Infinitesimal invariant and Massey products

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Abstract

In this work, we study the Griffiths infinitesimal invariant of the curve in the jacobian using secondary cohomology maps. In order to this, we construct a special differential graded algebra $\mathcal{A}$, quite similar to the Kodaira-Spencer algebra and we define a natural triple Massey product on it. This allows us to give a description of the infinitesimal invariant in terms of Massey products and, by the way, to study the formality of $\mathcal{A}$.

Introduction

In Hodge theory, algebraic cycles can be studied using the "cycle map": this map can be seen as the "first" link between the theory of algebraic cycles and the study of Hodge structures. Its main refinament is the Abel-Jacobi cycle map, which involves a secondary class with values into the complex torus given by the intermediate jacobian. However, the integral structure on cohomology is trascendental and, especially in higher codimension, this approach can be very difficult. For this reason, Griffiths introduced a more algebraic tool, the infinitesimal variation of Hodge structures ([7], [8]). In this setting, it is possible to construct infinitesimal invariant associated to families of algebraic cycles in families of varieties, to study behaviours and properties of the cycles. Otherwise, it is easy to expect that one-parameter deformations of algebraic structures take into account certain conditions on cohomology classes: often, these conditions can be expressed in terms of secondary maps, as for example in [10].

The aim of this paper is to study the Griffiths infinitesimal invariant of the curves in their jacobians using the language of Massey products on a special differential graded algebra.

Let C be a smooth complete connected curve of genus $g > 2$ and denote with J its jacobian variety. Chosen a base point $p \in C$, consider the canonical morphism $C \longrightarrow J$: the image of C in J is an algebraic cycle W of codimension $g-1$. The basic cycle $Z \in CH^{g-1}(J)$ associated to C is defined by

$$Z = W - W^-$$

where W^- is the image of W under the involution $j : J \longrightarrow J$. It is well known that Z is homologically equivalent to zero but W is not algebraically equivalent to W^-, for the generic curve C (see [1]).

The idea to use special secondary maps in cohomology to study the algebraically inequivalence of W and W^- was suggested to us by paper of B. Harris [10]. In fact, he introduced the tecniques of harmonic volumes and iterated integrals to detect non trivial cycles in the Griffiths group. On a non singular algebraic curve of genus $g \geq 3$, consider three real harmonic 1-forms with integer

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periods $\vartheta_1, \vartheta_2, \vartheta_3$ on C. Suppose that $\int_C \vartheta_i \wedge \vartheta_j = 0$ for $i, j = 1, 2, 3$ and $i \neq j$. The iterated integral associates to the triple of harmonic forms a point in the real torus $\mathbb{R}/\mathbb{Z}$

$$\int_\gamma (h_1 \vartheta_2 - \eta_{12}) \text{ modulo } \mathbb{Z},$$

where γ is a path in C, dual to the cohomology class of ϑ_3, η_{12} is a 1-form on C such that $d\eta_{12} = \vartheta_1 \wedge \vartheta_2$ and h_1 is a function on γ obtained by integrating ϑ_1. Moreover, it is possible to interprete the harmonic volume in terms of iterated integrals, showing, by its explicit computation, that Z is not algebraically trivial.

This suggests us the possibility to interprete the Griffiths infinitesimal invariant using the techniques of Massey products. In order to this, we will associate to the curve C, a special differential graded algebra $\mathcal{A}$ on which it is possible to define a triple Massey product.

Here we give a brief sketch of the construction of $\mathcal{A}$ and of the triple Massey product on it.

Consider a smooth complex non-hyperelliptic curve C of genus $g > 2$ and let ω_C be the canonical bundle on C. The idea of the construction of a dga $\mathcal{A}$ associated to ω_C is the following: we define as vector spaces, in degree even, the spaces of $\mathcal{C}^\infty$-sections of ω_C^k, $A^0(\omega_C^k)$, while in degree odd, the spaces of holomorphic forms of ω_C^k, $A^1(\omega_C^k)$. Moreover, the differential is given by the Dolbeault operator and the zero map in alternating cases. Note that, in this case, our dga is similar to the Kodaira-Spencer algebra.

The definition of the Massey product on $\mathcal{A}$ is quite natural. Consider w_1, w_2 two holomorphic forms in $H^0(\omega_C)$ such that both are orthogonal to a third form $\overline{\sigma}$ with $\sigma \in H^1(\mathcal{O}_C)$. Since $[w_i]\cdot[\sigma] = 0$ for $i = 1, 2$, this means there exist two elements ρ_1, ρ_2 in $A^0(\omega_C)$ such that $w_i \cdot \sigma = \overline{\partial}\rho_i$. It is possible to define the triple Massey product

$$\mathcal{M}(w_1, \sigma, w_2) = \rho_1 w_2 - \rho_2 w_1.$$

This element lies in $H^0(\omega_C^2)$ but, obviously, it depends on the choice of ρ_i. Let $\mathcal{I}([w_1], [w_2])$ be the image of $H^0(\omega_C) \otimes < [w_1], [w_2] >$ in $H^0(\omega_C^2)$. Then $\mathcal{M}(w_1, \sigma, w_2)$ is a well defined element in $H^0(\omega_C)/\mathcal{I}([w_1], [w_2])$.

We will show the following

Theorem 1. *For smooth non-hyperelliptic curves of genus $g > 2$, the Griffiths' infinitesimal invariant of the cycle can be computed in terms of the triple Massey product.*

More precisely, let $\xi \in H^1(T_C)$ be a first order deformation of C: the infinitesimal invariant $\phi(\xi \otimes w_1 \wedge w_1 \wedge \sigma)$ is equal, up to a constant, to $< \xi, \mathcal{M}(w_1, \sigma, w_2) >$ where $< \, , \, >$ denotes the Poincarè duality. Moreover, it is possible to construct a secondary map in cohomology which involves into the Massey product we had defined: it is easy to see that this map, defined in section 3, is again strictly linked to the infinitesimal invariant.

Theorem 1 allows us also to study and recognize some properties of the dga $\mathcal{A}$. In fact, Massey products could be used to study the formality of the dga. Formality is an important homotopy property, since the rational homotopy type of any nilpotent formal space can be reconstructed by some formal procedure from its cohomology algebra. The first characterization of formal spaces was given in [3] and states that a space is formal if and only if all rational Massey products vanish. Examples of formal spaces are compact Kahler manifolds and compact symmetric spaces (see [3]).

Then we are able to provide an example of non formal algebra, showing the following

Theorem 2. *For the generic non-hyperelliptic curve C of genus $g > 2$, the differential graded algebra $\mathcal{A}$ associated to ω_C is not formal.*

The computation of the infinitesimal invariant of the Ceresa cycle has been performed in [2]: we will use our interpretation of the Griffiths invariant to show that, for smooth non-hyperelliptic curves of genus $g > 2$, the Massey product on $\mathcal{A}$ does not vanish.

