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“Francesco Brioschi”
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ENVIRONMENT ON \mathbb{N} AND ON
GALTON-WATSON TREES

BERTACCHI, D.; ZUCCA, F.

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Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

RUMOR PROCESSES IN RANDOM ENVIRONMENT ON \mathbb{N} AND ON GALTON-WATSON TREES

DANIELA BERTACCHI AND FABIO ZUCCA

ABSTRACT. The aim of this paper is to study rumor processes in random environment. In a rumor process a signal starts from the stations of a fixed vertex (the root) and travels on a graph from vertex to vertex. We consider two rumor processes. In the firework process each station, when reached by the signal, transmits it up to a random distance. In the reverse firework process, on the other hand, stations do not send any signal but they “listen” for it up to a random distance. The first random environment that we consider is the deterministic 1-dimensional tree \mathbb{N} with a random number of stations on each vertex; in this case the root is the origin of \mathbb{N} . We give conditions for the survival/extinction on almost every realization of the sequence of stations. Later on, we study the processes on Galton-Watson trees with random number of stations on each vertex. We show that if the probability of survival is positive, then there is survival on almost every realization of the infinite tree such that there is at least one station at the root. We characterize the survival of the process in some cases and we give sufficient conditions for survival/extinction.

Keywords: rumor process, random environment, labelled Galton-Watson tree, multitype Galton-Watson tree, inherited event.

AMS subject classification: 60K35, 60G50.

1. INTRODUCTION

Rumor processes are models for the propagation of signals or information on a net of stations. There is a large literature on these processes, which takes into account deterministic and stochastic variants both in space with no structure or on graphs, complex networks and grids (see e.g. [11, 13, 14, 17] or [7] and the references therein). More variants include competing rumors and rumors spreading in a moving population (see [8, 9, 10]). So far, up to our knowledge, no rumor models on random environment have been studied. In this paper we study two kinds of discrete-time rumor processes (the *firework* and *reverse firework* processes, introduced in [7]) with a random number of stations at each vertex of a graph $\mathbf{X} := (X, E(X))$ which can be either \mathbb{N} or a Galton-Watson tree (briefly, GW tree). The use of a Galton-Watson tree (see for instance [15]) allows to study a random process on a random and heterogeneous graph structure still retaining some probabilistic homogeneity. The main question about this model is to understand under which conditions, the signal, starting from one vertex of the graph, will spread indefinitely with positive probability or die out almost surely in a finite number of steps. It is worth noting that rumor processes can describe the behavior of other models such as the frog model (see [2, 12]): for instance the firework process on \mathbb{N} can describe the local survival of a frog model with immortal particles with left drift (where the radius of a “station” or “frog” at x is the distance between x and the rightmost vertex ever reached by the frog).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider two families of random variables $\{N_x\}_{x \in X}$ and $\{R_{x,i}\}_{i \in \mathbb{N}^*, x \in X}$ such that $\{N_x, R_{x,i}\}_{i \in \mathbb{N}^*, x \in X}$ are independent and $\{R_{x,i}\}_{i \in \mathbb{N}^*}$ are identically distributed for all $x \in X$, that is $R_{x,i} \sim R_x$ (where $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$). We choose an origin $o \in X$ (which is the origin of \mathbb{N} or the root of the GW-tree) and we suppose that $\mathbb{P}(N_o > 0) > 0$. If $N_x \sim N$ and $R_x \sim R$ for all $x \in X$ (resp. for all $x \in X \setminus \{o\}$) then the firework (resp. reverse firework) process is called *homogeneous*; *heterogeneous* otherwise. N_x represents the random number of radio stations at x and $\{R_{x,i}\}_{i=1}^{N_x}$ are the operating radii of the stations. In order to avoid trivial cases, we assume that $\mathbb{P}(R_x < 1) \in (0, 1)$. By $\mathcal{N} = \{N_x\}_{x \in X}$ we denote the random sequence of the numbers of stations.

We describe now the firework process starting from the root o on a tree $(X, E(X))$, with deterministic number of stations $\{n_x\}_{x \in X}$ and radii $\{r_{x,i}\}_{i \in \mathbb{N}^*, x \in X}$. If either the number of stations or the radii are random (or the tree itself is a random graph), then this evolution applies to every fixed configuration $\omega \in \Omega$. At time 0 only the stations at o , if any, are active. At time 1 all the stations at x such that the distance between o and x is less than or equal to $\max_{i=1, \dots, n_o} r_{o,i}$ are activated. Given the set A_n of active stations at time n then $x \in A_{n+1}$ if and only if there exists $y \in A_n$ such that

- (a) x belongs to the subtree branching from y
- (b) the distance between y and x is less than or equal to $\max_{i=1, \dots, n_y} r_{y,i}$.

Clearly if $n_o = 0$ the process does not start at all. We say that the process survives if and only if the stations on an infinite number of vertices are activated. When the number of stations and the radii are random variables and the tree \mathbf{X} is a random graph, we say that there is survival if the event

$V = \{\omega \in \Omega: \text{the firework process on } \mathbf{X}(\omega) \text{ with radii } \{R_{x,i}(\omega)\} \text{ and } \{N_x(\omega)\} \text{ stations survives}\}$ has positive probability. We call this *annealed survival* (see Section 2.3 for the definition of the annealed counterpart of the process). We are mainly interested in the *quenched survival*, that is, $\mathbb{P}(V | (\mathbf{X}, \mathcal{N}) = (\mathbf{T}, \mathbf{n})) > 0$ for almost every tree \mathbf{T} and almost every sequence \mathbf{n} of stations in a fixed set (see Section 2 for details).

The reverse firework process starting from the root o on a tree $(X, E(X))$, with $\{n_x\}_{x \in X}$ stations and radii $\{r_{x,i}\}_{i \in \mathbb{N}^*, x \in X}$ evolves in a slightly different way. We consider o as an active vertex at time 0 disregarding its number of stations. The number of stations at o is not important because, while in the firework process the stations are actively sending and passively listening, in the reverse firework process it is the other way around. More precisely, at time 1 all the stations at x such that the distance between o and x is less than or equal to $\max_{i=1, \dots, n_x} r_{x,i}$ are activated. Given the set A_n of active stations at time n then $x \in A_{n+1}$ if and only if there exists $y \in A_n$ such that

- (a) x belongs to the subtree branching from y
- (b) the distance between y and x is less than or equal to $\max_{i=1, \dots, n_y} r_{x,i}$.

When the number of stations and the radii are random variables (or the tree is a random graph), the annealed and quenched survival for the reverse firework process are defined analogously to the firework case.

It has been shown in [7] that, on \mathbb{N} with a deterministic number of stations, even if the two processes look similar, they behave differently. The same phenomenon can be observed in random environment.

There is a simple way to describe the survival of our processes by using an auxiliary graph. Consider a tree $(X, E(X))$, a root o and the sequence of *annealed radii* $\{\tilde{R}_x\}_{x \in X}$ where $\tilde{R}_x := \mathbb{1}_{\{N_x \geq 1\}} \cdot \max\{R_{x,j} : j = 1, \dots, N_x\}$. We associate a new graph, that we call *F-graph*, with the firework process as follows: the set of vertices is X and we draw an edge (w, w') if and only if w' belongs to the subtree branching from w and the distance between w and w' is less than or equal to \tilde{R}_w ; the F-graph is the connected component containing o . A similar graph, the *RF-graph*, can be constructed for the reverse firework process: we draw an edge (w, w') if and only if w' belongs to the subtree branching from w and the distance between w and w' is less than or equal to $\tilde{R}_{w'}$; as before, we consider just the connected component containing o . It is clear that there is survival for the firework (resp. reverse firework) process if and only if the F-graph (resp. RF-graph) is infinite. In Section 5 the tree and the sequence $\{\tilde{R}_w\}_{w \in \mathbb{T}}$ are both random, thus the F-graph and RF-graph are random as well.

Here is a brief outline of the paper. Section 2 is devoted primarily to the construction of the probability space, starting with the space of labelled GW-trees, where our processes live (Section 2.1). The main result of this section, Lemma 2.2, is the key to obtain quenched results from the annealed ones for the processes on GW-trees. In Section 2.2 we discuss the important case where the GW-tree is the deterministic 1-dimensional tree \mathbb{N} . In Section 2.3 we introduce the notion of annealed counterpart of the firework and reverse firework processes. The results on survival and extinction for the firework process and the reverse firework process on \mathbb{N} with random number of stations can be found in Sections 3 and 4 respectively. Theorem 3.1 gives a characterization for the extinction of the homogeneous firework process. More explicit conditions for survival/extinction can be found in Proposition 3.2 and Corollary 3.3: these are conditions on the tails of the distributions R and N . In Remark 3.4 we show that, given any law R for the radii (resp. N for the number of stations) there exist a law N for the number of stations (resp. R for the radii) such that the firework process survives. In particular, unlike the deterministic case, survival is possible even if the expected value of R is finite. The possible behaviors of the firework process when $\mathbb{E}[R]$ and $\mathbb{E}[N]$ are finite/infinite are discussed in Remark 3.5. Theorem 3.6 gives a sufficient condition for the survival of the heterogeneous firework process. A characterization of survival for the homogeneous reverse firework process is given by Theorem 4.1 while some sufficient conditions for survival/extinction (on the tails of the distributions R and N) can be found in Corollary 5.5. Remark 4.4 is the reverse firework counterpart of Remark 3.4. Theorem 4.5 gives a sufficient condition for the survival

of the heterogeneous firework process. Section 5 is devoted to homogeneous firework and reverse firework processes on GW-trees. Lemma 5.1 derives quenched results from annealed ones by using Lemma 2.2. Conditions for survival/extinction can be found in Theorem 5.2 and Corollary 5.3 for the firework process and in Theorem 5.4 and Corollary 5.5 for the reverse firework process. In Example 5.7 we apply all these results to the case where, independently, at each vertex of the GW-tree there is a station according to a Bernoulli distribution. All the proofs, along with some technical lemmas, can be found in Section 6.

Finally it is worth noting that some results extend by coupling. Suppose that we have two families of random variables $\{N_x, R_{x,i}\}_{i \in \mathbb{N}^*, x \in X}$ and $\{N'_x, R'_{x,i}\}_{i \in \mathbb{N}^*, x \in X}$ such that $N_x \succeq N'_x$ and $R_{x,i} \succeq R'_{x,i}$ for all $i \in \mathbb{N}^*, x \in X$. By coupling, if the (firework or reverse firework) process associated with $\{N_x, R_{x,i}\}_{i \in \mathbb{N}^*, x \in X}$ dies out almost surely then so does the process associated with $\{N'_x, R'_{x,i}\}_{i \in \mathbb{N}^*, x \in X}$.

2. PRELIMINARIES AND CONSTRUCTION OF THE PROCESS

In this section we construct the space of the processes and we establish the notation that we use in the paper. This section is organized as follows. We start in Section 2.1 by constructing a *random labelled GW-tree* or *multitype GW-tree* (which generalizes the well-known GW-tree) and the probability space of our processes. Section 2.2 is devoted to the special case where the GW-tree is simply \mathbb{N} ; this is all we need in Sections 3 and 4. In Section 2.3 we introduce the *annealed counterpart* of our processes which is used throughout the whole paper.

2.1. Random labelled Galton-Watson trees. The idea is to construct a probability space for our processes on GW-trees with random radii and random number of stations. To this aim we construct the space of labelled GW-trees which can be seen as the genealogy tree associated with a generic discrete-time multitype branching process (or MBP). Even though it can be constructed for a generic MBP (or a generic branching random walk as described in [3, 4, 5, 19]), for sake of simplicity we write the explicit construction for the particular case that we need. Here the *labels* of the vertices are the number of stations.

The reader who is just interested in firework and reversed firework processes on a random environment on \mathbb{N} can skip this section and go to Section 2.2.

Let us define the space of unlabelled GW-trees (the usual GW-trees). Consider a GW-process, with offspring distribution ρ . We denote by $\varphi(z) := \sum_{n \in \mathbb{N}} \rho(n)z^n$ the generating function of the GW-process and we write $m := \sum_{n \in \mathbb{N}} n\rho(n)$. We consider the set of finite words \mathcal{W} (including the empty word, \emptyset , which is the root) on the infinite alphabet $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Given two words $w, w' \in \mathcal{W}$ we denote by ww' the extension of w obtained by attaching w' to the right. A tree \mathbf{T} on \mathcal{W} is a tree with vertices in \mathcal{W} such that every child of a vertex $w \in \mathcal{W}$ is in the set $\{wi : i \in \mathbb{N}\}$. We denote by \mathbb{T} the set of these trees. The length of a word $w \in \mathcal{W}$ will be denoted by $|w|$.

Let $\{T_w\}_{w \in \mathcal{W}}$ be a sequence of independent random variables with law ρ . For every realization of $\{T_w\}_{w \in \mathcal{W}}$ we draw an edge from a word w to the word wi (where $i \in \mathbb{N}^*$) if and only if $i \leq T_w$. This is a forest; we denote by τ the random GW-tree, that is the connected component of the forest containing the root \emptyset . Consider the space \mathbb{T} endowed with the minimal σ -algebra containing all sets $A_{\mathbf{T}} := \{\mathbf{T}' \in \mathcal{W} : \mathbf{T}' \supseteq \mathbf{T}\}$ where \mathbf{T} is a finite tree. The GW-tree τ is a \mathbb{T} -valued random variable; we denote by $\bar{\mu}$ its law and we call it *GW-measure*. The probability that τ is a finite tree is the smallest fixed point α of φ in $[0, 1]$. Moreover, if $\rho(1) = 1$ then τ^L is the 1-dimensional tree \mathbb{N} and $\alpha = 0$. If $\rho(1) < 1$ then $\alpha = 1$ (that is, τ^L is a finite tree a.s.) if and only if $m \leq 1$.

The set of vertices of a generic labelled tree is $\mathcal{V} := \mathcal{W} \times J$, where J is the at most countable set of labels. We consider only those trees Υ on \mathcal{V} such that (1) every child of $(w, j) \in \Upsilon$ is in the set $\{(wi, j') : i \in \mathbb{N}^*, j' \in J\}$, (2) if $(w, j), (w, j') \in \Upsilon$ then $j = j'$, (3) the root of Υ belongs to $\{(\emptyset, j) : j \in J\}$. Denote by \mathbb{LT} the set of all these trees on \mathcal{V} . The natural projection π of \mathcal{V} onto \mathcal{W} extends to a projection from \mathbb{LT} onto \mathbb{T} .

