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$C^{k,\alpha}$-regularity of solutions to quasilinear equations structured on Hörmander's vector fields*

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Abstract

For a linear nonvariational operator structured on smooth Hörmander's vector fields, with Hölder continuous coefficients, we prove a regularity result in the scale of $C_X^{k,\alpha}$ spaces. We deduce an analogous regularity result for nonvariational degenerate quasilinear equations.

Introduction

Let $X_1, X_2, ..., X_q$ be a system of smooth Hörmander's vector fields in a bounded smooth domain Ω of $\mathbb{R}^n$, with $q < n$ (see §1.1 for precise definitions). Nonvariational operators of the kind

$$L = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + \sum_{i=1}^{q} b_i(x) X_i + c(x),$$

with $\{a_{ij}\}$ real symmetric uniformly positive matrix, have been studied by several authors, establishing in particular local a priori estimates on $X_i X_j u$ in Hölder or L^p spaces, in terms of Lu and u, and assuming the coefficients a_{ij} bounded and, respectively, Hölder continuous or VMO: see [1], [3] for L^p estimates and [2], [3], [4] for Schauder estimates. In particular, in [2] evolution operators of the kind

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x) X_i X_j + \sum_{i=1}^{q} b_i(t,x) X_i + c(t,x)$$

have been studied, and local a priori Schauder estimates of the following kind have been proved: if $a_{ij}, b_i, c \in C^{k,\alpha}(U)$ for some integer $k \geqslant 0$ and some

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$\alpha \in (0,1)$, then for every domain $U' \Subset U$, $u \in C_{loc}^{k+2,\alpha}(U)$ with $Hu \in C^{k,\alpha}(U)$,

$$\|u\|_{C^{k+2,\alpha}(U')} \leqslant c\left\{\|Hu\|_{C^{k,\alpha}(U)} + \|u\|_{L^\infty(U)}\right\}.$$

Here the Hölder spaces $C^{k,\alpha}$ are those defined by means of derivatives with respect to the X_i's and the distance induced by the vector fields (more precisely, the parabolic version of these spaces, with the time derivative weighting as a second order derivative, see §1.1 and §4 for precise definitions).

Note that the previous estimate assumes *a priori* that $u \in C_{loc}^{k+2,\alpha}(U)$. A more subtle problem is that of proving a regularity result of the kind: if $u \in C^{2,\alpha}(U)$ solves $Hu = f$ and $a_{ij}, b_i, c, f \in C^{k,\alpha}(U)$ then actually $u \in C_{loc}^{k+2,\alpha}(U)$ (and therefore the above a priori estimate holds). In [2] this regularity result is actually proved, applying the classical strategy of regularizing the coefficients and data of the equation, solving the regularized Dirichlet problem and exploiting the a priori estimate to build a sequence of smooth functions converging in $C_{loc}^{k+2,\alpha}(U)$ to the solution of $Hu = f$. However, using this approach in [2] the boundedness of the approximating sequence is proved only for k even, hence the regularity result has been proved so far only for k even. In the present paper, exploiting the a priori estimates proved in [2], we find a different way of proving a regularity result which holds for all k (see Theorem 2.1), based on the Banach-Caccioppoli fixed point theorem. The above result and a standard bootstrap argument enable us to prove a Schauder regularity result for quasilinear equations of the kind

$$Qu \equiv \sum_{i,j=1}^{q} a_{ij}(x, u, Xu) X_i X_j u + b(x, u, Xu) = 0,$$

with $\{a_{ij}\}$ uniformly positive, concluding that, in particular, any $C^{2,\alpha}(\Omega)$ solution to $Qu = 0$ is smooth as soon as a_{ij}, b are smooth (see Theorem 3.1). Finally, in view of the results in [2], both the linear and the quasilinear regularity results described above can be easily extended to evolution operators $\partial_t - L$, $\partial_t - Q$ (see Theorems 4.1, 4.2). Actually, we have written our proofs in the stationary case just to simplify notation.

We mention that in the paper [7] this regularity result is also stated, but the Author makes an extra assumption on the structure of the X_i's (which does not cover general Hörmander's vector fields) and he actually proves only local a priori estimates, not a regularity result.

1 Preliminaries and known results

1.1 Hörmander's vector fields, control distance and Hölder spaces

Let $X_1, \ldots, X_q$ be a system of real smooth vector fields,

$$X_i = \sum_{j=1}^{n} b_{ij}(x)\, \partial_{x_j}, \quad i = 1, 2, ..., q$$

$(q < n)$ defined in some bounded, open and connected subset Ω_0 of $\mathbb{R}^n$. For any multiindex

$$I = (i_1, i_2, ..., i_k), \quad 1 \leq i_j \leq q$$

we set:

$$X_{[I]} = \left[X_{i_1}, \left[X_{i_2}, \ldots \left[X_{i_{k-1}}, X_{i_k}\right] \ldots\right]\right],$$

where $[X, Y] = XY - YX$ for any couple of vector fields X, Y. We will say that $X_{[I]}$ is a *commutator of length* $|I| = k$. As usual, X_i can be seen either as a differential operator or as a vector field. We will write $X_i f$ to denote the differential operator X_i acting on a function f, and $(X_i)_x$ to denote the vector field X_i evaluated at the point $x \in \Omega_0$. We shall say that $X_1, \ldots, X_q$ satisfy *Hörmander's condition of step* s in Ω_0 if these vector fields, together with their commutators of length $\leq s$, span the tangent space at every point $x \in \Omega_0$.

One can now define the control distance induced by these vector fields, as in [5]:

Definition 1.1 *For any $\delta > 0$, let $C(\delta)$ be the class of absolutely continuous mappings $\varphi : [0, 1] \to \Omega_0$ which satisfy*

$$\varphi'(t) = \sum_{i=1}^{q} \lambda_i(t)\, (X_i)_{\varphi(t)} \;\; a.e. \;\; t \in (0, 1)$$

with $|\lambda_j(t)| \leq \delta$ for $j = 1, ..., q$. We define

$$d_X(x, y) = \inf\left\{\delta : \exists \varphi \in C(\delta) \;\; with \;\; \varphi(0) = x, \varphi(1) = y\right\}.$$

Note that the finiteness of $d_X(x, y)$ for any two points $x, y \in \Omega_0$ is not a trivial fact, but depends on a connectivity result ("Chow's theorem"); moreover, it can be proved that d_X is a distance. It is also well-known that this distance is topologically equivalent to the Euclidean one (see [5] for all these facts).

