more profitably. First of all recall that, thanks to the conservation of mass (4.1),

$$\|u_\varepsilon(t)\|_{1,-\gamma} = \mu(\mathbb{R}^d) \quad \forall t > 0 \, , \tag{4.72}$$

while from the smoothing effect (4.38) (for $p_0 = 1$) we deduce that

$$\|u_\varepsilon(t)\|_\infty \le K \, t^{-\alpha} \, \mu(\mathbb{R}^d)^\beta \quad \forall t > 0 \, . \tag{4.73}$$

This means that, for any $\tau > 0$, both $\sup_{t\ge\tau} \|u_\varepsilon(t)\|_{1,-\gamma}$ and $\sup_{t\ge\tau} \|u_\varepsilon(t)\|_\infty$ are uniformly bounded w.r.t. ε. Since $\gamma < d-2s$, applying Lemma 4.8 we infer that also $\sup_{t\ge\tau} \|U_\varepsilon(t)\|_{W^{r,p}(\mathbb{R}^d)}$ is uniformly bounded w.r.t. ε for any $r \in (0,2s)$ and p satisfying (4.52). From standard Hölder embeddings for fractional Sobolev spaces (see e.g. [18, Th. 8.2]), for r and p such that $r > d/p$ and λ defined as $\lambda = r - \frac{d}{p}$, we can uniformly (still w.r.t. ε) bound $\sup_{t\ge\tau} \|U_\varepsilon(t)\|_{C^\lambda(\Omega)}$ for any $\tau > 0$ and any $\Omega \Subset \mathbb{R}^d$. Notice that, in order to ensure that $r > d/p$ as requested, it is enough to choose r sufficiently close to $2s$ and p sufficiently close to d/γ (this is feasible because $\gamma < 2s$).
As concerns the time behaviour of $U_\varepsilon(x,t)$, the differential equation (4.62) and the smoothing estimate (4.73) imply that, for any $\tau > 0$, $\sup_{t\ge\tau} \|(U_\varepsilon)_t\|_{L^\infty(\mathbb{R}^d\times(\tau,\infty))}$ is also uniformly bounded w.r.t. ε. We can therefore conclude that, for any $t_2 > t_1 > 0$ and any $\Omega \Subset \mathbb{R}^d$, there holds

$$\sup_{\varepsilon>0} \|U_\varepsilon\|_{C^\lambda(\Omega\times(t_1,t_2))} < \infty \, . \tag{4.74}$$

In particular, (4.74) guarantees the existence of a function $U \in C^\lambda_{loc}(\mathbb{R}^d \times (0,\infty))$ such that, up to subsequences,

$$\lim_{\varepsilon\to 0} U_\varepsilon(x,t) = U(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times (0,\infty) \, . \tag{4.75}$$

As discussed above, we know that $u_\varepsilon(x,t)$ converges pointwise a.e. (still up to subsequences) to a function $u(x,t)$ which satisfies (3.1), (3.2) and (3.3). Thanks to the smoothing estimate (4.73), by dominated convergence we deduce that for a.e. $t > 0$ such convergence also takes place locally in $L^1_{-\gamma}(\mathbb{R}^d)$, which of course implies that

$$\lim_{\varepsilon\to 0} |x|^{-\gamma} u_\varepsilon(t) = |x|^{-\gamma} u(t) \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d)) \, .$$

Since $\|u_\varepsilon(t)\|_{1,-\gamma} = \mu(\mathbb{R}^d)$ (in particular, $\|u_\varepsilon(t)\|_{1,-\gamma}$ is uniformly bounded w.r.t. ε), using (4.75) and applying again Theorem 3.8 of [28] we deduce that

$$\lim_{\varepsilon\to 0} U_\varepsilon(x,t) = U(x,t) = \left(I_{2s} * \left(|y|^{-\gamma} u(t)\right)\right)(x) \quad \text{for a.e. } x \in \mathbb{R}^d \, , \tag{4.76}$$

namely it is possible to identify (almost everywhere) $U(x,t)$ with the potential of $|x|^{-\gamma} u(x,t)$.
Our aim is to take advantage of the properties of U_ε and U in order to deal with the initial condition assumed by $u(x,t)$. Proceeding as in Section 6 of [41], let us multiply (4.62) by $|x|^{-\gamma}$ and integrate in $\mathbb{R}^d \times (t_1,t_2)$. We obtain:

$$\int_{\mathbb{R}^d} |U_\varepsilon(x,t_2) - U_\varepsilon(x,t_1)| \, |x|^{-\gamma} \mathrm{d}x = \int_{t_1}^{t_2} \left(\int_{\mathbb{R}^d} u_\varepsilon^m(x,t) \, |x|^{-\gamma} \mathrm{d}x \right) \mathrm{d}t \, .$$

The r.h.s. can be controlled by exploiting (4.72) and (4.73) as follows:

$$\int_{\mathbb{R}^d} |U_\varepsilon(x,t_2) - U_\varepsilon(x,t_1)|\, |x|^{-\gamma} \mathrm{d}x \le K^{m-1}\, \mu(\mathbb{R}^d)^{1+\beta(m-1)} \int_{t_1}^{t_2} t^{-\alpha(m-1)} \mathrm{d}t \tag{4.77}$$

(note that $\alpha(m-1) < 1$ thanks to (4.39) evaluated at $p_0 = 1$). Letting $t_1 \to 0$ in (4.77), exploiting (4.71) and Fatou's Lemma on the l.h.s., we get:

$$\int_{\mathbb{R}^d} |U_\varepsilon(x,t_2) - U^{\mu_\varepsilon}(x)|\, |x|^{-\gamma} \mathrm{d}x \le K^{m-1}\, \mu(\mathbb{R}^d)^{1+\beta(m-1)}\, \frac{t_2^{1-\alpha(m-1)}}{1-\alpha(m-1)} \,. \tag{4.78}$$

It is straightforward to check that $\mu_\varepsilon \to \mu$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ as $\varepsilon \to 0$, so that as a direct consequence of [28, Th. 3.8] there holds

$$\liminf_{\varepsilon \to 0} U^{\mu_\varepsilon}(x) = U^\mu(x) \quad \text{for a.e. } x \in \mathbb{R}^d \,. \tag{4.79}$$

Thanks to (4.76), (4.79) and to the fact that $U_\varepsilon(x,t)$ is nonincreasing w.r.t. t, we get:

$$\liminf_{\varepsilon \to 0} |U_\varepsilon(x,t_2) - U^{\mu_\varepsilon}(x)| = U^\mu(x) - U(x,t_2) = |U(x,t_2) - U^\mu(x)| \quad \text{for a.e. } x \in \mathbb{R}^d \,. \tag{4.80}$$

Hence, by Fatou's Lemma and (4.80), letting $\varepsilon \to 0$ in (4.78) yields

$$\int_{\mathbb{R}^d} |U(x,t_2) - U^\mu(x)|\, |x|^{-\gamma} \mathrm{d}x \le K^{m-1}\, \mu(\mathbb{R}^d)^{1+\beta(m-1)}\, \frac{t_2^{1-\alpha(m-1)}}{1-\alpha(m-1)} \,. \tag{4.81}$$

From (4.81) we then deduce that the difference $U(x,t) - U^\mu(x)$ converges to zero in $L^1_{-\gamma}(\mathbb{R}^d)$ as $t \to 0$. In particular,

$$\lim_{t \to 0} U(x,t) = U^\mu(x) \quad \text{for a.e. } x \in \mathbb{R}^d \,, \tag{4.82}$$

where the pointwise limit on the l.h.s. exists by monotonicity (trivially, also $U(x,t)$ is decreasing in t). Passing to the limit in (4.72) as $\varepsilon \to 0$ we obtain

$$\|u(t)\|_{1,-\gamma} \le \mu(\mathbb{R}^d) \quad \text{for a.e. } t > 0 \,. \tag{4.83}$$

By the results recalled in Section 2, (4.83) implies that (almost) every sequence $t_n \to 0$ admits a subsequence $\{t_{n_k}\}$ such that $|x|^{-\gamma} u(t_{n_k})$ converges in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ to a certain positive, finite measure ν (which, a priori, depends on the particular subsequence). From (4.82) and [28, Th. 3.8] we infer that necessarily

$$U^\nu(x) = U^\mu(x) \quad \text{for a.e. } x \in \mathbb{R}^d \,.$$

The uniqueness Theorem 1.12 of [28] ensures that two positive finite measures whose potentials are equal almost everywhere must coincide. Hence, $\nu = \mu$ and the limit measure does not depend on the particular subsequence, so that

$$\operatorname{ess}\lim_{t \to 0} |x|^{-\gamma} u(t) = \mu \quad \text{in } \sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d)) \,.$$

In order to prove that such convergence also takes place in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$, it is enough to show that

$$\operatorname{ess}\lim_{t \to 0} \|u(t)\|_{1,-\gamma} = \mu(\mathbb{R}^d) \,. \tag{4.84}$$

By the convergence of $|x|^{-\gamma} u(t)$ to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ we already know that

$$\mu(\mathbb{R}^d) \le \operatorname{ess}\liminf_{t \to 0} \|u(t)\|_{1,-\gamma} \tag{4.85}$$

(see again Section 2). Combining (4.85) with (4.83) we easily get (4.84).

Finally, the validity of the smoothing estimate (3.5) is just a consequence of passing to the limit in (4.73) as $\varepsilon \to 0$ (recall that $\{u_\varepsilon\}$ converges pointwise to u).