The paper is organized as follows.

In Section 1 we recall definitions and main results concerning differential graded algebras and we introduce triple Massey products. Section 2 is dedicated to construct the dga $\mathcal{A}$ for a generic line bundle L and we describe the dga of its cohomology. Moreover, we define a triple Massey product on $\mathcal{A}$ and we underline some properties of it. In section 3, we will construct a secondary map in cohomology which takes into account the Massey product we had defined. Finally, in Section 4, we recall the definition of Griffiths' infinitesimal invariant of the Ceresa cycle and we explain how the infinitesimal invariant can be computed using Massey products. This allows us to prove Theorem 1 and Theorem 2.

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1 Preliminaries on formality and Massey Products

In this section, we want to give a survey of differential graded algebras and Massey products on them. Then we introduce the concept of formality and we show how it is possible to prove the formality using Massey products ([12], [3], [4]).

Definition 1.1. *A differential graded algebra $\mathcal{A}$ (dga for short) is an algebra*

$$\mathcal{A} = \bigoplus_{k \in \mathbb{N}} \mathcal{A}_k$$

over a field $\mathbb{K}$ endowed with a product $\mathcal{A}_p \times \mathcal{A}_q \longrightarrow \mathcal{A}_{p+q}$ and a differential $d : \mathcal{A}_p \longrightarrow \mathcal{A}_{p+1}$ of degree 1 such that:

- *the product is graded commutative, i.e. for $a \in \mathcal{A}_p$ and $b \in \mathcal{A}_q$, $a \cdot b = (-1)^{pq} b \cdot a$;*
- *the differential satisfies $d^2 = 0$ and it is a derivation: for $a \in \mathcal{A}_p$ and $b \in \mathcal{A}_q$,*

$$d(a \cdot b) = d(a) \cdot b + (-1)^p a \cdot d(b).$$

It is possible to define in the standard way the cohomology $H^\bullet(\mathcal{A})$ of the dga $(\mathcal{A}, d)$, where

$$H^p(\mathcal{A}) = \frac{Ker\{\, d : \mathcal{A}_p \longrightarrow \mathcal{A}_{p+1}\}}{Im\ \{d : \mathcal{A}_{p-1} \longrightarrow \mathcal{A}_p\}}.$$

Note that the cohomology of a dga is itself an example of differential graded algebra as soon as we define the differential to be the zero map $(H^\bullet(\mathcal{A}), d = 0)$.

An algebra $(\mathcal{A}, d)$ is *connected* if $H^0(\mathcal{A}) = \mathbb{K}$; if, in addition $H^1(\mathcal{A}) = 0$, then we say that $\mathcal{A}$ is *simply connected*. A morphism of dga is a map which preserves the structure of degree, products and differentials. We say that a morphism is a quasi-isomorphism if the induced map in cohomology is an isomorphism.

Given $(\mathcal{A}, d)$, one would construct another differential graded algebra which is minimal in the following sense.

Definition 1.2. *A dga $(\mathcal{A}, d)$ is said to be minimal if:*

1. *it is free as an algebra, that is, it is a tensor product of polynomial algebras on generators of even degrees and exterior algebras on generators of odd degrees.*
2. *d is decomposable, that is $d(\mathcal{A}) \subseteq \mathcal{A}^+ \cdot \mathcal{A}^+$, where $\mathcal{A}^+ = \bigoplus_{i>0} \mathcal{A}_i$.*

A minimal model for a dga $(\mathcal{A}, d)$ is a minimal dga $\mathcal{B}$ together with a map of differential graded algebras $\phi : \mathcal{B} \longrightarrow \mathcal{A}$ which is a quasi-isomorphism.
It is well known ([12], [3]) that any connected algebra $\mathcal{A}$ having finite dimensional cohomology in each degree has minimal model. Moreover, this model is unique up to isomorphism.

A dga $(\mathcal{A}, d)$ is called **formal** if there is a map $\psi : \mathcal{A} \longrightarrow H^\bullet(\mathcal{A})$ of degree zero which is a quasi-isomorphism. Note that this is equivalent to saying that $\mathcal{A}$ and $H^\bullet(\mathcal{A})$ have isomorphic minimal models. An easy way to study the formality of a dga is to compute Massey products on it. So we are going to introduce Massey products.

If a is an element in $\mathcal{A}$, we write $[a]$ for the corresponding cohomology class while if $a \in \mathcal{A}_p$ is a element of degree p, we write $\overline{a}$ for $(-1)^p a$. Let $[a] \in H^p(\mathcal{A}), [b] \in H^q(\mathcal{A}), [c] \in H^s(\mathcal{A})$ be cohomology classes and let $a \in \mathcal{A}^p$, $b \in \mathcal{A}^q$ and $c \in \mathcal{A}^s$ be cocycles representing these cohomology classes in $\mathcal{A}$. Suppose that

$$[a][b] = [b][c] = 0.$$

These conditions mean that there are two elements $x \in \mathcal{A}_{p+q-1}$ and $y \in \mathcal{A}_{p+s-1}$ such that $dx = \overline{a}\, b$ and $dy = \overline{b}\, c$. One can check that the element $\overline{a}\; y + \overline{x}\; c$ is closed and therefore it determines a cohomology class

$$[\overline{a}\; y + \overline{x}\; c] \in H^{p+q+s-1}(\mathcal{A}).$$

Note that this class depends on the choices of x and y in $H^{p+q-1}(\mathcal{A})$ and $H^{p+s-1}(\mathcal{A})$ respectively. The indeterminacy of $[\overline{a}\; y + \overline{x}\; c]$ is the set of elements of the form $a\, u + c\, v$ where $u, v \in H^\bullet(\mathcal{A})$ are arbitrary elements with $\deg(a\; u) = \deg(c\; v) = \deg(a) + \deg(b) + \deg(c) - 1$. Denote with $\mathcal{I}([a], [c])$ the ideal of $H^{p+q+s-1}(\mathcal{A})$ generated by elements $[a]$ and $[c]$. The **triple Massey product** of the cohomology classes $[a], [b], [c]$ is the coset

$$[\overline{a}\; y + \overline{x}\; c] \in H^\bullet(\mathcal{A})/\mathcal{I}([a], [c])$$

and it is denoted by $\mathcal{M}(a, b, c)$.

The Massey products are natural in the following sense.

Remark 1.3. *Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a morphism of differential graded algebras. For every classes $a_1, a_2, a_3 \in H^\bullet(\mathcal{A})$ for which $\mathcal{M}(a_1, a_2, a_3)$ is defined, we have*

$$f_* \mathcal{M}(a_1, a_2, a_3) \subseteq \mathcal{M}(f_* a_1, f_* a_2, f_* a_3).$$

Note that if f is a quasi-isomorphism, then we have the equality in the previous relation. Thus if $\mathcal{M}$ is the minimal model for $\mathcal{A}$, the Massey product in $H^\bullet(\mathcal{A})$ can be computed on the basis of this minimal model. The proof follows directly from the definition.

