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Quantum continuous measurements: The stochastic Schrödinger equations and the spectrum of the output

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The stochastic Schrödinger equation, of classical or quantum type, allows to describe open quantum systems under measurement in continuous time. In this paper we review the link between these two descriptions and we study the properties of the output of the measurement. For simplicity we deal only with the diffusive case. Firstly, we discuss the quantum stochastic Schrödinger equation, which is based on quantum stochastic calculus, and we show how to transform it into the classical stochastic Schrödinger equation by diagonalization of suitable quantum observables, based on the isomorphism between Fock space and Wiener space. Then, we give the a posteriori state, the conditional system state at time t given the output up to that time and we link its evolution to the classical stochastic Schrödinger equation. Finally, we study the output of the continuous measurement, which is a stochastic process with probability distribution given by the rules of quantum mechanics. When the output process is stationary, at least in the long run, the spectrum of the process can be introduced and its properties studied. In particular we show how the Heisenberg uncertainty relations give rise to characteristic bounds on the possible spectra and we discuss how this is related to the typical quantum phenomenon of squeezing. We use a simple quantum system, a two-level atom stimulated by a laser, to discuss the differences between homodyne and heterodyne detection and to explicitly show squeezing and anti-squeezing and the Mollow triplet in the fluorescence spectrum.

I. INTRODUCTION

A big achievement in the 70's-80's was to show that, inside the axiomatic formulation of quantum mechanics, based on *positive operator valued measures* and *instruments* [1, 2], a consistent formulation of the theory of measurements in continuous time (*quantum continuous measurements*) was possible [2–8]. Starting from the 80's, two other very flexible and powerful formulations of continuous measurement theory were developed. The first one is often referred as *quantum trajectory theory* and it is based on the *the stochastic Schrödinger equation* (SSE), a stochastic differential equation of classical type (commuting noises, Itô calculus) [6, 7, 9–17]. The second formulation is based on quantum stochastic calculus [18–20] and the *quantum SSE* (non commuting noises, Bose fields, Hudson-Parthasarathy equation) [4–6, 8, 12, 15, 17, 21, 22]. The main applications of quantum continuous measurements are in the photon detection theory in quantum optics (*direct, heterodyne, homodyne detection*) [9–17, 21–25]. While the classical SSE gives a differential description of the joint evolution of the observed signal and of the measured system, in agreement with the axiomatic formulation of quantum mechanics, the quantum SSE gives a dilation of the measurement process, explicitly introducing an environment which interacts with the system and mediates the observations.

In this paper we start by giving a short presentation of continuous measurement theory based on the quantum SSE (Secs. II and III). We consider only the type of observables relevant for the description of homodyne/heterodyne detection and we make the mathematical simplification of introducing only bounded operators on the Hilbert space of the quantum system of interest and a finite number of noises; for the case of unbounded operators see [26–28].

In Sec. III we show how to derive the classical SSE and the related stochastic master equation (SME). The key point in the step from the quantum SSE to the classical SSE is the introduction of an Hilbert space isomorphism which diagonalizes a suitable complete set of quantum observables. The classical SSE and the SME give both the probability distribution for the observed output and the *a posteriori state*, the conditional system state given a realization of the output. These equations are driven by classical noises, but, in spite of this, they are fully quantum as they are equivalent to the formulation of continuous measurements based on quantum fields. It is just in this formulation that the probabilistic structure of the output current becomes very transparent.

In Sec. IV we introduce the spectrum of the classical stochastic process which represents the output and we study the general properties of the spectra of such processes by proving characteristic bounds due to the Heisenberg uncertainty principle. This bound is one of the evidences that the whole theory of continuous measurements is fully quantum, independently of the adopted formulation.

As an application, in Sec. V we present the case of a two-level atom, which is measured in continuous time by detection of its fluorescence light. The spectral analysis of the output can reveal the phenomenon of squeezing of the fluorescence light, a phenomenon related to the

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uncertainty relations. We use this example also to illustrate the differences between homodyning and heterodyning and between the *spectrum of the squeezing* and the *power spectrum*. Finally we show how Mollow triplet appears in the power spectrum in the case of an intense stimulating laser. Section VI contains our conclusions.

II. THE QUANTUM STOCHASTIC SCHRÖDINGER EQUATION

Quantum stochastic calculus is based on the use of some Bose fields playing the role of non-commuting noises and satisfying the canonical commutation relations (CCR) with a Dirac delta in time (1). This calculus and the unitary dynamics based on it were developed by Hudson and Parthasarathy [18], while Bose fields with delta-commutations in time were already found by Yuen and Shapiro [29] in their study of the quasi–monochromatic paraxial approximation of the electromagnetic field.

A. Quantum stochastic calculus and unitary dynamics

Quantum stochastic calculus and the Hudson-Parthasarathy equation [18, 20] allow the evolution of a Markovian open quantum system, which we call system S, to be represented as a unitary evolution for system S interacting with some quantum fields. For a short review see [22, Sec. 2] or [21, Secs. 11.1, 11.2]; for a discussion of the physical approximations see [19, 21].

1. Bose fields

Let us start by introducing the formal fields $b_k(t)$, $b_k^\dagger(t)$ satisfying the CCR

$$\left[b_i(s), b_k^\dagger(t)\right] = \delta_{ik}\delta(t-s), \qquad [b_i(s), b_k(t)] = 0. \quad (1)$$

In this paper we consider only the representation of the CCR (1) on the Fock space, the one characterized by the existence of the vacuum state.