In the beginning of the proof we required μ to be compactly supported. If μ does not meet this assumption, one can take a sequence of compactly supported measures $\{\mu_n\}$ converging to μ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ (for instance, $\mathrm{d}\mu_n = \chi_{B_n(0)} \mathrm{d}\mu$) and consider the corresponding sequence of solutions $\{u_n\}$ to (1.1), which exist thanks to the first part of the proof. The fundamental estimates

(4.58), (4.59), (4.72) and (4.73) are clearly stable as $\varepsilon \to 0$, thus they also hold true upon replacing u_ε with u_n and μ_ε with μ_n. It is then just a matter of using exactly the same techniques as above to prove that $\{u_n\}$ converges to a solution u of (1.1) starting from μ. □

4.1. **Strong solutions.** In order to justify rigorously some of the above computations (in particular, we refer to the proofs of Lemma 4.3 and Proposition 4.6), it is essential to show that the weak solutions constructed in Lemma 4.3 are actually strong. By a "strong solution", following [17, Sect. 6.2], we mean a weak solution u (in the sense of Definition 3.1) having the property

$$u_t \in L^\infty((\tau,\infty), L^1_{-\gamma}(\mathbb{R}^d)) \quad \forall \tau > 0\,. \tag{4.86}$$

Here we shall only give a sketch of how it is possible to prove that our solutions are indeed strong, as the techniques are analogous to the ones used in [17, Sect. 8.1]. The first step consists in showing that $|x|^{-\gamma} u_t(t)$ is a bounded Radon measure which satisfies the estimate

$$\left\| |x|^{-\gamma} u_t(t) \right\|_{\mathcal{M}(\mathbb{R}^d)} \le \frac{2\,\|u_0\|_{1,-\gamma}}{(m-1)t} \quad \forall t > 0\,, \tag{4.87}$$

where now we mean $\mathcal{M}(\mathbb{R}^d)$ as the Banach space of *signed* measures on $\mathbb{R}^d$, equipped with the usual norm of the variation. This can be proved proceeding exactly as in [42, Lem. 8.5], by exploiting in a crucial way the validity of the $L^1_{-\gamma}(\mathbb{R}^d)$ contraction principle

$$\int_{\mathbb{R}^d} [u(x,t) - \tilde{u}(x,t)]_+ \, |x|^{-\gamma} \mathrm{d}x \le \int_{\mathbb{R}^d} [u_0(x) - \tilde{u}_0(x)]_+ \, |x|^{-\gamma} \mathrm{d}x \quad \forall t > 0\,, \tag{4.88}$$

where u and $\tilde{u}$ are the solutions to (4.7) *constructed in Lemma 4.3* corresponding to the initial data u_0 and $\tilde{u}_0$, respectively. Such principle does hold for the approximate solutions u_η and $\tilde{u}_\eta$ used in the proof of Lemma 4.3:

$$\int_{\mathbb{R}^d} [u_\eta(x,t) - \tilde{u}_\eta(x,t)]_+ \, \rho_\eta(x) \mathrm{d}x \le \int_{\mathbb{R}^d} [u_0(x) - \tilde{u}_0(x)]_+ \, \rho_\eta(x) \mathrm{d}x \quad \forall t > 0\,. \tag{4.89}$$

This is proved in [36, Prop. 3.3]. Hence, (4.88) is just a consequence of passing to the limit in (4.89) as $\eta \to 0$.

Afterwards one proves that $z = u^{\frac{m+1}{2}}$ is a function satisfying estimate (4.21), with a constant C which a priori depends on

$$\int_{t_*}^{t^*} \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{s}{2}} \left(u^m\right)(x,t) \right|^2 \mathrm{d}x \, \mathrm{d}t$$

for some $t_* < t_1 < t_2 < t^*$. In order to do that, one can just repeat the proof of [17, Lem. 8.1] (the idea is to use Steklov averages). In particular,

$$z_t \in L^2_{loc}((0,\infty); L^2_{-\gamma}(\mathbb{R}^d))\,. \tag{4.90}$$

The dependence of the constant C in (4.21) on the initial datum as in (4.22) is then a consequence of the energy identity (4.20) (the proof of which requires however that solutions are strong, see Section 4.2 below). Having at our disposal (4.87) and (4.90) we apply the general result [6, Th. 1.1], which ensures that u_t is actually a function satisfying

$$u_t \in L^1_{loc}((0,\infty); L^1_{-\gamma}(\mathbb{R}^d))\,. \tag{4.91}$$

Thanks to (4.87) and (4.91) we then get the estimate

$$\|u_t(t)\|_{1,-\gamma} \le \frac{2\,\|u_0\|_{1,-\gamma}}{(m-1)t}\,.$$

In particular, (4.86) holds true and solutions are strong.

Remark 4.9. We have just shown that the weak solutions to (4.7) constructed in Lemma 4.3 are strong. Since, for any $\tau > 0$, every weak solution u to (1.1) provided by Theorem 3.2 can be seen as a weak solution to (4.7) corresponding to the initial datum $u(\tau) \in L^1_{-\gamma}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, one may claim that also such u is a strong solution. This is actually true: however, in order to prove it rigorously, we first need a uniqueness result (see Section 5) which ensures that u coincides (up to time shifts) with the weak solution starting from $u(\tau)$ constructed in Lemma 4.3.
Knowing that also the weak solutions provided by Theorem 3.2 are strong allows us (a posteriori) to state properties of such solutions *for all* $t > 0$ rather than only *for a.e.* $t > 0$, which we do in the corresponding statement.

4.2. **Decrease of the norms.** A very important consequence of the fact that the solutions constructed in Lemma 4.3 are strong is the decrease of their $L^p_{-\gamma}$ norms for any $p \in [1, \infty]$. Indeed, thanks to (4.86), we are allowed to multiply the differential equation in (4.7) by u^{p-1} and integrate in $\mathbb{R}^d \times [t_1, t_2]$. Exploiting the Stroock-Varopoulos inequality (4.30) (let $v = u^m$ and $q = (p+m-1)/m$), we obtain:

$$\int_{\mathbb{R}^d} u^p(x,t_2)\,|x|^{-\gamma}\mathrm{d}x - \int_{\mathbb{R}^d} u^p(x,t_1)\,|x|^{-\gamma}\mathrm{d}x = -p\int_{t_1}^{t_2}\int_{\mathbb{R}^d} u^{p-1}(x,t)(-\Delta)^s(u^m)(x,t)\,\mathrm{d}x\,\mathrm{d}t \le 0 \tag{4.92}$$

for all $t_2 > t_1 > 0$. In order to retrieve the case $t_1 = 0$ we cannot simply let $t_1 \to 0$ in (4.92), since a priori we have no information about the continuity of $\|u(t)\|_{p,-\gamma}$ down to $t = 0$. However, reasoning exactly as above, we can prove that also the approximate solutions u_η of Lemma 4.3 are strong and hence satisfy

$$\int_{\mathbb{R}^d} u_\eta^p(x,t_2)\,\rho_\eta(x)\mathrm{d}x \le \int_{\mathbb{R}^d} u_\eta^p(x,t_1)\,\rho_\eta(x)\mathrm{d}x\,. \tag{4.93}$$

Moreover, from the results of [36], we are allowed to let $t_1 \to 0$ in (4.93), which yields

$$\int_{\mathbb{R}^d} u_\eta^p(x,t)\,\rho_\eta(x)\mathrm{d}x \le \int_{\mathbb{R}^d} u_0^p(x)\,\rho_\eta(x)\mathrm{d}x \quad \forall t > 0\,. \tag{4.94}$$

This can be proved by exploiting the fact that $u_\eta \in C([0,\infty); L^1_{\rho_\eta}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (0,\infty))$ (see [36, Th. 3.1]). Hence, letting $\eta \to 0$ and using for instance Fatou's Lemma on the l.h.s. of (4.94), we get that

$$\int_{\mathbb{R}^d} u^p(x,t_2)\,|x|^{-\gamma}\mathrm{d}x \le \int_{\mathbb{R}^d} u^p(x,t_1)\,|x|^{-\gamma}\mathrm{d}x \tag{4.95}$$

holds true for all $t_2 > t_1 \ge 0$.

Notice that, when $p = m + 1$, (4.92) becomes exactly the energy identity (4.20).

Of course the above computations are rigorous provided $p \in (1, \infty)$. Nevertheless, we already know that $\|u(t)\|_{1,-\gamma}$ is preserved, while the case $p = \infty$ can be handled by taking limits.

5. Uniqueness of weak solutions

As in the previous section, we shall prove the results only for weak solutions to (1.1), but notice once again that the modifications required to deal with (1.2) (provided ρ complies with (1.3)) are inessential.

Prior to the proof of the uniqueness Theorem 3.3, we need some technical lemmas. We use some of the ideas of the pioneering paper [33], which need to be carefully modified in order to deal with our fractional, weighted problem. The Markov property for the linear semigroup associated to the operator $A = |x|^\gamma(-\Delta)^s$ will have a crucial role in our strategy.

Hereafter, we shall always refer to a "weak solution" to (1.1) in the sense of Definition 3.1.

Lemma 5.1. *Let* $\gamma \le d - 2s$. *Let* u *be a weak solution to* (1.1). *Then the potential* $U(t)$ *of* $|x|^{-\gamma}u(t)$ *admits an absolutely continuous version (w.r.t.* $t > 0$, *for instance in* $L^1_{loc}(\mathbb{R}^d)$*), which is nonincreasing in* t.

Proof. One proceeds just as in the proof of Theorem 3.2, in particular using the same technique we exploited to prove (4.62). □

Lemma 5.2. *Let $\gamma \le d-2s$. Let u be a weak solution to* (1.1)*, starting from the initial datum μ whose potential is U^μ. There holds*

$$\lim_{t\downarrow 0} U(x,t) = U^\mu(x) \quad \forall x \in \mathbb{R}^d . \tag{5.1}$$

Proof. Thanks to Theorem 3.8 of [28] and the monotonicity ensured by Lemma 5.1, we have that the limit (5.1) is taken at least for a.e. $x \in \mathbb{R}^d$. However, for what follows it will be crucial to prove that this relation holds *for every* $x \in \mathbb{R}^d$. To this end we make use again of the monotonicity property given in Lemma 5.1. In fact, Lemma 1.12 of [28] shows that, as a consequence of monotonicity of potentials, there exist a positive finite measure ν, whose potential is denoted by U^ν, and a constant $A \ge 0$ such that

$$\lim_{t\downarrow 0} U(x,t) = U^\nu(x) + A \quad \forall x \in \mathbb{R}^d .$$

Since (5.1) holds almost everywhere,

$$U^\mu(x) = U^\nu(x) + A \quad \text{for a.e. } x \in \mathbb{R}^d . \tag{5.2}$$

But using the corollary at p. 129 of [28], from (5.2) we deduce that necessarily $A = 0$. Hence, (5.2) implies that $U^\nu = U^\mu$ almost everywhere, and from Theorem 1.12 of [28] we know that two potentials coinciding a.e. are in fact equal everywhere, whence (5.1) follows. □

Now let u_1 and u_2 be two weak solutions to (1.1) (in the sense of Definition 3.1), such that they both take a common positive, finite measure μ as initial datum. We denote as $U_1(t)$ and $U_2(t)$ the corresponding potentials of $|x|^{-\gamma}u_1(t)$ and $|x|^{-\gamma}u_2(t)$, respectively. Fix once for all the parameters $h, T > 0$ and consider the function

$$g(x,t) = U_2(x,t+h) - U_1(x,t) \quad \forall (x,t) \in \mathbb{R}^d \times (0,T] . \tag{5.3}$$

Proceeding again as in the proof of Theorem 3.3 (under the hypothesis $\gamma \le d-2s$, see in particular the proof of (4.62)), we get that $g(t)$ is an absolutely continuous curve (for instance in $L^1_{loc}(\mathbb{R}^d)$) satisfying

$$|x|^{-\gamma} g_t(x,t) = |x|^{-\gamma}\left(u_1^m(x,t) - u_2^m(x,t+h)\right) = -a(x,t)(-\Delta)^s(g)(x,t) \ \text{ for a.e. } (x,t) \in \mathbb{R}^d \times (0,T) , \tag{5.4}$$

where we used the fact that, thanks to the properties of Riesz potentials,

$$(-\Delta)^s(g)(x,t) = |x|^{-\gamma}u_2(x,t+h) - |x|^{-\gamma}u_1(x,t) ,$$

and we defined the function a as

$$a(x,t) = \begin{cases} \frac{u_1^m(x,t)-u_2^m(x,t+h)}{u_1(x,t)-u_2(x,t+h)} & \text{if } u_1(x,t) \ne u_2(x,t+h) , \\ 0 & \text{if } u_1(x,t) = u_2(x,t+h) . \end{cases} \tag{5.5}$$

Note that, since $m > 1$ and $u_1, u_2 \in L^\infty(\mathbb{R}^d \times (\tau,\infty))$ for all $\tau > 0$, a is a nonnegative function belonging to $L^\infty(\mathbb{R}^d \times (\tau,\infty))$ for all $\tau > 0$.