The property of Massey operators, that is the most important one for this paper, is expressed by the following result.

Theorem 1.4. *Let $\mathcal{A}$ be a formal differential graded algebra. Then all triple Massey products in $H^\bullet(\mathcal{A})$ are trivial.*

Proof. Let $(\mathcal{M}, d)$ be the minimal model for $\mathcal{A}$ and let $\mathcal{H} = H^\bullet(\mathcal{A})$. By the formality of $\mathcal{A}$, there is a quasi-isomorphism $f : (\mathcal{M}, d) \longrightarrow (\mathcal{H}, d_\mathcal{H} = 0)$. So we can compute the Massey products in $\mathcal{H}$ from $(\mathcal{M}, d)$ alone: let $\mathcal{M}(a_1, a_2, a_3)$ be a triple Massey product, then $\mathcal{M}(a_1, a_2, a_3) = \mathcal{M}(a_1, a_2, a_3)_\mathcal{H}$ where the second member of the equality denotes a Massey product in $\mathcal{H}$. But, since $d_\mathcal{H} = 0$, it is clear that all these products are zero. □

2 Construction of a differential graded algebra

This section is devoted to construct a special differential graded algebra $(\mathcal{A}, d)$.

Let C be a smooth complete algebraic curve of genus g. Consider a non trivial line bundle L on C and denote with $\mathcal{L}$ the sheaf of the holomorphic forms of L. Recall that $A^0(L^k) = A^{0,0}(L^k)$ is the vector space of the $\mathcal{C}^\infty$ sections on the sheaf $\mathcal{L}^k$ while $A^1(L^k) = A^{0,1}(L^k)$ denotes the space of the $(0,1)$-forms with coefficients in $\mathcal{L}^k$. Moreover, it is well known that the map $\overline{\partial}_{L^k} : A^{0,0}(L^k) \longrightarrow A^{0,1}(L^k)$ is defined by the Dolbeault operator and that the kernel of this map is given by the holomorphic sections of L^k.

We define the graded algebra $\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ as the family of vector spaces

$$\mathcal{A}_n = \begin{cases} A^0(L^k) & \text{if } n = 2k \\ A^1(L^k) & \text{if } n = 2k+1 \end{cases} \qquad \forall k \in \mathbb{N}.$$

We have to introduce an inner product $\mathcal{A}_i \times \mathcal{A}_j \longrightarrow \mathcal{A}_{i+j}$ which has to be graded commutative. There are three cases to consider corresponding to products between spaces with even or odd index. We begin by considering the following product $\mathcal{A}_{2k} \times \mathcal{A}_{2h} \longrightarrow \mathcal{A}_{2(k+h)}$. Using the definition of $\mathcal{A}_n$, this is equivalent to consider the product of holomorphic sections

$$A^0(L^k) \times A^0(L^h) \longrightarrow A^0(L^{k+h})$$

which is commutative. In the case of product between spaces of different indexes $\mathcal{A}_{2k} \times \mathcal{A}_{2h+1} \longrightarrow \mathcal{A}_{2(k+h)+1}$, we have a similar situation as soon as we rewrite this product in the following way

$$A^0(L^k) \times A^1(L^h) \longrightarrow A^1(L^{k+h}).$$

Finally, the third case is given by the product between a couple of spaces with odd index $\mathcal{A}_{2k+1} \times \mathcal{A}_{2h+1} \longrightarrow \mathcal{A}_{2(k+h+1)}$. We define this product to be zero, since the product between element in $A^1(L^k)$ and $A^1(L^h)$ should be a $(0,2)$-form with coefficient in $\mathcal{L}^{k+h}$ but forms of such type don't exist on a curve.

So we have just shown the following

Proposition 2.1. *The inner product $\mathcal{A}_i \times \mathcal{A}_j \longrightarrow \mathcal{A}_{i+j}$ is graded commutative.*

Now we need to introduce the differential d. For convenience, we denote with d_n the differential operator $d_n : \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$. We define it in the following way

$$d_n = \begin{cases} \overline{\partial}_{L^k} & \text{if } n = 2k \\ 0 & \text{if } n = 2k+1 \end{cases} \qquad \forall k \in \mathbb{N} \tag{1}$$

where $\overline{\partial}_{L^k} : A^0(L^k) \longrightarrow A^{0,1}(L^k)$ is the differential induced by the Dolbeault operator.

We have to check that d is a derivation. Note that the only case we need to consider is the following $d(a \cdot b) = \overline{\partial}_{L^{k+h}}(a \cdot b)$ where $a \in \mathcal{A}_{2k}$ and $b \in \mathcal{A}_{2h}$. In fact, in the other cases, the proof is obvious since if $c \in \mathcal{A}_{2k+1}$, by definition we have $d(c) = 0$. Now the element $a \in \mathcal{A}_{2k} = A^0(L^k)$ can be written as $f\alpha^k$ where α is an holomorphic section of $A^0(L)$; in the same way, we have $b = g\alpha^h \in A^0(L^h)$. Then showing the Leibnitz rule is easy because the product becomes

$$\overline{\partial}(fg\,\alpha^{k+h}) = \overline{\partial}(fg)\,\alpha^{k+h} = [\overline{\partial}(f)\,g + \overline{\partial}(g)\,f]\,\alpha^{k+h}.$$

To visualize the chain complex, we can write it in the following way

$$\mathcal{A}_0 \xrightarrow{\overline{\partial}} \mathcal{A}_1 \xrightarrow{0} \mathcal{A}_2 \xrightarrow{\overline{\partial}_L} \mathcal{A}_3 \longrightarrow \cdots \longrightarrow \mathcal{A}_{2k} \xrightarrow{\overline{\partial}_{L^k}} \mathcal{A}_{2k+1} \xrightarrow{0} \mathcal{A}_{2k+2} \xrightarrow{\overline{\partial}_{L^{k+1}}} \mathcal{A}_{2n+3} \longrightarrow \cdots$$

We have shown the following

Theorem 2.2. *The graded algebra $\mathcal{A}$ endowed with the inner product $\mathcal{A}_i \times \mathcal{A}_j \longrightarrow \mathcal{A}_{i+j}$, described before, and the differential d, defined by (1), is a differential graded algebra.*

It is easy to compute the cohomology of this dga $\mathcal{A}$.