We denote by $\Gamma \equiv \Gamma\big(L^2(\mathbb{R};\mathbb{C}^d)\big)$ the *symmetric Fock space* over the "one–particle space" $L^2(\mathbb{R}) \otimes \mathbb{C}^d = L^2(\mathbb{R};\mathbb{C}^d)$, and by $e(f)$, $f \in L^2(\mathbb{R};\mathbb{C}^d)$, the *coherent vectors,* whose components in the $0, 1, \ldots, n, \ldots$ particle spaces are

$$e(f) := \mathrm{e}^{-\frac{1}{2}\,\|f\|^2}\left(1, f, (2!)^{-1/2} f\otimes f, \ldots, (n!)^{-1/2} f^{\otimes n}, \ldots\right). \quad (2)$$

Note that $e(0)$ represents the vacuum state and that $\langle e(g)|e(f)\rangle = \exp\left\{-\frac{1}{2}\|f\|^2 - \frac{1}{2}\|g\|^2 + \langle g|f\rangle\right\}$.

Let $\{z_k,\ k\geq 1\}$ be the canonical basis in $\mathbb{C}^d$ and for any $f \in L^2(\mathbb{R};\mathbb{C}^d)$ let us set $f_k(t) := \langle z_k|f(t)\rangle_{\mathbb{C}^d}$. Then we have

$$b_k(t)\, e(f) = f_k(t)\, e(f). \quad (3)$$

It is a property of the Fock spaces the fact that the action on the coherent vectors uniquely determines a densely defined linear operator.

2. Factorization properties of the Fock space

A symmetric Fock space $\Gamma(\mathcal{K})$ can be defined for every Hilbert space $\mathcal{K}$; coherent vectors are defined always by (2). When the one-particle space is given by a direct sum ($\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$), one has the factorization property $\Gamma(\mathcal{K}_1 \oplus \mathcal{K}_2) = \Gamma(\mathcal{K}_1) \otimes \Gamma(\mathcal{K}_2)$.

In our set up, for every time interval A, let us denote by $\Gamma[A] \equiv \Gamma\big(L^2(A;\mathbb{C}^d)\big)$ the symmetric Fock space over $L^2(A;\mathbb{C}^d)$; in particular, we have $\Gamma = \Gamma[\mathbb{R}]$. Then, for any $s<t$, we have $L^2(\mathbb{R};\mathbb{C}^d) = L^2\big((-\infty,s);\mathbb{C}^d\big) \oplus L^2\big((s,t);\mathbb{C}^d\big) \oplus L^2\big((t,+\infty);\mathbb{C}^d\big)$ and

$$\Gamma[\mathbb{R}] = \Gamma\big[(-\infty,s)\big] \otimes \Gamma\big[(s,t)\big] \otimes \Gamma\big[(t,+\infty)\big]. \quad (4)$$

Moreover, each space $\Gamma[A]$ can be identified with a subspace of the full Fock space $\Gamma[\mathbb{R}]$ by taking the tensor product of a generic vector in $\Gamma[A]$ with the vacuum of $\Gamma[\mathbb{R}\setminus A]$. Then, for every $f \in L^2(\mathbb{R};\mathbb{C}^d)$, we have the identification

$$e(f|_A) \in \Gamma[A] \mapsto e(1_A f) \in \Gamma[\mathbb{R}].$$

We are denoting by $1_A(\cdot)$ the indicator function of the set A and by $f|_A$ the restriction of the function f to the set A. With an abuse of notation we write

$$e(f) = e\big(1_{(-\infty,s)}f\big) \otimes e\big(1_{(s,t)}f\big) \otimes e\big(1_{(t,+\infty)}f\big).$$

In particular, $e\big(1_{(s,t)}f\big)$ can represent a vector in $\Gamma[\mathbb{R}]$ or in $\Gamma[(s,t)]$ and we have the identification $e\big(1_{(s,t)}f\big) = e(0) \otimes e\big(1_{(s,t)}f\big) \otimes e(0)$.

3. Temporal modes and Weyl operators.

The free evolution of the bose fields is represented by the left shift in Γ

$$\Theta_t\, e(f) = e(\theta_t f), \quad \big(\theta_t f\big)(s) = f(s+t).$$

Then, the action of the shift on the fields is given by

$$\Theta_t^\dagger\, b_k(s)\, \Theta_t = b_k(s+t).$$

By using the energy representation of the fields,

$$\hat b_k(\nu) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\nu t} b_k(t)\,\mathrm{d}t,$$

we get $\Theta_t^\dagger\, \hat b_k(\nu)\, \Theta_t = \mathrm{e}^{-\mathrm{i}\nu t}\hat b_k(\nu)$, which shows that indeed Θ_t is the usual free dynamics. Therefore, the argument

t in the fields $b_k(t)$ has a double role: it is the time, because it appears in the evolution operator, and it is a field degree of freedom, the conjugate momentum of the free field energy.

When the fields $b_k(t)$ represent the electromagnetic field, the index k can be used to denote the polarization, the direction of propagation (discretized) or other spatial degrees of freedom [30].