Let us briefly describe the strategy of the proof. A particular role will be played by a suitable family of positive finite measures $\{\nu(t)\}$, which is somehow related to equation (5.4). More precisely, $\nu(t)$ is the limit in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ as $\varepsilon \to 0$ of $\{|x|^{-\gamma}\psi_\varepsilon(t)\}$. Here, $\psi_\varepsilon(t)$ is in turn the weak limit in $L^2_{-\gamma}(\mathbb{R}^d \times (\tau,T))$ (for all $\tau \in (0,T)$) as $n \to \infty$ of a suitable sequence $\{\psi_{n,\varepsilon}(t)\}$. Such a $\psi_{n,\varepsilon}$ is defined, for every $n \in \mathbb{N}$ and $\varepsilon > 0$, to be a solution (in a sense which will be clarified later) to the problem

$$\begin{cases} |x|^{-\gamma}\left(\psi_{n,\varepsilon}\right)_t = (-\Delta)^s\left[(a_n+\varepsilon)\,\psi_{n,\varepsilon}\right] & \text{in } \mathbb{R}^d \times (0,T) , \\ \psi_{n,\varepsilon} = \psi & \text{on } \mathbb{R}^d \times \{T\} . \end{cases} \tag{5.6}$$

The sequence $\{a_n\}$ is a suitable approximation of the function a defined in (5.5). In particular we suppose that, for every $n \in \mathbb{N}$, $a_n(x,t)$ is a piecewise constant function of t (regular in x) on the time intervals $(T-(k+1)T/n, T-kT/n]$, for any $k \in \{0, \dots n-1\}$. This allows to treat problem (5.6) by means of standard semigroup theory, after having shown some preliminary results which are contained in Appendix B. In [33, Theorem 1] such difficulty is not present because of the parabolic regularity of the equation dealt with there.

5.1. **Construction and properties of the family $\{\psi_{n,\varepsilon}\}$.**

Lemma 5.3. *Let $d > 2s$ and $\gamma \in (0, 2s)$. Let $\{a_n\}$ be a sequence of functions converging a.e. to the function a as in (5.5) such that:*

- *for any $n \in \mathbb{N}$ and $t > 0$, $a_n(x,t)$ is a regular function of x;*
- *for any $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$, $a_n(x,t)$ is a piecewise constant function of t on the time intervals $(T-(k+1)T/n, T-kT/n]$, for any $k \in \{0, \dots n-1\}$;*
- *$\{\|a_n\|_{L^\infty(\mathbb{R}^d \times (\tau,\infty))}\}$ is uniformly bounded in n for any $\tau > 0$.*

Then, for any $\varepsilon > 0$ and any nonnegative $\psi \in \mathcal{D}(\mathbb{R}^d)$, there exists a nonnegative solution $\psi_{n,\varepsilon}$ to problem (5.6), in the sense that $\psi_{n,\varepsilon}(t)$ is a continuous curve in $L^p_{-\gamma}(\mathbb{R}^d)$ for all $p \in (1,\infty)$ satisfying $\psi_{n,\varepsilon}(0) = \psi(0)$ and it is absolutely continuous on $(T-(k+1)T/n, T-kT/n)$ for all $k \in \{0, \dots n-1\}$, so that the identity

$$\psi_{n,\varepsilon}(t_2) - \psi_{n,\varepsilon}(t_1) = \int_{t_1}^{t_2} |x|^\gamma (-\Delta)^s \left[(a_n+\varepsilon)\,\psi_{n,\varepsilon}\right](\tau)\,\mathrm{d}\tau \qquad (5.7)$$
$$\forall t_1, t_2 \in \left(T - \frac{(k+1)T}{n}, T - \frac{kT}{n}\right), \ \forall k \in \{0, \dots n-1\}$$

holds true in $L^p_{-\gamma}(\mathbb{R}^d)$ for all $p \in (1,\infty)$. Moreover,

$$\psi_{n,\varepsilon} \in L^\infty((0,T); L^p_{-\gamma}(\mathbb{R}^d)) \ \ \forall p \in [1,\infty] \quad \text{and} \quad \|\psi_{n,\varepsilon}(t)\|_{1,-\gamma} \le \|\psi\|_{1,-\gamma} \quad \forall t \in [0,T]. \qquad (5.8)$$

Proof. To construct $\psi_{n,\varepsilon}$ as in the statement, we first define ζ_1 as the solution of

$$\begin{cases} |x|^{-\gamma}\,(\zeta_1)_t = (-\Delta)^s\left[(a_n(T)+\varepsilon)\,\zeta_1\right] & \text{in } \mathbb{R}^d \times \left(T-\frac{T}{n}, T\right), \\ \zeta_1 = \psi & \text{on } \mathbb{R}^d \times \{T\}. \end{cases} \qquad (5.9)$$

In fact, to obtain ζ_1, one can for instance exploit the change of variable

$$\phi_1(x,t) = (a_n(x,T)+\varepsilon)\,\zeta_1(x,t) \qquad (5.10)$$

and consider ϕ_1 as the solution of the problem

$$\begin{cases} (\phi_1)_t = (a_n(T)+\varepsilon)\,|x|^\gamma(-\Delta)^s(\phi_1) & \text{in } \mathbb{R}^d \times \left(T-\frac{T}{n}, T\right), \\ \phi_1 = (a_n(T)+\varepsilon)\,\psi & \text{on } \mathbb{R}^d \times \{T\}. \end{cases} \qquad (5.11)$$

Problem (5.11) is indeed solvable by standard semigroup theory. Indeed, letting

$$\rho_1(x) = (a_n(x,T)+\varepsilon)^{-1}\,|x|^{-\gamma} \quad \forall x \in \mathbb{R}^d, \qquad (5.12)$$

we have that the operator

$$\rho_1^{-1}(-\Delta)^s,$$

with domain $X_{s,\gamma}$ (see Appendix B for the definition of $X_{s,\gamma}$), is positive, self-adjoint and generates a Markov semigroup on $L^2_{\rho_1}(\mathbb{R}^d)$. All these properties have been analysed in Appendix B for the case $\rho_1(x) = |x|^{-\gamma}$, but notice that the discussion there also applies to a weight ρ_1 as in (5.12) with inessential modifications. We do not claim that our initial datum ϕ_1 belongs to $X_{s,\gamma}$. However, it clearly belongs to $L^p_{\rho_1}(\mathbb{R}^d)$ for all $p \in [1,\infty]$, and this is enough in order to have a solution to (5.11) which is continuous up to $t = T$ and absolutely continuous in $\left(T-\frac{T}{n}, T\right)$ in $L^p_{\rho_1}(\mathbb{R}^d)$ for all $p \in (1,\infty)$. In fact, as recalled in Theorem B.2, the semigroup associated to A is Markov and therefore, as a consequence of [12, Theorems 1.4.1, 1.4.2], it is extendible to a contraction semigroup on $L^p_{\rho_1}(\mathbb{R}^d)$ (consistent with the original semigroup on $L^2_{\rho_1}(\mathbb{R}^d) \cap L^p_{\rho_1}(\mathbb{R}^d)$) for all $p \in [1,\infty]$, which is analytic with a suitable angle $\theta_p > 0$ when $p \in (1,\infty)$. By classical results (see e.g. [32, Th. 5.2

at p. 61]) the latter property ensures in particular that problem (5.11) is solved by a *differentiable* curve $\phi_1(t)$ in $L^p_{\rho_1}(\mathbb{R}^d)$ for all $p \in (1,\infty)$.
Going back to the original variable ζ_1 through (5.10), we deduce that it solves (5.9) in the same sense in which ϕ_1 solves (5.11). Having at our disposal such a ζ_1, we can then solve the problem

$$\begin{cases} |x|^{-\gamma}\left(\zeta_2\right)_t = (-\Delta)^s\left[\left(a_n\left(T-\frac{T}{n}\right)+\varepsilon\right)\zeta_2\right] & \text{in } \mathbb{R}^d \times \left(T-\frac{2T}{n}, T-\frac{T}{n}\right), \\ \zeta_2 = \psi_1 = \left(a_n\left(x,T\right)+\varepsilon\right)^{-1}\phi_1 & \text{on } \mathbb{R}^d \times \left\{T-\frac{T}{n}\right\}, \end{cases}$$

just by proceeding as above. That is, we perform the change of variable

$$\phi_2(x,t) = \left(a_n\left(x, T-\frac{T}{n}\right)+\varepsilon\right)\zeta_2(x,t)$$

and take ϕ_2 as the solution of the problem

$$\begin{cases} \left(\phi_2\right)_t = \left(a_n(T-\frac{T}{n})+\varepsilon\right)|x|^{\gamma}(-\Delta)^s(\phi_2) & \text{in } \mathbb{R}^d \times \left(T-\frac{2T}{n}, T-\frac{T}{n}\right), \\ \phi_2 = \left(a_n(T-\frac{T}{n})+\varepsilon\right)\zeta_1 = \frac{\left(a_n(T-\frac{T}{n})+\varepsilon\right)}{\left(a_n(T)+\varepsilon\right)}\phi_1 & \text{on } \mathbb{R}^d \times \left\{T-\frac{T}{n}\right\}. \end{cases}$$