Given a probability space where we have a realization of the GW-tree τ and, independently, a family of independent J -valued random variables $\{N_w\}_{w \in \mathcal{W}}$, we define an \mathbb{LT} -valued random variable $\tau^L(\omega) := \{(w, j) \in \mathcal{V} : w \in \tau(\omega), j = N_w(\omega)\}$ that we call *labelled GW-tree*. Its law, which is uniquely determined by ρ and by the laws of $\{N_w\}_{w \in \mathcal{W}}$, will be denoted by μ_ζ (where $\zeta = \mathbb{P}_{N_\emptyset}$ is the law of N_\emptyset) or simply by μ . Roughly speaking, in this special case, the random labelled GW-tree can be obtained by generating a GW-tree in the first place and then by placing a random type (in our case, a random number of stations) on each vertex w independently with law N_w . The projected random variable $\pi \circ \tau^L$ is the unlabelled GW-tree τ and $\bar{\mu}(\cdot) = \mu(\pi^{-1}(\cdot))$. Hence, given a measurable set $A \subset \mathbb{T}$, the measure of the set $\pi^{-1}(A) \subset \mathbb{LT}$ does not depend on $\{N_w\}_{w \in \mathcal{W}}$. In particular if E is the set of finite labelled trees then $\mu_\zeta(E) = \alpha$ for every law ζ , since it is independent of $\{N_w\}_{w \in \mathcal{W}}$.

On the probability space (\mathbb{LT}, μ_ζ) there is a canonical process: given $\Upsilon \in \mathbb{LT}$, define $Z_n(\Upsilon, j)$ as the total number of vertices $(w, j) \in \Upsilon$ for all words $w \in \mathcal{W}$ of length $n \geq 0$. If $\{N_w\}_{w \in \mathcal{W} \setminus \{\emptyset\}}$ are i.i.d. then $\{Z_n\}_{n \in \mathbb{N}}$ is a discrete-time MBP starting from one particle of random type N_\emptyset and τ^L is its genealogy tree.

We denote by $l(\Upsilon)$ the label of the root of Υ ; l is a random variable on \mathbb{LT} with law ζ . By construction l and $\tau = \pi \circ \tau^L$ are independent. The measure μ_ζ depends on the initial distribution ζ ; a particular case is $\zeta := \delta_j$ where we denote the measure μ_{δ_j} simply by μ_j . Clearly $\mu_\zeta = \sum_{j \in J} \zeta(j) \mu_j$ and, for every measurable set $A \subseteq \mathbb{LT}$, $\mathbb{E}[A | l = j] = \mu_j(A) = \mathbb{E}[\tau^L \in A | Z_0 = \mathbb{1}_j]$. The supports $\mathbb{LT}_j := \{\Upsilon \in \mathbb{LT} : l(\Upsilon) = j\}$ of the measures μ_j induces a natural partition $\mathbb{LT} = \bigcup_{j \in J} \mathbb{LT}_j$.

We note that given a labelled tree Υ and a vertex $(w, j) \in \Upsilon$ there is a natural identification of the subtree branching from (w, j) and a labelled tree in \mathbb{LT}_j . Analogously, every branching random subtree of a random labelled GW-tree τ^L can be identified with a random labelled GW-tree.

Definition 2.1. Given $\bar{J} \subseteq J$, a couple (A, \tilde{A}) of measurable sets of trees $A, \tilde{A} \subseteq \mathbb{LT}$ is called inherited with respect to \bar{J} if and only if

- (1) $\mu_j(E \setminus A) = \mu_j(A \Delta \tilde{A}) = 0$ for all $j \in \bar{J}$;
- (2) if $\Upsilon \in A$ then all subtrees branching from the children of the root belong to \tilde{A} .

For instance, the first condition is satisfied if $E \subseteq A = \tilde{A}$; in this case, being inherited is just a set property which does not depend on $j \in J$. When $A = \tilde{A}$ and $\bar{J} = J$ we simply say that A is inherited (when J is a singleton this is the usual definition, see e.g. [16]). The following lemma holds.

Lemma 2.2. Let $\{N_w\}_{w \in \mathcal{W}}$ be an independent family of random variables such that $N_w \sim N$ for all $w \in \mathcal{W} \setminus \{\emptyset\}$ and define $J_N := \{j \in J : \mathbb{P}(N = j) > 0\}$. Suppose that $J_N \subseteq \bar{J} \subseteq J$. If (A, \tilde{A}) is inherited w.r. to \bar{J} then either $\mu_j(A \Delta E) = \mu_j(\tilde{A} \Delta E) = 0$ for all $j \in \bar{J}$ or $\mu_j(A) = \mu_j(\tilde{A}) = 1$ for all $j \in J_N$. Moreover, in the first case if $\text{supp}(\varsigma) \subseteq \bar{J}$, we have $\mu_\varsigma(A \Delta E) = \mu_\varsigma(\tilde{A} \Delta E) = 0$, while in the second case if $\text{supp}(\varsigma) \subseteq J_N$, we have $\mu_\varsigma(A) = \mu_\varsigma(\tilde{A}) = 1$.

Note that in general, given an inherited set A , it is not true that if for some $j \in J$, $\mu_j(A) > \mu_j(E)$ then $\mu_j(A) = 1$ for all $j \in J$. Indeed, suppose that $J = \{1, 2\}$ and $N = 2$ almost surely. If $m > 1$ then the smallest fixed point α of φ is strictly smaller than 1. Let A be the collection of all trees which are either finite or with root of type 2. Then A is inherited and $\mu_1(A) = \alpha$, $\mu_2(A) = 1$. When J is a singleton, Lemma 2.2 applies to classical (unlabelled) GW-trees.

From now on, $J = \mathbb{N}$ and the *environment* is a realization of the labelled GW-tree τ^L , that is, a choice of the random GW-tree τ along with the number of stations at each vertex.

After the construction of the environment, we construct the probability space for our processes. Let (\mathbb{LT}, μ) be as before and $\nu = \prod_{w \in \mathcal{W}, i \in \mathbb{N}^*} \mathbb{P}_{R_{w,i}}$ be the product measure of the laws of $\{R_{w,i}\}_{w \in \mathcal{W}, i \in \mathbb{N}^*}$ on the canonical product space $\mathcal{O} := [0, +\infty)^{\mathcal{W} \times \mathbb{N}^*}$. Henceforth we do not need the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we consider $\Omega := \mathbb{LT} \times \mathcal{O}$ endowed with the usual product σ -algebra and $\mathbb{P} := \mu \times \nu$. With a slight abuse of notation we denote again by τ^L , N_w and R_w the natural counterparts of the original random variables which are now defined on the “new” space Ω . For every $\omega \in \Omega$ the processes evolve according to the sequence of radii $\{\max_{i \in \mathbb{N}^*} R_{w,i}(\omega) \mathbb{1}_{[1, N_w(\omega)]}(i)\}_{x \in X}$ (see also Section 2.3). Extinction and survival are measurable sets with respect to the product σ -algebra. By standard measure theory, for every event A we have

$$\mathbb{P}(A) = \int_{\mathbb{LT}} \mathbb{P}(A | \tau^L = \Upsilon) \mu(d\Upsilon) \tag{2.1}$$

where $\mathbb{P}(A | \tau^L = \Upsilon) = \nu(\mathbf{r} \in \mathcal{O} : (\Upsilon, \mathbf{r}) \in A)$ since $\tau^L(\Upsilon, \mathbf{r}) \equiv \Upsilon$. Using equation (2.1), we have that $\mathbb{P}(A) = 0$ (resp. $\mathbb{P}(A) = 1$) if and only if $\mathbb{P}(A | \tau^L = \Upsilon) = 0$ (resp. $\mathbb{P}(A | \tau^L = \Upsilon) = 1$) μ -almost surely. It is clear that $\mathbb{P}(A) > 0$ if and only if $\mu(\Upsilon : \mathbb{P}(A | \tau^L = \Upsilon) > 0) > 0$. Quenched results, when A is the event “the process survives”, can be obtained from Lemma 2.2 as shown by Lemma 5.1.

2.2. Random environment on \mathbb{N} . We discuss here the case where $\rho(1) = 1$. The projection of the resulting labelled GW-tree is the set of words (empty word included) where 1 is the only letter. We identify this projected tree with \mathbb{N} . To stress the fact that we are dealing with this special case, we adopt a different notation. The tree τ^L can be identified with $\mathbb{N}^{\mathbb{N}}$ -valued random variable \mathcal{N} (representing the sequence of labels of τ^L) where $\mathcal{N}(\omega) := \{N_i(\omega)\}_{i \in \mathbb{N}}$. Remember that $\{N_i\}_{i \in \mathbb{N}}$ is a family of independent random variables (possibly with different distributions).

In this case the environment is represented by every fixed realization of \mathcal{N} . Instead of (\mathbb{LT}, μ) , here we use the space $(\mathbb{N}^{\mathbb{N}}, \mu)$ and μ is the law of \mathcal{N} , that is, the product measure $\prod_{i \in \mathbb{N}} \mathbb{P}_{N_i}$ of the laws of $\{N_i\}_{i \in \mathbb{N}}$ on $\mathbb{N}^{\mathbb{N}}$. The measure $\mathbb{P} = \mu \times \nu$ is defined on $\Omega = \mathbb{N}^{\mathbb{N}} \times \mathcal{O}$. Equation (2.1) becomes $\mathbb{P}(A) = \int_{\mathbb{N}^{\mathbb{N}}} \mathbb{P}(A | \mathcal{N} = \mathbf{n}) \mu(d\mathbf{n})$. With a slight abuse of notation we denote by $\{N_x\}_{x \in X}$ the realization of the sequence \mathcal{N} on $\mathbb{N}^{\mathbb{N}} \times \mathcal{O}$.

2.3. The annealed counterpart. On a deterministic graph or on any given realization of a random graph, we associate with our (firework or reverse firework) process with random numbers of stations, a (firework or reverse firework) process with one station per vertex. Indeed, consider the process with one station on each vertex x and radius $\tilde{R}_x = \mathbb{1}_{\{N_x \geq 1\}} \cdot \max\{R_{x,j} : j = 1, \dots, N_x\}$ at $x \in X$. We call this process, the *deterministic counterpart* or *annealed counterpart* of the original process. The annealed counterpart does not retain any information about the environment, nevertheless the probability of annealed survival of the processes are the same. In the following we use extensively the cumulative distribution function of \tilde{R}_x which can be easily computed as $\mathbb{1}_{(0, +\infty)}(t) G_{N_x}(\mathbb{P}(R_x \leq t))$ where $G_{N_x}(t) := \mathbb{E}[t^{N_x}] = \sum_{j=0}^{\infty} \mathbb{P}(N_x = j) t^j$. As a consequence we have $\mathbb{P}(\tilde{R}_x < t) = \mathbb{1}_{(0, +\infty)}(t) G_{N_x}(\mathbb{P}(R_x < t))$. Since we assumed that $\mathbb{P}(R_x < 1) \in (0, 1)$ then $\mathbb{P}(\tilde{R}_x < 1) \in (0, 1)$.

3. FIREWORK PROCESS ON \mathbb{N}

According to Section 2.2, the environment is the random sequence \mathcal{N} (where $J = \mathbb{N}$), its law is μ and \mathbb{P} is the probability measure on $\Omega = \mathbb{N}^{\mathbb{N}} \times \mathcal{O}$. We denote by V the event “the firework process survives”, that is, “all the vertices are reached by a signal”. $\mathbb{P}(V)$ is also the probability of survival of the annealed counterpart of the reverse firework process; on the other hand $\mathbb{P}(V | \mathcal{N} = \mathbf{n})$ is the probability of survival of the firework process conditioned on a specific realization \mathbf{n} of the sequence of numbers of stations. Results for the deterministic case with k stations per site can be retrieved by using $G_N(t) \equiv t^k$.

Theorem 3.1. *In the homogeneous case ($R_{i,j} \sim R$ and $N_i \sim N$ for all $i \in \mathbb{N}$, $j \in \mathbb{N}^*$),*

(1) *if*

$$\sum_{n=0}^{\infty} \prod_{i=0}^n G_N(\mathbb{P}(R < i + 1)) = +\infty \tag{3.2}$$

then there is extinction for μ -almost all \mathcal{N} ;

(2) if

$$\sum_{n=0}^{\infty} \prod_{i=0}^n G_N(\mathbb{P}(R < i + 1)) < +\infty \quad (3.3)$$

then $\mathbb{P}(V) > 0$ and $\mu(\mathbf{n}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0) = \mathbb{P}(N > 0)$.

Since there is no survival if there are no stations at the origin, then the probability that the environment \mathcal{N} can sustain a surviving firework process cannot exceed $\mathbb{P}(N > 0)$. The previous theorem tells us that when the above probability is not 0 then it attains its maximum value.

A function $L : (0, +\infty) \rightarrow \mathbb{R}$ is *slowly varying* if and only if for all $x > 0$ (\iff for all $x > 1$) $\lim_{t \rightarrow +\infty} L(xt)/L(t) = 1$. It is clear that if a slowly varying function L does not vanish on $(0, +\infty)$ then $1/L$ is a slowly varying function. Examples of slowly varying functions are $(\ln p(\cdot))^\alpha$ where $p(\cdot)$ is a polynomial and $\alpha \in \mathbb{R}$.

The following proposition gives sufficient conditions for extinction or survival.

Proposition 3.2. *Suppose that $R_{i,j} \sim R$ and $N_i \sim N$ for all $i \in \mathbb{N}$, $j \in \mathbb{N}^*$.*

- (1) *If $\liminf_{n \rightarrow \infty} n(1 - G_N(\mathbb{P}(R < n))) > 1$ then $\mathbb{P}(V) > 0$ and $\mu(\mathbf{n}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0) = \mathbb{P}(N > 0)$.*
- (2) *If $\limsup_{n \rightarrow \infty} n(1 - G_N(\mathbb{P}(R < n))) < 1$ then $\mathbb{P}(V) = 0$ and $\mathbb{P}(V|\mathcal{N} = \mathbf{n}) = 0$ for μ -almost all configurations \mathbf{n} .*
- (3) *If $\mathbb{E}[N] < +\infty$ and $\limsup_{n \rightarrow \infty} n\mathbb{P}(R \geq n) < 1/\mathbb{E}[N]$ then $\mathbb{P}(V) = 0$ and $\mathbb{P}(V|\mathcal{N} = \mathbf{n}) = 0$ for μ -almost all configurations \mathbf{n} .*
- (4) *If $\mathbb{E}[N] < +\infty$ and $\mathbb{E}[R] < +\infty$ then $\mathbb{P}(V) = 0$ and $\mathbb{P}(V|\mathcal{N} = \mathbf{n}) = 0$ for μ -almost all configurations \mathbf{n} .*
- (5) *If $\liminf_{n \rightarrow \infty} n\mathbb{P}(R \geq n)G'_N(\mathbb{P}(R < n)) > 1$ then $\mathbb{P}(V) > 0$ and $\mu(\mathbf{n}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0) = \mathbb{P}(N > 0)$. In particular this holds if $\liminf_{n \rightarrow \infty} n\mathbb{P}(R \geq n) > 1/\mathbb{E}[N]$ (where $1/\mathbb{E}[N] := 0$ if $\mathbb{E}[N] = +\infty$).*

The following corollary gives sufficient conditions for extinction or survival where N and R play disjoint roles.