If we take a function $g \in L^2(\mathbb{R})$ we can define the annihilation operator

$$c_k(g) := \int_{-\infty}^{+\infty} \overline{g(t)}\, b_k(t)\, \mathrm{d}t. \tag{5}$$

By Eq. (3), its action on the coherent vectors is given by

$$c_k(g)\, e(f) = \int_{-\infty}^{+\infty} \overline{g(t)}\, f_k(s)\, \mathrm{d}s\, e(f) \equiv \langle g|f_k\rangle_{L^2(\mathbb{R})}\, e(f).$$

If we take a complete orthonormal system g^i, $i = 1, 2, \ldots$, in $L^2(\mathbb{R})$, we can define the annihilation operators $c_k(g^i)$. Together with their adjoint operators, they satisfy the usual CCR. We can say that the upper index i denotes the *temporal modes*, while the lower index k denotes the polarization/spatial modes.

An important technical tool is represented by the *Weyl operators* $\mathcal{W}(q)$, $q \in L^2(\mathbb{R};\mathbb{C}^d)$, the unitary operators defined by: $\forall f \in L^2(\mathbb{R};\mathbb{C}^d)$,

$$\mathcal{W}(q)e(f) = \exp\left\{\mathrm{i}\,\mathrm{Im}\langle f|q\rangle_{L^2(\mathbb{R};\mathbb{C}^d)}\right\} e(f+q);$$

this is nothing but the *displacement operator* for the field. By using the notation (5) we can write

$$\mathcal{W}(q) = \exp\left\{\sum_k \left(c_k^\dagger(q_k) - \text{h.c.}\right)\right\}, \tag{6}$$

while, by using the discrete modes introduced above, we have

$$\mathcal{W}(q) = \exp\left\{\sum_{ki} \left(\langle g^i|q_k\rangle_{L^2(\mathbb{R})} c_k^\dagger(g^i) - \text{h.c.}\right)\right\}.$$

By h.c. we denote the Hermitian conjugate operator.

4. *The Hudson-Parthasarathy equation*

Let $\mathcal{H}$ be the *system space,* the complex separable Hilbert space associated to the quantum system S, and we take Γ, with its free evolution Θ_t, as the environment space. Now we want to construct the unitary evolution of the composite system on $\mathcal{H} \otimes \Gamma$.

By formally writing

$$B_k(t) = \int_0^t b_k(s)\mathrm{d}s, \qquad B_k^\dagger(t) = \int_0^t b_k^\dagger(s)\mathrm{d}s, \tag{7}$$

we get the *annihilation* and *creation processes*, families of mutually adjoint operators, whose actions on the coherent vectors are given by

$$B_k(t)\, e(f) = \int_0^t f_k(s)\, \mathrm{d}s\, e(f)\,,$$
$$\langle e(g)|B_k^\dagger(t)e(f)\rangle = \int_0^t \overline{g_k(s)}\, \mathrm{d}s\, \langle e(g)|e(f)\rangle.$$

The overline denotes the complex conjugation.

For $t > 0$, the annihilation and creation processes are adapted, in the sense that they factorizes, with respect to (4), as

$$B_k^{(\dagger)}(t) = \mathbb{1}_{(-\infty,0)} \otimes B_k^{(\dagger)}(t) \otimes \mathbb{1}_{(t,+\infty)},$$

and they satisfy a variant of the CCR, namely

$$[B_k(t), B_l^\dagger(s)] = \delta_{kl} t \wedge s, \tag{8}$$
$$[B_k(t), B_l(s)] = 0, \qquad [B_k^\dagger(t), B_l^\dagger(s)] = 0;$$

$t \wedge s$ is the minimum between t and s and $\mathbb{1}_A$ is the identity operator on $\Gamma[A]$.

By defining integrals of Itô type with respect to the increments of the quantum processes B_k, $B_k^\dagger$, it is possible to construct adapted operator processes on $\mathcal{H} \otimes \Gamma$ and to develop a quantum stochastic calculus, whose rules are summarized, at a heuristic level, by the quantum Itô table

$$\mathrm{d}B_k(t)\, \mathrm{d}B_l^\dagger(t) = \delta_{kl}\, \mathrm{d}t, \qquad \mathrm{d}B_k^\dagger(t)\, \mathrm{d}B_l(t) = 0, \tag{9a}$$
$$\mathrm{d}B_k(t)\, \mathrm{d}B_l(t) = 0, \qquad \mathrm{d}B_k^\dagger(t)\, \mathrm{d}B_l^\dagger(t) = 0, \tag{9b}$$
$$\mathrm{d}B_k^\dagger(t)\, \mathrm{d}t = 0, \qquad \mathrm{d}B_k(t)\, \mathrm{d}t = 0, \qquad (\mathrm{d}t)^2 = 0. \tag{9c}$$

Let H_0, R_k, $k, l = 1, \ldots, d$, be bounded operators on $\mathcal{H}$ such that $H_0^\dagger = H_0$. We set also

$$K := -\mathrm{i}H_0 - \frac{1}{2}\sum_k R_k^\dagger R_k. \tag{10}$$

Then, the quantum stochastic differential equation (quantum stochastic Schrödinger equation or Hudson-Parthasarathy equation) [18, 20]

$$\mathrm{d}U_t = \left\{\sum_k R_k\, \mathrm{d}B_k^\dagger(t) - \sum_k R_k^\dagger\, \mathrm{d}B_k(t) + K\, \mathrm{d}t\right\} U_t, \tag{11}$$

with the initial condition $U_0 = \mathbb{1}$, has a unique solution, which is a strongly continuous adapted family of unitary operators on $\mathcal{H} \otimes \Gamma$, representing a system-field dynamics in the interaction picture with respect to the free field evolution [31].