It is clear how the procedure goes on and allows to obtain a solution $\psi_{n,\varepsilon}$ to (5.6) in the sense of the statement, just by defining it as

$$\psi_{n,\varepsilon}(t) = \zeta_{k+1}(t) \quad \forall t \in \left(T-\frac{(k+1)T}{n}, T-\frac{kT}{n}\right], \quad \forall k \in \{0,\dots n-1\}.$$

Finally, since

$$\rho_{k+1}^{-1}(-\Delta)^s$$

generates a contraction semigroup on $L^p_{\rho_{k+1}}(\mathbb{R}^d)$ for all $p \in [1,\infty]$, where

$$\rho_{k+1}(x) = \left(a_n\left(x, T-\frac{kT}{n}\right)+\varepsilon\right)^{-1}|x|^{-\gamma} \quad \forall x \in \mathbb{R}^d, \tag{5.13}$$

the inequalities

$$\begin{gathered} \left\|\phi_{k+1}(t)\right\|_{p,\rho_{k+1}} \le \left\| \frac{\left(a_n(T-\frac{kT}{n})+\varepsilon\right)}{\left(a_n(T-\frac{(k-1)T}{n})+\varepsilon\right)}\phi_k\left(T-\frac{kT}{n}\right)\right\|_{p,\rho_{k+1}} \\ \forall t \in \left(T-\frac{(k+1)T}{n}, T-\frac{kT}{n}\right], \ \forall p \in [1,\infty] \end{gathered} \tag{5.14}$$

hold true for any $k \in \{0,\dots n-1\}$ (on the r.h.s. of (5.14), for $k=0$ we conventionally set $\phi_0 = \psi$ and $a_n(T+T/n)+\varepsilon = 1$). Going back to the variables ζ_{k+1} and recalling (5.13), from (5.14) one deduces (5.8): in fact, for $p=1$ it is easy to see that the terms containing a_n cancel out and give the right inequality in (5.8), while for $p>1$ such terms remain and one obtains an inequality of the type $\|\psi_{n,\varepsilon}(t)\|_{p,-\gamma} \le C(n,\varepsilon)\|\psi\|_{p,-\gamma}$, where $C(n,\varepsilon)$ is a suitable positive constant depending on n,ε. □

Lemma 5.4. *Let $d > 2s$ and $\gamma \in (0,2s) \cap (0, d-2s]$. Let g be as in* (5.3), *a as in* (5.5) *and a_n, $\psi_{n,\varepsilon}$, ψ as in Lemma 5.3. Then the identity*

$$\begin{aligned} &\int_{\mathbb{R}^d} g(x,T)\psi(x)\,|x|^{-\gamma}\,\mathrm{d}x - \int_{\mathbb{R}^d} g(x,t)\psi_{n,\varepsilon}(x,t)\,|x|^{-\gamma}\,\mathrm{d}x \\ &= \int_t^T \int_{\mathbb{R}^d} \left(a_n(x,\tau)+\varepsilon-a(x,\tau)\right)(-\Delta)^s(g)(x,\tau)\,\psi_{n,\varepsilon}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \end{aligned} \tag{5.15}$$

holds true for all $t \in (0,T]$.

Proof. To begin with, let us set

$$t_k = T - \frac{kT}{n} \quad \forall k \in \{0, \dots n\}. \tag{5.16}$$

Recall that, from Lemma 5.3, we have that $\psi_{n,\varepsilon}(t)$ is a continuous curve in $L^p_{-\gamma}(\mathbb{R}^d)$ on $(0,T]$, absolutely continuous on any interval (t_{k+1}, t_k) for $k \in \{0, \dots n-1\}$ and satisfying the differential equation in (5.6) on those intervals, for all $p \in (1, \infty)$. Moreover, $g(t)$ is an absolutely continuous curve in $L^p_{-\gamma}(\mathbb{R}^d)$ on $(0,T]$ for all p such that

$$p \in \left(\frac{d-\gamma}{d-2s}, \infty \right). \tag{5.17}$$

In fact, to prove that $g(t) \in L^p_{-\gamma}(\mathbb{R}^d)$ for all p as in (5.17), it suffices to show that $g(t) \in L^p_{-\gamma}(B_1^c)$ for such p, since $g(x,t)$ is a continuous function of x (recall Lemma 4.8) and the weight $|x|^{-\gamma}$ is locally integrable. Still Lemma 4.8 ensures that $g(t) \in L^p(\mathbb{R}^d)$ for all p satisfying (4.50): the latter property and Hölder inequality imply that $g(t) \in L^p_{-\gamma}(B_1^c)$ for all p satisfying (5.17).
The fact that $g(t)$ is also absolutely continuous in $L^p_{-\gamma}(\mathbb{R}^d)$ for such p on the time interval $(0,T]$ is then a consequence of (5.4) and of the integrability properties of u_1, u_2. Hence, thanks to Lemma 5.3, we get that the product

$$\int_{\mathbb{R}^d} g(x,t)\psi_{n,\varepsilon}(x,t)\,|x|^{-\gamma}\mathrm{d}x \tag{5.18}$$

is a continuous function on $(0,T]$, absolutely continuous on each (t_{k+1}, t_k) and satisfying

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} g(x,t)\psi_{n,\varepsilon}(x,t)\,|x|^{-\gamma}\mathrm{d}x \\ =&\int_{\mathbb{R}^d} \left(-a(x,t)(-\Delta)^s(g)(x,t)\,\psi_{n,\varepsilon}(x,t) + g(x,t)\,(-\Delta)^s\left[(a_n+\varepsilon)\,\psi_{n,\varepsilon}\right](x,t)\right)\mathrm{d}x \end{aligned} \tag{5.19}$$

on such intervals. As we have just seen, $g(t) \in L^p_{-\gamma}(\mathbb{R}^d)$ for all p complying with (5.17) and $|x|^\gamma(-\Delta)^s(g(t)) \in L^p_{-\gamma}(\mathbb{R}^d)$ for all $p \in [1,\infty]$. Moreover, as a consequence of Lemma 5.3, we have that $(a_n(t)+\varepsilon)\psi_{n,\varepsilon}(t) \in L^p_{-\gamma}(\mathbb{R}^d)$ for all $p \in [1,\infty]$ and $|x|^\gamma(-\Delta)^s[(a_n(t)+\varepsilon)\psi_{n,\varepsilon}(t)] \in L^p_{-\gamma}(\mathbb{R}^d)$ for all $p \in (1,\infty)$. We are therefore in position to apply Proposition B.3 to the r.h.s. of (5.19) (the interval $((d-\gamma)/(d-2s), \infty) \cap [2, 2(d-\gamma)/(d-2s))$ is never empty) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} g(x,t)\psi_{n,\varepsilon}(x,t)\,|x|^{-\gamma}\mathrm{d}x = \int_{\mathbb{R}^d} \left(a_n(x,t)+\varepsilon-a(x,t)\right)(-\Delta)^s(g)(x,t)\,\psi_{n,\varepsilon}(x,t)\,\mathrm{d}x\,. \tag{5.20}$$

But the r.h.s. of (5.20) is $L^1((\tau,T))$ for any $\tau \in (0,T)$, from which (5.18) is absolutely continuous on the whole of $(0,T]$ and not only on (t_{k+1}, t_k). Formula (5.15) is then just a consequence of integrating (5.20) between t and T. □

In the next Lemma we introduce the Riesz potential of $|x|^{-\gamma}\psi_{n,\varepsilon}(t)$, which will take a fundamental role from now on.

Lemma 5.5. *Let $d > 2s$ and $\gamma \in (0,2s) \cap (0, d-2s]$. Let a_n, $\psi_{n,\varepsilon}$ and ψ be as in Lemma 5.3. We denote as $H_{n,\varepsilon}(t)$ the Riesz potential of $\psi_{n,\varepsilon}(t)$, that is*

$$H_{n,\varepsilon}(t) = I_{2s} * \left(|x|^{-\gamma}\psi_{n,\varepsilon}(t)\right) \quad \forall t \in (0,T]\,.$$

Then $H_{n,\varepsilon}(t) \in \dot{H}^s(\mathbb{R}^d)$ and the identity

$$\left\| I_{2s} * (|x|^{-\gamma}\psi) \right\|^2_{\dot{H}^s} = \|H_{n,\varepsilon}(t)\|^2_{\dot{H}^s} + 2\int_t^T \int_{\mathbb{R}^d} \left(a_n(x,\tau)+\varepsilon\right)\psi^2_{n,\varepsilon}(x,\tau)\,|x|^{-\gamma}\mathrm{d}x\,\mathrm{d}\tau \tag{5.21}$$

holds true for all $t \in (0,T]$.

Proof. First notice that $|x|^\gamma(-\Delta)^s(H_{n,\varepsilon}(t)) = \psi_{n,\varepsilon}(t) \in L^p_{-\gamma}(\mathbb{R}^d)$ for all $p \in [1,\infty]$ (recall (5.8)) and $H_{n,\varepsilon}(t) \in L^p_{-\gamma}(\mathbb{R}^d)$ for all p satisfying (5.17) (this can be proved by exploiting Lemma 4.8 exactly as in the proof of Lemma 5.4). Again, since the interval $((d-\gamma)/(d-2s),\infty) \cap [2, 2(d-\gamma)/(d-2s))$ is never empty, by applying Proposition B.3 we get that $H_{n,\varepsilon}(t) \in \dot{H}^s(\mathbb{R}^d)$ and the identity

$$\|H_{n,\varepsilon}(t)\|^2_{\dot{H}^s} = \int_{\mathbb{R}^d} H_{n,\varepsilon}(x,t)(-\Delta)^s\left(H_{n,\varepsilon}\right)(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^d} H_{n,\varepsilon}(x,t)\psi_{n,\varepsilon}(x,t)\,|x|^{-\gamma}\mathrm{d}x \tag{5.22}$$

holds true. Thanks to the validity of the differential equation

$$\left(H_{n,\varepsilon}\right)_t(x,t) = \left(a_n(x,t)+\varepsilon\right)\psi_{n,\varepsilon}(x,t) \quad \text{for a.e. } (x,t) \in \mathbb{R}^d \times (0,T)\,, \tag{5.23}$$

which can be justified as (5.4) for $\gamma \le d-2s$, taking the time derivative of (5.22) in the intervals (t_{k+1},t_k) (let $\{t_k\}$ be as in (5.16)), using (5.23), (5.6) and again Proposition B.3 we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|H_{n,\varepsilon}(t)\|^2_{\dot{H}^s} = 2\int_{\mathbb{R}^d}\left(a_n(x,t)+\varepsilon\right)\psi^2_{n,\varepsilon}(x,t)\,|x|^{-\gamma}\mathrm{d}x\,. \tag{5.24}$$