Corollary 3.3. *Suppose that $R_{i,j} \sim R$ and $N_i \sim N$ for all $i \in \mathbb{N}$, $j \in \mathbb{N}^*$.*

If $\mathbb{P}(N > n) \sim n^{-\alpha}L(n)$ (as $n \rightarrow \infty$) for some $\alpha \in (0, 1)$ and a slowly varying function L then

- (1) *$\liminf_{n \rightarrow \infty} n\mathbb{P}(R \geq n)^\alpha L(1/\mathbb{P}(R \geq n))\Gamma(1 - \alpha) > 1$ implies $\mathbb{P}(V) > 0$ and $\mu(\mathbf{n}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0) = \mathbb{P}(N > 0)$;*
- (2) *$\limsup_{n \rightarrow \infty} n\mathbb{P}(R \geq n)^\alpha L(1/\mathbb{P}(R \geq n))\Gamma(1 - \alpha) < 1$ implies $\mathbb{P}(V) = 0$ and $\mathbb{P}(V|\mathcal{N} = \mathbf{n}) = 0$ for μ -almost all configurations \mathbf{n} .*

If $\mathbb{P}(N > n) \sim cn^{-1}$ (for some $c > 0$) then

- (3) *$\liminf_{n \rightarrow \infty} n \ln(1/\mathbb{P}(R \geq n))\mathbb{P}(R \geq n) > 1/c$ implies $\mathbb{P}(V) > 0$ and $\mu(\mathbf{n}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0) = \mathbb{P}(N > 0)$;*

(4) $\limsup_{n \rightarrow \infty} n \ln(1/\mathbb{P}(R \geq n))\mathbb{P}(R \geq n) < 1/c$ implies $\mathbb{P}(V) = 0$ and $\mathbb{P}(V|\mathcal{N} = \mathbf{n}) = 0$ for μ -almost all configurations \mathbf{n} .

Observe that, by coupling, if $\mathbb{P}(N > n) \geq n^{-\alpha}L(n)$ (resp. $\mathbb{P}(N > n) \leq n^{-\alpha}L(n)$) when $\alpha \in (0, 1)$ then the conclusion of Corollary 3.3(1) (resp. Corollary 3.3(2)) still holds. An analogous result holds for the case $\alpha = 1$.

Remark 3.4. (i) For every fixed unbounded random variable R (with finite or infinite expected value), there exists a random variable N such that the firework process (with $R_{i,j} \sim R$ and $N_i \sim N$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}^*$) survives. Let us fix $\varepsilon > 0$, $\delta \in (0, 1)$ and choose a variable N such that

$$\mathbb{P}\left(N \geq \frac{\ln(1-\delta)}{\ln(\mathbb{P}(R < n))}\right) \geq \frac{1+\varepsilon}{n\delta}$$

for every sufficiently large n . Such N exists since $\mathbb{P}(R < n) < 1$ for all $n \in \mathbb{N}$. Indeed consider

$$\begin{aligned} n(1 - G_N(\mathbb{P}(R < n))) &= n\mathbb{E}[1 - \mathbb{P}(R < n)^N] \\ &\geq n\left(1 - \mathbb{P}(R < n)^{\ln(1-\delta)/\ln(\mathbb{P}(R < n))}\right)\mathbb{P}\left(N \geq \frac{\ln(1-\delta)}{\ln(\mathbb{P}(R < n))}\right) \\ &= n\delta\mathbb{P}\left(N \geq \frac{\ln(1-\delta)}{\ln(\mathbb{P}(R < n))}\right) \geq 1 + \varepsilon \end{aligned}$$

thus Proposition 3.2(1) applies.

(ii) For every fixed (bounded or unbounded) N such that $\mathbb{P}(N = 0) < 1$, there exists R such that the firework process (with $R_{i,j} \sim R$ and $N_i \sim N$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}^*$) survives.

Indeed, define $p_n := \inf\{t \geq 0: G_N(1-t) \leq 1 - 2/n\}$. Since $G_N(1-t) < 1$ for all $t > 0$ we have that $p_n \downarrow 0$ as $n \rightarrow \infty$; moreover, by continuity, $G_N(1-p_n) \leq 1 - 2/n$. By construction $\liminf_{n \rightarrow \infty} n(1 - G_N(1-p_n)) \geq 2$, hence if $\mathbb{P}(R \geq n) = p_n$ then Proposition 3.2(1) applies.

Another proof can be derived by coupling from the following example. Let N be a Bernoulli random variable with parameter $p > 0$. In this case $n(1 - G_N(\mathbb{P}(R < n))) = np\mathbb{P}(R \geq n)$, hence if, for instance, $\mathbb{P}(R \geq n) = 2/(pn)$ then according to Proposition 3.2(1) there is survival. Since every nontrivial N stochastically dominates a Bernoulli variable with parameter $p = \mathbb{P}(N > 0)$, by coupling, we have survival of the homogeneous firework process associated with N and R (where $\mathbb{P}(R \geq n) = 2/(pn)$).

We note that if $\mathbb{P}(N = 0) \in (0, 1)$ in almost every realization there are connected sequences of empty vertices of arbitrarily large length and nevertheless the process may survive with positive probability. This happens in particular in the Bernoulli case where we have at most one station per site. This proves, for instance, that the sufficient conditions of [7, Theorem 2.2] are not necessary.

Remark 3.5. Here we consider the possible behaviors of the system depending on the convergence/divergence of the expected values $\mathbb{E}[N]$ and $\mathbb{E}[R]$.

- If $\mathbb{E}[N] < +\infty$ and $\mathbb{E}[R] < +\infty$ then there is a.s. extinction for almost every configuration \mathbf{n} (see Proposition 3.2(4)).
- If $\mathbb{E}[N] = +\infty$ and $\mathbb{E}[R] < +\infty$ then both survival and extinction are possible. Indeed Remark 3.4 proves that survival for almost every configuration \mathbf{n} is possible. On the other hand if $\mathbb{P}(N \geq n) \sim n^{-\alpha}$ (for some $\alpha \in (0, 1)$) and $\mathbb{P}(R \geq n) \sim n^{-\frac{1}{\alpha}-\epsilon}$, according to Corollary 3.3(2), we have a.s. extinction for almost every configuration \mathbf{n} .
- If $\mathbb{E}[N] < +\infty$ and $\mathbb{E}[R] = +\infty$ then both survival and extinction are possible. Indeed fix any N such that $0 < \mathbb{E}[N] < +\infty$ and suppose that $\mathbb{P}(R \geq n) \sim \alpha/n$; then, according to Proposition 3.2, if $\alpha > 1/\mathbb{E}[N]$ there is survival for almost every configuration \mathbf{n} while if $\alpha < 1/\mathbb{E}[N]$ there is extinction for almost every configuration \mathbf{n} .
- If $\mathbb{E}[N] = +\infty$ and $\mathbb{E}[R] = +\infty$ then, again, both survival and extinction are possible. Survival is easy: take $\mathbb{P}(R \geq n) = p_n \vee 1/n$ (where p_n is defined as in Remark 3.4(ii)) and apply Proposition 3.2(1).

As for the extinction consider N and R such that $\mathbb{P}(N \geq n) \sim 1/n$ as $n \rightarrow \infty$ and $\mathbb{P}(R \geq n) = 1/(n \ln(n) \ln(\ln(n)))$ for every sufficiently large n . Clearly $\sum_{n \in \mathbb{N}} \mathbb{P}(R \geq n) = \sum_{n \in \mathbb{N}} \mathbb{P}(N \geq n) = +\infty$; moreover

$$n \ln\left(\frac{1}{\mathbb{P}(R \geq n)}\right) \mathbb{P}(R \geq n) = \frac{\ln(n) + \ln(\ln(n)) + \ln(\ln(\ln(n)))}{\ln(n) \cdot \ln(\ln(n))} \rightarrow 0$$

as $n \rightarrow \infty$, hence Corollary 3.3(4) applies and there is extinction for almost every configuration \mathbf{n} .

The previous remark is summarized by the following table.

	$\mathbb{E}[N] < +\infty$	$\mathbb{E}[N] = +\infty$
$\mathbb{E}[R] < +\infty$	extinction	extinction/survival
$\mathbb{E}[R] = +\infty$	extinction/survival	extinction/survival

In the heterogeneous case we have a sufficient condition for survival.

Theorem 3.6. *In the heterogeneous case, if*

$$\sum_{n=0}^{\infty} \prod_{i=0}^n G_{N_i}(\mathbb{P}(R_i < n - i + 1)) < +\infty \quad (3.4)$$

then $\mathbb{P}(V) > 0$. Moreover $\mathbb{P}(V | \mathcal{N} = \mathbf{n}) > 0$ for μ -almost all configurations \mathbf{n} .

4. REVERSE FIREWORK PROCESS ON \mathbb{N}

We use the same notation as in Section 3. We denote by S the event “the reverse process survives”. Here we suppose that there is always one station at 0. In the general case of a random number N_0 of station at 0 we can condition on the events $\{N_0 = 0\}$ and $\{N_0 > 0\}$: in the first case we have extinction, in the second one we apply the results of this section. As before, the results for the deterministic case with one station per site can be retrieved by using $G_N(t) \equiv t$.

Theorem 4.1. Let $\{R_{i,j}\}_{i,j \in \mathbb{N}^*}$ and $\{N_i\}_{i \in \mathbb{N}^*}$ be two i.i.d. families. Define $W := \sum_{n \in \mathbb{N}} (1 - G_N(\mathbb{P}(R < n)))$ (or $W := \int_0^\infty (1 - G_N(\mathbb{P}(R < t))) dt$).

- (1) If $W = +\infty$ then $\mathbb{P}(S|\mathcal{N} = \mathbf{n}) = 1$ for μ -almost all configurations \mathbf{n} .
- (2) If $W < +\infty$ then $\mathbb{P}(S|\mathcal{N} = \mathbf{n}) = 0$ for μ -almost all configurations \mathbf{n} .

Remark 4.2. (i) If the homogeneous reverse firework process associated with N and R dies out almost surely, so does the homogeneous firework process. Indeed, in this case $\sum_{n \in \mathbb{N}} (1 - G_N(\mathbb{P}(R < n))) < +\infty$, hence $\lim_{n \rightarrow \infty} n(1 - G_N(\mathbb{P}(R < n))) = 0$ (see Lemma 6.3) and Proposition 3.2(ii) applies. Hence if there is survival for the firework process then there is survival for the reverse firework process.

- (ii) For every fixed unbounded random variable R (with finite or infinite expected value), there exists a random variable N such that the reverse firework process (with $R_{i,j} \sim R$ and $N_i \sim N$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}^*$) survives. Take the example in Remark 3.4(i).
- (iii) For every fixed (bounded or unbounded) N such that $\mathbb{P}(N = 0) < 1$, there exists R such that the reverse firework process (with $R_{i,j} \sim R$ and $N_i \sim N$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}^*$) survives. Consider the example in Remark 3.4(ii).

Theorem 4.1 gives a necessary and sufficient condition for (almost sure) survival for the homogeneous reverse firework process in RE, that is, $W = +\infty$. This condition, implicitly involves both N and R . What we want to do now, is to find conditions involving separately N and R .

Theorem 4.3. Let $\{R_{i,j}\}_{i,j \in \mathbb{N}^*}$ and $\{N_i\}_{i \in \mathbb{N}^*}$ be two i.i.d. families ($R_{i,j} \sim R$ and $N_i \sim N$) and L a positive slowly varying function. Define W as in Theorem 4.1.

- (1) If $\mathbb{E}[N] < +\infty$ then $W < +\infty$ if and only if $\mathbb{E}[R] < +\infty$.
- (2) If there exists $\varepsilon > 0$, $\alpha \in (0, 1)$ such that $n^\alpha \mathbb{P}(N \geq n)/L(n) \geq \varepsilon$ for all $n \geq 1$ and $\int_0^{+\infty} \mathbb{P}(R \geq t)^\alpha L(1/\mathbb{P}(R \geq t)) dt = +\infty$ then $W = +\infty$.
- (3) If there exists $M > 0$, $\alpha \in (0, 1)$ such that $n^\alpha \mathbb{P}(N \geq n)/L(n) \leq M$ for all $n \geq 1$ and $\int_0^{+\infty} \mathbb{P}(R \geq t)^\alpha L(1/\mathbb{P}(R \geq t)) dt < +\infty$ then $W < +\infty$.
- (4) If there exists $M > 0$ such that $\mathbb{P}(N \geq n) \leq M/n$ for all $n \geq 1$ and $\int_0^{+\infty} \mathbb{P}(R \geq t) \ln(1/\mathbb{P}(R \geq t)) dt < +\infty$ then $W < +\infty$.
- (5) If there exists $\varepsilon > 0$ such that $\mathbb{P}(N \geq n) \geq \varepsilon/n$ for all $n \geq 1$ and $\int_0^{+\infty} \mathbb{P}(R \geq t) \ln(1/\mathbb{P}(R \geq t)) dt = +\infty$ then $W = +\infty$.

We note that, according to Theorem 4.3(2–3), if there exists $\alpha \in (0, 1)$ such that $\mathbb{P}(N \geq n) \asymp L(n)/n^\alpha$ then $\int_0^{+\infty} \mathbb{P}(R \geq t)^\alpha L(1/\mathbb{P}(R \geq t)) dt < +\infty$ if and only if $W < +\infty$. Analogously, using Theorem 4.3(4–5), if $\mathbb{P}(N \geq n) \asymp 1/n$ then $\int_0^{+\infty} \mathbb{P}(R \geq t) \ln(1/\mathbb{P}(R \geq t)) dt < +\infty$ if and only if $W < +\infty$.

Remark 4.4. As in Remark 3.5, we consider the possible behaviors of the reverse firework process depending on the convergence/divergence of the expected values $\mathbb{E}[N]$ and $\mathbb{E}[R]$.