Then, for $t \geq 0$, the dynamics in the Schrödinger picture is $\mathrm{e}^{-\mathrm{i}H_{\mathrm{TOT}}t} = \Theta_t\, U_t$, a strongly continuous unitary group whose Hamiltonian H_{TOT} is a singular perturbation of the unbounded generator of Θ_t [32, 33]. Roughly speaking, the system S is hit by a flow of bosons which can have only a singular interaction with S; then, they are carried away by their free dynamics and never come back. The physical approximations involved in Eq. (11) are discussed in [21, Sec. 11.1.1]. Note that the interaction picture with respect to the free field dynamics coincides with the Schrödinger picture when only reduced system states and observables are considered.

B. The reduced dynamics of the system

The states of a quantum system are represented by statistical operators, positive trace-class operators with trace one; let us denote by $\mathcal{S}(\mathcal{H})$ the set of statistical operators on $\mathcal{H}$. For every composed state Σ in $\mathcal{S}(\mathcal{H}\otimes\Gamma)$, the partial trace Tr_Γ (resp. $\operatorname{Tr}_\mathcal{H}$) with respect to the field (resp. system) Hilbert space gives the reduced system (resp. field) state $\operatorname{Tr}_\Gamma \Sigma$ in $\mathcal{S}(\mathcal{H})$ (resp. $\operatorname{Tr}_\mathcal{H} \Sigma$ in $\mathcal{S}(\Gamma)$).

1. The initial state and the reduced states

As initial state of the composed system "S plus fields" we take $\rho\otimes\varrho_\Gamma(f)\in\mathcal{S}(\mathcal{H}\otimes\Gamma)$, where $\rho\in\mathcal{S}(\mathcal{H})$ is generic and $\varrho_\Gamma(f)$ is a coherent state, $\varrho_\Gamma(f):=|e(f)\rangle\langle e(f)|$. Then, the system-field state at time t, in the field interaction picture, is

$$\Sigma_f(t):=U_t\left(\rho\otimes\varrho_\Gamma(f)\right)U_t^\dagger. \qquad (12)$$

We introduce also the *reduced system state* and the *reduced field state*:

$$\eta_t:=\operatorname{Tr}_\Gamma\left\{\Sigma_f(t)\right\}, \qquad \Pi_f(t):=\operatorname{Tr}_\mathcal{H}\left\{\Sigma_f(t)\right\}. \qquad (13)$$

2. The master equation

One of the main properties of the Hudson-Parthasarathy equation is that, with the initial state introduced above, the reduced dynamics of system S exactly obeys a quantum master equation [18, 20, 22]. Indeed, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\eta_t=\mathcal{L}(t)[\eta_t], \qquad (14)$$

where the Liouville operator $\mathcal{L}(t)$ turns out to be given by

$$\mathcal{L}(t)[\rho]=-\mathrm{i}\left[H_0+H_f(t),\,\rho\right] + \sum_k\left(R_k\rho R_k^\dagger-\frac{1}{2}R_k^\dagger R_k\rho-\frac{1}{2}\rho R_k^\dagger R_k\right), \qquad (15)$$

$$H_f(t):=\mathrm{i}\sum_k\overline{f_k(t)}R_k-\mathrm{i}\sum_k f_k(t)R_k^\dagger. \qquad (16)$$

Therefore, S is an open system, as it interacts with the fields in Γ, and its evolution turns out to be Markovian thanks to the properties of the interaction and of the initial state of the environment.

It is useful to introduce also the evolution operator from s to t by

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Upsilon(t,s)=\mathcal{L}(t)\circ\Upsilon(t,s), \qquad \Upsilon(s,s)=\mathbb{1}. \qquad (17)$$

With this notation we have $\eta_t=\Upsilon(t,0)[\rho]$.

III. CONTINUOUS MONITORING

The connections among quantum stochastic calculus, quantum Langevin equations and input and output fields were developed by Gardiner and Collet in [19]. Then, in [5] these notions were connected to the unitary evolution (11) and to continuous measurements. Indeed, another fundamental property of the Hudson-Parthasarathy equation is that it allows for a fully quantum description of a continuous measurement of the system S: the measurement is obtained by a detection of the bosons that have already interacted with S. Of course such a measurement acquires information on both S and the detected bosons.

A. Input and output fields

Let us call "input fields" the fields $B_k(t)$, $B_k^\dagger(t),\ldots$ when they are considered as operators in interaction picture at time t, with respect to Θ_t, and let us call "output fields" the same fields in the Heisenberg picture:

$$B_k^{\mathrm{out}}(t):=U_t^\dagger B_k(t)U_t \qquad (18)$$

and a similar definition for $B_k^{\mathrm{out}\,\dagger}(t)$. By the properties of the Fock space Γ and of the unitary operators U_t, it is possible to prove that

$$B_k^{\mathrm{out}}(t)=U_T^\dagger B_k(t)U_T\,, \qquad \forall T\geq t. \qquad (19)$$

This equation is of fundamental importance and it immediately implies that the output fields satisfy the same commutation rules of the input fields, for instance the CCR (8): the output fields remain Bose free fields. By applying the formal rules of QSC (9), we can express the output fields as the quantum stochastic integrals [5]

$$B_k^{\mathrm{out}}(t)=B_k(t)+\int_0^t U_s^\dagger R_k U_s\,\mathrm{d}s; \qquad (20)$$

$B_k^{\mathrm{out}\,\dagger}(t)$ is given by the adjoint expression.