A priori, from (5.22), we have that $\|H_{n,\varepsilon}(t)\|^2_{\dot{H}^s}$ is continuous on $(0,T]$ and absolutely continuous only on (t_{k+1},t_k); but the r.h.s. of (5.24) is $L^1((\tau,T))$ for any $\tau \in (0,T)$. Thus, (5.21) just follows by integrating (5.24) from t to T. □

Lemma 5.6. *Let $d > 2s$ and $\gamma \in (0,2s)$. Let $\psi_{n,\varepsilon}$ and ψ be as in Lemma 5.3. Then the $L^1_{-\gamma}$ norm of $\psi_{n,\varepsilon}(t)$ is preserved, that is*

$$\int_{\mathbb{R}^d}\psi_{n,\varepsilon}(x,t)\,|x|^{-\gamma}\mathrm{d}x = \int_{\mathbb{R}^d}\psi(x)\,|x|^{-\gamma}\mathrm{d}x \quad \forall t \in (0,T]\,. \tag{5.25}$$

Proof. Multiplying (5.7) by any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and integrating in $\mathbb{R}^d$, we obtain:

$$\begin{aligned}&\int_{\mathbb{R}^d}\psi_{n,\varepsilon}(x,t_2)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x - \int_{\mathbb{R}^d}\psi_{n,\varepsilon}(x,t_1)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x\\ &= \int_{\mathbb{R}^d}(-\Delta)^s(\varphi)(x)\left(\int_{t_1}^{t_2}\left(a_n(x,\tau)+\varepsilon\right)\psi_{n,\varepsilon}(x,\tau)\,\mathrm{d}\tau\right)\mathrm{d}x\end{aligned} \tag{5.26}$$

for all $t_1,t_2 \in (t_{k+1},t_k)$. Since the $L^1_{-\gamma}$ norm of $\psi_{n,\varepsilon}(t)$ is controlled by the $L^1_{-\gamma}$ norm of the final datum ψ (recall (5.8)), from (5.26) we get:

$$\left|\int_{\mathbb{R}^d}\psi_{n,\varepsilon}(x,t_2)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x - \int_{\mathbb{R}^d}\psi_{n,\varepsilon}(x,t_1)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x\right| \le C\,|t_2-t_1|\,\|\psi\|_{1,-\gamma}\,\||x|^\gamma(-\Delta)^s(\varphi)\|_\infty \tag{5.27}$$

for all $t_1,t_2 \in (t_{k+1},t_k)$, where $C = \|a_n+\varepsilon\|_{L^\infty(\mathbb{R}^d\times(t_1\wedge t_2,T))}$, namely it is a positive constant independent of n and ε. Replacing φ with the cut-off function ξ_R (defined as in Lemma A.3 of Appendix A) yields

$$\left|\int_{\mathbb{R}^d}\psi_{n,\varepsilon}(x,t_2)\xi_R(x)|x|^{-\gamma}\mathrm{d}x - \int_{\mathbb{R}^d}\psi_{n,\varepsilon}(x,t_1)\xi_R(x)|x|^{-\gamma}\mathrm{d}x\right| \le C|t_2-t_1|\,\|\psi\|_{1,-\gamma}\,\frac{\||x|^\gamma(-\Delta)^s(\xi)\|_\infty}{R^{2s-\gamma}} \tag{5.28}$$

for all $R > 0$ and $t_2,t_1 \in (t_{k+1},t_k)$. Recalling that $\psi_{n,\varepsilon}(t)$ is a continuous curve (for instance in $L^2_{-\gamma}(\mathbb{R}^d)$) on $(0,T]$, we can extend the validity of (5.28) (and (5.27)) to any $t_1,t_2 \in (0,T]$. By choosing $t_1 = T$ an letting $R \to \infty$ in (5.28) we finally get (5.25). □

5.2. **Passing to the limit as $n \to \infty$.** The goal of the next lemma is to show that, as $n \to \infty$, $\{\psi_{n,\varepsilon}\}$ suitably converges to a limit function ψ_ε that enjoys some crucial properties.

Lemma 5.7. *Let $d > 2s$ and $\gamma \in (0,2s) \cap (0,d-2s]$. Let u_1 and u_2 be two weak solutions to problem* (1.1)*, taking the common positive, finite measure μ as initial datum. Let g be as in* (5.3)*, a as in* (5.5) *and $\psi_{n,\varepsilon},\psi$ as in Lemma 5.3. Then, up to subsequences, $\{\psi_{n,\varepsilon}\}$ converges weakly in $L^2_{-\gamma}(\mathbb{R}^d\times(\tau,T))$ (for all $\tau \in (0,T)$) to a suitable nonnegative function ψ_ε and $\{|x|^{-\gamma}\psi_{n,\varepsilon}(t)\}$*

converges to $|x|^{-\gamma}\psi_\varepsilon(t)$ *in* $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$ *for a.e.* $t \in (0,T)$*. Moreover, such a* ψ_ε *enjoys the following properties:*

$$\int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\,|x|^{-\gamma}\mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\,|x|^{-\gamma}\mathrm{d}x\,, \tag{5.29}$$

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi(x)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x - \int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x \\ &= \int_{\mathbb{R}^d} (-\Delta)^s(\varphi)(x) \left(\int_t^T (a(x,\tau)+\varepsilon)\,\psi_\varepsilon(x,\tau)\,\mathrm{d}\tau \right) \mathrm{d}x\,,\end{aligned} \tag{5.30}$$

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} g(x,T)\psi(x)\,|x|^{-\gamma}\mathrm{d}x - \int_{\mathbb{R}^d} g(x,t)\psi_\varepsilon(x,t)\,|x|^{-\gamma}\mathrm{d}x \right| \\ \le &\varepsilon\,(T-t)\,\|\psi\|_{1,-\gamma}\,\|u_2(\tau+h)-u_1(\tau)\|_{L^\infty(\mathbb{R}^d\times(t,T))}\end{aligned} \tag{5.31}$$

for a.e. $t \in (0,T)$*, where* φ *is any function of* $\mathcal{D}(\mathbb{R}^d)$*.*

Proof. From (5.21) one gets that, up to subsequences, $\{\psi_{n,\varepsilon}\}$ converges weakly in $L^2_{-\gamma}\left(\mathbb{R}^d \times (\tau,T)\right)$ (for all $\tau \in (0,T)$) to a suitable ψ_ε. Moreover, thanks to the uniform boundedness of $\{|x|^{-\gamma}\psi_{n,\varepsilon}(t)\}$ in $L^1(\mathbb{R}^d)$ (see (5.8)), for every $t \in (0,T)$ there exists a subsequence (which a priori may depend on t) such that $\{|x|^{-\gamma}\psi_{n,\varepsilon}(t)\}$ converges in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ to some positive, finite measure $\nu(t)$ (recall the preliminary results of Section 2). We aim at identifying (al least for almost every $t \in (0,T)$) $\nu(t)$ with $|x|^{-\gamma}\psi_\varepsilon(t)$, so that a posteriori the subsequence does not depend on t. In order to do that, let $t \in (0,T)$ be a Lebesgue point of $\psi_\varepsilon(t)$ (as a curve in $L^1((\tau,T); L^2_{-\gamma}(\mathbb{R}^d)$, for instance). Taking any $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and using (5.27), we obtain:

$$\begin{aligned}&\left| \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,\tau)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x\,\mathrm{d}\tau - \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x\,\mathrm{d}\tau \right| \\ \le &\int_t^{t+\delta} \left| \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,\tau)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x - \int_{\mathbb{R}^d} \psi_{n,\varepsilon}(x,t)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x \right| \mathrm{d}\tau \\ \le &\int_t^{t+\delta} C(\tau-t)\,\|\psi\|_{1,-\gamma}\,\||x|^\gamma(-\Delta)^s(\varphi)\|_\infty\,\mathrm{d}\tau = \frac{\delta^2}{2} C\,\|\psi\|_{1,-\gamma}\,\||x|^\gamma(-\Delta)^s(\varphi)\|_\infty\,,\end{aligned} \tag{5.32}$$

for all δ sufficiently small. Letting $n \to \infty$ (up to suitable subsequences) in (5.32) yields

$$\left| \int_t^{t+\delta} \int_{\mathbb{R}^d} \psi_\varepsilon(x,\tau)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x\,\mathrm{d}\tau - \delta \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}\nu(t) \right| \le \frac{\delta^2}{2} C\,\|\psi\|_{1,-\gamma}\,\||x|^\gamma(-\Delta)^s(\varphi)\|_\infty\,. \tag{5.33}$$

Dividing (5.33) by δ and letting $\delta \to 0$ one deduces that (recall that t is a Lebesgue point for $\{\psi_\varepsilon(t)\}$)

$$\int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\,\varphi(x)\,|x|^{-\gamma}\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}\nu(t)\,,$$

which is valid for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, whence $|x|^{-\gamma}\psi_\varepsilon(x,t)\,\mathrm{d}x = \mathrm{d}\nu(t)$.
We now prove the claimed properties of ψ_ε. Letting $n \to \infty$ in (5.28) (with $t_1 = T$ and $t_2 = t$) and using the just proved convergence of $\{|x|^{-\gamma}\psi_{n,\varepsilon}(t)\}$ to $|x|^{-\gamma}\psi_\varepsilon(t)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ we get

$$\left| \int_{\mathbb{R}^d} \psi(x)\,\xi_R(x)\,|x|^{-\gamma}\mathrm{d}x - \int_{\mathbb{R}^d} \psi_\varepsilon(x,t)\,\xi_R(x)\,|x|^{-\gamma}\mathrm{d}x \right| \le C(T-t)\,\|\psi\|_{-1,\gamma}\,\frac{\||x|^\gamma(-\Delta)^s(\xi)\|_\infty}{R^{2s-\gamma}} \tag{5.34}$$

for a.e. $t \in (0,T)$. Letting $R \to \infty$ in (5.34) we deduce (5.29). Thanks to (5.25) and (5.29) we infer in particular that

$$\lim_{n\to\infty} \|\psi_{n,\varepsilon}(t)\|_{1,-\gamma} = \|\psi_\varepsilon(t)\|_{1,-\gamma}\,,$$