- If $\mathbb{E}[N] < +\infty$ and $\mathbb{E}[R] < +\infty$ then there is a.s. extinction for almost every configuration \mathbf{n} (see Theorem 4.3(1)).
- If $\mathbb{E}[N] = +\infty$ and $\mathbb{E}[R] < +\infty$ then both survival and extinction are possible. Indeed suppose that $\mathbb{P}(N \geq n) \asymp n^{-\alpha}$ (for some $\alpha \in (0, 1)$) and $\mathbb{P}(R \geq n) \asymp n^{-\beta}$ (where by $a_n \asymp b_n$ we mean that there exist $m, M \in (0, +\infty)$ such that $a_n/b_n \in [m, M]$ for every sufficiently large n). According to the discussion after Theorem 4.3, if $\beta \in (1, 1/\alpha]$ then there is survival for almost every configuration \mathbf{n} , while if $\beta > 1/\alpha$ there is a.s. extinction for almost every configuration \mathbf{n} .
- If $\mathbb{E}[N] < +\infty$ and $\mathbb{E}[R] = +\infty$ then there is survival for almost every configuration \mathbf{n} (see Theorem 4.3(1)).
- If $\mathbb{E}[N] = +\infty$ and $\mathbb{E}[R] = +\infty$ then, due to a coupling with the case $\mathbb{E}[N] < +\infty$ and $\mathbb{E}[R] = +\infty$, only survival is possible.

The previous remark is summarized by the following table.

	$\mathbb{E}[N] < +\infty$	$\mathbb{E}[N] = +\infty$
$\mathbb{E}[R] < +\infty$	extinction	extinction/survival
$\mathbb{E}[R] = +\infty$	survival	survival

In the heterogeneous case we have the following result.

Theorem 4.5. *Consider the heterogeneous reversed firework process on \mathbb{N} .*

- (1) $\sum_{k \geq 1} (1 - G_{N_{n+k}}(\mathbb{P}(R_{n+k} < k))) = +\infty$ for all $n \in \mathbb{N}$ if and only if $\mathbb{P}(S|\mathcal{N} = \mathbf{n}) = 1$ for μ -almost all configurations \mathbf{n} .
- (2) If $\sum_{n \in \mathbb{N}} \prod_{k=1}^{\infty} G_{N_{n+k}}(\mathbb{P}(R_{n+k} < k)) < +\infty$ then $\mathbb{P}(S|\mathcal{N} = \mathbf{n}) > 0$ for μ -almost all configurations \mathbf{n} .

5. FIREWORK AND REVERSE FIREWORK PROCESSES ON GALTON-WATSON TREES

Let us consider a GW-process with offspring distribution ρ . We know that if $\rho(1) = 1$ then the resulting random tree is \mathbb{N} . This particular case has been studied in Sections 3 and 4. In the rest of the paper we assume that $\rho(1) < 1$ and we suppose that $m := \sum_{n \in \mathbb{N}} n\rho(n) > 1$. The underlying random graph will be a GW-tree generated by this process. Henceforth, to avoid a cumbersome notation, we use the same symbol to denote a tree (as a graph) and its set of vertices. Let $\varphi(z) := \sum_{n \in \mathbb{N}} \rho(n)z^n$ be the generating function of ρ and let $\alpha \in [0, 1]$ be the smallest nonnegative fixed point of φ . When $\sum_{i=0}^k \rho(i) = 1$ for some k we say that the GW-tree has maximum degree k or that it is k -bounded.

In this section we consider just the homogeneous case: the set of labels J is \mathbb{N} and the random number of stations $\{N_w\}_{w \in \mathcal{W}}$ are independent \mathbb{N} -valued random variables, $N_w \sim N$ for all $w \in \mathcal{W} \setminus \{\emptyset\}$. We also assume that, in the case of the firework process, $N_\emptyset \sim N$, while in the case of the reverse firework process the number of stations at the root does not matter as long as it is positive,

hence we take a deterministic $N_\emptyset = \min\{n \in \mathbb{N}^*: \mathbb{P}(N = n) > 0\}$ (the minimum positive value of N). In both cases the support of the law of N_\emptyset is a subset of the support of N . The environment is a random labelled GW-tree τ^L defined in Section 2.1 where the label of each vertex w is the number of stations at w . As in Section 2.1, the law of τ^L is denoted by μ and \mathbb{P} is the probability measure on $\Omega = \mathbb{LT} \times \mathcal{O}$. Remember that $\mathbb{P}(\tau^L \text{ is finite}) = \mathbb{P}(\tau \text{ is finite}) = \mu(E) = \mu_j(E) = \alpha < 1$ for all $j \in \mathbb{N}$. The radii $\{R_{w,i}\}_{w \in \mathcal{W}, i \in \mathbb{N}^*}$ of the stations are independent and identically distributed (with distribution R).

Two interesting particular cases are when there is one station per site ($N = 1$ a.s.) and when N is a Bernoulli variable. The first case can be easily retrieved from the general results by taking $G_N(t) \equiv t$ (see the comments along this section), while the second case is discussed in Example 5.7.

The strategy is to study the annealed counterpart of the process on a GW-tree, with one station per site and radii $\tilde{R}_w = \mathbb{1}_{N_w \geq 1} \max\{R_{w,i}: i = 1, \dots, N_w\}$ (see Section 2.3). We prove that under suitable conditions the probability of survival of the annealed counterpart is 0 (resp. > 0). This implies that the annealed probability of survival of the original process is 0 (resp. > 0): quenched results then follow as we explain in the following lemma (remember the definition of $\mu, \bar{\mu}$ as the laws of τ^L and τ respectively and recall that $l(\Upsilon)$ is the number of stations at the root of $\Upsilon \in \mathbb{LT}$).

Lemma 5.1. *Consider a homogeneous firework process or a homogeneous reverse firework process.*

- (1) *If $\mathbb{P}(\text{survival}) = 0$ then $\mathbb{P}(\text{survival}|\tau^L = \Upsilon) = 0$ for almost every $\Upsilon \in \mathbb{LT}$ (that is, for almost every tree and every sequence of stations).*
- (2) *If $\mathbb{P}(\text{survival}|\tau \text{ is infinite}) = 1$ then $\mathbb{P}(\text{survival}|\tau^L = \Upsilon) = 1$ for almost every infinite $\Upsilon \in \mathbb{LT}$.*
- (3) *If $\mathbb{P}(\text{survival}) > 0$ and $\mathbb{P}(N = 0) = 0$ then $\mathbb{P}(\text{survival}|\tau^L = \Upsilon) > 0$ for almost every infinite $\Upsilon \in \mathbb{LT}$ such that $l(\Upsilon) \geq 1$ (that is, for almost every realization of the environment such that the underlying tree is infinite and there is at least one station at the root).*
- (4) *If $\mathbb{P}(\text{survival}) > 0$ and $\mathbb{P}(N = 0) > 0$ then $\mathbb{P}(\text{survival}|\tau = \mathbf{T}, N_\emptyset = n) > 0$ for almost every $(\mathbf{T}, n) \in \mathbb{T} \times \mathbb{N}$ such that the tree \mathbb{T} is infinite and $n \geq 1$.*

Moreover, $\mu(\Upsilon: \mathbb{P}(\text{survival}|\tau^L = \Upsilon) > 0)$ and $\bar{\mu} \times \mathbb{P}_{N_\emptyset}((\mathbf{T}, n): \mathbb{P}(\text{survival}|\tau = \mathbf{T}, N_\emptyset = n) > 0)$ are both either 0 or $(1 - \alpha)\mathbb{P}(N_\emptyset \geq 1)$.

The third case of the previous lemma applies, for instance, to any annealed counterpart of a process (since in the annealed counterpart there is one station per vertex).

Note the difference between “ $\mathbb{P}(\text{survival}|\tau^L = \Upsilon) > 0$ for almost every $\Upsilon \in \mathbb{LT}$ such that the underlying tree is infinite and $l(\Upsilon) \geq 1$ ” and “ $\mathbb{P}(\text{survival}|\tau = \mathbf{T}, N_\emptyset = n) > 0$ for almost every infinite tree $\mathbf{T} \in \mathbb{T}$ and for \mathbb{P}_{N_\emptyset} -almost all $n \geq 1$ ”. In both cases there is at least one station at the root, but the second assertion is weaker than the first one since in the former the tree and the number of stations at each vertex are fixed while in the latter just the tree and the number of

stations at the root are fixed. Indeed we show that, when $\mathbb{P}(N = 0) > 0$, the second assertion may hold while the first does not (see Example 5.8).

Finally, since there is no survival if there are no stations at the root or the tree is finite, then the probability that the environment can sustain a surviving process cannot exceed the probability of the event “the tree is infinite and there are stations at the root”, that is, $(1 - \alpha)\mathbb{P}(N_\emptyset \geq 1)$. Lemma 5.1 tells us that when the probability that the environment can sustain a surviving process is not 0 then it attains the maximum admissible value.

Theorem 5.2. *Consider a homogeneous firework process. Define $\Phi(t) := G_N(\mathbb{P}(R < 1)) + \sum_{n=1}^{\infty} (G_N(\mathbb{P}(R < n+1)) - G_N(\mathbb{P}(R < n)))t^n \in [0, +\infty]$ ($t \in [0, +\infty)$).*

- (1) *If $\Phi(m) - 1 > \Phi(0) = G_N(\mathbb{P}(R < 1))$ and $\mathbb{P}(N = 0) = 0$ then for the firework process there is survival with positive probability for almost every realization of the environment such that the underlying tree is infinite and there is at least one station at the root.*
- (2) *If $\Phi(m) - 1 > \Phi(0) = G_N(\mathbb{P}(R < 1))$ and $\mathbb{P}(N = 0) > 0$ then for the firework process $\mathbb{P}(\text{survival} | \tau = \mathbf{T}, N_\emptyset = n) > 0$ for almost every $(\mathbf{T}, n) \in \mathbb{T} \times \mathbb{N}$ such that \mathbf{T} is an infinite (unlabelled) tree and $n \geq 1$.*
- (3) *If the GW-tree is k -bounded and $\Phi(k) - 1 \leq 1 - 1/k$ then the firework process becomes extinct a.s. for almost every realization of the environment.*

It is clear, from the previous theorem, that the radius of convergence of Φ will play an important role. Elementary computations show that $\limsup_{n \rightarrow \infty} \sqrt[n]{G_N(\mathbb{P}(R < n+1)) - G_N(\mathbb{P}(R < n))} = \limsup_{n \rightarrow \infty} \sqrt[n]{1 - G_N(\mathbb{P}(R < n))}$. Moreover, in the case $N = 1$ a.s., $\Phi(t) := \mathbb{E}[t^{\lfloor R \rfloor}] = \sum_{n=0}^{\infty} \mathbb{P}(n \leq R < n+1)t^n$.

Corollary 5.3. *Define $\bar{m}_c := \sup\{m > 0: \text{the firework process dies out a.s.}\}$. We have that $\limsup_{n \rightarrow \infty} \sqrt[n]{1 - G_N(\mathbb{P}(R < n))} = 1$ implies $\bar{m}_c = 1$.*

Consider the case $N = 1$ almost surely. In this case, if $\mathbb{E}[R^a] = +\infty$ for some $a > 0$ we have $m_c = 1$. An example where $\mathbb{E}[R] < +\infty$ and $\bar{m}_c = 1$ is given by any law R such that $\mathbb{P}(n \leq R < n+1) \asymp n^{-\alpha}$ ($\alpha > 2$).

The next result deals with the behavior of the reverse homogeneous firework process. By definition, $\prod_{j=1}^0 \alpha_n := 1$ for every sequence $\{\alpha_n\}_{n \in \mathbb{N}}$.

Theorem 5.4. *Consider a homogeneous reverse firework process. Let $\phi_1(m) := \sum_{n=1}^{\infty} (1 - G_N(\mathbb{P}(R < n)))m^n$ and $\phi_2(m) := \sum_{i=1}^{\infty} (1 - G_N(\mathbb{P}(R < i)))m^i \prod_{j=1}^{i-1} G_N(\mathbb{P}(R < j))$. The following hold*

- (1) *if $\phi_1(m) = +\infty$ then there is survival with probability 1 for the reverse firework process for almost all realizations of the environment such that the underlying tree is infinite;*
- (2) *if $\mathbb{P}(N = 0) = 0$, $\phi_1(m) < +\infty$ and $\phi_2(m) > 1$ then there is survival with positive probability (strictly smaller than 1) for the reverse firework process for almost all realizations of the environment such that the underlying tree is infinite;*

- (3) if $\mathbb{P}(N = 0) > 0$, $\phi_1(m) < +\infty$ and $\phi_2(m) > 1$ then $\mathbb{P}(\text{survival} | \tau = \mathbf{T}) \in (0, 1)$ for almost every infinite (unlabelled) tree $\mathbf{T} \in \mathbb{T}$;
- (4) if $\phi_1(m) < +\infty$ and $\phi_2(m) \leq 1$ then there is a.s. extinction for the reverse firework process for almost all realizations of the environment;

We note that when $N = 1$ a.s. then $\phi_1(m)$ and $\phi_2(m)$ become $\sum_{n=1}^{\infty} m^n \mathbb{P}(R \geq n)$ and $\sum_{n=1}^{\infty} m^n \mathbb{P}(R \geq n) \prod_{j=1}^{n-1} \mathbb{P}(R < j)$ respectively.

Corollary 5.5. Define $M_c := 1 / \limsup_{n \rightarrow \infty} \sqrt[n]{1 - G_N(\mathbb{P}(R < n))}$. There exists a critical value $m_c \in [1, +\infty)$, $m_c \leq M_c$ such that

- (1) $m < m_c$ implies a.s. extinction for almost all realizations of the environment;
- (2) $m_c < m < M_c$ and $\mathbb{P}(N = 0) = 0$ implies survival with positive probability for almost all realizations of the environment such that the underlying tree is infinite;
- (3) $m_c < m < M_c$ and $\mathbb{P}(N = 0) > 0$ implies survival with positive probability for almost every infinite (unlabelled) tree;
- (4) $M_c < m$ implies survival with probability 1 for almost all realizations of the environment such that the underlying tree is infinite.

Moreover, if $m = m_c < M_c$ then there is a.s. extinction for almost all realizations of the environment.

Again the previous corollary applies, in particular, in the case $N = 1$ a.s. by using $\mathbb{P}(R \geq n)$ instead of $1 - G_N(\mathbb{P}(R < n))$.

Remark 5.6. If $\phi_1(1) = +\infty$, (that is, $\mathbb{E}[R] = +\infty$ if $N = 1$ a.s.) then $M_c = m_c = 1$ and there is survival with probability 1 for every realization of the environment such that the underlying tree is infinite (when $m > 1$).

If $M_c = +\infty$ then the probability of survival is smaller than 1 for almost every infinite (unlabelled) tree (it might be 0 of course).

If $M_c \in (1, +\infty]$ and $\phi_1(M_c) = +\infty$ then $m_c < M_c$, thus there is a.s. extinction for almost all realizations of the environment when $m = m_c$ (see the details in Section 6).