Remark 2.3. *The cohomology of the differential graded algebra $\mathcal{A}$ is the following*

$$H^{2k}(\mathcal{A}) = H^0(L^k) \qquad H^{2k+1}(\mathcal{A}) = H^1(L^k).$$

The proof is quite simple: it is enough to recall the definition of the Dolbeaut cohomology. In fact, by definition of $H^p(\mathcal{A})$, we have

$$H^{2k}(\mathcal{A}) = Ker\{\, \overline{\partial}_{L^k} : A^0(L^k) \longrightarrow A^1(L^k)\} = H^0(L^k)$$

$$H^{2k+1}(\mathcal{A}) = \frac{A^1(L^k)}{\overline{\partial}_{L^k}(A^0(L^k))} = H^1(L^k)$$

2.1 A new Massey product

Now we are going to construct a triple Massey product on our dga $\mathcal{A}$.

Let $[s_1], [s_2]$ be two elements in $H^2(\mathcal{A})$ and let $[\tau]$ be a class in $H^1(\mathcal{A})$. Denote with s_1, s_2 two representants of the cohomology classes in $\mathcal{A}_2$, while τ is a representant in $\mathcal{A}_1$. Suppose that $[s_i]\cdot[\tau] = 0$ for $i = 1, 2$ in $H^3(\mathcal{A})$. This means that $s_i \cdot \tau$ is an exact element. Then, there is an element $r_i \in \mathcal{A}_2$ such that

$$s_i \cdot \tau = \overline{\partial} r_i \qquad \text{for} \quad i = 1, 2.$$

The triple Massey product of $[s_1], [\tau], [s_2]$ is the cocycle in $H^4(\mathcal{A})$

$$\mathcal{M}(s_1, \tau, s_2) = r_1 s_2 - r_2 s_1$$

defined modulo the ideal spanned by $[s_1]$ and $[s_2]$.

2.1.1 Computations on the Massey product

We would give an "interpretation" of this Massey product when $L = \omega_C$ with the language of vector bundles. This will allow us to show the relation with the Griffiths invariant and to detect directly whether the Massey product vanish.

Let $[w_1], [w_2]$ be two elements in $H^2(\mathcal{A})$ and let $[\sigma]$ be a class in $H^1(\mathcal{A})$. We begin by noting that $H^4(\mathcal{A})$ corresponds to $H^0(\omega_C^2)$. Let $[\xi]$ be an element in $H^1(T_C)$. So, using the Poincaré duality, we can contract our Massey product against $[\xi]$: explicity we have

$$< \xi, \mathcal{M}(w_1, \sigma, w_2) >= \int_C \xi \wedge (r_1 w_2 - r_2 w_1) = \int_C r_1 \wedge \xi \cdot w_2 - r_2 \wedge \xi \cdot w_1. \qquad (2)$$

Since $H^1(T_C) \simeq Ext^1(\omega_C, \mathcal{O}_C)$, the element $[\xi]$ can be seen as the extension class of the following exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow \omega_C \longrightarrow 0, \qquad (3)$$

where E is a rank 2 vector bundle. Since we have fixed $[\xi]$, the coboundary map induced in cohomology $\delta_\xi : H^0(\omega_C) \longrightarrow H^1(\mathcal{O}_C)$ is given by the cup product with the extension class. Let $\mathcal{K} = Ker(\delta_\xi)$. Assume that $\dim(\mathcal{K}) \geq 2$ and suppose that w_1 and w_2 lie in this kernel, i.e. $[\xi]\cdot[s_i] = 0$ for $i = 1, 2$. This means that there are two elements h_i for $i = 1, 2$ in $A^0(\mathcal{O}_C)$ such that

$$\xi \cdot s_i = \overline{\partial} h_i.$$

So the integral (2) can be rewritten in the following way

$$\begin{aligned}\int r_1(\overline{\partial} h_2) - r_2(\overline{\partial} h_1) &= \int h_1(\overline{\partial} r_2) - h_2(\overline{\partial} r_1) = \\ &= \int h_1(w_2 \cdot \sigma) - h_2(w_1 \cdot \sigma) = \int (h_1 w_2 - h_2 w_1) \cdot \sigma.\end{aligned}$$

Note that $(h_1 w_2 - h_2 w_1)$ is a section in $H^0(\omega_C)$. We would interprete this element in terms of a special map, the *"adjunction"* map, introduced in [2], to study the infinitesimal invariant. (For more details, see also [11])

Consider again the sequence (3). The vector bundle E has canonical determinant: then it is well defined the determinant map $\Delta : \bigwedge^2 H^0(E) \longrightarrow H^0(\bigwedge^2 E) \longrightarrow H^0(\omega_C)$. Fix a 2-dimensional subspace $\mathcal{U}$ in $\mathcal{K}$. Set $\rho^{-1}(\mathcal{U}) = W \subset H^0(E)$ where $\rho : H^0(E) \longrightarrow H^0(\omega_C)$. So we can consider the restriction of the map Δ to $\bigwedge^2 W$

$$\Delta_W : \bigwedge^2 W \hookrightarrow \bigwedge^2 H^0(E) \longrightarrow H^0(\omega_C). \tag{4}$$

Let V be the space generated by all the images of W in $H^0(\omega_C)$ and consider the map $\beta : \bigwedge^2 W \longrightarrow H^0(\omega_C)/V$ as the composition of Δ_W with the quotient map. It is easy to show that β factors through $\bigwedge^2 \mathcal{U}$ to give a map

$$\alpha_\xi : \bigwedge^2 \mathcal{U} \longrightarrow H^0(\omega_C)/V. \tag{5}$$

Suppose that w_1, w_2 lie in $\mathcal{U}$. We can lift these two sections to $\tilde{w}_1, \tilde{w}_2 \in H^0(E)$. In local coordinates, they can be written as $\tilde{w}_i = w_i + h_i d\epsilon$ where $h_i \in A^0(\mathcal{O}_C)$ for $i = 1, 2$. The map α acts on the element $w_1 \wedge w_2$ as

$$det \begin{bmatrix} w_1 & w_2 \\ h_1 & h_2 \end{bmatrix}.$$

Hence we can conclude that

$$< \xi, \mathcal{M}(w_1, \sigma, w_2) >= \int \sigma \wedge \alpha_\xi(w_1 \wedge w_2). \tag{6}$$

2.1.2 Some remarks on the dga $\mathcal{A}$ and its Massey products

The definition of a dga can be, in some way, generalized if we define the algebra $\mathcal{A}$ as a $\mathbb{Z}$ graded vector space. We have to say that the theory of minimal models of dga is developed especially for algebras graded on $\mathbb{N}$. For this reason, studying these algebras could not be easy, althought interesting. However, in this paper, we want to show an other way to compute the infinitesimal invariant using a dga graded on $\mathbb{Z}$: this allows us to underline a kind of "duality" between Massey products.