so that the convergence of $\{|x|^{-\gamma}\psi_{n,\varepsilon}(t)\}$ to $|x|^{-\gamma}\psi_\varepsilon(t)$ takes place also in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$. Recalling that $g(t)$ belongs to $C_b(\mathbb{R}^d)$ (Lemma 4.8), we can let $n \to \infty$ in (5.15) to obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^d} g(x,T)\psi(x)\,|x|^{-\gamma}\,\mathrm{d}x - \int_{\mathbb{R}^d} g(x,t)\psi_\varepsilon(x,t)\,|x|^{-\gamma}\,\mathrm{d}x \\
&= \lim_{n\to\infty}\left(\int_t^T \int_{\mathbb{R}^d} (a_n(x,\tau)+\varepsilon - a(x,\tau))\,(-\Delta)^s(g)(x,\tau)\,\psi_{n,\varepsilon}(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau\right) \\
&= \lim_{n\to\infty}\left(\int_t^T \int_{\mathbb{R}^d} (a_n(x,\tau)+\varepsilon - a(x,\tau))\,(u_2(x,\tau+h)-u_1(x,\tau))\,\psi_{n,\varepsilon}(x,\tau)\,|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}\tau\right) \\
&= \varepsilon \int_t^T \int_{\mathbb{R}^d} (u_2(x,\tau+h)-u_1(x,\tau))\,\psi_\varepsilon(x,\tau)\,|x|^{-\gamma}\,\mathrm{d}x\,\mathrm{d}\tau \quad \text{for a.e. } t\in(0,T)\,,
\end{aligned} \tag{5.35}$$

where in the last integral we can pass to the limit since $\{\psi_{n,\varepsilon}\}$ tends to ψ_ε in $L^2_{-\gamma}(\mathbb{R}^d \times (t,T))$, $\{a_n\}$ tends to a in $L^\infty(\mathbb{R}^d \times (t,T))$ and u_1, u_2 belong to $L^p_{-\gamma}(\mathbb{R}^d \times (t, T+h))$ for all $p \in [1,\infty]$. In particular, from (5.35) and (5.29) we get (5.31). Notice that, in a similarly way, we can pass to the limit in (5.26) (which actually holds for any $t_1, t_2 \in (0,T)$) and obtain (5.30). □

5.3. **Passing to the limit as $\varepsilon \to 0$ and proof of Theorem 3.3.** We are now in position to prove Theorem 3.3, using the strategy of [33]: we give some detail for the reader's convenience.

Proof of Theorem 3.3. To begin with, we introduce the Riesz potential $H_\varepsilon(t)$ of $|x|^{-\gamma}\psi_\varepsilon(t)$. Since we only know that $|x|^{-\gamma}\psi_\varepsilon(t) \in L^1(\mathbb{R}^d)$, we have no information over the integrability of $H_\varepsilon(t)$ other than $L^1_{loc}(\mathbb{R}^d)$ (by classical results, see e.g. [28]). However, exploiting (5.30) and proceeding once again as in the proof of (4.62), we obtain

$$I_{2s} * \left(|x|^{-\gamma}\psi\right) - H_\varepsilon(t) = \int_t^T (a(\tau)+\varepsilon)\,\psi_\varepsilon(\tau)\,\mathrm{d}\tau \ge 0 \quad \text{for a.e. } t \in (0,T)\,,$$

whence, in particular,

$$0 \le H_\varepsilon(x,t_1) \le H_\varepsilon(x,t_2) \le H_\varepsilon(x,T) = I_{2s} * \left(|y|^{-\gamma}\psi\right)(x) \quad \forall x \in \mathbb{R}^d\,, \text{ for a.e. } 0 < t_1 \le t_2 \le T\,. \tag{5.36}$$

The above inequality shows that $H_\varepsilon(t)$ belongs to $L^p(\mathbb{R}^d)$ at least for the same p for which $H_\varepsilon(x,T)$ does, namely for any $p \in (d/(d-2), \infty]$. The fact that (5.36) holds for *every* x rather than for *almost every* x follows by standard potential theory: one uses the strategy of [28, Th. 1.12] and exploits [28, Lem. 1.1] to find that two potentials ordered for a.e. x are actually ordered for every x.
Our next goal is to let $\varepsilon \to 0$. Thanks to the boundedness of $\{|x|^{-\gamma}\psi_\varepsilon(t)\}$ in $L^1(\mathbb{R}^d)$ (trivial consequence of (5.29)), for a.e. $t \in (0,T)$ there exists a subsequence $\{\varepsilon_n\}$ (a priori depending on t) such that $\{|x|^{-\gamma}\psi_{\varepsilon_n}(t)\}$ converges to a positive, finite measure $\nu(t)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$. In order to overcome the possible dependence of $\{\varepsilon_n\}$ on t, we exploit the properties of $\{H_\varepsilon\}$. First notice that (5.36) ensures the uniform boundedness of $\{H_\varepsilon\}$ in $L^p(\mathbb{R}^d \times (0,T))$ for any $p \in (d/(d-2), \infty]$. Such boundedness entails the existence of a subsequence $\{\varepsilon_m\}$ (which can be taken to be decreasing) such that $\{H_{\varepsilon_m}\}$ converges weakly in $L^p(\mathbb{R}^d \times (0,T))$ to a suitable limit H. But Mazur's Lemma implies that there exists a sequence $\{H_k\}$ of convex combinations of $\{H_{\varepsilon_m}\}$ that converges *strongly* to H in $L^p(\mathbb{R}^d \times (0,T))$. By definition, the sequence $\{H_k\}$ is of the form

$$H_k = \sum_{m=1}^{M_k} \lambda_{m,k} H_{\varepsilon_m}\,, \quad \sum_{m=1}^{M_k} \lambda_{m,k} = 1$$

for some sequence $\{M_k\} \subset \mathbb{N}$ and a suitable choice of the coefficients $\lambda_{m,k} \in [0,1]$. With no loss of generality we shall also assume that

$$\lim_{k\to\infty}\left(\sum_{m=1}^{M_k} \varepsilon_m \lambda_{m,k}\right) = 0\,.$$

This can be justified by applying iteratively Mazur's Lemma on suitable subsequences of $\{H_{\varepsilon_m}\}$. Now notice that the function whose Riesz potential is H_k is nothing but

$$f_k(x,t) = \sum_{m=1}^{M_k} \lambda_{m,k} \, |x|^{-\gamma} \psi_{\varepsilon_m}(x,t) \, .$$

Multiplying (5.31) (with $\varepsilon = \varepsilon_m$) by $\lambda_{m,k}$ and summing over k, one gets that f_k satisfies

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} g(x,T)\psi(x) \, |x|^{-\gamma} \, \mathrm{d}x - \int_{\mathbb{R}^d} g(x,t) f_k(x,t) \, \mathrm{d}x \right| \\ & \le \left(\sum_{m=1}^{M_k} \varepsilon_m \lambda_{m,k} \right) (T-t) \, \|\psi\|_{1,-\gamma} \, \|u_2(\tau+h) - u_1(\tau)\|_{L^\infty(\mathbb{R}^d \times (t,T))} \end{aligned} \tag{5.37}$$

for a.e. $t \in (0,T)$, whereas from (5.29) and (5.34) one infers that

$$\left| \int_{\mathbb{R}^d} \psi(x) \, \xi_R(x) \, |x|^{-\gamma} \mathrm{d}x - \int_{\mathbb{R}^d} f_k(x,t) \, \xi_R(x) \, \mathrm{d}x \right| \le C(T-t) \, \|\psi\|_{1,-\gamma} \, \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_\infty}{R^{2s-\gamma}} \tag{5.38}$$

for a.e. $t \in (0,T)$ and

$$\int_{\mathbb{R}^d} \psi(x) \, |x|^{-\gamma} \mathrm{d}x = \int_{\mathbb{R}^d} f_k(x,t) \, \mathrm{d}x \quad \text{for a.e. } t \in (0,T) \, . \tag{5.39}$$

Letting $k \to \infty$ we find that, for a.e. $t \in (0,T)$, there exists a subsequence of $\{f_k(t)\}$ (a priori depending on t) that converges in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$ to a positive, finite measure $\nu(t)$. But the fact that $\{H_k\}$ converges strongly in $L^p(\mathbb{R}^d \times (0,T))$ to H forces the potential of $\nu(t)$ to coincide a.e. with $H(t)$. This is a consequence of [28, Th. 3.8]. By [28, Th. 1.12] we therefore deduce that the limit $\nu(t)$ is uniquely determined by its potential $H(t)$. This identification allows to assert that for a.e. $t \in (0,T)$ the *whole* sequence $\{f_k(t)\}$ converges to $\nu(t)$ in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$.
Passing to the limit in (5.36) (after having set $\varepsilon = \varepsilon_m$, multiplied by $\lambda_{m,k}$ and summed over k) one gets that also the potentials $H(t)$ of $\nu(t)$ are ordered and bounded above by $I_{2s} * (|x|^{-\gamma}\psi)$:

$$0 \le H(x,t_1) \le H(x,t_2) \le I_{2s} * \left(|y|^{-\gamma} \psi \right)(x) \quad \forall x \in \mathbb{R}^d \, , \text{ for a.e. } 0 < t_1 \le t_2 \le T \, . \tag{5.40}$$

Letting $k \to \infty$ in (5.38) yields

$$\left| \int_{\mathbb{R}^d} \psi(x) \, \xi_R(x) \, |x|^{-\gamma} \mathrm{d}x - \int_{\mathbb{R}^d} \xi_R(x) \, \mathrm{d}\nu(t) \right| \le C(T-t) \, \|\psi\|_{1,-\gamma} \, \frac{\| |x|^\gamma (-\Delta)^s(\xi) \|_\infty}{R^{2s-\gamma}} \tag{5.41}$$

for a.e. $t \in (0,T)$, whence, letting $R \to \infty$ in (5.41), we obtain

$$\int_{\mathbb{R}^d} \psi(x) \, |x|^{-\gamma} \mathrm{d}x = \int_{\mathbb{R}^d} \mathrm{d}\nu(t) \quad \text{for a.e. } t \in (0,T) \, . \tag{5.42}$$

Gathering (5.39) and (5.42) we infer that $\{f_k(t)\}$ converges to $\nu(t)$ also in $\sigma(\mathcal{M}(\mathbb{R}^d), C_b(\mathbb{R}^d))$: this allows to pass to the limit in (5.37) to get the identity (by exploiting (5.3) as well)

$$\int_{\mathbb{R}^d} g(x,T)\psi(x) \, |x|^{-\gamma} \mathrm{d}x = \int_{\mathbb{R}^d} g(x,t) \, \mathrm{d}\nu(t) \quad \text{for a.e. } t \in (0,T) \, . \tag{5.43}$$

Note that, as a consequence of the monotonicity given by (5.40) and thanks to (5.41)-(5.43), the curve $\nu(t)$ can be extended to *every* $t \in (0,T]$ so that it still complies with (5.40)-(5.43).
Recalling that $g(x,t) = U_2(x,t+h) - U_1(x,t)$ and that potentials do not increase in time (Lemma 5.1), we have that $g(x,t) \le U_2(x,h) - U_1(x,t_0)$ holds for all $x \in \mathbb{R}^d$ and all $t_0 > t$. Because $\nu(t)$ is a positive, finite measure, this fact and (5.43) imply that

$$\int_{\mathbb{R}^d} g(x,T)\psi(x) \, |x|^{-\gamma} \mathrm{d}x \le \int_{\mathbb{R}^d} \left(U_2(x,h) - U_1(x,t_0) \right) \mathrm{d}\nu(t) \quad \forall t_0 > t \, . \tag{5.44}$$