Note that according to Theorems 5.2 and 5.4 the firework and reverse firework processes on a non-trivial GW-tree can survive even if the radius R is bounded. Indeed, if $\mathbb{P}(N = 0) < 1 - 1/m$ and $\mathbb{P}(R < 1)$ is sufficiently small then we have $1 - G_N(\mathbb{P}(R < 1)) > 1/m$ and the conditions for survival given in Theorems 5.2 and 5.4 hold. According to Theorems 3.1(1) and 4.1(2), this is not possible on \mathbb{N} . As an example we discuss the Bernoulli case.

Example 5.7. We consider $N \sim B(p)$, a Bernoulli variable with parameter $p \in (0, 1)$. In this framework, each vertex is occupied by one station independently with probability p and unoccupied with probability $1 - p$. Let $R_w \sim R$ for all vertices $w \in \mathcal{W}$. Clearly $G_N(t) = pt + 1 - p$.

Consider the firework process. Applying Theorem 5.2, we have that

- (1) $\sum_{n=1}^{\infty} \mathbb{P}(n \leq R < n+1)m^n > 1/p$ implies survival with positive probability for almost every infinite (unlabelled) tree if there is at least one station at the root;
- (2) GW-tree k -bounded and $\sum_{n=1}^{\infty} \mathbb{P}(n \leq R < n+1)k^n - \mathbb{P}(R \geq 1) \leq (k-1)/(kp)$ implies a.s. extinction for almost every realization of the environment.

Consider the reverse firework process. According to Theorem 5.4, there is survival with probability 1 (for almost all realizations of the environment such that the underlying tree is infinite) if and only if $\sum_{n=1}^{\infty} m^n \mathbb{P}(R \geq n) = +\infty$ (note that this condition does not depend on $p > 0$). If $\sum_{n=1}^{\infty} m^n \mathbb{P}(R \geq n) < +\infty$ then the probability of survival is strictly less than 1 for every (unlabelled) tree. In particular there is a.s. extinction for almost all realizations of the environment if and only if $\sum_{n=1}^{\infty} m^n \mathbb{P}(R \geq n) \prod_{j=1}^{n-1} (1 - p \mathbb{P}(R \geq j)) \leq 1/p$. We observe that M_c does not depend on p . As for the critical value $m_c = m_c(p)$ it is not difficult to show that $p \mapsto m_c(p)$ is nonincreasing and continuous. In particular $\lim_{p \rightarrow 0} m_c(p) = M_c$. Indeed, if $M_c = 1$ there is nothing to prove. Otherwise, suppose that $1 < r < M_c$; since $\lim_{p \rightarrow 0} p \sum_{n=1}^{\infty} m^n \mathbb{P}(R \geq n) \prod_{j=1}^{n-1} (1 - p \mathbb{P}(R \geq j)) = 0$ according to Theorem 5.4 and Corollary 5.5 we have that $\liminf_{p \rightarrow 0} m_c(p) \geq r$.

When $\mathbb{P}(N = 0) > 0$ and $\mathbb{P}(\text{survival}) > 0$ it could happen that the firework and reverse firework processes die out almost surely on a μ -positive set of infinite labelled trees with at least one station at the root. This implies that, even when the probability of survival is positive, it is not possible, in general, to have positive probability of survival for almost every realization of the environment (but only for almost every realization of the unlabelled GW-tree) as the following example shows.

Example 5.8. Suppose that $\mathbb{P}(N = 0) \in (0, 1)$, $m > 1$ and $\mathbb{P}(R \leq M) = 1$ (for some $M \in \mathbb{N}^*$). Under these conditions, it is an easy exercise to prove that the set of infinite labelled trees Υ with $l(\Upsilon) \geq 1$ such that every vertex between distance 1 and M from the root has label 0, has always μ -positive measure. On this set the firework and reverse firework processes die out almost surely. Nevertheless, according to Theorem 5.2 for every sufficiently large m the firework process survives with (strictly) positive probability, Similarly, according to Theorem 5.4 there is a positive (strictly smaller than 1) probability of survival for the reverse firework process (note that, if $\mathbb{P}(R \leq M) = 1$, then $\phi_1(m) < +\infty$ for all m). This happens, for instance, in the Bernoulli model described in Example 5.7.

6. PROOFS

Proof of Lemma 2.2. In the following we consider the canonical MBP $\{Z_n\}_{n \in \mathbb{N}}$ defined on the probability space \mathbb{LT} endowed with the probability measure μ_c . Hence $\tau^L(\Upsilon) = \Upsilon$ for all $\Upsilon \in \mathbb{LT}$. Remember the definition of the generating function $\varphi(z) = \sum_{n \in \mathbb{N}} \rho(n)z^n$ and its smallest fixed point $\alpha \in [0, 1]$. Define $S_J := \{g \in \mathbb{N}^J : |g| := \sum_{j \in J} g(j) < +\infty\}$ and denote by \mathbb{P}_N the common

law of $\{N_w\}_{w \in \mathcal{W} \setminus \{\emptyset\}}$. Given an inherited couple (A, \tilde{A}) where $A, \tilde{A} \subset \mathbb{LT}$, if the expected value m of the law ρ satisfies $m \leq 1$ then $\mu_j(A) \geq \mu_j(E) = \alpha = 1$ for all $j \in \bar{J}$, and the conclusion follows.

Suppose that $m > 1$. By conditioning on Z_1 , using the conditional independence and the Markov property for labelled GW-trees we have, for all $j \in \bar{J}$,

$$\begin{aligned}
\mu_j(\tilde{A}) &= \mu_j(A) = \mathbb{E}^{\mu_j}[\mathbb{1}_A] = \mathbb{E}^{\mu_j}[\mu_j(\Upsilon \in A | Z_1)] \leq \mathbb{E}^{\mu_j}[\mu_j(\Upsilon: \Upsilon^{(1)} \in \tilde{A}, \dots, \Upsilon^{(|Z_1|)} \in \tilde{A} | Z_1)] \\
&= \sum_{n \in \mathbb{N}} \sum_{g \in S_J: |g|=n} \mu_j(\Upsilon: \Upsilon^{(1)} \in \tilde{A}, \dots, \Upsilon^{(|g|)} \in \tilde{A} | Z_1 = g) \mu_j(Z_1 = g) \\
&= \sum_{n \in \mathbb{N}} \sum_{g \in S_J: |g|=n} \mu_j(Z_1 = g) \prod_{i=1}^{|g|} \mu_j(\Upsilon: \Upsilon^{(i)} \in \tilde{A} | Z_1 = g) \\
&= \sum_{n \in \mathbb{N}} \sum_{g \in S_J: |g|=n} \mu_j(Z_1 = g) \prod_{j' \in J} \mu_{j'}(\Upsilon \in \tilde{A})^{g(j')} \\
&= \sum_{n \in \mathbb{N}} \sum_{g \in S_J: |g|=n} \rho(|g|) \frac{|g|!}{\prod_{j' \in J} g(j')!} \prod_{j' \in J} \mathbb{P}_N(j')^{g(j')} \prod_{j' \in J} \mu_{j'}(\tilde{A})^{g(j')} \\
&= \sum_{n \in \mathbb{N}} \rho(n) \left(\sum_{j' \in J} \mu_{j'}(\tilde{A}) \mathbb{P}_N(j') \right)^n = \varphi \left(\sum_{j' \in J_N} \mu_{j'}(\tilde{A}) \mathbb{P}_N(j') \right),
\end{aligned}$$

where $|Z_1| := \sum_{j \in J} Z_1(j)$ is the total number of descendants of the root.

Take a generic $h \in [0, 1]^J$ and suppose that $\alpha \leq h(j) \leq \varphi(\sum_{j' \in J_N} h(j') \mathbb{P}_N(j'))$ for all $j \in \bar{J}$. Define $\bar{h} := \sum_{j' \in J_N} h(j') \mathbb{P}_N(j')$. Hence

$$\bar{h} \leq \sup_{j \in \bar{J}} h(j) \leq \varphi(\bar{h}).$$

Thus, either $\bar{h} = \alpha$ or $\bar{h} = 1$. This implies $\bar{h} \leq \sup_{j \in \bar{J}} h(j) \leq \varphi(\bar{h}) = \bar{h}$. Since $h(j) \in [\alpha, 1]$ for all $j \in \bar{J}$, then $\bar{h} = \alpha$ implies $h(j) = \alpha \equiv \mu_j(E)$ for all $j \in \bar{J}$, while $\bar{h} = 1$ implies $h(j) = 1$ for all $j \in J_N$.

By Definition 2.1, $\mu_j(A) = \mu_j(\tilde{A})$, $\mu_j(E \setminus A) = \mu_j(E \setminus \tilde{A}) = 0$ and $\mu_j(A \setminus E) = \mu_j(\tilde{A} \setminus E)$ for all $j \in \bar{J}$. If $h(j) := \mu_j(\tilde{A})$ for all $j \in \bar{J}$ then we have that either $\mu_j(\tilde{A}) = \mu_j(A) = 1$ for all $j \in J_N$ or $\mu_j(\tilde{A}) = \mu_j(A) = \mu_j(E)$ for all $j \in \bar{J}$ and the conclusion follows.

If ς (the law of N_\emptyset) satisfies $\text{supp}(\varsigma) \subseteq J'$ then $\mu_\varsigma = \sum_{j \in J'} \varsigma(j) \mu_j$, whence, for every measurable $B \subseteq \mathbb{LT}$, $\mu_\varsigma(B) = \sum_{j \in J'} \varsigma(j) \mu_j(B)$ which yields the conclusion (by taking $J' = \bar{J}$, $B := A \triangle E$ and $J' = J_N$, $B := A$). \square

Lemma 6.1. *If $\{t_{i,n}\}_{i,n \in \mathbb{N}, i \leq n}$ is an arbitrary nonnegative sequence and*

$$\sum_{n=0}^{\infty} \prod_{i=0}^n G_{N_i}(t_{i,n}) < +\infty \tag{6.5}$$

then

$$\mathbb{P} \left(\sum_{n=0}^{\infty} \prod_{i=0}^n t_{i,n}^{N_i} < +\infty \right) = 1.$$

In particular if equation (6.5) holds

$$\mathbb{P}\left(\sum_{n=0}^{\infty} \prod_{i=0}^n t_{i,n}^{N_i} < +\infty \mid \mathcal{N} = \mathbf{n}\right) = 1$$

for μ -almost all $\mathbf{n} = \{n_i\}_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$.

Remark 6.2. Suppose that $t_{i,n} \rightarrow 1$ as $n \rightarrow \infty$ for all $i \in \mathbb{N}$. We observe that the event $\{\sum_{n=0}^{\infty} \prod_{i=0}^n t_{i,n}^{N_i} < +\infty\}$ is a tail event with respect to $\{\sigma(N_i: i \leq n)\}_n$. Hence its probability is either 0 or 1. Indeed, for all $i_0 \in \mathbb{N}$, $\prod_{i=0}^n t_{i,n}^{N_i} \sim \prod_{i=i_0}^n t_{i,n}^{N_i}$ as $n \rightarrow \infty$ hence $\{\sum_{n=0}^{\infty} \prod_{i=0}^n t_{i,n}^{N_i} < +\infty\} = \{\sum_{n=i_0}^{\infty} \prod_{i=i_0}^n t_{i,n}^{N_i} < +\infty\}$.

Proof of Lemma 6.1. Let $\xi := \sum_{n=0}^{\infty} \prod_{i=0}^n t_{i,n}^{N_i}$. Note that, by the Monotone Convergence Theorem and using the independence of $\{N_i\}_{i \in \mathbb{N}}$,

$$\begin{aligned} \mathbb{E}[\xi] &= \mathbb{E}_{\mu}[\xi] = \int \sum_{n=0}^{\infty} \prod_{i=0}^n t_{i,n}^{N_i} d\mu = \sum_{n=0}^{\infty} \mathbb{E}_{\mu} \left[\prod_{i=0}^n t_{i,n}^{N_i} \right] \\ &= \sum_{n=0}^{\infty} \prod_{i=0}^n G_{N_i}(t_{i,n}) < +\infty \end{aligned}$$

thus $\mathbb{P}(\xi < +\infty) \equiv \mu(\xi < +\infty) = 1$. The last equality is an easy consequence of equation (2.1). \square

Proof of Theorem 3.1. We investigate the behavior of the deterministic counterpart of our process. Since $R_{i,j} \sim R$ and $N_i \sim N$ for all i, j then $\tilde{R}_i \sim \tilde{R}$ where

$$\mathbb{P}(\tilde{R} < i) = \sum_{j \in \mathbb{N}} \mathbb{P}(R < i)^j \mathbb{P}(N = j) = G_N(\mathbb{P}(R < i)).$$

By [7, Theorem 2.1] the a.s. extinction of the deterministic counterpart is equivalent to

$$\sum_{n=1}^{\infty} \prod_{i=0}^n \mathbb{P}(\tilde{R} < i+1) = +\infty.$$

Hence equation (3.3) is equivalent to $\mathbb{P}(V) = 0$ which, in turn, is equivalent to $\mathbb{P}(V \mid \mathcal{N} = \mathbf{n}) = 0$ for μ -almost all configurations \mathbf{n} .

We are left to prove that $\mathbb{P}(V) > 0$ implies $\mu(\mathbf{n} \in \overline{\mathbb{N}^{\mathbb{N}}}: \mathbb{P}(V \mid \mathcal{N} = \mathbf{n}) > 0) = \mathbb{P}(N > 0)$. Note that $\mathbb{P}(V) = \mathbb{P}(V \mid N_0 > 0) \mathbb{P}(N_0 > 0)$ since $\mathbb{P}(V \mid N_0 = 0) = 0$; moreover $\mathbb{P}(V \mid N_0 > 0) > 0$ if and only if $\mathbb{P}(V) > 0$. We condition now on the event $\{N_0 > 0\}$. We denote by $\overline{\mathbb{N}^{\mathbb{N}}}$ the space $\{\mathbf{n}: n_0 > 0\}$ and by $\bar{\mu}$ the measure μ conditioned on $\overline{\mathbb{N}^{\mathbb{N}}}$ (clearly $\mu(\overline{\mathbb{N}^{\mathbb{N}}}) = \mathbb{P}(N_0 > 0)$). Observe that $\mu(\mathbf{n}: \limsup_i n_i > 0) = 1$, $\mu(\mathbf{n}: n_0 > 0) = \mathbb{P}(N_0 > 0)$ and that the variables $\{N_i\}_{i \geq 1}$ are independent with respect to the conditioned probability $\mathbb{P}(\cdot \mid N_0 > 0)$. Denote by

$$W_k := \{\mathbf{n} \in \overline{\mathbb{N}^{\mathbb{N}}}: \mathbb{P}(\text{"}V \text{" starting from } \mathbf{n}(k) \text{ stations at } k \mid \mathcal{N} = \mathbf{n}) > 0\}$$

the set of realizations of the environment such that the firework process starting from vertex k (and moving rightwards) survives with positive probability. Note that $\mu(W_0) = \mu(\mathbf{n} \in \overline{\mathbb{N}^{\mathbb{N}}}: \mathbb{P}(V \mid \mathcal{N} = \mathbf{n}) > 0)$

$\mathbf{n}) > 0$). Clearly $W_k \in \sigma(N_i: i \geq k)$ where, in this case, $\{N_i\}_{i \in \mathbb{N}}$ is the canonical realization of \mathcal{N} on $\mathbb{N}^{\mathbb{N}}$.