We define the extended version of the dga $\mathcal{A}$ simply adding the negative degree parts of our algebra. Then let $\hat{\mathcal{A}}$ be the $\mathbb{Z}$ graded vector space defined by

$$\mathcal{A}_n = \begin{cases} A^0(\omega_C^k) & \text{if } n = 2k \\ A^1(\omega_C^k) & \text{if } n = 2k+1 \end{cases} \qquad \forall k \in \mathbb{Z}.$$

This algebra is endowed with the usual inner product and the differential d exactly given by (1) extended also to negative degree spaces.

$$\cdots \longrightarrow \mathcal{A}_{-2} \xrightarrow{\overline{\partial}} \mathcal{A}_{-1} \xrightarrow{0} \mathcal{A}_0 \xrightarrow{\overline{\partial}} \mathcal{A}_1 \xrightarrow{0} \mathcal{A}_2 \longrightarrow \cdots \longrightarrow \mathcal{A}_{2k} \xrightarrow{\overline{\partial}} \mathcal{A}_{2k+1} \xrightarrow{0} \mathcal{A}_{2k+2} \xrightarrow{\overline{\partial}} \mathcal{A}_{2k+3} \longrightarrow \cdots$$

Note that on $\hat{\mathcal{A}}$, the Massey product $\mathcal{M}(w_1, \sigma, w_2)$ is again well defined. We would like to define an other triple Massey product on the dga $\hat{\mathcal{A}}$ in the following way.

Let $[w_1], [w_2]$ be two representatives in $H^2(\mathcal{A})$ and let $[\xi]$ be a class in $H^{-1}(\mathcal{A})$. Assume that $[\xi] \cdot [w_i] = 0$ for $i = 1, 2$. Consider the bilinear product $\mathcal{A}_2 \times \mathcal{A}_{-1} \longrightarrow \mathcal{A}_1$: then we can find two functions on C, h_1 and h_2 in $\mathcal{A}_0$, such that

$$\xi \cdot w_i = \overline{\partial} h_i \quad \text{for} \quad i = 1, 2.$$

The triple Massey product $\mathcal{M}(w_1, \xi, w_2)$ is exactly the cocycle

$$\mathcal{M}(w_1, \xi, w_2) = h_1 w_2 - h_2 w_1.$$

Note that this form lies in $H^2(\mathcal{A})$ and it is well defined modulo the ideal generated by $[w_1], [w_2]$ exactly as in the previous case.

Observe that if $\xi \cdot w_i = 0$, the Massey product $\mathcal{M}(w_1, \xi, w_2)$ is exactly given by the adjuction map described in the previous section.

Remark 2.4. *Consider w_i for $i = 1, 2$ two holomorphic forms in $H^0(\omega_C)$ and σ a form in $H^1(\mathcal{O}_C)$. Let ξ be an element in $H^1(T_C)$ such that $\xi \cdot w_i = 0$ for $i = 1, 2$. Then it holds the following formula*

$$< \xi, \mathcal{M}(w_1, \sigma, w_2) > = < \mathcal{M}(w_1, \xi, w_2), \sigma > .$$

3 The harmonic variation

In this section, we are going to define a secondary map in cohomology, called harmonic variation, which takes into account the Massey product we had defined. The construction of this map was inspired by the definition of the Massey products: for this reason, it can be considered a kind of "generalization" of Massey products. The harmonic variation shows two important aspects: on one side, it is involved into the computation of the infinitesimal invariant, on the other side, this map allows us to underline some properties of simmetry of the Griffiths' invariant.

Let w_1, w_2 be two holomorphic forms in $H^0(\omega_C)$ and consider $\xi \in H^1(T_C)$. The element $\xi \cdot w_i$ is an element of $H^1(\mathcal{O}_C)$ where $\cdot$ stays for the cup product. Consider now the element

$$\psi = w_1 \wedge \xi \cdot w_2 - w_2 \wedge \xi \cdot w_1.$$

Clearly, ψ is a $(1,1)$-form such that $\int_C \psi = 0$, through the canonical isomorphism $H^1(\omega_C) \simeq \mathbb{C}$.
Now we analize more in details the meaning of this form.
Consider the harmonic representative of $\xi \cdot w_i$, i.e. the image of the element $\xi \cdot w_i$ in $\mathcal{H}^{0,1}$ under the map $\mathcal{H} : A^{0,1}(C) \longrightarrow \mathcal{H}^{0,1}(C)$, and we denote again with $\xi \cdot w_i$ its harmonic representative. Then we set

$$\xi \cdot w_1 = \theta w_1 - \overline{\partial} h_1, \qquad \xi \cdot w_2 = \theta w_2 - \overline{\partial} h_2,$$

where h_i are $\mathcal{C}^\infty$ functions on C and θw_i is Dolbeaut representative of $\xi \cdot w_i$. Then we have

$$\psi = \overline{\partial}(h_2 w_1 - h_1 w_2).$$

Since the form ψ is a $\overline{\partial}$ exact form, then $\overline{\partial}\psi = 0$. The $\partial\overline{\partial}$-lemma assures that there is a $\mathcal{C}^\infty$ function on C $\beta_{1,2}$ such that $\psi = \overline{\partial}\partial\beta_{1,2}$. Then, starting from a couple of holomorphic forms w_1, w_2 we construct an holomorphic $(1,0)$-form

$$\alpha_3 = \partial\beta_{1,2} - h_2 w_1 + h_1 w_2.$$

Remark 3.1. *This definition holds also if ξ vanishes both w_1 and w_2. In this case, the form ψ is zero as $(1,1)$-form, since $\xi \cdot w_i = 0$ for $i = 1, 2$. Then it is easy to see that $\partial\beta_{1,2}$ is zero and hence α_3 is the image of $w_1 \wedge w_2$ under the map α_ξ or, in the language of our Massey product,*

$$\alpha_3 = \mathcal{M}(w_1, \xi, w_2)$$

Now we define the harmonic variation.

Consider three holomorphic forms w_i in $H^0(\omega_C)$ and $\xi \in H^1(T_C)$. We can repeat the same construction for every pair (w_i, w_j) with $i \neq j$ and we obtain three $(1,0)$ holomorphic forms $\alpha_k = \partial\beta_{i,j} - h_j w_i + h_i w_j$, for $k \neq i, j$ where $\beta_{i,j} \in \mathcal{C}^\infty(C)$. This allows us to define a map

$$\mathcal{L} : H^1(T_C) \otimes H^0(\omega_C) \otimes H^0(\omega_C) \otimes H^0(\omega_C) \longrightarrow H^0(\omega_C^2),$$

which associates to the element $(\xi \otimes w_1 \otimes w_2 \otimes w_3)$ the following sum $\sum_{i=1}^3 w_i\alpha_i$. An equivalent definition of this map is the following $\mathcal{L}(\xi \otimes w_1 \otimes w_2 \otimes w_3) = \sum_{i,j,k=1}^3 w_i \partial\beta_{j,k}$ for $i \neq j, k$, as we can show by a direct computation.

The map $\mathcal{L}$ is well defined as we show in the following proposition.