Our next goal is to let t tend to zero in (5.44). Since the mass of $\nu(t)$ is constant (formula (5.42)), up to subsequences $\nu(t)$ converges to a suitable positive, finite measure ν in $\sigma(\mathcal{M}(\mathbb{R}^d), C_c(\mathbb{R}^d))$. Moreover, by (5.40) we know that the potentials $H(t)$ of $\nu(t)$ are nonincreasing as $t \downarrow 0$: hence, Theorem 3.10 of [28] ensures that $H(t)$ converges a.e. to the potential H_0 of the limit measure ν (which therefore does not depend on the subsequence). We can then pass to the limit in the integral

$$\int_{\mathbb{R}^d} U_1(x, t_0)\, \mathrm{d}\nu(t)\,. \tag{5.45}$$

Indeed, Fubini's Theorem ensures that (5.45) coincides with

$$\int_{\mathbb{R}^d} u_1(x, t_0) H(x, t)\, |x|^{-\gamma} \mathrm{d}x\,. \tag{5.46}$$

Passing to the limit in (5.46) as $t \downarrow 0$ we get that

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} u_1(x, t_0) H(x, t)\, |x|^{-\gamma} \mathrm{d}x = \int_{\mathbb{R}^d} u_1(x, t_0) H_0(x)\, |x|^{-\gamma} \mathrm{d}x\,, \tag{5.47}$$

for instance by dominated convergence. Recalling that H_0 is the potential of ν, and using again Fubini's Theorem, (5.47) can be rewritten as

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} U_1(x, t_0)\, \mathrm{d}\nu(t) = \int_{\mathbb{R}^d} U_1(x, t_0)\, \mathrm{d}\nu\,.$$

One proceeds similarly for the integral

$$\int_{\mathbb{R}^d} U_2(x, h)\, \mathrm{d}\nu(t)\,.$$

Hence, passing to the limit as $t \downarrow 0$ in (5.44) yields

$$\int_{\mathbb{R}^d} g(x, T)\psi(x)\, |x|^{-\gamma} \mathrm{d}x \le \int_{\mathbb{R}^d} (U_2(x, h) - U_1(x, t_0))\, \mathrm{d}\nu \quad \forall t_0 > 0\,. \tag{5.48}$$

Now we let $t_0 \downarrow 0$ in (5.48). By monotone convergence (see Lemmas 5.1 and 5.2) we obtain

$$\int_{\mathbb{R}^d} g(x, T)\psi(x)\, |x|^{-\gamma} \mathrm{d}x \le \int_{\mathbb{R}^d} (U_2(x, h) - U^\mu(x))\, \mathrm{d}\nu\,. \tag{5.49}$$

In this step it is crucial that the limit of $U_1(x, t_0)$ to $U^\mu(x)$ is taken *for all* $x \in \mathbb{R}^d$ (Lemma 5.2), because we have no information over ν besides the fact that it is a positive, finite measure. Still by monotonicity we have that $U_2(x, h) \le U^\mu(x)$ for all $x \in \mathbb{R}^d$. Thus, from (5.49) it follows that

$$\int_{\mathbb{R}^d} g(x, T)\psi(x)\, |x|^{-\gamma} \mathrm{d}x \le 0\,. \tag{5.50}$$

Since (5.50) holds for any $h, T > 0$ and any nonnegative $\psi \in \mathcal{D}(\mathbb{R}^d)$, we infer that $U_2 \le U_1$. Interchanging the role of u_1 and u_2 we also get that $U_1 \le U_2$, whence $U_1 = U_2$ and $u_1 = u_2$. □

Appendix A

We recall here some basic properties of the fractional Laplacian (and a similar nonlocal, nonlinear operator) of compactly supported, regular functions.

Lemma A.1. *The fractional s-Laplacian $(-\Delta)^s(\phi)(x)$ of any $\phi \in \mathcal{D}(\mathbb{R}^d)$ is a regular function which decays (together with its derivatives) at least like $|x|^{-d-2s}$ as $|x| \to \infty$.*

Proof. We sketch the easy proof for the convenience of the reader, following the strategy of [5, Lemma 2.1], Take $r > 0$ such that $\operatorname{supp}\phi \Subset B_r$. We can split the principal value in (2.4) as follows:

$$(-\Delta)^s(\phi)(x) = p.v.\ C_{d,s} \int_{B_{2r}} \frac{\phi(x) - \phi(y)}{|x - y|^{d+2s}}\, \mathrm{d}y + C_{d,s}\, \phi(x) \int_{B_{2r}^c} \frac{1}{|x - y|^{d+2s}}\, \mathrm{d}y \quad \forall x \in \overline{B}_r\,, \tag{A.1}$$

$$(-\Delta)^s(\phi)(x) = -C_{d,s} \int_{\operatorname{supp}\phi} \frac{\phi(y)}{|x - y|^{d+2s}}\, \mathrm{d}y \quad \forall x \in B_r^c\,. \tag{A.2}$$

Given the integrability of y^{-d-2s} as $|y| \to \infty$ and the regularity of $\phi(x)$, the second term on the r.h.s. of (A.1) gives rise to a continuous function of x in $\overline{B}_r$. The same holds true for the principal

value in (A.1). In fact, for $y \to x$, one can replace $\phi(x) - \phi(y)$ with $(x-y)'\nabla_x^2(\phi)(x-y)$, which makes the singularity integrable.
As for (A.2), we have:

$$\int_{\operatorname{supp}\phi} \frac{\phi(y)}{|x-y|^{d+2s}}\,\mathrm{d}y = \frac{1}{|x|^{d+2s}} \int_{\operatorname{supp}\phi} \frac{\phi(y)}{\big|x|x|^{-1} - y|x|^{-1}\big|^{d+2s}}\,\mathrm{d}y \quad \forall x \in B_r^c\,, \tag{A.3}$$

where the integral on the r.h.s. of (A.3) is a continuous and bounded function of x in B_r^c.

The fact that $(-\Delta)^s(\phi)$ is $C^\infty(\mathbb{R}^d)$ (and all of its derivatives decay at least like $|x|^{-d-2s}$ as $|x| \to \infty$) just follows by the identity

$$\frac{\mathrm{d}^k(-\Delta)^s(\phi)}{\mathrm{d}x^k} = (-\Delta)^s\left(\frac{\mathrm{d}^k\phi}{\mathrm{d}x^k}\right) \quad \forall k \in \mathbb{N}\,.$$

□

Lemma A.2. *For any $\phi \in \mathcal{D}(\mathbb{R}^d)$, the function*

$$l_s(\phi)(x) = \int_{\mathbb{R}^d} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}y \quad \forall x \in \mathbb{R}^d$$

is regular and decays (together with its derivatives) at least like $|x|^{-d-2s}$ as $|x| \to \infty$.

Proof. Proceeding exactly as in the proof of Lemma A.1, we obtain:

$$l_s(\phi)(x) = \int_{B_{2r}} \frac{(\phi(x)-\phi(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}y + \phi^2(x)\int_{B_{2r}^c} \frac{1}{|x-y|^{d+2s}}\,\mathrm{d}y \quad \forall x \in \overline{B}_r\,, \tag{A.4}$$

$$l_s(\phi)(x) = \int_{\operatorname{supp}\phi} \frac{\phi^2(y)}{|x-y|^{d+2s}}\,\mathrm{d}y \quad \forall x \in B_r^c\,. \tag{A.5}$$

The second integral on the r.h.s. of (A.4) and (A.5) can be dealt with exactly as in the proof of the quoted lemma. As for the first integral on the r.h.s. of (A.4) just notice that, for $y \to x$, $(\phi(x)-\phi(y))^2$ behaves like $(\nabla_x(\phi)\cdot(x-y))^2$, and this also makes the singularity integrable.

The fact that $l_s(\phi)$ is $C^\infty(\mathbb{R}^d)$ (and derivatives decay at least like $|x|^{-d-2s}$ as $|x| \to \infty$) cannot be proved as in Lemma A.1, since l_s is nonlinear. However, performing the change of variable $z = x - y$, one has:

$$l_s(\phi)(x) = \int_{\mathbb{R}^d} \frac{(\phi(x)-\phi(x-z))^2}{|z|^{d+2s}}\,\mathrm{d}z \quad \forall x \in \mathbb{R}^d\,, \tag{A.6}$$

and taking derivatives of (A.6) one sees that they can be written as sums of terms whose expression is similar to (A.6). □

Lemma A.3. *For any $R > 0$, let ξ_R be the cut-off function*

$$\xi_R(x) = \xi\left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^d\,,$$

where $\xi(x)$ is a positive, regular function such that $\|\xi\|_\infty \le 1$, $\xi = 1$ in B_1 and $\xi = 0$ in B_2^c. Then, $(-\Delta)^s(\xi_R)$ and $l_s(\xi_R)$ enjoy the following scaling properties:

$$(-\Delta)^s(\xi_R)(x) = \frac{1}{R^{2s}}(-\Delta)^s(\xi)\left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^d\,,$$

$$l_s(\xi_R)(x) = \frac{1}{R^{2s}} l_s(\xi)\left(\frac{x}{R}\right) \quad \forall x \in \mathbb{R}^d\,.$$

Proof. We just show the result for $l_s(\xi_R)$, but notice that for $(-\Delta)^s(\xi_R)$ the proof is identical. Letting $\widetilde{y} = y/R$, one has:

$$\begin{aligned} l_s(\xi_R)(x) =& \int_{\mathbb{R}^d} \frac{(\xi_R(x)-\xi_R(y))^2}{|x-y|^{d+2s}}\,\mathrm{d}y \\ =& R^d \int_{\mathbb{R}^d} \frac{(\xi(x/R)-\xi(\widetilde{y}))^2}{|x-R\widetilde{y}|^{d+2s}}\,\mathrm{d}\widetilde{y} = \frac{1}{R^{2s}}\int_{\mathbb{R}^d} \frac{(\xi(x/R)-\xi(\widetilde{y}))^2}{|x/R-\widetilde{y}|^{d+2s}}\,\mathrm{d}\widetilde{y} = \frac{1}{R^{2s}} l_s(\xi)\left(\frac{x}{R}\right) \end{aligned}$$

for all $x \in \mathbb{R}^d$. □

Appendix B

This section is devoted to the statement (and a sketch of the corresponding proof) of a technical result which is a crucial tool in Section 5.