If for a fixed sequence in $\overline{\mathbb{N}^{\mathbb{N}}}$ there is survival for the firework process starting from n_{i_0} stations at i_0 then, by the FKG inequality, there is survival starting from every $i \leq i_0$ (note that if equation (3.2) holds then the radii are unbounded variables, hence the event $\{R_0 > i_0\}$ has a positive probability). Whence $W_k \supseteq W_{k+1}$ for all $k \in \mathbb{N}$.

Moreover, if there is survival for a sequence \mathbf{n} then for all $j \in \mathbb{N}$ there exists $i_0 \geq j$ such that there is survival starting from $\mathbf{n}(i_0)$ particles at i_0 . Hence $W_0 \subseteq \limsup_k W_k$ which implies that $W_0 = \limsup_k W_k = W_i$ for all $i \in \mathbb{N}$. Hence W_0 is a tail event, namely it belongs to $\bigcap_{k \in \mathbb{N}} \sigma(N_i: i \geq k)$. Thus $\bar{\mu}(W_0)$ is either 0 or 1. This implies that $\mu(W_0)$ is either 0 or $\mathbb{P}(N_0 > 0)$.

Equation (3.2) implies $\mathbb{P}(V) > 0$ or, equivalently, $\mu(\mathbf{n} \in \mathbb{N}^{\mathbb{N}}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0) > 0$. Since $\mu(W_0) = \mu(\mathbf{n} \in \mathbb{N}^{\mathbb{N}}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0)$ we have that $\mu(\mathbf{n} \in \mathbb{N}^{\mathbb{N}}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0) = \mathbb{P}(N_0 > 0)$. \square

Before proving Proposition 3.2 we need another lemma.

Lemma 6.3. (1) Consider a sequence $\{x_i\}_{i \in \mathbb{N}}$ of nonnegative real numbers. If $\{y_i\}_{i \in \mathbb{N}}$ is a nondecreasing sequence such that $y_0 > 0$ and $\limsup_n \sum_{i=0}^n x_i/y_n > 0$ then

$$\sum_{i=1}^{\infty} x_i = +\infty \iff \sum_{i=0}^{\infty} \frac{x_i}{y_i} = +\infty.$$

In particular, if $x_0 > 0$,

$$\sum_{i=1}^{\infty} x_i = +\infty \iff \sum_{i=0}^{\infty} \frac{x_i}{\sum_{j=0}^i x_j} = +\infty.$$

(2) If $\{\alpha_i\}_{i \in \mathbb{N}}$ is a nondecreasing sequence of strictly positive real numbers such that $\liminf_n n/\alpha_n > 0$ then for every nonincreasing, nonnegative sequence $\{z_i\}_{i \in \mathbb{N}}$ such that $\sum_{n=0}^{\infty} z_n < +\infty$ we have $\lim_{n \rightarrow \infty} z_n \alpha_n = 0$.

Proof. (1) Observe that $x_i/y_i \leq x_i/y_0$, this implies $\sum_{i=0}^{\infty} x_i/y_i \leq \sum_{i=1}^{\infty} x_i/y_0$. Whence $\sum_{i=0}^{\infty} x_i/y_i = +\infty$ implies $\sum_{i=1}^{\infty} x_i = +\infty$.

Conversely, suppose that $\sum_{i=1}^{\infty} x_i = +\infty$. Fix $\delta \in (0, \limsup_n \sum_{i=0}^n x_i/y_n)$. For all $m \in \mathbb{N}$ there exists $n > m$ such that $\sum_{i=m}^n x_i/y_n > \delta$. By induction we can find a strictly increasing sequence $\{n_j\}_{j \in \mathbb{N}}$ of natural numbers such that $\sum_{i=n_{j+1}}^{n_{j+1}} x_i/y_{n_{j+1}} > \delta$. Clearly

$$\sum_{i=0}^{\infty} \frac{x_i}{y_i} = \sum_{j=0}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} \frac{x_i}{y_i} \geq \sum_{j=0}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} \frac{x_i}{y_{n_{j+1}}} = +\infty.$$

(2) By contradiction, suppose that, for some increasing sequence $\{n_j\}_{j \in \mathbb{N}}$ and $\delta > 0$, we have $z_{n_j} \alpha_{n_j} \geq \delta$ for all $j \in \mathbb{N}$. Then for all $n \in (n_j, n_{j+1}]$ we have $z_n \geq z_{n_{j+1}} \geq \delta/\alpha_{n_{j+1}}$. Thus, $\sum_{n=0}^{\infty} z_n \geq \sum_{j=0}^{\infty} (n_{j+1} - n_j) \delta/\alpha_{n_{j+1}}$. If we define $x_j := n_{j+1} - n_j$ and $y_j := \alpha_{n_{j+1}}$ then

$\sum_{j \in \mathbb{N}} x_j = +\infty$ and $\limsup_{j \rightarrow \infty} \sum_{i=0}^j x_i/y_j \geq \liminf_{n \rightarrow \infty} n/\alpha_n > 0$. Thus, according to (1), we have

$$\sum_{n=0}^{\infty} z_n \geq \sum_{j=0}^{\infty} (n_{j+1} - n_j) \delta / \alpha_{n_{j+1}} = \delta \sum_{j=0}^{\infty} \frac{x_j}{y_j} = +\infty$$

which contradicts $\sum_{n=0}^{\infty} z_n < +\infty$. □

A particular case in Lemma 6.3 is $\alpha_n = n$.

Proof of Proposition 3.2. (1) In order to prove that equation (3.3) holds we use the Kummer's test. According to Kummer's result, if $\{a_n\}_{n \in \mathbb{N}}$ and $\{p_n\}_{n \in \mathbb{N}}$ are two positive sequences, $\alpha := \liminf_n (p_n a_n / a_{n+1} - p_{n+1})$ and $\beta := \limsup_n (p_n a_n / a_{n+1} - p_{n+1})$ then $\alpha > 0$ implies $\sum_n a_n < +\infty$, while $\sum_n 1/p_n = +\infty$ and $\beta < 0$ implies $\sum_n a_n = +\infty$.

If we take $p_n = n + 2$ and $a_n = \prod_{i=0}^n G_N(\mathbb{P}(R < i + 1))$ then $\alpha > 0$ hence $\mathbb{P}(V) > 0$ (and $\mu(\mathbf{n}: \mathbb{P}(V|\mathcal{N} = \mathbf{n}) > 0) > 0$).

- (2) As in the previous case, it follows immediately from Theorem 3.1 and Kummer's Test with $p_n = n + 2$.
- (3) If $\mathbb{E}[N] < +\infty$ then $1 - G_N(\mathbb{P}(R < n)) \sim \mathbb{E}[N]\mathbb{P}(R \geq n)$ as $n \rightarrow \infty$. The result follows from (2).
- (4) Trivially, $\mathbb{E}[R] < +\infty$ if and only if $\sum_{n \in \mathbb{N}} \mathbb{P}(R \geq n) < +\infty$. From Lemma 6.3(2) with $\alpha_n = n$ and $z_n = \mathbb{P}(R \geq n)$ we have that $\mathbb{E}[R] < +\infty$ implies $\lim_{n \rightarrow \infty} n\mathbb{P}(R \geq n) = 0 < 1/\mathbb{E}[N]$ and the previous part of the theorem applies.
- (5) In this case, since G' is a power series with nonnegative coefficients, $1 - G_N(\mathbb{P}(R < n)) \geq \mathbb{P}(R \geq n)G'_N(\mathbb{P}(R < n))$. Thus, the result follows from (1). □

Proof of Corollary 3.3. The main idea of the proof is to compute a suitable asymptotic estimate $1 - G_N(x) \sim (1 - x)f(x)$ as $x \rightarrow 1^-$.

- (1) Note that, using the Cauchy product of power series, $(1 - G_N(x))/(1 - x) = \sum_{n=0}^{\infty} \mathbb{P}(N > n)x^n$ for all $|x| < 1$. From a well-known Tauberian theorem (see e.g. [1, Theorem 9] or [6, Sec. XIII.5, Theorem 5]), we have that $\mathbb{P}(N > n) \sim n^{-\alpha}L(n)$ if and only if $\sum_{n=0}^{\infty} \mathbb{P}(N > n)x^n \sim \Gamma(1 - \alpha)L(1/(1 - x))/(1 - x)^{1 - \alpha}$ as $x \rightarrow 1^-$. Hence $n(1 - G_N(\mathbb{P}(R < n))) \sim n\mathbb{P}(R \geq n)^{\alpha}L(1/\mathbb{P}(R \geq n))\Gamma(1 - \alpha)$ and Proposition 3.2(1) yields the result.
- (2) It follows analogously from Proposition 3.2(2).
- (3) If $\mathbb{P}(N \geq n) \sim cn^{-1}$, since $\sum_{n=0}^{\infty} \mathbb{P}(N \geq n)$ is divergent, then $\sum_{n=0}^{\infty} \mathbb{P}(N \geq n)x^n \sim c \ln(1/(1 - x))$. Indeed, for any $\varepsilon > 0$, there exists \bar{n} such that for all $n \geq \bar{n}$, we have

$$\begin{aligned} \mathbb{P}(N \geq n) &\in (c(1 - \varepsilon)/n, c(1 + \varepsilon)/n). \text{ Hence, eventually as } x \rightarrow 1^-, \\ \frac{\sum_{n=0}^{\infty} \mathbb{P}(N \geq n)x^n}{c \ln(1/(1-x))} &= \frac{\sum_{n=0}^{\infty} \mathbb{P}(N \geq n)x^n}{c \sum_{n=1}^{\infty} x^n/n} = \frac{\sum_{n=0}^{\bar{n}-1} \mathbb{P}(N \geq n)x^n + \sum_{n=\bar{n}}^{\infty} \mathbb{P}(N \geq n)x^n}{c \sum_{n=1}^{\bar{n}-1} x^n/n + c \sum_{n=\bar{n}}^{\infty} x^n/n} \\ &\sim \frac{\sum_{n=\bar{n}}^{\infty} \mathbb{P}(N \geq n)x^n}{c \sum_{n=\bar{n}}^{\infty} x^n/n} \in (1 - \varepsilon, 1 + \varepsilon). \end{aligned}$$

This implies immediately that $n(1 - G_N(\mathbb{P}(R < n))) \sim n c \ln(1/\mathbb{P}(R \geq n))\mathbb{P}(R \geq n)$ and, again, Proposition 3.2(1) yields the result.

(4) It follows analogously from Proposition 3.2(2). □

Proof of Theorem 3.6. We study the behavior of the deterministic counterpart. Note that $\mathbb{P}(\tilde{R}_i < n - i + 1) = G_{N_i}(\mathbb{P}(R_i < n - i + 1))$ is the probability that the $n + 1$ -vertex does not belong to the radius of influence of the i -th vertex. Hence $\prod_{i=0}^n G_{N_i}(\mathbb{P}(R_i < n - i + 1))$ is the probability that the $n + 1$ -vertex does not belong to the radius of influence of any vertex to its left. Denote this event by E_n : by Borel-Cantelli $\mathbb{P}(\limsup_n E_n) = 1$. Whence, there exists n_0 such that for all $\mathbb{P}\left(\bigcap_{k \geq n_0} E_k^c\right) > 0$. Since $\mathbb{P}(V_{n_0}) > 0$, where V_{n_0} is the event “all the stations at $0, 1, \dots, n_0$ are activated”, we have (using the FKG inequality)

$$\mathbb{P}(V) \geq \mathbb{P}\left(\bigcap_{k \geq n_0} E_k^c \mid V_{n_0}\right) \mathbb{P}(V_{n_0}) = \mathbb{P}\left(\bigcap_{k \geq n_0} E_k^c \cap V_{n_0}\right) \geq \mathbb{P}\left(\bigcap_{k \geq n_0} E_k^c\right) \mathbb{P}(V_{n_0}) > 0.$$

In particular if we have a deterministic environment, say $N_i := m_i \in \mathbb{N}$ for all $i \in \mathbb{N}$, and

$$\sum_{n=0}^{\infty} \prod_{i=0}^n \mathbb{P}(R_i < n - i + 1)^{m_i} < +\infty \tag{6.6}$$

then, since $G_{N_i}(x) = x^{m_i}$, equation (3.4) holds and $\mathbb{P}(V) > 0$.

Finally, by Lemma 6.1 (using $t_i := \mathbb{P}(R_i < n - i + 1)$) we see that equation (6.6) holds for μ -almost all configurations \mathbf{n} and this yields the result. □

Proof of Theorem 4.1. Apply [7, Theorem 2.8] to the deterministic annealed process. Trivially $\mathbb{E}[\tilde{R}] < +\infty$ if and only if $W < +\infty$. The results follow immediately from the equivalence between $\mathbb{P}(S) = 1$ (resp. $\mathbb{P}(S) = 0$) and $\mathbb{P}(S|\mathcal{N} = \mathbf{n}) = 1$ (resp. $\mathbb{P}(S|\mathcal{N} = \mathbf{n}) = 0$) for μ -almost all configurations \mathbf{n} . □

In the following, by $f \gtrsim g$ as $x \rightarrow x_0$ we mean that $\liminf_{x \rightarrow x_0} f(x)/g(x) \geq 1$.

Proof of Theorem 4.3. (1) We start by noting that, given any \mathbb{N} -valued random variable N such that $0 < \mathbb{E}[N] < +\infty$ and an arbitrary sequence $\{t_n\}_{n \in \mathbb{N}}$ in $[0, 1]$, then $\sum_{n \in \mathbb{N}} (1 - G_N(t_n)) < +\infty$ if and only if $\sum_{n \in \mathbb{N}} (1 - t_n) < +\infty$. This follows easily from (a) $\lim_{n \rightarrow \infty} (1 - t_n) = 0 \iff \lim_{n \rightarrow \infty} (1 - G_N(t_n)) = 0$ and (b) $1 - G_N(x) \sim \mathbb{E}[N](1 - x)$ as $x \rightarrow 1^-$. The result follows by taking $t_n := \mathbb{P}(R < n)$.