Now, let $\Omega \subset \Omega_0$ be another fixed domain. For any $x \in \Omega$, we set

$$B_r(x) = \left\{y \in \Omega_0 : d_X(x, y) < r\right\}.$$

Let us define several types of Hölder spaces that we will need in the following:

Definition 1.2 *For any* $\alpha \in (0,1), u : \Omega \to \mathbb{R}$, *let:*

$$|u|_{C_X^\alpha(\Omega)} = \sup \left\{ \frac{|u(x) - u(y)|}{d_X(x,y)^\alpha} : x, y \in \Omega, x \neq y \right\},$$
$$\|u\|_{C_X^\alpha(\Omega)} = |u|_{C^\alpha(\Omega)} + \|u\|_{L^\infty(\Omega)},$$
$$C_X^\alpha(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \|u\|_{C^\alpha(\Omega)} < \infty \right\}.$$

Also, for any positive integer k, *let*

$$C_X^{k,\alpha}(\Omega) = \left\{ u : \Omega \to \mathbb{R} : \|u\|_{C^{k,\alpha}(\Omega)} < \infty \right\},$$

with

$$\|u\|_{C_X^{k,\alpha}(\Omega)} = \sum_{l=1}^{k} \sum_{j_i=1}^{q} \|X_{j_1} \dots X_{j_l} u\|_{C^\alpha(\Omega)} + \|u\|_{C^\alpha(\Omega)}.$$

We will set $C_{X,0}^\alpha(\Omega)$ *and* $C_{X,0}^{k,\alpha}(\Omega)$ *for the subspaces of* $C_X^\alpha(\Omega)$ *and* $C_X^{k,\alpha}(\Omega)$ *of functions which are compactly supported in* Ω, *and* $C_{X,loc}^{k,\alpha}(\Omega)$ *for the space of functions belonging to* $C_X^{k,\alpha}(\Omega')$ *for every* $\Omega' \Subset \Omega$.

Finally, we will write $C_{X,*}^{k,\alpha}(\Omega)$ *to denote the subspace of* $C_X^{k,\alpha}(\Omega)$ *consisting of functions* u *such that both* u *and all the derivatives* $X_{i_1} X_{i_2} ... X_{i_l} u$ $(l \leq k)$ *vanish on* $\partial\Omega$.

Proposition 1.3 *The spaces* $C_{X,*}^{k,\alpha}(\Omega)$ *are complete. Moreover, if* $u \in C_{X,*}^{k,\alpha}(B_r(x_0))$ *and* $R > r$ *for some* $B_R(x_0) \subset \Omega$, *then*

$$\overline{u}(x) = \begin{cases} u(x) & \text{in } B_r(x_0) \\ 0 & \text{in } B_R(x_0) \setminus B_r(x_0) \end{cases}$$

belongs to $C_{X,0}^{k,\alpha}(B_R(x_0))$.

Proof. We leave to the reader to check the completeness of these spaces. Let us prove the second assertion for $k = 0$, since the same argument applies to the derivatives. It is enough to check that

$$|\overline{u}(x) - \overline{u}(y)| \leq c d_X(x,y)^\alpha \text{ for any } x \in B_r(x_0), y \in B_R(x_0) \setminus B_r(x_0),$$

the other cases being obvious. By definition of d_X, for any fixed $\varepsilon > 0$, we can pick a curve $\gamma : [0,1] \to \Omega$ such that

$$\gamma(0) = x, \gamma(1) = y$$
$$\gamma'(t) = \sum_{i=1}^{q} \lambda_i(t) (X_i)_{\gamma(t)} \text{ with } |\lambda_i(t)| \leq d_X(x,y) + \varepsilon.$$

Since $d_X(x_0, \gamma(0)) < r$ and $d_X(x_0, \gamma(1)) > r$, there exists a point $z = \gamma(\overline{t})$ such that $d(x_0, z) = r$ and $u(z) = 0$. Then

$$|\overline{u}(x) - \overline{u}(y)| = |u(x)| = |u(x) - u(z)|$$
$$\leq |u|_{C^\alpha} d_X(x,z)^\alpha \leq |u|_{C^\alpha} (d_X(x,y) + \varepsilon)^\alpha.$$

Since this is true for every $\varepsilon > 0$, we are done. $\blacksquare$

The following easy properties of our function spaces will be useful:

Proposition 1.4 *(See [2, Proposition 4.2], also [3, Prop.3.27]) Let $B_R(\overline{x}) \subset \Omega$, then*

(i) For any $f \in C^1_{X,0}(B_R(\overline{x}))$, one has

$$|f(x) - f(y)| \leq d_X(x,y) \sum_{i=1}^{q} \sup_{B_R(\overline{x})} |X_i f| \tag{1.1}$$

for any $x, y \in B_R(\overline{x})$.

(ii) For any couple of functions $f, g \in C^\alpha_X(B_R(\overline{x}))$, one has

$$|fg|_{C^\alpha_X(B_R(\overline{x}))} \leq |f|_{C^\alpha_X(B_R(\overline{x}))} \|g\|_{L^\infty(B_R(\overline{x}))} + |g|_{C^\alpha_X(B_R(\overline{x}))} \|f\|_{L^\infty(B_R(\overline{x}))}$$

and

$$\|fg\|_{C^\alpha_X(B_R(\overline{x}))} \leq 2 \|f\|_{C^\alpha_X(B_R(\overline{x}))} \|g\|_{C^\alpha_X(B_R(\overline{x}))}. \tag{1.2}$$

1.2 Lifted vector fields and integral operators

Throughout the paper we will make use of some results and techniques originally introduced by Rothschild-Stein [6] and then adapted to nonvariational operators in [1], [2], [3]. However, in order to understand the proofs in the present paper, it is not necessary for the reader to know in detail all the background which is implicitly involved here. Therefore, to reduce the length of this paper we will content ourselves of pointing out the facts which will be explicitly used, giving to the interested reader all the relevant references.