Proposition 3.2. *The map $\mathcal{L}$ does not depend on the choice of the Dolbeault representation of $\xi \cdot w_i$.*

Proof. Let $\xi \cdot w_1 = \theta w_1 - \overline{\partial} h_1$ be a representation of $\xi \cdot w_1$, with $h_1 \in \mathcal{C}^\infty(C)$. Suppose that there is a second description of $\xi \cdot w_1$, given by $\theta' w_1 - \overline{\partial} g_1$. Observe that $\theta - \theta' = \overline{\partial}\tau$ where τ is a $\mathcal{C}^\infty$ function on T_C. The difference of the two representations is

$$(\theta - \theta')w_1 - \overline{\partial}(h_1 - g_1) = \overline{\partial}\tau \cdot w_1 - \overline{\partial}(h_1 - g_1) = 0;$$

this implies that the function $\tau \cdot w_1 - h_1 + g_1 = c_1$ is holomorphic on T_C. We have two different $(1,0)$-forms α_3 corresponding to the two different representations

$$\partial\beta_{1,2} + h_1 w_2 - h_2 w_1 = \alpha_3 \qquad \partial\beta'_{1,2} + g_1 w_2 - g_2 w_1 = \alpha'_3.$$

We compute the difference between α_3 and α'_3. Since, up to a constant, we can identify $\partial\beta_{1,2} = \partial\beta'_{1,2}$, we obtain $\alpha_3 - \alpha'_3 = c_2 w_1 - c_1 w_2$.
In similar way, we proceed also for the other forms. If we denote with α'_i the form corresponding to the second representation, we have $\alpha_1 - \alpha'_1 = c_3 w_2 - c_2 w_3$ and $\alpha_2 - \alpha'_2 = c_1 w_3 - c_3 w_1$, with $c_i \in A^1(T_C)$. Now a straight computation shows that $\sum_{i=1}^3 \alpha_i w_i = \sum_{i=1}^3 \alpha'_i w_i$. □

Moreover, the map $\mathcal{L}$ is alternant with respect to forms w_i: then it is actually defined on the wedge product $\bigwedge^3 H^0(\omega_C)$.

Lemma 3.3. *The map $\mathcal{L}$ is invariant under cyclic permutations of forms in $H^0(\omega_C)$.*

Proof. Let w_1, w_2, w_3 be three forms in $H^0(\omega_C)$. Suppose to apply the map $\mathcal{L}$ to the element $\xi \otimes (w_2 \otimes w_1 \otimes w_3)$. We want to show that the result is the opposite of $\mathcal{L}(\xi \otimes (w_1 \otimes w_2 \otimes w_3))$, that is, the map $\mathcal{L}$ changes sign under the transposition (12). Note that

$$\overline{\partial}\partial\beta_{k,j} = -\overline{\partial}\partial\beta_{j,k}.$$

Denote with $\hat{\alpha}_i$ the "new" form; it is easy to see that $\hat{\alpha}_i = -\alpha_i$ for $i = 1, 2, 3$. This shows that $\mathcal{L}(\xi \otimes (w_1 \otimes w_2 \otimes w_3)) = -\mathcal{L}(\xi \otimes (w_2 \otimes w_1 \otimes w_3))$. In similar way, it is possible to show that the map $\mathcal{L}$ changes sign under the transposition (13) and this concludes the proof. □

Definition 3.4. *The map*

$$\mathcal{L} : H^1(T_C) \otimes \bigwedge^3 H^0(\omega_C) \longrightarrow H^0(\omega_C^2)$$

which sends $\xi \otimes w_1 \wedge w_2 \wedge w_3$ in the holomorphic form $\sum_{i=1}^3 w_i\alpha_i$, is called harmonic variation.

3.0.3 Properties of the map $\mathcal{L}$

In this paragraph, we study some properties of the harmonic variation: in particular, we show the simmetry of the map $\mathcal{L}$. Let $\xi_1, \xi_2 \in H^1(T_C)$. We have the following equality

$$\xi_1 \cdot \mathcal{L}(\xi_2 \otimes w_1 \wedge w_2 \wedge w_3) = \xi_2 \cdot \mathcal{L}(\xi_1 \otimes w_1 \wedge w_2 \wedge w_3). \tag{7}$$

In order to show (7), we have to evaluate $\mathcal{L}(\xi_j \otimes w_1 \wedge w_2 \wedge w_3)$ for $j = 1, 2$. We recall the description of $\xi_j \cdot w_i$ in terms of harmonic representatives $\xi_1 w_i = \theta_1 w_i - \overline{\partial} h_i$ and $\xi_2 w_i = \theta_2 w_i - \overline{\partial} g_i$. Suppose that $\mathcal{L}(\xi_1 \otimes w_1 \wedge w_2 \wedge w_3) = \sum_{i,j,k=1}^{3} w_i \partial\beta_{j,k}$ with $i \neq j, k$ and we compute explicitally $< \xi_2, \mathcal{L}(\xi_1 \otimes w_1 \wedge w_2 \wedge w_3) >$. Assume that, in every sum, $i \neq j, k$. We have

$$\begin{aligned}
\int \theta_2 \wedge \sum_{i,j,k=1}^{3} w_i \partial\beta_{j,k} &= \sum_{i,j,k=1}^{3} \int \theta_2 \wedge (w_i \partial\beta_{j,k}) = \sum_{i,j,k=1}^{3} \int \theta_2 w_i \wedge \partial\beta_{j,k} = \\
&= \sum_{i,j,k=1}^{3} \int (\xi_2 w_i + \overline{\partial} g_i) \wedge \partial\beta_{j,k} = \sum_{i,j,k=1}^{3} \int \overline{\partial} g_i \wedge \partial\beta_{j,k}.
\end{aligned}$$

since we have $\int \xi_2 w_i \wedge \partial\beta_{j,k} = 0$. Since, by property of the differential of wedge product, we have $\overline{\partial} g_i \wedge \partial\beta_{j,k} = d(g_i \wedge \partial\beta_{j,k}) - g_i \wedge \overline{\partial}\partial\beta_{j,k}$, we obtain $\sum_{i,j,k=1}^{3} \int \overline{\partial} g_i \wedge \partial\beta_{j,k} = -\sum_{i,j,k=1}^{3} \int g_i \wedge \overline{\partial}\partial\beta_{j,k}$. It's easy to rewrite the second member in the following way

$$\int (g_3 \overline{\partial} h_1 - g_1 \overline{\partial} h_3) \wedge w_2 + (g_1 \overline{\partial} h_2 - g_2 \overline{\partial} h_1) \wedge w_3 + (g_2 \overline{\partial} h_3 - g_3 \overline{\partial} h_2) \wedge w_1.$$

The proof is ended as soon as we compute the element $< \xi_1, \mathcal{L}(\xi_2 \otimes w_1 \wedge w_2 \wedge w_3) >$. It's enought to repeat the same subject, changing the role of ξ_1 and ξ_2, to obtain exactly the same expression.