- (2) Note that $W < +\infty$ if and only if $\mathbb{E}[\tilde{R}] < +\infty$ where \tilde{R} is the law of the radius of the deterministic annealed counterpart of the reverse firework process. Moreover

$$\begin{aligned}
\int_0^{+\infty} (1 - G_N(\mathbb{P}(R < t))) dt &= \int_0^{+\infty} \sum_{n \in \mathbb{N}} (1 - \mathbb{P}(R < t)^n) \mathbb{P}(N = n) dt \\
&= \int_0^{+\infty} \sum_{n=1}^{\infty} (1 - \mathbb{P}(R < t)) \sum_{i=0}^{n-1} \mathbb{P}(R < t)^i \mathbb{P}(N = n) dt \\
&= \int_0^{+\infty} \mathbb{P}(R \geq t) \sum_{i=0}^{\infty} \mathbb{P}(R < t)^i \mathbb{P}(N \geq i + 1) dt \\
&= \int_0^{+\infty} f_{N,R}(t) \mathbb{P}(R \geq t) dt
\end{aligned}$$

where $f_{N,R}(t) := \sum_{i=0}^{\infty} \mathbb{P}(R < t)^i \mathbb{P}(N \geq i + 1)$. By the Monotone Convergence Theorem $f_{N,R}(t) \uparrow \mathbb{E}[N]$ as $t \uparrow +\infty$.

Since $\{\mathbb{P}(N \geq n)\}_{n \in \mathbb{N}}$ is a nonincreasing sequence, then from [6, Sec. XIII.5, Theorem 5], if $\mathbb{P}(N \geq n) \geq \varepsilon L(n)/n^\alpha$ we have that

$$f_{N,R}(t) \geq \varepsilon \sum_{i=0}^{\infty} \mathbb{P}(R < t)^i L(i + 1)/(i + 1)^\alpha \sim \varepsilon \frac{\Gamma(1 - \alpha)}{(1 - \mathbb{P}(R < t))^{1 - \alpha}} L(1/(1 - \mathbb{P}(R < t)))$$

as $t \rightarrow +\infty$. This, in turn, implies

$$f_{N,R}(t) \mathbb{P}(R \geq t) \gtrsim \varepsilon \Gamma(1 - \alpha) \mathbb{P}(R \geq t)^\alpha L(1/\mathbb{P}(R \geq t))$$

(remember that $\mathbb{P}(R < t) \rightarrow 1$ as $t \rightarrow \infty$). Hence $\int_0^{+\infty} \mathbb{P}(R \geq t)^\alpha L(1/\mathbb{P}(R \geq t)) dt = +\infty$ implies $W = +\infty$.

- (3) Analogously

$$f_{N,R}(t) \mathbb{P}(R \geq t) \lesssim M \cdot \Gamma(1 - \alpha) \mathbb{P}(R \geq t)^\alpha L(1/\mathbb{P}(R \geq t)),$$

Hence $\int_0^{+\infty} \mathbb{P}(R \geq t)^\alpha L(1/\mathbb{P}(R \geq t)) dt < +\infty$ implies $W < +\infty$.

- (4) In this case we have $\mathbb{P}(N \geq n + 1) \leq M/(n + 1)$ for all $n \in \mathbb{N}$, whence

$$f_{N,R}(t) \mathbb{P}(R \geq t) \lesssim M \cdot \mathbb{P}(R \geq t) \ln(1/\mathbb{P}(R \geq t)).$$

Thus, $\int_0^{+\infty} \mathbb{P}(R \geq t) \ln(1/\mathbb{P}(R \geq t)) dt < +\infty$ implies $W < +\infty$.

- (5) Finally $\mathbb{P}(N \geq n + 1) \geq \varepsilon/(n + 1)$ for all $n \in \mathbb{N}$, whence

$$f_{N,R}(t) \mathbb{P}(R \geq t) \gtrsim \varepsilon \cdot \mathbb{P}(R \geq t) \ln(1/\mathbb{P}(R \geq t)).$$

Thus, $\int_0^{+\infty} \mathbb{P}(R \geq t) \ln(1/\mathbb{P}(R \geq t)) dt = +\infty$ implies $W = +\infty$.

□

Proof of Theorem 4.5. If we define

$$\xi_n := \sum_{k \geq 1} (1 - \mathbb{P}(R_{n+k} < k))^{N_{n+k}}, \quad \zeta := \sum_{n \in \mathbb{N}} \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k)^{N_{n+k}}$$

then, by the Monotone Convergence Theorem and the Bounded Convergence Theorem,

$$\mathbb{E}_\mu[\xi_n] = \sum_{k \geq 1} (1 - G_{N_{n+k}}(\mathbb{P}(R_{n+k} < k))), \quad \mathbb{E}_\mu[\zeta] = \sum_{n \in \mathbb{N}} \prod_{k=1}^{\infty} G_{N_{n+k}}(\mathbb{P}(R_{n+k} < k)).$$

According to [7, Theorem 2.4(i)], $\mathbb{E}_\mu[\xi_n] = +\infty$ if and only if $\mathbb{P}(S) = 1$ (almost sure survival of the deterministic annealed counterpart) which is equivalent to $\mathbb{P}(S|\mathcal{N} = \mathbf{n}) = 1$ for μ -almost all configurations \mathbf{n} . Moreover, $\mathbb{E}_\mu[\zeta] < +\infty$ implies $\zeta < +\infty$ for μ -almost all configurations \mathbf{n} and then, according to [7, Theorem 2.4(ii)], $\mathbb{P}(S|\mathcal{N} = \mathbf{n}) > 0$ for μ -almost all configurations \mathbf{n} . \square

Before proving the results of Section 5 we discuss again the role of the annealed counterpart introduced in Section 2.3. By using the equality $\mathbb{P}(\tilde{R} < t) = \mathbb{1}_{(0,+\infty)}(t)G_N(\mathbb{P}(R < t))$ we have that if the original process satisfies the assumptions in one of the results of Section 5 (Theorems 5.2 and 5.4 or Corollaries 5.3 and 5.5) then the annealed counterpart satisfies the same conditions by taking $N = 1$ a.s. and \tilde{R} instead of R (observe that, in this case, $G_N(t) \equiv t$). This means that proving these results in the special case $N = 1$ a.s. is equivalent to proving them for every annealed counterpart of a generic process.

Proof of Lemma 5.1. (1) and (2) are consequences of equation (2.1) (in the second case one applies equation (2.1) to $\mathbb{P}(\cdot|\tau \text{ is infinite})$).

Let us prove (3) and (4). Given $\{R_{w,i}\}_{w \in \mathcal{W}, i \in \mathbb{N}^*}$, denote by $q(\Upsilon) = \mathbb{P}(\text{extinction}|\tau^L = \Upsilon)$ the probability of extinction of the firework (resp. reverse firework) process on a fixed labelled tree $\Upsilon \in \mathbb{LT}$ (observe that, when conditioning on $\{\tau = \Upsilon\}$, the radii are still random variables). Remember that μ (defined in Section 2.1) is the law of τ^L depending on N_\emptyset (the law of the random label l of the root). By hypothesis, the support of the law of N_\emptyset is a subset of J_N (the support of the law of N).

Denote by O the set $\{\Upsilon : q(\Upsilon) = 1\}$ of labelled trees where the process dies out almost surely. Clearly if $A := \{\Upsilon : q(\Upsilon) = 1, l(\Upsilon) \geq 1\}$ then $O = A \cup \mathbb{LT}_0$ (where $l(\Upsilon)$ is the label of the root of Υ and \mathbb{LT}_0 are the trees with no stations at the root). Note that $O^c = \{q(\Upsilon) < 1\} \subseteq \{\Upsilon \text{ is infinite}, l(\Upsilon) \geq 1\}$.

On a finite labelled tree Υ the firework and the reverse firework processes become extinct almost surely, hence $E \subseteq O$. Using the notation $\mu_j(\cdot) = \mu(\cdot|l = j)$ (see Section 2.1), we have $\mu_j(E \setminus A) = \mu_j(A \triangle O) = 0$ for all $j \in \mathbb{N}^*$. If a process becomes extinct on a labelled tree $\Upsilon \in A$ (since there is a positive probability that it reaches each child of the root) then it becomes extinct on every labelled subtree branching from a child of the root. Since each subtree can be identified with a labelled tree and the sequence of radii is i.i.d., we have that (A, O) is inherited with respect to $\bar{J} = \mathbb{N}^*$. To be precise, for the above identification, in the case of the reverse firework, we just need that $\{R_{w,i}\}_{w \in \mathcal{W} \setminus \{\emptyset\}, i \in \mathbb{N}^*}$ is an i.i.d. family. If the annealed probability of survival is positive then by equation (2.1) we have that $\mu(O) \equiv \mathbb{P}(q(\tau^L) = 1) < 1$ which implies $\mathbb{P}(q(\tau^L) = 1|N_\emptyset \geq 1) < 1$, where $q(\tau^L) = \mathbb{P}(\text{extinction}|\tau^L)$. Note that $\{q(\tau^L) = 1\} = O^c \times \mathcal{O}$.

(3) Suppose that $\mathbb{P}(N = 0) = 0$. Thus, $J_N \subseteq \bar{J}$ and, applying Lemma 2.2, we have that either $\mu_j(O \triangle E) = 0$ for all $j \in \mathbb{N}^*$ or $\mu_j(O) = 1$ for all $j \in J_N$. Whence $\mu_j(O) \in \{\alpha, 1\}$ for all $j \in J_N$. In particular, recalling that $\mu(l = j) = \mathbb{P}(N_\emptyset = j)$, if $\bar{\zeta}(i) := \mu(l = i | l \geq 1) \equiv \mathbb{P}(N_\emptyset = i | N_\emptyset \geq 1)$ then we have $\mu_{\bar{\zeta}}(\cdot) = \sum_{j \in \mathbb{N}} \bar{\zeta}(j) \mu_j(\cdot) = \sum_{j \in J_N} \bar{\zeta}(j) \mu_j(\cdot) = \mu(\cdot | l \geq 1)$. From Lemma 2.2 either $\mu_{\bar{\zeta}}(O \triangle E) = 0$ or $\mu_{\bar{\zeta}}(O) = 1$. This implies $\mu_{\bar{\zeta}}(O) = \mu(O | l \geq 1) \in \{\alpha, 1\}$ (since $\alpha = \mu_j(E) = \mu(E) = \mu_{\bar{\zeta}}(E)$ for all $j \in \mathbb{N}$). Observe that $\mu(O) < 1$ if and only if $\mu_{\bar{\zeta}}(O) = \alpha$; in this case $\mu(O \cap \{l \geq 1\}) = \alpha \mu(l \geq 1) = \mu(E \cap \{l \geq 1\})$, since the label of the root and the finiteness of the tree are independent by construction. Hence, when $\mu(O) < 1$ we have $O \cap \{l \geq 1\} = E \cap \{l \geq 1\}$ except for a μ -null set (since $O \supseteq E$, that is, there is no survival on a finite tree). Thus either $\mu(O) = 1$ or $O = \mathbb{L}\mathbb{T}_0 \cup E$ except for a μ -null set (in this case $\mu(O) = \mathbb{P}(N_\emptyset = 0) + \mathbb{P}(N_\emptyset \geq 1)\alpha$). Equivalently, either $\mu(O^{\mathbb{L}}) = 0$ or $O^{\mathbb{L}} = \{\Upsilon \text{ is infinite}, l(\Upsilon) \geq 1\}$ except for a μ -null set. Equivalently, $\{q(\tau^{\mathbb{L}}) < 1\} = \{\tau^{\mathbb{L}} \text{ is infinite}, N_\emptyset \geq 1\}$ except for a \mathbb{P} -null set. This means that there is a positive probability of survival for the process for μ -almost every realization of the environment (that is, the labelled GW-tree) such that the underlying tree is infinite and there is at least one station at the root. Easy computations shows that $\mathbb{P}(\tau^{\mathbb{L}} \text{ is infinite}, N_\emptyset \geq 1) = \mathbb{P}(N_\emptyset \geq 1)(1 - \alpha)$. This completes the proof of (3).

(4) If $\mathbb{P}(N = 0) > 0$ then consider the annealed process with one station per site and radii $\{\tilde{R}_w\}_{w \in \mathcal{W}}$ (see Section 2.3). The environment of this process can be identified with the unlabelled tree τ . Define $q_2(\mathbf{T}) = \mathbb{P}(\text{extinction} | \tau = \mathbf{T})$ the probability of extinction of the annealed process on $\mathbf{T} \in \mathbb{T}$. By reasoning as in Section 2.3, for every fixed realization \mathbf{T} of τ , $q_2(\mathbf{T})$ is the same for the annealed or the original process. Since the annealed process has one station per site, we apply (3) obtaining that $\mathbb{P}(\text{survival}) > 0$ implies $\{q_2(\tau) < 1\} = \{\tau \text{ is infinite}\}$ except for a \mathbb{P} -null set.

Denote now by $q_1(\mathbf{T}, n) = \mathbb{P}(\text{extinction} | \tau = \mathbf{T}, N_\emptyset = n)$ the probability of extinction of the firework (resp. reverse firework) process on a fixed unlabelled tree $\mathbf{T} \in \mathbb{T}$ with $n \in \mathbb{N}$ stations at the root (observe that, when conditioning on $\{\tau = \mathbf{T}, N_\emptyset = n\}$ the radii of all stations and the numbers of stations outside the root are still random variables). Clearly, in the firework process, the probability of survival starting from n stations at the root is less than or equal to n times the probability of survival starting from 1 station (since it is necessary and sufficient that at least one of the n stations triggers a surviving process). Hence $1 - q_1(\tau, 1) \leq 1 - q_1(\tau, n) \leq n(1 - q_1(\tau, 1))$ for all $n \geq 1$. For the reverse firework process the first inequality turns into an equality, since the behavior of the process does not depend on the number of stations at the root as long as they are positive. Since $1 - q_2(\tau) = \sum_{n \in \mathbb{N}^*} (1 - q_1(\tau, n)) \mathbb{P}_{N_\emptyset}(n)$ then, using the previous inequalities, $q_2(\tau) < 1 \iff q_1(\tau, n) < 1$ for some $n \in \text{supp}(N_\emptyset) \iff q_1(\tau, n) < 1, \forall n \in \text{supp}(N_\emptyset)$. This implies that $\{q_1(\tau, N_\emptyset) < 1, N_\emptyset = n\} = \{\tau \text{ is infinite}, N_\emptyset = n\}$ for all $n \geq 1$ except for a \mathbb{P} -null set. This means that $\mathbb{P}(\text{survival} | \tau = \mathbf{T}, N_\emptyset = n) > 0$ for $\bar{\mu}$ -almost every infinite (unlabelled) tree $\mathbf{T} \in \mathbb{T}$ and for \mathbb{P}_{N_\emptyset} -almost all $n \geq 1$. Observe that $q_1(\tau, N_\emptyset) = \mathbb{P}(\text{extinction} | \tau, N_\emptyset)$; again $\mathbb{P}(\tau \text{ is infinite}, N_\emptyset \geq 1) = \mathbb{P}(N_\emptyset \geq 1)(1 - \alpha)$. \square

Proof of Theorem 5.2. The proof is divided into two main parts. We start in part (a) by proving the results in the case $N = 1$ a.s.: the general case will be considered in (b).