First of all, much of the proof of our main results lives in the space of "lifted variables", as in [6]. This basically means what follows. For every point $\overline{x} \in \Omega$ there exists a neighborhood $B_R(\overline{x}) \subset \Omega$ and, in terms of new variables, $h_{n+1}, \ldots, h_N$, there exist smooth functions $\lambda_{il}(x,h)$ $(1 \leq i \leq q,\ n+1 \leq l \leq N)$ defined in a neighborhood $\widetilde{U}$ of $\overline{\xi} = (\overline{x}, 0) \in \mathbb{R}^N$ such that the vector fields $\widetilde{X}_i$ given by

$$\widetilde{X}_i = X_i + \sum_{l=n+1}^{N} \lambda_{il}(x,h) \frac{\partial}{\partial h_l}, \quad i = 1, \ldots, q$$

still satisfy Hörmander's condition of step s in $\widetilde{U}$ and possesses further properties, some of which we are going to recall.

Let us first fix some notation. We will denote by $d_{\widetilde{X}}$ the control distance induced by the vector fields $\widetilde{X}_i$ in $\widetilde{U}$, by $\widetilde{B}_R(\overline{\xi})$ the corresponding balls, and we will denote by $C^\alpha_{\widetilde{X}}\left(\widetilde{B}_R(\overline{\xi})\right)$, $C^{k,\alpha}_{\widetilde{X}}\left(\widetilde{B}_R(\overline{\xi})\right)$, $C^\alpha_{\widetilde{X},0}\left(\widetilde{B}_R(\overline{\xi})\right)$ and $C^{k,\alpha}_{\widetilde{X},0}\left(\widetilde{B}_R(\overline{\xi})\right)$ the function spaces over $\widetilde{B}_R(\overline{\xi})$ defined by the $\widetilde{X}_i$'s as in §1.1.

The following relation between the spaces $C^\alpha_X(B_R(\overline{x}))$ and $C^\alpha_{\widetilde{X}}\left(\widetilde{B}_R(\overline{\xi})\right)$ is crucial for us:

Proposition 1.5 *(See [2, Prop. 8.3], [3, Prop. 3.28]) Let* $\widetilde{B}_r\left(\overline{\xi}\right)$ *be a lifted ball, with* $\overline{\xi}=(\overline{x},0)$. *If* f *is a function defined in* $B_{2r}(\overline{x})$ *and* $\widetilde{f}(x,h)=f(x)$ *is regarded as a function defined on* $\widetilde{B}_r\left(\overline{\xi}\right)$, *then the following inequalities hold true*

$$\left|\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{B}_r\left(\overline{\xi}\right)\right)}\leqslant|f|_{C^{\alpha}_{X}(B_r(\overline{x}))}\leqslant c\left|\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{B}_{2r}\left(\overline{\xi}\right)\right)}.$$

Moreover,

$$\left|\widetilde{X}_{i_1}\widetilde{X}_{i_2}...\widetilde{X}_{i_k}\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{B}_r\left(\overline{\xi}\right)\right)}\leqslant|X_{i_1}...X_{i_k}f|_{C^{\alpha}_{X}(B_r(\overline{x}))}\leqslant c\left|\widetilde{X}_{i_1}...\widetilde{X}_{i_k}\widetilde{f}\right|_{C^{\alpha}_{\widetilde{X}}\left(\widetilde{B}_{2r}\left(\overline{\xi}\right)\right)}$$

for $i_j=1,2,...,q$.

The main tool to prove a priori estimates (in [6], [1], [2], [3]) is the combination of some abstract theory of singular integrals with some representation formulas for the second order derivatives $\widetilde{X}_i\widetilde{X}_j u$ of any test function by means of suitable integral operators. The reason why this is performed in the spaces of lifted variables is that this allows to make use of singular integral operators with better properties. The key notion here is that of *frozen operator of type zero over a ball* $\widetilde{B}_R(\xi_0)$, first introduced in [1] adapting the notion of operator of type zero given in [6]. We will not recall the definition of this concept (see [2, Definition 6.3]) because it involves several other notions that we will not use explicitly. It is enough to say that a frozen operator of type zero over $\widetilde{B}_R\left(\overline{\xi}\right)$ is an integral operator $T(\xi_0)$ (depending on some point $\xi_0\in\widetilde{B}_R\left(\overline{\xi}\right)$ like a parameter), and that the following two results hold:

Theorem 1.6 *(see [2, Thm.6.6], see also [3, Thm.5.1]). There exists* $C_R>0$ *depending on* R,Ω *and the vector fields* X_i *(but not on* ξ_0*) such that for every* $r\leq R, f\in C^{\alpha}_{X,0}\left(\widetilde{B}_r\left(\overline{\xi}\right)\right)$,

$$\|T(\xi_0)f\|_{C^{\alpha}_{X}\left(\widetilde{B}_r\left(\overline{\xi}\right)\right)}\leq C_R\|f\|_{C^{\alpha}_{X}\left(\widetilde{B}_r\left(\overline{\xi}\right)\right)}.$$

Theorem 1.7 *For any* $k=1,2,...,q$ *there exist* $q+1$ *frozen operators of type zero over* $\widetilde{B}_R\left(\overline{\xi}\right)$, $T^k_h(\xi_0)$ *for* $k=0,1,2,...,q$, *such that for any* $f\in C^1_X\left(\widetilde{B}_r\left(\overline{\xi}\right)\right)$ *one has:*

$$\widetilde{X}_k T(\xi_0)f=\sum_{h=1}^{q}T^k_h(\xi_0)\widetilde{X}_h f+T^0_k(\xi_0)f$$

Proof of Theorem 1.7. In [2, Prop.6.9] an analogous formula is stated for $T(\xi_0),T^k_h(\xi_0)$ frozen operators *of type one.* Exploiting the fact that any frozen operator of type zero involved in our argument is actually of the kind $\widetilde{X}_iS(\xi_0)$ with $S(\xi_0)$ frozen operator of type one, and *viceversa* for every frozen operator of type one $S(\xi_0)$ the derivative $\widetilde{X}_iS(\xi_0)$ is a frozen operator of type zero, we can write (denoting frozen operators of type zero or one with the letters $T,S,$

respectively):

$$\widetilde{X}_k T(\xi_0) f = \widetilde{X}_k \widetilde{X}_i S(\xi_0) f = \widetilde{X}_k \left(\sum_{h=1}^{q} S_h^i(\xi_0) \widetilde{X}_h f + S_k^0(\xi_0) f \right)$$

$$= \sum_{h=1}^{q} T_h^k(\xi_0) \widetilde{X}_h f + T_k^0(\xi_0) f,$$

which gives the assertion. ■

2 The linear regularity theory

Let us consider the linear operator

$$L = \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j + \sum_{i=1}^{q} b_i(x) X_i + c(x)$$

where:

$X_1, X_2, ..., X_q$ are a system of Hörmander's vector fields in a neighborhood Ω_0 of some bounded domain $\Omega \subset \mathbb{R}^n$, as described at the beginning of § 1.1;

$a_{ij}, b_i, c \in C_X^\alpha(\Omega)$ for some $\alpha \in (0,1)$, $a_{ij} = a_{ji}$ satisfying for some constant $\Lambda > 0$ the condition

$$\Lambda|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x) \xi_i \xi_j \leq \Lambda^{-1}|\xi|^2 \ \forall \xi \in \mathbb{R}^q, x \in \Omega. \tag{2.1}$$

The aim of this section is to prove the following result:

Theorem 2.1 *Under the above assumptions, let $u \in C_X^{2,\alpha}(\Omega)$ satisfy the equation*

$$Lu = f \text{ in } \Omega$$

and assume that for some integer $k = 1, 2, 3, ...$ we have:

$$a_{ij}, b_i, c, f, \in C_X^{k,\alpha}(\Omega).$$

Then

$$u \in C_{X,loc}^{k+2,\alpha}(\Omega).$$

In particular, if

$$a_{ij}, b_i, c, f, \in C^\infty(\Omega)$$

then

$$u \in C^\infty(\Omega).$$

In virtue of the results in [2], the regularity $u \in C_{X,loc}^{k+2,\alpha}(\Omega)$ also implies the validity of local a priori estimates

$$\|u\|_{C_X^{k+2,\alpha}(\Omega')} \leqslant c \left\{ \|Lu\|_{C_X^{k,\alpha}(\Omega)} + \|u\|_{L^\infty(\Omega)} \right\}$$

for any $\Omega' \Subset \Omega$, with constant c independent of u.

Remark 2.2 *Clearly, it is enough to prove the theorem for* $b_i = c = 0$*, because assuming this we can proceed as follows: let* $u \in C_X^{2,\alpha}(\Omega)$ *satisfy the equation*

$$Lu = f \text{ in } \Omega$$

and assume that

$$a_{ij}, b_i, c, f, \in C_X^{1,\alpha}(\Omega).$$

Then

$$\sum_{i,j=1}^{q} a_{ij}(x) X_i X_j u = f - \sum_{i=1}^{q} b_i(x) X_i u - c(x) u \equiv \widetilde{f} \in C_X^{1,\alpha}(\Omega)$$

and by the result that we suppose already proved for the principal part operator we conclude $u \in C_{X,loc}^{3,\alpha}(\Omega)$*. Iterating this argument gives the general result for any* k*. Hence, we need to prove Theorem 2.1 only for* $b_i = c = 0$*.*

Now, fix $x_0 \in \Omega$ and a small ball $B_R(x_0) \subset \Omega$ where the lifting procedure is applicable. Let $\xi_0 = (x_0, 0), \xi = (x, h)$, and define, for $\xi \in \widetilde{B}_R(\xi_0)$,

$$\widetilde{a}_{ij}(\xi) = a_{ij}(x)$$
$$\widetilde{L}u(\xi) = \sum_{i,j=1}^{q} \widetilde{a}_{ij}(\xi) \widetilde{X}_i \widetilde{X}_j u(\xi).$$

Next, let us freeze the coefficients $\widetilde{a}_{ij}$ at ξ_0, and let

$$\widetilde{L}_0 u(\xi) = \sum_{i,j=1}^{q} \widetilde{a}_{ij}(\xi_0) \widetilde{X}_i \widetilde{X}_j u(\xi)$$

For this frozen lifted operator the following representation formula holds true:

Theorem 2.3 (Representation of $\widetilde{X}_m \widetilde{X}_l u$ by frozen operators) *([2, p.211]) Given* $a \in C_0^\infty\left(\widetilde{B}_R(\xi_0)\right)$*, there exist frozen operators* $T_{lm}(\xi_0)$ *over the ball* $\widetilde{B}_R(\xi_0)$ $(m, l = 1, 2, ..., q)$*, such that for any* $u \in C_{X,0}^{2,\alpha}\left(\widetilde{B}_R(\xi_0)\right)$

$$\widetilde{X}_m \widetilde{X}_l(au) = T_{lm}(\xi_0) \mathcal{L}_0 u + \sum_{i,j=1}^{q} \widetilde{a}_{ij}(\xi_0) \left\{ \sum_{k=1}^{q} T_{lm,k}^{ij}(\xi_0) \widetilde{X}_k u + T_{lm}^{ij}(\xi_0) u \right\}.$$

Also,

$$\widetilde{X}_m \widetilde{X}_l(au) = T_{lm}(\xi_0) \mathcal{L} u + T_{lm}(\xi_0) \left(\sum_{i,j=1}^{q} [\widetilde{a}_{ij}(\xi_0) - \widetilde{a}_{ij}(\cdot)] \widetilde{X}_i \widetilde{X}_j u \right) +$$
$$+ \sum_{i,j=1}^{q} \widetilde{a}_{ij}(\xi_0) \left\{ \sum_{k=1}^{q} T_{lm,k}^{ij}(\xi_0) \widetilde{X}_k u + T_{lm}^{ij}(\xi_0) u \right\}.$$

In order to make more readable the previous formulas, let us define:

$$T_{lm,k}^{A}(\xi_0)=\sum_{i,j=1}^{q}\widetilde{a}_{ij}(\xi_0)\,T_{lm,k}^{ij}(\xi_0)$$

$$T_{lm}^{A}(\xi_0)=\sum_{i,j=1}^{q}\widetilde{a}_{ij}(\xi_0)\,T_{lm}^{ij}(\xi_0)$$

hence

$$\widetilde{X}_m\widetilde{X}_l(au)=T_{lm}(\xi_0)\,\widetilde{\mathcal{L}}u+T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\widetilde{X}_i\widetilde{X}_ju\right)+\sum_{k=1}^{q}T_{lm,k}^{A}(\xi_0)\,\widetilde{X}_ku+T_{lm}^{A}(\xi_0)\,u. \tag{2.2}$$