A consequence of this simmetry is the following

Proposition 3.5. *Let w_i be three forms in $H^0(\omega_C)$. Consider two deformations ξ_j for $j = 1, 2$ in $H^1(T_C)$ such that for $i = 1, 2$ and $j = 1, 2$, $\xi_j \cdot w_i = 0$. The harmonic variation acts as follows*

$$\xi_2 \cdot \mathcal{L}(\xi_1 \otimes w_1 \wedge w_2 \wedge w_3) = \int \alpha_{\xi_1}(w_1 \wedge w_2) \wedge \xi_2 \cdot w_3. \tag{8}$$

Proof. Let $\xi_1, \xi_2 \in H^1(T_C)$ such that $\xi_j \cdot w_i = 0$ for $j = 1, 2$ and $i = 1, 2$. We have to evaluate $< \xi_2, \mathcal{L}(\xi_1 \otimes w_1 \wedge w_2 \wedge w_3) >$. Thus, we consider

$$< \xi_2, \sum_{i=1}^{3} \alpha_i w_i > = < \xi_2, \alpha_1 w_1 > + < \xi_2, \alpha_2 w_2 > + < \xi_2, \alpha_3 w_3 > .$$

The first two summands are zero, because for $i = 1, 2$ $< \xi_2, \alpha_i w_i > = \int \xi_2 \wedge \alpha_i w_i = \int \alpha_i \wedge \xi_2 \cdot w_i = 0$. Then it remains only the third part $< \xi_2, \alpha_3 w_3 >$. Remark (3.1) assures that if $\xi_1 \cdot w_1 = \xi_1 \cdot w_2 = 0$, then α_3 is exactly the adjunction image $\alpha_{\xi_1}(w_1 \wedge w_2)$. Then, the final result is $< \xi_2, \alpha_{\xi_1}(w_1 \wedge w_2)\ w_3 >$, that is, $\int \alpha_{\xi_1}(w_1 \wedge w_2) \wedge \xi_2 \cdot w_3$. □

4 The infinitesimal invariant

In this section, we want to show that the infinitesimal invariant $\delta(\nu)$ can be rewritten using the triple Massey product on the differential graded algebra $\mathcal{A}$ constructed before in the case $L = \omega_C$. As a corollary, we deduce that for smooth non-hyperelliptic curves of genus $g > 2$, the differential graded algebra $\mathcal{A}$ associated to ω_C is not formal.

4.1 The Griffiths' infinitesimal invariant

The first definition of the infinitesimal invariant is given by Griffiths in [8] to determine if normal functions are locally constant. Later, the theory of the infinitesimal invariant was completed by Green and Voisin (see for example [5], [14]). Here we present the definition given by Green in [6].

Let $f : \mathcal{C} \longrightarrow S$ be a family of smooth complete connected curves of genus $g > 2$ over a smooth irreducible variety S of dimension n. Suppose there is a section of $\mathcal{C}$ over S, and then, by means of Abel-Jacobi map, we define

$$i : \mathcal{C} \longrightarrow \mathcal{J}.$$

The image of this morphism is an algebraic cycle $\mathcal{W}$ of codimension $g-1$ in the family of jacobians $\mathcal{J}$. Let $\mathcal{W}^-$ be the image of $\mathcal{W}$ under the involution $j : \mathcal{J} \longrightarrow \mathcal{J}$, $j(u) = -u$. The cycles $\mathcal{W}$ and $\mathcal{W}^-$ are homologically equivalent on $\mathcal{J}$ but not algebraically equivalent as Ceresa shows in [1]. We have the Ceresa cycle $\mathcal{Z} = \mathcal{W} - \mathcal{W}^- \in CH^{g-1}(\mathcal{J})_{hom}$. Consider the higher Abel-Jacobi map

$$AJ^{g-1}_{\mathcal{J}} : CH^{g-1}(\mathcal{J})_{hom} \longrightarrow \mathcal{J}^{g-1}$$

where $\mathcal{J}^{g-1}$ is the family of the intermediate jacobians.

In order to study the Abel-Jacobi map, Griffiths introduced the concept of normal functions in a general context of a family of algebraic varieties. In our case, the normal function associated to the family of cycles $\mathcal{Z}$ is a holomorphic section $\nu : S \longrightarrow \mathcal{J}^{g-1}$ given by $\nu(s) = AJ_{J_s}(Z_s) \in J^{g-1}(C)$, where $Z_s = \mathcal{Z} \cap J_s$ is a cycle homologous to zero in J_s, fiber of $\mathcal{J}$ over the generic point $s \in S$.

The main tool to determine if normal functions are locally constant is the infinitesimal invariant. We define $(g-1)$-th Koszul complex associated to the family $\mathcal{C}$ the complex

$$K^k_{g-1} = \Omega^k_S \otimes F^{g-1-k}\mathcal{R}^{2g-3}_{\pi_*}\mathbb{C}$$

with the boundary map ∇ given by the Gauss-Manin connection. Let $\tilde{\nu}$ be a local lifting of ν on a open set $U \subset S$. We have

$$\nabla\tilde{\nu} \in \Omega^1_S \otimes F^{g-2}\mathcal{R}^{2g-3}_{\pi_*}\mathbb{C} = K^1_{g-1}.$$

The flatness of the Gauss-Manin connection assures that $\nabla\tilde{\nu}$ vanishes in $K^2_{g-1} = \Omega^2_S \otimes F^{g-3}\mathcal{R}^{2g-3}_{\pi_*}\mathbb{C}$. Moreover, it is possible to show that $\nabla\tilde{\nu}$ doesn't depend on the choice of the lifting. So, we can conclude that $\nabla\tilde{\nu}$ determines a well defined element

$$\delta(\nu) \in H^1(K^\bullet_{g-1}).$$

Definition 4.1. *The Griffiths' infinitesimal invariant $\delta(\nu)$ of the normal function ν is an element of the cohomology of the following Koszul complex*

$$F^{g-1}\mathcal{R}^{2g-3}_{\pi_*}\mathbb{C} \longrightarrow \Omega^1_S \otimes F^{g-2}\mathcal{R}^{2g-3}_{\pi_*}\mathbb{C} \longrightarrow \Omega^2_S \otimes F^{g-3}\mathcal{R}^{2g-3}_{\pi_*}\mathbb{C}.$$

It is easy to show that a normal function ν is locally constant if and only if $\delta(\nu) = 0$.