B. The field observables

The key point of the theory of continuous measurements is to consider field observables represented by time dependent, commuting selfadjoint operators in the Heisenberg picture [4, 5, 22]. Being commuting at different times, these observables represent outputs produced at different times which can be obtained in the same experiment. Here we present a very special case of observables, some field quadratures. Let us start by introducing the selfadjoint operators

$$Q(t;\vartheta,h)=\mathrm{e}^{-\mathrm{i}\vartheta}\int_0^t h(s)\,\mathrm{d}B_1^\dagger(s)+\text{h.c.},\qquad t\geq 0; \quad (21)$$

the phase $\vartheta\in(-\pi,\pi]$ and the function h, with $|h(t)|=1$, are fixed. The operators (21) have to be interpreted as linear combinations of the formal increments $\mathrm{d}B_1^\dagger(s)$, $\mathrm{d}B_1(s)$ which represent field operators in the interaction picture. The corresponding operators in the Heisenberg picture are

$$\begin{aligned}Q^{\text{out}}(t;\vartheta,h) &:= U_t^\dagger Q(t;\vartheta,h)U_t\\ &= U_T^\dagger Q(t;\vartheta,h)U_T,\quad \forall T\geq t,\end{aligned} \qquad (22)$$

where the second equality follows from Eq. (19). These "output" quadratures are our observables.

Each quadrature $Q^{\text{out}}(t;\vartheta,h)$ is observed at time t and it regards those bosons in "field 1" which have eventually interacted with S between time 0 and time t, so it can be interpreted as an indirect measurement performed on the system S.

By using CCRs, one can check that the operators (21) commute: $[Q(t;\vartheta,h),Q(s;\vartheta,h)]=0$. The important point is that, thanks to Eq. (22), these operators commute for different times also in the Heisenberg picture. Therefore, the observables $Q^{\text{out}}(t;\vartheta,h)$, $t\geq 0$, can be jointly measured for every interaction (11). The output is a (random) number at every time t, that is a signal depending on time, a stochastic process, which is the result of a continuous indirect monitoring of the system S. Its probability distribution is given by the usual postulates of quantum mechanics trough the joint diagonalization of the operators $Q^{\text{out}}(t;\vartheta,h)$. Actually, always thanks to Eq. (22), it will be enough to jointly diagonalize the operators $Q(t;\vartheta,h)$.

Let us stress that quadratures of type (21) with different phases and h functions represent incompatible observables, because they do not commute but satisfy

$$[Q(t;\vartheta,h),Q(s;\varphi,g)]=2\mathrm{i}\int_0^{t\wedge s}\mathrm{d}r\;\mathrm{Im}\left(\mathrm{e}^{\mathrm{i}(\vartheta-\varphi)}\overline{h(r)}\,g(r)\right).$$

Note that for $g=h$ we get

$$[Q(t;\vartheta,h),Q(s;\varphi,h)]=2\mathrm{i}\,(t\wedge s)\sin(\vartheta-\varphi)\,, \qquad (23)$$

and for $\varphi=\vartheta$ they commute as anticipated.

When "field 1" represents the electromagnetic field, a physical realization of a measurement of the observables (22) is implemented by what is called balanced heterodyne/homodyne detection [34–36], [21, Sec. 8.4.4]. The light emitted by the system in the "channel 1" interferes with an intense laser beam represented by h, the *local oscillator*. The mathematical description of the apparatus is given in [22, Sec. 3.5].

Let us note that the operator $Q^{\text{out}}(t;\vartheta,h)$ involves the whole time interval $[0,t]$ and has to be interpreted as cumulated output. The instantaneous output current is represented by its formal time derivative $\hat I^{\text{out}}(t):=\dot Q^{\text{out}}(t;\vartheta,h)$. From (7), (19), (20), (22) we get

$$\hat I^{\text{out}}(t)=\mathrm{e}^{\mathrm{i}\vartheta}\,\overline{h(t)}\left(b_1(t)+U_t^\dagger R_1U_t\right)+\text{h.c.} \qquad (24)$$

C. The stochastic representation

The commuting selfadjoint operators (21) have a joint projection valued measure (pvm) E_ϑ^h, which gives the probability distribution for the output of the continuous measurement. Moreover, via the partial trace on the fields, E_ϑ^h gives also the *instruments* describing the transformations of S from time 0 to an arbitrary time t, conditioned on the information acquired up to time t. Furthermore, via joint diagonalization and conditioning, the pvm E_ϑ^h even gives the stochastic evolution of the conditional state ρ_t (or *a posteriori state*), the state of S at time t given the observed signal from time 0 to time t. This evolution turns out to satisfy a stochastic differential equation (SSE or SME), with classical driving noises. The introduction of such stochastic evolution equations for the conditional state was an achievement of the *quantum filtering theory* [7, 37–40].

The passage from the formulation with quantum fields and Hudson-Parthasarathy equation to the one based on classical stochastic differential equations can be done by different techniques. The technique based on the use of isomorphisms between the Fock space and the Wiener space is very powerful and clear; here we present a variant of the construction given in [12].