Remark 2.4 *We note that by Theorem 1.6, the operators* $T_{lm,k}^{A}(\xi_0)\,,T_{lm}^{A}(\xi_0)$ *satisfy the estimate:*

$$\left\|T_{\cdots}^{A}(\xi_0)\,f\right\|_{C^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)}\leq C_{R,\Lambda}\left\|f\right\|_{C^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)} \tag{2.3}$$

for any $f\in C_{X,0}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)$*, where now the constant* $C_{R,\Lambda}$ *also depends on the number* Λ *in (2.1).*

We are going to see (2.2) as an identity involving a suitable integral operator, to which apply the Banach-Caccioppoli fixed point theorem. To this aim, for a fixed $v\in C_{X,0}^{2,\alpha}\left(\widetilde{B}(\xi_0,R)\right)$ let

$$G_{l,m}=T_{lm}(\xi_0)\,\widetilde{\mathcal{L}}v+\sum_{k=1}^{q}T_{lm,k}^{A}(\xi_0)\,\widetilde{X}_kv+T_{lm}^{A}(\xi_0)\,v \tag{2.4}$$

hence by (2.3)

$$G_{l,m}\in C_X^{\alpha}\left(\widetilde{B}(\xi_0,R)\right)$$

and

$$\widetilde{X}_m\widetilde{X}_l(av)=G_{l,m}+T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\widetilde{X}_i\widetilde{X}_jv\right).$$

Now, for a number $r<R$ to be fixed later, pick another $\beta\in C_0^{\infty}\left(\widetilde{B}_r(\xi_0)\right)$ such that $\beta=1$ in $\widetilde{B}_{r/2}(\xi_0)$, and write

$$\beta\widetilde{X}_m\widetilde{X}_l(av)=\beta G_{l,m}+\beta T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\widetilde{X}_i\widetilde{X}_jv\right). \tag{2.5}$$

Now, for any $F=(F_{ij})_{i,j=1}^{q}\in\left(C_{X,*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}$, let us define the operator

$$\mathcal{T}(F)=\beta G_{l,m}+\beta T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]F_{ij}\right).$$

Theorem 2.5 *For $r>0$ small enough, the operator $\mathcal{T}$ is a contraction of $\left(C_{X,*}^{\alpha}(B_r(\xi_0))\right)^{q\times q}$ in itself.*

Proof. Since $F_{ij}\in C_{*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)$, F_{ij} can be extended to zero in $\widetilde{B}_R(\xi_0)$, hence (see Proposition 1.3)

$$\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]F_{ij}\in C_{X,0}^{\alpha}\left(\widetilde{B}_R(\xi_0)\right),$$

and by Theorem 1.6,

$$T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]F_{ij}\right)\in C_{X}^{\alpha}\left(\widetilde{B}_R(\xi_0)\right)$$

and

$$\beta T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]F_{ij}\right)\in C_{X,*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right).$$

Since also $G_{l,m}\in C_{X}^{\alpha}\left(\widetilde{B}_R(\xi_0)\right)$ and $\beta G_{l,m}\in C_{X,*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)$, we conclude that $\mathcal{T}$ maps $\left(C_{X,*}^{\alpha}(B_r(\xi_0))\right)^{q\times q}$ in itself. In order to show that $\mathcal{T}$ is a contraction, let $F^{(1)},F^{(2)}\in\left(C_{X,*}^{\alpha}(B_r(\xi_0))\right)^{q\times q}$. We have, by Theorem 1.6 and (1.2):

$$\begin{aligned}
&\left\|\mathcal{T}F^{(1)}-\mathcal{T}F^{(2)}\right\|_{\left(C_X^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}}\\
&\le\sum_{l,m=1}^{q}\left\|\beta T_{lm}(\xi_0)\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\left(F_{ij}^{(1)}-F_{ij}^{(2)}\right)\right\|_{C_X^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)}\\
&\le\sum_{l,m=1}^{q}\sum_{i,j=1}^{q}c\left\|\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\left(F_{ij}^{(1)}-F_{ij}^{(2)}\right)\right\|_{C_X^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)}\\
&\le c\omega(r)\left\|F^{(1)}-F^{(2)}\right\|_{\left(C_X^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}}
\end{aligned}$$

with

$$\omega(r)=\sup_{i,j=1,2,\dots,q}\left|\widetilde{a}_{ij}\right|_{C_X^{\alpha}\left(\widetilde{B}_R(\xi_9)\right)}r^{\alpha}$$

Hence for r small enough $\mathcal{T}$ is a contraction of $\left(C_{X,*}^{\alpha}(B_r(\xi_0))\right)^{q\times q}$ in itself. ∎

Next, we need the following similar result:

Theorem 2.6 *If $v \in C^{2,\alpha}_{X,0}\left(\widetilde{B}(\xi_0,R)\right)$, $\widetilde{a}_{ij}, \widetilde{L}v \in C^{1,\alpha}_{X}\left(\widetilde{B}_R(\xi_0)\right)$ and r is small enough, the operator $\mathcal{T}$ is also a contraction of $\left(C^{1,\alpha}_{X,*}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}$ in itself.*

Proof. We already know that

$$\beta G_{l,m} = \beta T_{lm}(\xi_0)\widetilde{\mathcal{L}}v + \sum_{k=1}^{q} \beta T^{A}_{lm,k}(\xi_0)\widetilde{X}_k v + \beta T^{A}_{lm}(\xi_0) v \in C^{\alpha}_{X,0}\left(\widetilde{B}_r(\xi_0)\right).$$