It is possible to give a more detailed description of the Koszul complex which defines the infinitesimal invariant. In fact, the complex $K^\bullet_{g-1}$ can be filtered by the subcomplexes $K^k_{g-1,m} = \Omega^k_S \otimes F^{m-k}\mathcal{R}^{2g-3}_{\pi_*}\mathbb{C}$ for $m \geq g-1$. Using the definition of Hodge filtration, we can note that, for every k, $\hat{K}^k_{g-1,m} = K^k_{g-1,m}/K^k_{g-1,m+1} = \Omega^k_S \otimes H^{m-k,2g-3-m+k}(C)$. Observe that if $H^1(\hat{K}^\bullet_{g-1,m}) = 0$ for all $m \geq g-1$ then $H^1(K^\bullet_{g-1}) = 0$. So the Griffiths' infinitesimal invariant can be seen as an element of the cohomology of the complex $\hat{K}^\bullet_{g-1,g-1}$,

$$H^{g-2,g-1} \longrightarrow \Omega^1_S \otimes H^{g-1,g-2} \longrightarrow \Omega^2_S \otimes H^{g,g-3}. \tag{9}$$

If we dualize this complex, $\delta(\nu)$ can be seen as a linear map $Ker(\gamma)/Im(\beta) \longrightarrow \mathbb{C}$, where γ and β are the maps into the dual complex

$$\bigwedge^2 T_S \otimes H^{3,0} \xrightarrow{\beta} T_S \otimes H^{2,1} \xrightarrow{\gamma} H^{1,2}. \tag{10}$$

Now the problem is whether it is possible to "compute" the infinitesimal invariant. In the next section, we will report the result of Collino and Pirola [2]: they give a kind of formula to compute the infinitesimal invariant of the Ceresa cycle which allows them to recovering the cycle from its infinitesimal invariant.

4.1.1 The formula

Let $\mathcal{C} \longrightarrow S = Spec\mathbb{C}\,[\varepsilon]/(\varepsilon^2)$ be a family of smooth curves of genus g over a smooth variety S of dimension 1. Denote with C the fiber of the family $\mathcal{C}$ over the generic point $s \in S$. Consider the following natural exact sequence

$$0 \longrightarrow N^* \longrightarrow \Omega^1_{\mathcal{C}}|_C \longrightarrow \omega_C \longrightarrow 0, \tag{11}$$

where $\Omega^1_{\mathcal{C}}|_C$ is the cotangent bundle of the family $\mathcal{C}$ restricted to C and N^* is the conormal bundle. Chosen a basis for the tangent space $T_{S,s}$ at the point $s \in S$, it is easy to see that N^* is trivial.

The class of extension $[\xi]$ of (11) lies in $H^1(T_C)$: hence, geometrically, it represents a first order deformation of C. So the map (5) is the adjunction map $\alpha_\xi : \bigwedge^2 \mathcal{U} \longrightarrow H^0(\omega_C)/V$, where, as in section 2.1.1, $\mathcal{U}$ is a 2-dimensional subspace of the kernel of the coboundary map of (11).

Consider w_1 and w_2 two forms in $\mathcal{U}$ i.e. such that $\xi \cdot w_i = 0$ for $i = 1, 2$ and let σ be a third form in $H^1(\mathcal{O}_C)$ such that $\overline{\sigma}$ is orthogonal to w_1 and w_2. Note that the element $\xi \otimes w_1 \wedge w_2 \wedge \sigma$ lies in the kernel of γ; then it is possible to calculate the infinitesimal invariant $\delta(\nu)(\xi \otimes w_1 \wedge w_2 \wedge \sigma)$.

Collino and Pirola showed the following

Theorem 4.2 ([2]). $\delta(\nu)(\xi \otimes w_1 \wedge w_2 \wedge \sigma) = -2 \int_C \alpha_\xi(w_1 \wedge w_2) \wedge \sigma$.

4.2 The main Theorem

This section is dedicated to showing that how it is possible to compute the Griffiths' infinitesimal invariant using Massey products and the harmonic variation.

Let $\xi \in H^1(T_C)$ and $\sigma \in H^1(\mathcal{O}_C)$. Consider two holomorphic forms $w_1, w_2 \in H^0(\omega_C)$ such that $\xi \cdot w_i = 0$ for $i = 1, 2$: we can suppose that w_1, w_2 lie in $\mathcal{U}$. Moreover, assume that w_i are orthogonal to $\overline{\sigma}$. This condition allows us to construct a triple Massey product on $\mathcal{A}$. In fact, in the language of our dga, $[w_1], [w_2]$ are two elements in $H^2(\mathcal{A})$ and $[\sigma]$ is a class in $H^1(\mathcal{A})$. Since $[w_i] \cdot [\sigma] = 0$ for $i = 1, 2$, then there are two elements $\rho_i \in \mathcal{A}_2$ such that $w_i \wedge \sigma = \overline{\partial} \rho_i$ for $i = 1, 2$. The triple Massey product of $[w_1], [\sigma], [w_2]$ is the cocycle in $H^4(\mathcal{A})$

$$\mathcal{M}(w_1, \sigma, w_2) = \rho_1 w_2 - \rho_2 w_1$$

defined modulo the ideal spanned by $[w_1]$ and $[w_2]$.

Now if we consider the formula (6) and apply the theorem (4.2), we are able to show the following

Theorem 4.3. *Let w_1, w_2 be forms in $H^0(\omega_C)$, σ a form in $H^1(\mathcal{O}_C)$ and $\xi \in H^1(T_C)$. Suppose that $\overline{\sigma}$ is orthogonal to w_i for every i and $\xi \cdot w_i = 0$ for $i = 1, 2$. Assume it is possible to compute the infinitesimal invariant $\delta(\nu)$ of $\xi \otimes w_1 \wedge w_2 \wedge \sigma$. Then we have*

$$\delta(\nu)(\xi \otimes w_1 \wedge w_2 \wedge \sigma) = -2 < \xi, \mathcal{M}(w_1, \sigma, w_2) > .$$

Finally, proposition (3.5) and again theorem (4.2) show the following

Corollary 4.4. *Under the hypoteses of theorem (4.3), if we assume that* $\xi \cdot \overline{\sigma} = \sigma$*, we have*

$$\delta(\nu)(\xi \otimes w_1 \wedge w_2 \wedge \sigma) = -2\xi \cdot \mathcal{L}(\xi \otimes w_1 \wedge w_2 \wedge \overline{\sigma}).$$

4.3 Non-Formality of $\mathcal{A}$

The construction of the triple Massey on $\mathcal{A}$ suggests us to study the formality of the dga. We change point of view: it is well known that, for a smooth complete non-hyperelliptic curve of genus $g > 2$, the infinitesimal invariant of the Ceresa cycle is not trivial (see for example [2]). Then theorem (4.3) assures that also the Massey product $\mathcal{M}(w_1, \sigma, w_2)$ is not trivial: since we have found a non-vanishing Massey product on $\mathcal{A}$, we can conclude, by theorem (1.4), that $\mathcal{A}$ is a non formal differential graded algebra. Hence we have shown the following

Theorem 4.5. *Let* C *be a smooth non-hyperelliptic curve of genus* $g > 2$*. Consider* $\mathcal{A}$ *the differential graded algebra defined in section 2 with* $L = \omega_C$*. Then the algebra* $(\mathcal{A}, d)$ *is not formal.*

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