Let us note that the observation we consider is not complete, because it regards only field 1 and involves only positive times. To make unique the isomorphism which diagonalizes the self-adjoint operators (21), we need to add fictitious observations, involving quadratures of the fields $2,\ldots,d$ too. So, we take a function $\ell\in L^\infty(\mathbb{R};\mathbb{C}^d)$ such that

$$|\ell_k(t)|=1,\ \forall t\in\mathbb{R},\ \forall k,\quad \text{and}\quad \ell_1(t)=\mathrm{e}^{-\mathrm{i}\vartheta}h(t),\ \forall t\geq 0;$$

then, we introduce the field quadratures: for $k=1,\ldots,d$,

$$Q_k(t):=\int_0^t\ell_k(s)\mathrm{d}B_k^\dagger(s)+\text{h.c.} \qquad (25)$$

We use this definition for positive and negative times by taking the convention $\int_0^t=-\int_t^0$ for a negative t. These quadratures form a complete set of compatible observables on the Fock space Γ. Note that $Q_1(t)=Q(t;\vartheta,h)$. In the following subsection we jointly diagonalize all the observables (25) by introducing an explicit isomorphism between Fock and Wiener spaces.

1. Spectral representation on the Wiener space

Fixed the functions $\ell_1, \dots, \ell_d$, that is the field quadratures (25), we look for a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, a unitary operator $J : \Gamma[L^2(\mathbb{R}; \mathbb{C}^d)] \to L^2(\Omega, \mathcal{F}, \mathbb{Q})$ (the Hilbert space of the complex square integrable random variables on the given probability space), and a family of random variables $W_k(t)$ on Ω such that

$$\big(JQ_k(t)\Psi\big)(\omega) = W_k(t;\omega)\,\big(J\Psi\big)(\omega), \tag{26}$$

for all t, k, and for all Ψ in the domain of the selfadjoint operator $Q_k(t)$. This means that each $Q_k(t)$ is represented in $L^2(\Omega, \mathcal{F}, \mathbb{Q})$ as the multiplication operator by $W_k(t)$. We can get such a joint diagonalization on the space of the canonical representation of the Wiener process; the canonical Wiener process is presented, for instance, in [16, Secs. A.2.4, A.2.6].

Let $\Omega = C_0(\mathbb{R}; \mathbb{R}^d)$ be the space of the continuous functions $\omega : \mathbb{R} \to \mathbb{R}^d$ such that $\omega(0) = 0$. We define the d-dimensional process $W(t) : \Omega \to \mathbb{R}^d$, $t \in \mathbb{R}$, by $W(t,\omega) = \omega(t)$ and we denote by $\mathcal{F}$ the smallest σ-algebra of subsets of Ω for which these functions $W(t)$ are measurable: $\mathcal{F} = \sigma\big(W(t) : t \in \mathbb{R}\big)$. Then, there exists a unique probability measure $\mathbb{Q}$ on the measurable space $(\Omega, \mathcal{F})$, the Wiener measure, such that the processes $W_k(t)$, $W_k(-t)$, $t \geq 0$, $k = 1, \dots, d$ are $2d$ independent standard Wiener processes. Moreover, for positive times we introduce the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the process W: $\mathcal{F}_t = \sigma(W(s) : s \in [0,t])$. Let us also recall that, if $\phi, \psi \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$, then their inner product is given by the $\mathbb{Q}$-expectation $\mathbb{E}_\mathbb{Q}$:

$$\langle \psi | \phi \rangle = \mathbb{E}_\mathbb{Q}[\overline{\psi}\,\phi] = \int_\Omega \overline{\psi(\omega)}\,\phi(\omega)\mathbb{Q}(\mathrm{d}\omega).$$

Let $J : \Gamma[L^2(\mathbb{R}; \mathbb{C}^d)] \to L^2(\Omega, \mathcal{F}, \mathbb{Q})$ be the linear operator defined by: $\quad \forall g \in L^2(\mathbb{R}; \mathbb{C}^d)$,

$$\begin{aligned} J\,e(g) = &\exp\left\{\sum_{k=1}^d \int_{-\infty}^{+\infty} \overline{\ell_k(s)}\,g_k(s)\,\mathrm{d}W_k(s)\right\} \\ &\times\, \exp\left\{-\frac{1}{2}\sum_{k=1}^d \int_{-\infty}^{+\infty} \Big(\overline{\ell_k(s)}\,g_k(s)\Big)^2\,\mathrm{d}s\right\} \\ &\times \exp\left\{-\frac{1}{2}\sum_{k=1}^d \int_{-\infty}^{+\infty} |g_k(s)|^2\,\mathrm{d}s\right\}. \end{aligned} \tag{27}$$

In particular we have

$$J\,e(0) = 1, \qquad J\,e(1_{(0,t)}f) \in L^2(\Omega, \mathcal{F}_t, \mathbb{Q}).$$

The operator J turns out to be an isomorphism and it realizes the representation (26): $J\,Q_k(t)\,J^{-1} = W_k(t)$, i.e. the field quadratures are mapped into the operators "multiplication by the Wiener processes". Because the isomorphism J jointly diagonalizes all the observables (25), then their joint pvm on the Fock space is $J^{-1}1_A J$, $\forall A \in \mathcal{F}$.

a. The distribution of the output. Let us restrict now to the observed quadrature (21); the σ-algebra $\mathcal{G}_\infty = \sigma(W_1(t) : 0 \leq t < +\infty)$ is the space of all the events regarding our observables $Q(t;\vartheta,h)$, $t \geq 0$. Then, the joint pvm E^h_ϑ of the observed quadratures is defined on the measurable space $(\Omega, \mathcal{G}_\infty)$ by

$$E^h_\vartheta(G) = J^{-1}1_G J, \qquad \forall G \in \mathcal{G}_\infty. \tag{28}$$