To show that also $\widetilde{X}_h(\beta G_{l,m}) \in C^{\alpha}_{X,*}\left(\widetilde{B}_r(\xi_0)\right)$, let us compute

$$\begin{aligned}\widetilde{X}_h(\beta G_{l,m}) = &\left(\widetilde{X}_h\beta\right) G_{l,m} + \beta\Big\{\widetilde{X}_h T_{lm}(\xi_0)\widetilde{\mathcal{L}}v + \\ &+ \sum_{k=1}^{q} \widetilde{X}_h T^{A}_{lm,k}(\xi_0)\widetilde{X}_k v + \widetilde{X}_h T^{A}_{lm}(\xi_0) v\Big\}\end{aligned}$$

exploiting Theorem 1.7

$$\begin{aligned}&= \left(\widetilde{X}_h\beta\right) G_{l,m} + \beta\Bigg\{\left(\sum_{s=1}^{q} T^{s}_{lm}(\xi_0)\widetilde{X}_s + T^{0}_{lm}(\xi_0)\right)\widetilde{\mathcal{L}}v + \\ &+ \sum_{k=1}^{q}\left(\sum_{s=1}^{q} T^{A,s}_{lm,k}(\xi_0)\widetilde{X}_s + T^{A,0}_{lm,k}(\xi_0)\right)\widetilde{X}_k v \\ &+ \left(\sum_{s=1}^{q} T^{A,s}_{lm}(\xi_0)\widetilde{X}_s + T^{A,0}_{lm}(\xi_0)\right) v\Bigg\}.\end{aligned}$$

Recalling that $v \in C^{2,\alpha}_{X,0}\left(\widetilde{B}_R(\xi_0)\right)$ by (2.3) we get that the quantity in $\{...\}$ belongs to $C^{\alpha}_{X}\left(\widetilde{B}_R(\xi_0)\right)$, hence by our choice of β,

$$\widetilde{X}_h(\beta G_{l,m}) \in C^{\alpha}_{X,0}\left(\widetilde{B}_r(\xi_0)\right) \subset C^{\alpha}_{X,*}\left(\widetilde{B}_r(\xi_0)\right).$$

As to the other term of $\mathcal{T}(F)$,

$$\beta T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0) - \widetilde{a}_{ij}(\cdot)\right] F_{ij}\right),$$

for $\widetilde{a}_{ij} \in C^{1,\alpha}_{X}\left(\widetilde{B}_R(\xi_0)\right), F_{ij} \in C^{1,\alpha}_{X,*}\left(\widetilde{B}_r(\xi_0)\right)$ we can compute, by Theorem 1.7:

$$\begin{aligned}&\widetilde{X}_k\left(\beta T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0) - \widetilde{a}_{ij}(\cdot)\right] F_{ij}\right)\right) \\ &= \beta\left\{\left(\sum_{s=1}^{q} T^{s}_{lm}(\xi_0)\widetilde{X}_s + T^{0}_{lm}(\xi_0)\right)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0) - \widetilde{a}_{ij}(\cdot)\right] F_{ij}\right)\right\} +\end{aligned}$$

$$+\left(\widetilde{X}_k\beta\right)T_{lm}\left(\xi_0\right)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\,F_{ij}\right)$$

$$=\beta\left\{\sum_{s=1}^{q}T_{lm}^{s}\left(\xi_0\right)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\widetilde{X}_sF_{ij}\right)-\sum_{s=1}^{q}T_{lm}^{s}\left(\xi_0\right)\left(\sum_{i,j=1}^{q}\left(\widetilde{X}_s\widetilde{a}_{ij}\right)F_{ij}\right)+\right.$$

$$\left.+\,T_{lm}^{0}\left(\xi_0\right)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]F_{ij}\right)\right\}+\left(\widetilde{X}_k\beta\right)T_{lm}\left(\xi_0\right)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\,F_{ij}\right).$$

Since, under our assumptions, all the functions:

$$\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\widetilde{X}_sF_{ij}$$

$$\sum_{i,j=1}^{q}\left(\widetilde{X}_s\widetilde{a}_{ij}\right)F_{ij}$$

$$\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]F_{ij}$$

belong to $C_{X,*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)\subset C_{X,0}^{\alpha}\left(\widetilde{B}_R(\xi_0)\right)$, by (2.3) and our choice of β we conclude

$$\widetilde{X}_k\left(\beta T_{lm}\left(\xi_0\right)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\,F_{ij}\right)\right)\in C_{X,*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right),$$

hence $\mathcal{T}$ maps $C_{X,*}^{1,\alpha}\left(\widetilde{B}_r\left(\xi_0\right)\right)$ in itself. Let us show that $\mathcal{T}$ is also a contraction in $C_{X,*}^{1,\alpha}\left(\widetilde{B}_r\left(\xi_0\right)\right)$. We already know that

$$\left\|\mathcal{T}F^{(1)}-\mathcal{T}F^{(2)}\right\|_{C_X^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)^{q\times q}}\le c\omega\left(r\right)\left\|F^{(1)}-F^{(2)}\right\|_{\left(C_X^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}}\tag{2.6}$$

so let us compute

$$\widetilde{X}_k\mathcal{T}F^{(1)}-\widetilde{X}_k\mathcal{T}F^{(2)}$$

$$=\beta\left\{\sum_{s=1}^{q}T_{lm}^{s}\left(\xi_0\right)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\left(\widetilde{X}_sF_{ij}^{(1)}-\widetilde{X}_sF_{ij}^{(2)}\right)\right)\right.$$

$$-\sum_{s=1}^{q}T_{lm}^{s}\left(\xi_0\right)\left(\sum_{i,j=1}^{q}\widetilde{X}_s\widetilde{a}_{ij}\left(F_{ij}^{(1)}-F_{ij}^{(2)}\right)\right)+$$

$$+ T^0_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\left(F^{(1)}_{ij}-F^{(2)}_{ij}\right)\right)\Bigg\} +$$

$$+\left(\widetilde{X}_k\beta\right)T_{lm}(\xi_0)\left(\sum_{i,j=1}^{q}\left[\widetilde{a}_{ij}(\xi_0)-\widetilde{a}_{ij}(\cdot)\right]\left(F^{(1)}_{ij}-F^{(2)}_{ij}\right)\right)$$

$$\equiv A+B+C+D.$$

Applying again Theorem 1.6 and (1.2),

$$\|A\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)} \le c\omega(r)\left\|XF^{(1)}_{ij}-XF^{(2)}_{ij}\right\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)} \tag{2.7}$$

$$\le c\omega(r)\left\|F^{(1)}_{ij}-F^{(2)}_{ij}\right\|_{C^{1,\alpha}_X\left(\widetilde{B}_r(\xi_0)\right)}$$

$$\|C\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)}+\|D\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)} \le c\omega(r)\left\|F^{(1)}_{ij}-F^{(2)}_{ij}\right\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)} \tag{2.8}$$

$$\|B\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)} \le c\sum_{s,i,j=1}^{q}\|X\widetilde{a}_{ij}\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)}\left\|F^{(1)}_{ij}-F^{(2)}_{ij}\right\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)}.$$