(a). Here the environment τ^L can be identified with a realization of the GW-tree τ (with no labels) and the generating function of N is $G_N(t) \equiv t$. As in Section 1 we denote by $\bar{\mu}$ the law of τ on \mathbb{T} .

(1-2). Consider at each step and for each station at a vertex w the border of the signal ∂_w , that is, the vertices at distance n from the source such that $n \leq R_w < n + 1$. Define a new process where at each step we activate only the stations in ∂_w where w is the site of a station activated in the previous step. This process is stochastically dominated by the original Firework process. The generating function of the law of $\#\partial_w$ can be computed as $\Psi(z) := \sum_{n \in \mathbb{N}} \mathbb{P}(n \leq R < n + 1) \varphi_\rho^{(n)}(z)$ where $\varphi_\rho^{(n)}$ is the generating function of the convolution $\underbrace{\rho * \dots * \rho}_{n \text{ times}}$ (and $\varphi_\rho^{(0)}(z) \equiv 1$), that is, of the law of the number of descendants in the n -th generation. The number of activated stations at each step is a GW-process with generating function Ψ hence it survives if and only if $\Psi'(1) > 1$ which is $\sum_{n=1}^{\infty} \mathbb{P}(n \leq R < n + 1) m^n > 1$. This inequality is clearly equivalent to $\Phi(m) - 1 > \Phi(0)$. Since the (annealed) probability of survival is positive then $\mathbb{P}(q(\tau) = 1) < 1$ hence, according to Lemma 5.1(3), $\{q(\tau) = 1\} = \{\tau \text{ is finite}\}$ except for a \mathbb{P} -null set.

(3). By using a coupling argument it is enough to prove the result for the process on the deterministic tree \mathbb{T}_k where each vertex has k children. Consider the following random tree constructed on the vertices of \mathbb{T}_k . We connect by a blue edge the root \emptyset with each vertex $w \in \mathbb{T}_k$ such that the length $|w| \leq R_\emptyset$. This is the first generation. Suppose we have defined the n -th generation \mathcal{H}_n . Take the vertices of \mathcal{H}_n , one at the time, in lexicographic order and connect $w \in \mathcal{H}_n$ to every vertex ww' in the subtree of \mathbb{T}_k branching from w which has not been previously connected and such that the length $|w'| \leq R_w$. The connected vertices ww' are the $n + 1$ -th generation. The vertices of this random tree are exactly the vertices of \mathbb{T}_k which are activated in the firework process. More precisely, this is a spanning tree of the (random) F-graph described in Section 1. Hence, by coupling, the firework process survives on \mathbb{T}_k if and only if the blue tree is infinite.

But the blue random tree is a subtree of a new GW-tree whose (random) number of offsprings has the same law as the number of descendants of any vertex of \mathbb{T}_k , say the root for instance, up to the random generation R (included). It is easy to check that the expected number of descendants up to the (random) R -th generation is $\sum_{n=1}^{\infty} \mathbb{P}(n \leq R < n + 1) (k^{n+1} - k) / (k - 1)$; thus if this expected value is less or equal than 1 the blue tree is finite almost surely and the probability of survival of the firework process is 0. The extinction for almost all the realizations of the GW-tree follows easily from equation (2.1). The proof is complete in the case where $N = 1$ almost surely.

(b). We consider now the general case. Let us denote by τ^L the labelled GW-tree and by τ the projection of τ^L on \mathcal{W} , that is, the underlying (unlabelled) GW-tree. Let μ be the law of τ^L on \mathbb{LT} . Given $\{R_{w,i}\}_{w \in \mathcal{W}, i \geq 1}$ and $\{N_w\}_{w \in \mathcal{W}}$, we consider, along with the firework process η on the

labelled GW-tree, the annealed counterpart, that is, the firework process $\tilde{\eta}$ with one station per vertex and radii $\tilde{R}_w := \max(R_{w,1}, \dots, R_{w,N_w})$ for all $w \in \mathcal{W}$. If η satisfies the conditions of the theorem then $\tilde{\eta}$ is a process with one station per site satisfying again the condition of the theorem (in the case studied in the previous part of the proof).

(1). Since $\mathbb{P}(\tilde{R}_w < t) = G_N(\mathbb{P}(R < t))$ according to (1-2) we have that if $\Phi(m) - 1 > \Phi(0)$ (or equivalently $\sum_{n=1}^{\infty} (G_N(\mathbb{P}(R < n+1)) - G_N(\mathbb{P}(R < n)))m^n > 1$) then the process $\tilde{\eta}$ survives with positive probability on almost every infinite GW-tree, that is, $\bar{\mu}(\mathbf{T}: \mathbb{P}(\tilde{S}|\tau = \mathbf{T}) > 0) = \bar{\mu}(\mathbf{T}: \mathbf{T} \text{ is infinite})$ (where \tilde{S} is the event “ $\tilde{\eta}$ survives” and $\bar{\mu}$ is the GW-probability measure on the space \mathbb{T} of unlabelled trees). This implies that $\mathbb{P}(\tilde{S}) > 0$. Since $\mathbb{P}(S) = \mathbb{P}(\tilde{S})$ (where S is the event “ η survives”) then we have $\mu(\Upsilon: \mathbb{P}(S|\tau^L = \Upsilon) > 0) > 0$. Since $\mu(\Upsilon: \mathbb{P}(S|\tau^L = \Upsilon) > 0) > 0$ then, according to Lemma 5.1 (remember that the support of the law N_\emptyset is equal to J_N), since $\mathbb{P}(N = 0) = 0$, we have $\{\mathbb{P}(S|\tau^L) > 0\} = \{\tau^L \notin E, l(\tau^L) > 0\}$ except for a \mathbb{P} -null measure set (where E is the set of all finite labelled trees).

(2). As in (1) of part (b) we have that $\mathbb{P}(S) = \mathbb{P}(\tilde{S}) > 0$ which implies, according to Lemma 5.1, $\{\mathbb{P}(S|\tau, N_\emptyset) > 0\} = \{\tau \text{ is infinite}, N_\emptyset \geq 1\}$ except for a \mathbb{P} -null set.

(3). If $\Phi(k) - 1 \leq 1 - 1/k$ then $\bar{\mu}(\mathbf{T}: \mathbb{P}(\tilde{S}|\tau = \mathbf{T}) = 0) = 1$ which implies $0 = \mathbb{P}(\tilde{S}) = \mathbb{P}(S)$. This, in turn, is equivalent to $\mu(\Upsilon: \mathbb{P}(S|\tau^L = \Upsilon) = 0) = 1$. \square

Proof of Corollary 5.3. Since $\mathbb{P}(R < 1) < 1$ then $\Phi(m) > \Phi(0) + 1$ eventually as $m \rightarrow \infty$; thus, according to Theorem 5.2, $\bar{m}_c < +\infty$. Since the GW-tree is a.s. finite when $m \leq 1$, we have that $\bar{m}_c \geq 1$. Moreover, if $m > 1/\limsup_{n \rightarrow \infty} \sqrt[n]{1 - G_N(\mathbb{P}(R < n))}$ (the latter being the radius of convergence of Φ according to the discussion before Corollary 5.3) then, by Theorem 5.2, there is survival with positive probability for the firework process. This implies that if $\limsup_{n \rightarrow \infty} \sqrt[n]{1 - G_N(\mathbb{P}(R < n))} = 1$ we have positive survival if and only if $m > 1$. Thus $\bar{m}_c = 1$. \square

Proof of Theorem 5.4. As in the proof of Theorem 5.2 we start with the case where $N = 1$ almost surely. In the first part of the proof, the environment τ^L is identified with the GW-tree τ (since we have one station per vertex).

After the construction of the GW-tree τ , we construct a new tree (which is not in general a subtree of τ) iteratively as follows: starting from the origin, (1) for every word i of length 1 of the tree draw a purple edge from \emptyset to i if and only if $R_i \geq 1$, (2) for all $n \geq 1$ suppose we finished connecting words of length n , take all words w' of length $n+1$ such that there are no ancestors already connected to the root and connect them to the root if and only if $R_{w'} \geq n+1$. This is the construction of the first purple generation. Now we construct the second generation applying (1) and (2) to the subtrees branching from each vertex of the 1st purple generation (in the construction, these vertices become the roots of the branching subtrees). The constructions on these subtrees are independent since the subtrees are pairwise disjoint. The construction of the

subsequent generations follows by iteration. Using this construction, if a vertex is able to listen to more than one station which can broadcast the signal then we are connecting it to the closest one. This is a spanning tree of the (random) RF-graph described in Section 1.

This new purple tree is a GW-tree and there is survival if and only if this tree is infinite. The expected number of purple edges from the origin is $\phi_2(m) = \sum_{i=1}^{\infty} m^i \mathbb{P}(R \geq i) \prod_{j=1}^{i-1} \mathbb{P}(R < j)$ where m is the expected number of children in the original GW-tree. Hence, the purple GW-tree is finite if and only if the probability that a vertex has at least one child is strictly smaller than 1 and the expected number of children of a vertex is smaller or equal than 1.

Given the original GW-process $\{Z_n\}_{n \in \mathbb{N}}$, the probability (conditioned on $\{Z_n\}_{n \in \mathbb{N}}$) that the root has no children in the purple process is $\prod_{n=1}^{\infty} \mathbb{P}(R < n)^{Z_n}$. Clearly $\prod_{n=1}^{\infty} \mathbb{P}(R < n)^{Z_n} = 0$ if and only if $\sum_{n=1}^{\infty} Z_n \mathbb{P}(R \geq n) = +\infty$. For almost every realization of $\{Z_n\}_{n \in \mathbb{N}}$ such that the GW-tree is infinite, we have $Z_n \sim m^n$ (where $m > 1$), thus $\sum_{n=1}^{\infty} Z_n \mathbb{P}(R \geq n) = +\infty$ is equivalent to $\phi_1(m) = \sum_{n=1}^{\infty} m^n \mathbb{P}(R \geq n) = +\infty$.

Hence there is annealed a.s. extinction if and only if $\phi_1(m) < +\infty$ and $\phi_2(m) \leq 1$; equation (2.1) implies a.s. extinction on almost every realization of the GW-tree. Moreover the annealed probability of survival conditioned of the event “the GW-tree is infinite” is 1 if and only if $\phi_1(m) = +\infty$; again equation (2.1) implies survival with probability 1 on almost every realization of the GW-tree. When $\phi_1(m) < +\infty$ and $\phi_2(m) > 1$ we apply Lemma 5.1 to obtain the quenched results.

As in the proof of Theorem 5.2, the results in the general case come from the first part of the proof and from Lemma 5.1 by using the reverse firework process η (associated with $\{R_{w,i}\}_{w \in \mathcal{W}, i \geq 1}$ and $\{N_w\}_{w \in \mathcal{W}}$) and its annealed counterpart $\tilde{\eta}$ (with one station per vertex and radii $\tilde{R}_w := \max(R_{w,1}, \dots, R_{w,N_w})$ for all $w \in \mathcal{W}$). \square

Proof of Corollary 5.5. Recall ϕ_1 and ϕ_2 defined in Theorem 5.4. From the probabilistic interpretation of $\phi_1(m)$ and $\phi_2(m)$ given in the proof of Theorem 5.4 we have that $\phi_2(m) < 1$ implies $\phi_1(m) < +\infty$. Moreover, $\phi_1 \geq \phi_2$. Since we assumed that $\mathbb{P}(R < 1) \in (0, 1)$ we have $G_N(\mathbb{P}(R < 1)) \in (0, 1)$ as well. In particular, $G_N(\mathbb{P}(R < 1)) < 1$, implies that ϕ_2 is strictly increasing and $\lim_{m \rightarrow \infty} \phi_2(m) = +\infty$. Define

$$M_c := \sup\{m \geq 0: \phi_1(m) < +\infty\}, \quad m_c := \sup\{m \geq 0: \phi_2(m) \leq 1\} \equiv \sup\{m \geq 0: \phi_2(m) < 1\}.$$

By the discussion above, $m_c \leq M_c$ and $m_c < +\infty$. Observe that, in general, $\sum_{n=1}^{\infty} (1 - \alpha_n) \prod_{j=1}^{n-1} \alpha_j = 1 - \lim_{n \rightarrow \infty} \prod_{j=1}^n \alpha_j$ when the limit exists. Hence $\phi_2(1) = 1 - \prod_{j=1}^{\infty} G_N(\mathbb{P}(R < j))$ which implies $m_c \geq 1$ (note that $\phi_2(1) < 1$ if and only if $\phi_1(1) < +\infty$).

Since ϕ_2 is a series with nonnegative coefficients, we have that $\phi_2(m_c) \leq 1$; thus if $m_c < M_c$ then $\phi_1(m_c) < +\infty$ and for $m = m_c$ there is almost sure extinction for almost every realization of the environment. \square

Details on Remark 5.6. Recall the definition of ϕ_1 and ϕ_2 given in the proof of Corollary 5.5. If $M_c = +\infty$ there is nothing to prove since, according to Corollary 5.5, $m_c < +\infty$. On the other hand, suppose that $1 < M_c < +\infty$. This implies immediately that $\phi_1(1) < +\infty$ hence $\prod_{j=1}^{\infty} \mathbb{P}(R < j) = \delta > 0$. Thus, $\phi_1(m) \geq \phi_2(m) \geq \delta\phi_1(m)$. Since ϕ_2 is a series with nonnegative coefficients, we have that $\phi_2(m_c) \leq 1$, whence $\phi_1(m_c) \leq 1/\delta < +\infty$. If, in addition, $\phi_1(M_c) = +\infty$ then we have that $M_c > m_c$. \square

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D. BERTACCHI, UNIVERSITÀ DI MILANO–BICOCCA DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, VIA COZZI 53, 20125 MILANO, ITALY

E-mail address: daniela.bertacchi@unimib.it

F. ZUCCA, DIPARTIMENTO DI MATEMATICA, POLITECNICO DI MILANO, PIAZZA LEONARDO DA VINCI 32, 20133 MILANO, ITALY.

E-mail address: fabio.zucca@polimi.it