Definition B.1. *Let $d > 2s$ and $\gamma \in (0, 2s)$. We denote as $X_{s,\gamma}$ the Hilbert space of all functions $v \in L^2_{-\gamma}(\mathbb{R}^d)$ such that $(-\Delta)^s(v)$ (as a distribution) belongs to $L^2_\gamma(\mathbb{R}^d)$, equipped with the norm*

$$\|v\|^2_{X_{s,\gamma}} = \|v\|^2_{2,-\gamma} + \|(-\Delta)^s(v)\|^2_{2,\gamma} \quad \forall v \in X_{s,\gamma}\,.$$

Theorem B.2. *Let $d > 2s$ and $\gamma \in (0, 2s)$. Let $A : D(A) = X_{s,\gamma} \subset L^2_{-\gamma}(\mathbb{R}^d) \to L^2_{-\gamma}(\mathbb{R}^d)$ be the operator*

$$A(v) = |x|^\gamma (-\Delta)^s(v) \quad \forall v \in X_{s,\gamma}\,.$$

Then, A is densely defined, positive and self-adjoint on $L^2_{-\gamma}(\mathbb{R}^d)$, and the quadratic form associated to it is

$$Q(v, v) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(v(x) - v(y))^2}{|x - y|^{d+2s}}\, \mathrm{d}x\, \mathrm{d}y\,,$$

with domain $D(Q) = L^2_{-\gamma}(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$.

Moreover, Q is a Dirichlet form on $L^2_{-\gamma}(\mathbb{R}^d)$ and A generates a Markov semigroup $S_2(t)$ on $L^2_{-\gamma}(\mathbb{R}^d)$. In particular for all $p \in [1, \infty]$ there exists a contraction semigroup $S_p(t)$ on $L^p_{-\gamma}(\mathbb{R}^d)$, consistent with $S_2(t)$ on $L^2_{-\gamma}(\mathbb{R}^d) \cap L^p_{-\gamma}(\mathbb{R}^d)$, which is furthermore analytic with a suitable angle $\theta_p > 0$ when $p \in (1, \infty)$.

Sketch of proof. As already remarked, here we shall just give an outline of the techniques that allow to prove the theorem. Full details are given in the note [30].

The basic idea is to start from the validity of the fractional "integration by parts" formula

$$C_{d,s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\phi(x) - \phi(y))(\psi(x) - \psi(y))}{|x - y|^{d+2s}}\, \mathrm{d}x\, \mathrm{d}y = \int_{\mathbb{R}^d} \phi(x)(-\Delta)^s(\psi)(x)\, \mathrm{d}x \tag{B.1}$$

for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$ and then to extend it to all functions of $X_{s,\gamma}$. In order to do it, the first step consists in showing that $C^\infty(\mathbb{R}^d) \cap X_{s,\gamma}$ is dense in $X_{s,\gamma}$. This can be done by mollification arguments, which however are slightly more complicated than the standard ones, since one works with $L^2_{-\gamma}(\mathbb{R}^d)$ and $L^2_\gamma(\mathbb{R}^d)$ instead of $L^2(\mathbb{R}^d)$. Hence, given $v, w \in C^\infty(\mathbb{R}^d) \cap X_{s,\gamma}$, one plugs the cut-off functions $\phi = \xi_R v$ and $\psi = \xi_R w$ into (B.1) and lets $R \to \infty$. The problem is that on the r.h.s. there appear terms involving $\|\xi_R w\|_{\dot{H}^s}$, and a priori one does not know whether $C^\infty(\mathbb{R}^d) \cap X_{s,\gamma}$ is continuously embedded in $\dot{H}^s(\mathbb{R}^d)$. But this turns out to be true: the inequality

$$C_{d,s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(w(x) - w(y))^2}{|x - y|^{d+2s}}\, \mathrm{d}x\, \mathrm{d}y \le \int_{\mathbb{R}^d} w(x)(-\Delta)^s(w)(x)\, \mathrm{d}x \quad \forall w \in C^\infty(\mathbb{R}^d) \cap X_{s,\gamma} \tag{B.2}$$

can be proved just by repeating the above scheme with $\phi = \psi = \xi_R w$. In fact, on the r.h.s. of (B.1) one still has terms involving $\|\xi_R w\|_{\dot{H}^s}$, but the latter are small and can be absorbed into the l.h.s.; passing to the limit as $R \to \infty$ yields (B.2). Therefore, having at our disposal (B.2), we can now let $R \to \infty$ safely in (B.1) (with $\phi = \xi_R v$ and $\psi = \xi_R w$) and obtain that

$$C_{d,s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(v(x) - v(y))(w(x) - w(y))}{|x - y|^{d+2s}}\, \mathrm{d}x\, \mathrm{d}y = \int_{\mathbb{R}^d} v(x)(-\Delta)^s(w)(x)\, \mathrm{d}x \tag{B.3}$$

for all $v, w \in C^\infty(\mathbb{R}^d) \cap X_{s,\gamma}$, which then shows that (B.2) is actually an equality. Notice that in all these approximation procedures using cut-off functions, to prove that "remainder" terms go to zero we deeply exploit the results provided by Lemmas A.1, A.2 and A.3. It is in fact here that the condition $\gamma < 2s$ plays a fundamental role: in particular, it ensures that both $\||x|^\gamma(-\Delta)^s(\xi_R)\|_\infty$ and $\||x|^\gamma l_s(\xi_R)\|_\infty$ vanish as $R \to \infty$. As already mentioned, we refer the reader to the note [30] for

the details; however, for similar computations involving $(-\Delta)^s(\xi_R)$ and $l_s(\xi_R)$, see also the proofs of Proposition 4.1, Lemma 4.4 and Lemma 5.6.
By the claimed density of $C^\infty(\mathbb{R}^d) \cap X_{s,\gamma}$, we are allowed to extend (B.3) to the whole of $X_{s,\gamma}$. Clearly, the r.h.s. of (B.3) can be rewritten as

$$\int_{\mathbb{R}^d} v(x)\, A(w)(x)\, |x|^{-\gamma} \mathrm{d}x \,,$$

and letting $v = w$ in particular entails that the operator A is positive. The fact that it is densely defined is trivial since, for instance, $\mathcal{D}(\mathbb{R}^d) \subset X_{s,\gamma}$. Because in (B.3) one can interchange the role of v and w, one also has that A is symmetric. In order to prove that it is self-adjoint one needs to show that $D(A^*) \subset D(A)$, namely that any function of $D(A^*)$ also belongs to $X_{s,\gamma}$. It is indeed straightforward to check this fact, and we leave the details to the reader.

We now deal with the quadratic form Q associated to A. Thanks to (B.3), we have that

$$Q(v,v) = C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x)-v(y))^2}{|x-y|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}y \quad \forall v \in D(A) \,. \tag{B.4}$$

As it is well-known (see e.g. [12]), the domain $D(Q)$ of Q is just the closure of $D(A)$ w.r.t. the norm

$$\|v\|_Q^2 = \|v\|_{2,-\gamma}^2 + Q(v,v) = \|v\|_{2,-\gamma}^2 + \|v\|_{\dot{H}^s}^2 \,.$$

It is then easy to see that such a closure is nothing but $L^2_{-\gamma}(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ and the quadratic form on $D(Q) = L^2_{-\gamma}(\mathbb{R}^d) \cap \dot{H}^s(\mathbb{R}^d)$ is still represented by (B.4).

By classical results (we refer again to [12]), proving that A generates a Markov semigroup is equivalent to proving that if v belongs to $D(Q)$ then both $v \vee 0$ and $v \wedge 1$ belong to $D(Q)$ and satisfy

$$Q(v \vee 0, v \vee 0) \le Q(v,v) \,, \quad Q(v \wedge 1, v \wedge 1) \le Q(v,v) \,.$$

But the latter properties are straightforward consequences of the characterization of Q given above.

The last assertions follow from the general theory of symmetric Markov semigroups (cf. [12, Sect. 1.4]) and from their known analiticity properties (cf. [12, Th. 1.4.2]). □

The next proposition extends the symmetry property of the operator $A = |x|^\gamma(-\Delta)^s$ to functions which belong to other suitable $L^p_{-\gamma}$ spaces. This is essential in proving our uniqueness Theorem 3.3 for certain values of γ in low dimensions $d \le 3$, more precisely whenever $(d-\gamma)/(d-2s) > 2$.

Proposition B.3. *Let $d > 2s$ and $\gamma \in (0, 2s)$. Let $p \in [2, 2(d-\gamma)/(d-2s))$ and $p' = p/(p-1)$ be its conjugate exponent. Suppose that $v, w \in L^p_{-\gamma}(\mathbb{R}^d)$ are such that $A(v), A(w) \in L^{p'}_{-\gamma}(\mathbb{R}^d)$. Then $v, w \in \dot{H}^s(\mathbb{R}^d)$ and the following formula holds true:*

$$\begin{aligned}\int_{\mathbb{R}^d} v(x)(-\Delta)^s(w)(x)\, \mathrm{d}x &= \int_{\mathbb{R}^d} (-\Delta)^s(v)(x)\, w(x)\, \mathrm{d}x \\ &= C_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{d+2s}} \, \mathrm{d}x\mathrm{d}y \,.\end{aligned}$$

Sketch of proof. The method of proof proceeds along the lines of the one of Theorem B.2. The main difference here lies in the fact that, when using the approximation procedure by cut-off functions mentioned above, if p is *strictly larger* than 2 in order to prove that "remainder" terms go to zero one cannot exploit the fact that $|x|^\gamma(-\Delta)^s(\xi_R)$ and $|x|^\gamma l_s(\xi_R)$ vanish in $L^\infty(\mathbb{R}^d)$ as $R \to \infty$. In fact, such remainder terms are of the form

$$\int_{\mathbb{R}^d} v^2(x)(-\Delta)^s(\xi_R)(x)\, \mathrm{d}x \quad \text{or} \quad \int_{\mathbb{R}^d} v^2(x) l_s(\xi_R)(x)\, \mathrm{d}x \,. \tag{B.5}$$

Thanks to Lemmas A.1, A.2 and A.3 it is direct to see that $\||x|^\gamma(-\Delta)^s(\xi_R)\|_{q,-\gamma}$ and $\||x|^\gamma l_s(\xi_R)\|_{q,-\gamma}$ vanish as $R \to \infty$ provided $q > (d-\gamma)/(2s-\gamma)$, whence the condition $p \in [2, 2(d-\gamma)/(d-2s))$ to ensure that also the integrals in (B.5) go to zero as $R \to \infty$. □

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