To complete the bound on B, let us note that if $g\in C^{1,\alpha}_{X,*}\left(\widetilde{B}_r(\xi_0)\right)$ we have

$$\|g\|_\infty \le \sup_{\xi,\eta\in\widetilde{B}_r(\xi_0)}|g(\xi)-g(\eta)| \le |g|_{C^\alpha\left(\widetilde{B}_r(\xi_0)\right)}(2r)^\alpha$$

and applying (1.1) (seeing g as a function in $C^{1,\alpha}_{X,0}\left(\widetilde{B}_R(\xi_0)\right)$),

$$|g|_{C^\alpha\left(\widetilde{B}_r(\xi_0)\right)} = \sup_{\xi,\eta\in\widetilde{B}_r(\xi_0)}\frac{|g(\xi)-g(\eta)|}{d_{\widetilde{X}}(\xi,\eta)^\alpha} \le \sup_{\widetilde{B}_r(\xi_0)}\left|\widetilde{X}g\right|(2r)^{1-\alpha},$$

hence

$$\|g\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)} \le \|g\|_{C^{1,\alpha}_X\left(\widetilde{B}_r(\xi_0)\right)}\left((2r)^\alpha+(2r)^{1-\alpha}\right)$$

and

$$\|B\|_{C^\alpha_X\left(\widetilde{B}_r(\xi_0)\right)} \le c\sum_{s,i,j=1}^{q}\left\|\widetilde{X}_s\widetilde{a}_{ij}\right\|_{C^\alpha_X(B_r(\xi_0))}\left((2r)^\alpha+(2r)^{1-\alpha}\right)\left\|F^{(1)}_{ij}-F^{(2)}_{ij}\right\|_{C^{1,\alpha}_X\left(\widetilde{B}_r(\xi_0)\right)}. \tag{2.9}$$

From (2.6), (2.7), (2.8), (2.9) we deduce that for r small enough

$$\left\|\mathcal{T}F^{(1)}-\mathcal{T}F^{(2)}\right\|_{\left(C^{1,\alpha}_X\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}} \le \delta\left\|F^{(1)}-F^{(2)}\right\|_{\left(C^{1,\alpha}_X(B_r(\xi_0))\right)^{q\times q}}$$

with $\delta<1$, and we are done. ■

We now come to the

Conclusion of the proof of Theorem 2.1. By Remark 2.2 it is enough to prove the theorem for $b_i = c = 0$. We will prove the regularity result for $k = 1$; an iterative argument gives the general case. Also, once the $C_{X,loc}^{k+2,\alpha}(\Omega)$ is proved for every k, Hörmander's condition implies that a solution $u \in C_{X,loc}^{k+2,\alpha}(\Omega)$ for any k is also smooth in Euclidean sense.

So, let $u \in C_X^{2,\alpha}(\Omega)$ satisfy the equation

$$Lu \equiv \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j u = f \text{ in } \Omega$$

and assume that

$$a_{ij}, f, \in C_X^{1,\alpha}(\Omega).$$

Fix $x_0 \in \Omega$ and a small ball $B_R(x_0) \subset \Omega$ where the lifting procedure is applicable. Let $\xi_0 = (x_0, 0), \xi = (x, h)$. Then by Proposition 1.5,

$$\begin{aligned}
\widetilde{u}(\xi) &= u(x) \in C_X^{2,\alpha}\left(\widetilde{B}_R(\xi_0)\right), \\
\widetilde{a}_{ij}(\xi) &= a_{ij}(x), \widetilde{f}(\xi) = f(x) \in C_X^{1,\alpha}\left(\widetilde{B}_R(\xi_0)\right) \\
\widetilde{L}\widetilde{u} &= \widetilde{f} \text{ in } \widetilde{B}_R(\xi_0).
\end{aligned}$$

Then, let $\phi \in C_0^\infty\left(\widetilde{B}_r(\xi_0)\right)$ such that $\phi = 1$ in $\widetilde{B}_{r/2}(\xi_0)$, hence $v = \phi\widetilde{u} \in C_{X,0}^{2,\alpha}\left(\widetilde{B}_R(\xi_0)\right)$ and

$$\widetilde{L}v = \phi\widetilde{f} + 2\sum_{i,j=1}^{q} \widetilde{a}_{ij}\widetilde{X}_i\widetilde{u}\widetilde{X}_j\phi + \widetilde{u}\widetilde{L}\phi \equiv g \in C_X^{1,\alpha}\left(\widetilde{B}_R(\xi_0)\right).$$

For this function v the representation formula (2.5) holds true, with G_{lm} given by (2.4). Let us define the number r, the cutoff function β and the operator $\mathcal{T}$ as in Theorems 2.5, 2.6. Since $\left(C_{X,*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}$ and $\left(C_{X,*}^{1,\alpha}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}$ are Banach spaces, by the Banach-Caccioppoli Theorem the operator $\mathcal{T}$ possesses a unique fixed point W both in $\left(C_{X,*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}$ and in $\left(C_{X,*}^{1,\alpha}\left(\widetilde{B}_r(\xi_0)\right)\right)^{q\times q}$. On the other hand, since $\widetilde{X}_i\widetilde{X}_j v \in C_{X,*}^{\alpha}\left(\widetilde{B}_r(\xi_0)\right)$, by (2.5) choosing $a(\xi) = 1$ in $\widetilde{B}_{R/2}(\xi_0) \supset \widetilde{B}_{r/2}(\xi_0)$, we get

$$\widetilde{X}_i\widetilde{X}_j v(\xi) = W(\xi) \text{ in } \widetilde{B}_{r/2}(\xi_0),$$

hence $\widetilde{X}_i\widetilde{X}_j v \in C_X^{1,\alpha}\left(\widetilde{B}_{r/2}(\xi_0)\right)$ and also $\widetilde{X}_i\widetilde{X}_j\widetilde{u} \in C_X^{1,\alpha}\left(\widetilde{B}_{r/2}(\xi_0)\right)$. By Proposition 1.5 this implies

$$X_i X_j u \in C_X^{1,\alpha}\left(B_{r/4}(x_0)\right)$$

therefore $u \in C_X^{3,\alpha}\left(B_{r/4}(x_0)\right)$ and by a covering argument $u \in C_{X,loc}^{3,\alpha}(\Omega)$. ■

3 Smoothness of solutions to quasilinear equations

Let us apply the previous linear theory to a regularity result for solutions to quasilinear equations.