Finally, we get the distribution of the output. By setting $\mathcal{G}_t = \sigma(W_1(s) : s \in [0,t])$, then, $\mathcal{G}_t \subset \mathcal{G}_\infty$, is the space of the observed events up to time t, associated to the observables $Q(s;\vartheta,h)$ for times from 0 to t, and, according to the usual rules of quantum mechanics, the probabilities of such events are given by

$$\mathbb{P}^{\vartheta,h}_{\rho,t}(G) = \operatorname{Tr}\left\{\big(\mathbb{1}_\mathcal{H} \otimes E^h_\vartheta(G)\big)\Sigma_f(t)\right\},\ \forall G \in \mathcal{G}_t,\ \forall t \geq 0. \tag{29}$$

Note that, when the field state is the vacuum and there is no interaction between system S and the fields, this probability reduces to $\mathbb{P}^{\vartheta,h}_{\rho,t}(G) = \langle e(0)|E^h_\vartheta(G)\,e(0)\rangle = \mathbb{E}_\mathbb{Q}[1_G] = \mathbb{Q}(G)$. This means that in this case the quadratures (21) are distributed as a standard Wiener process.

Let us stress that the pvm (28) depends on the parameters ϑ and h defining the quadrature (21); these parameters are contained in the definition of the isomorphism J (27). On the contrary, the choices of the trajectory space (the measurable space $(\Omega, \mathcal{G}_\infty)$) and of the process W_1 are independent of ϑ and h. With respect to the time dependence, the physical probabilities (29) are *consistent*, i.e.

$$0 \leq s \leq t, \quad G \in \mathcal{G}_s \ \Rightarrow\ \mathbb{P}^{\vartheta,h}_{\rho,t}(G) = \mathbb{P}^{\vartheta,h}_{\rho,s}(G). \tag{30}$$

This result is due to the factorization property (4) of the Fock space and to the localization properties of U_t [22, Theor. 2.3], which imply $U_t^\dagger\big(\mathbb{1}_\mathcal{H} \otimes E^h_\vartheta(G)\big)U_t = U_s^\dagger\big(\mathbb{1}_\mathcal{H} \otimes E^h_\vartheta(G)\big)U_s$ for $0 \leq s \leq t$ and $G \in \mathcal{G}_s$, cf. Eq. (22).

A more detailed study of the statistical properties of the output needs the introduction of the characteristic operator (Sec. III D).

2. The instruments

The observation of the emitted field can be interpreted as an indirect measurement on the system S and this is formalized by the concept of *instrument* [1, 2]. The family of instruments $\mathcal{I}_t$, $t > 0$, describing our measure is defined by: $\forall G \in \mathcal{G}_t$, $\forall \tau \in \mathcal{S}(\mathcal{H})$,

$$\mathcal{I}_t(G)[\tau] = \operatorname{Tr}_\Gamma\left\{\big(\mathbb{1}_\mathcal{H} \otimes E^h_\vartheta(G)\big)U_t\big(\tau \otimes \varrho_\Gamma(f)\big)U_t^\dagger\right\}. \tag{31}$$

For $\tau = \rho$, the initial system state, Eq. (31) gives the non-normalized state of S at time t conditioned on the information that the values of the signal in the time interval from 0 to t were in G. Of course we have

$$\operatorname{Tr}_\mathcal{H}\left\{\mathcal{I}_t(G)[\rho]\right\} = \mathbb{P}^{\vartheta,h}_{\rho,t}(G), \tag{32}$$

while the normalized conditioned state is given by $\mathcal{I}_t(G)[\rho]$ divided by its trace (32).

Let us remark that

$$\eta_t = \mathcal{I}_t(\Omega)[\rho], \tag{33}$$

so that the system reduced state at time t in the case of no observation (η_t) coincides with the so called *a priori state* ($\mathcal{I}_t(\Omega)[\rho]$), that is the system state at time t in the case of observation performed but not taken into account. This is in agreement with our rough picture of the measurement process: we observe fields which have already interacted with system S and which will never interact again with it. This means that we acquire information on S, as we have $\mathcal{I}_t(G)[\rho] \neq \mathbb{P}^{\vartheta,h}_{\rho,t}(G)\eta_t$, but we do not add any perturbation on its evolution as we have $\mathcal{I}_t(\Omega)[\rho] = \eta_t$.

a. The a posteriori states. Now we want to introduce ρ_t, the state of S at time t conditioned on the whole information supplied by our indirect measurement between time 0 and time t, that is by the signal produced by the measurement of $Q(s;\vartheta,h)$ for $s \in [0,t]$. Therefore, ρ_t has to be a random state depending on the output $W_1(s)$, $0 \leq s \leq t$, that is a random state measurable with respect to $\mathcal{G}_t$; in other terms, we have the functional dependence $\rho_t(\omega) = \rho_t\big(\omega_1(s),\, 0 \leq s \leq t\big)$. Such a state is called *a posteriori state* and it is determined by the initial state ρ and by the instrument $\mathcal{I}_t$: it is the unique $\mathcal{G}_t$-measurable random state such that

$$\mathcal{I}_t(G)[\rho] = \int_G \rho_t(\omega)\, \mathbb{P}^{\vartheta,h}_{\rho,t}(\mathrm{d}\omega), \qquad \forall G \in \mathcal{G}_t. \tag{34}$$

The definition of a posteriori state is not linked only to measurements in continuous time, but it has been introduced for a generic instrument [41].