Theorem 3.1 *Let*

$$Qu \equiv \sum_{i,j=1}^{q} a_{ij}(x,u,Xu)\, X_i X_j u + b(x,u,Xu)$$

where $X_1, X_2, ..., X_q$ *are as above,* $a_{ij} = a_{ji}$*,*

$$\Lambda|\xi|^2 \le \sum_{i,j=1}^{q} a_{ij}(x,u,p)\,\xi_i\xi_j \le \Lambda^{-1}|\xi|^2 \;\; \forall\xi \in \mathbb{R}^q, (x,u,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^q.$$

and assume that for some $k = 1,2,3,..., \alpha \in (0,1)$

$$a_{ij}, b \in C_X^{k,\alpha}(\Omega \times \mathbb{R} \times \mathbb{R}^q).$$

If $u \in C_X^{2,\alpha}(\Omega)$ *solves the equation* $Qu = 0$*, then* $u \in C_{X,loc}^{k+2,\alpha}(\Omega)$*. In particular, if* a_{ij}, b *are smooth, then* u *is also smooth.*

Proof. Under our assumptions we have that u is a solution to the linear equation

$$Lf(x) \equiv \sum_{i,j=1}^{q} \overline{a}_{ij}(x)\, X_i X_j f(x) = g(x)$$

where

$$\overline{a}_{ij}(x) = a_{ij}(x, u(x), Xu(x)) \in C_X^{1,\alpha}(\Omega)$$
$$g(x) = -b(x, u(x), Xu(x)) \in C_X^{1,\alpha}(\Omega)$$

hence by Theorem 2.1, $u \in C_{X,loc}^{3,\alpha}(\Omega)$; if $k = 1$ we are done, while if $k \ge 2$,

$$\text{since } u \in C_{X,loc}^{3,\alpha}(\Omega), \text{ then } \overline{a}_{ij}, g \in C_{X,loc}^{2,\alpha}(\Omega)$$

and by Theorem 2.1 $u \in C_{X,loc}^{4,\alpha}(\Omega)$. Iteration gives the desired result. Again, Hörmander's condition assures that if u belongs to $C_{X,loc}^{k+2,\alpha}(\Omega)$ for any k, then it is also smooth in the Euclidean sense. ∎

4 The evolution case

In virtue of the results contained in [2] all the previous theory can be developed also in the evolution case. Let us consider the linear operator

$$H = \partial_t - \sum_{i,j=1}^{q} a_{ij}(t,x)\, X_i X_j + \sum_{i=1}^{q} b_i(t,x)\, X_i + c(t,x)$$

where the X_i's are still a system of Hörmander's vector fields in a neighborhood Ω_0 of a bounded domain Ω, $Q = (0,T)\times\Omega$. We define in Q the parabolic distance

$$d_P\left((t,x),(s,y)\right) = \sqrt{d_X\left(x,y\right)^2 + |t-s|}$$

and define the spaces $C_P^\alpha(Q)$ of Hölder continuous functions of exponent α with respect to the distance d_P, and the spaces $C_P^{k,\alpha}(Q)$ of functions such that all the derivatives up to weight k with respect to the X_i's and ∂_t, with ∂_t weighting as a second order derivative, belong to $C_P^\alpha(Q)$. We assume with $a_{ij}, b_i, c \in C_P^\alpha(Q)$ for some $\alpha\in(0,1)$, $a_{ij} = a_{ji}$ satisfying the condition

$$\Lambda|\xi|^2 \le \sum_{i,j=1}^q a_{ij}(t,x)\,\xi_i\xi_j \le \Lambda^{-1}|\xi|^2 \;\forall\xi\in\mathbb{R}^q, (t,x)\in Q.$$

Then the same reasoning of §2 gives the following:

Theorem 4.1 *Under the above assumptions, let $u\in C_P^{2,\alpha}(Q)$ satisfy the equation*

$$Hu = f \text{ in } Q$$

and assume that for some integer $k = 1,2,3,...$we have:

$$a_{ij}, b_i, c, f, \in C_P^{k,\alpha}(Q).$$

Then

$$u\in C_{P,loc}^{k+2,\alpha}(Q).$$

In particular, in this case the a priori estimates proved in [2] apply:

$$\|u\|_{C_P^{k+2,\alpha}(Q')} \leqslant c\left\{\|Hu\|_{C_P^{k,\alpha}(Q)} + \|u\|_{L^\infty(Q)}\right\}$$

for any $Q'\Subset Q$, with constant c independent of u.

We also get the following quasilinear counterpart:

Theorem 4.2 *Let*

$$\mathcal{Q}u \equiv \partial_t u - \sum_{i,j=1}^q a_{ij}(t,x,u,Xu)\,X_iX_ju + b(t,x,u,Xu)$$

where $X_1, X_2, ..., X_q$ are as above, $a_{ij} = a_{ji}$,

$$\Lambda|\xi|^2 \le \sum_{i,j=1}^q a_{ij}(t,x,u,p)\,\xi_i\xi_j \le \Lambda^{-1}|\xi|^2$$

$\forall\xi\in\mathbb{R}^q, (t,x,u,p)\in(0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^q$, and assume that for some $k = 1,2,3,...,\alpha\in(0,1)$

$$a_{ij}, b\in C_P^{k,\alpha}\left((0,T)\times\Omega\times\mathbb{R}\times\mathbb{R}^q\right).$$

If $u\in C_P^{2,\alpha}(Q)$ solves the equation $\mathcal{Q}u = 0$, then $u\in C_{P,loc}^{k+2,\alpha}(Q)$. In particular, if a_{ij}, b are smooth, then u is also smooth.

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