As we have a reference probability $\mathbb{Q}$ on the output space $(\Omega, \mathcal{G}_t)$ we can equivalently look for the *non-normalized a posteriori state* σ_t, the unique $\mathcal{G}_t$-measurable random positive operator such that

$$\mathcal{I}_t(G)[\rho] = \int_G \sigma_t(\omega)\, \mathbb{Q}(\mathrm{d}\omega), \qquad \forall G \in \mathcal{G}_t. \tag{35}$$

Then, $\operatorname{Tr}\{\sigma_t\}$ is the probability density of $\mathbb{P}^{\vartheta,h}_{\rho,t}$ with respect to $\mathbb{Q}$ and we have $\rho_t = \sigma_t / \operatorname{Tr}\{\sigma_t\}$.

The non-normalized a posteriori state σ_t can be computed by using the spectral representation (26) of the operators Q_k and its evolution can be obtained by passing through the SSE.

3. *The stochastic Schrödinger equation*

In order to compute the a posteriori state of our instrument $\mathcal{I}_t$ (31), it is convenient to pass through two fictitious instruments: $\hat{\mathcal{J}}_t$, associated to a complete set of compatible observables in Γ, and $\mathcal{J}_t$, associated to a complete set of compatible observables in $\Gamma[(0,t)]$. This latter instrument has the simple a posteriori state (38), whose evolution is given by the SSE (40).

First of all, let us imagine, in the Heisenberg picture, that in the time interval $[0,t]$ we measure all the quadratures $Q_k^{\mathrm{out}}(s) = U_s^\dagger Q_k(s) U_s$, $k = 1,\dots,d$, $s \in [0,t]$, and moreover we conclude the measure by observing at time t also the field observables $\hat{Q}_k(u;t) = U_t^\dagger Q_k(u) U_t$, $k = 1,\dots,d$ and $u < 0$ or $u > t$. This is a family of commuting observables, thanks to (23) and to (19), that implies $Q_k^{\mathrm{out}}(s) = U_t^\dagger Q_k(s) U_t$. Then, the instrument $\hat{\mathcal{J}}_t$ associated to this fictitious measurement is given by an expression analogous to (31). If the system initial state is pure, $\rho = |r\rangle\langle r|$, $r \in \mathcal{H}$, $\|r\| = 1$, then $\forall F \in \mathcal{F}$,

$$\hat{\mathcal{J}}_t(F)[|r\rangle\langle r|] = \operatorname{Tr}_\Gamma \left\{ \big(\mathbb{1}_{\mathcal{H}} \otimes J^{-1} 1_F J\big) |\Psi_t\rangle\langle\Psi_t| \right\}, \tag{36}$$

where

$$\Psi_t = U_t\big(r \otimes e(f)\big).$$

To get the expression (36) we have changed picture and cycled U_t against the state; then we have used the isomorphism J (27) that diagonalizes all the quadratures $Q_k(u)$ (25).

The isomorphism J^{-1} does not involve the space $\mathcal{H}$ and it can be cycled after $\langle\Psi_t|$; in this way we get

$$\hat{\mathcal{J}}_t(F)[|r\rangle\langle r|] = \int_F |\varphi_t(\omega)\rangle\langle\varphi_t(\omega)|\, \mathbb{Q}(\mathrm{d}\omega), \tag{37}$$

where φ_t is the random $\mathcal{H}$-vector

$$\varphi_t = J\Psi_t = J\, U_t\Big(r \otimes e(f)\Big).$$

By comparing Eq. (37) with Eq. (35), we get that the non-normalized a posteriori state $\sigma_t^{\hat{\mathcal{J}}}$ associated to the instrument $\hat{\mathcal{J}}$ and to the premeasurement system state $\rho = |r\rangle\langle r|$ is

$$\sigma_t^{\hat{\mathcal{J}}}(\omega) = |\varphi_t(\omega)\rangle\langle\varphi_t(\omega)|.$$

By construction $\sigma_t^{\hat{\mathcal{J}}}$ is a random positive trace-class operator, which is $\mathcal{F}$-measurable.

Suppose now that we measure only the quadratures $Q_k^{\mathrm{out}}(s)$, $0 \leq s \leq t$, $k = 1,\dots,d$; note that this set of compatible observables is complete in $\Gamma[(0,t)]$, not in $\Gamma[\mathbb{R}]$. With respect to the previous case, we simply have to drop some commuting observables and thus the new instrument $\mathcal{J}_t$ is just the restriction of $\hat{\mathcal{J}}_t$ to the σ-algebra $\mathcal{F}_t \subset \mathcal{F}$ and, therefore, the new a posteriori state $\sigma_t^{\mathcal{J}}$ is the conditional expectation

$$\sigma_t^{\mathcal{J}} = \mathbb{E}_{\mathbb{Q}}\big[\sigma_t^{\hat{\mathcal{J}}} \big| \mathcal{F}_t\big],$$

which is an $\mathcal{F}_t$-measurable random positive operator. Thanks to the properties of the Hudson-Parthasarathy equation and to the choice of the observed quadratures (local in $(0,t)$ and no k neglected) the a posteriori state is still almost surely pure:

$$\sigma_t^{\mathcal{J}}(\omega) = |\phi_t(\omega)\rangle\langle\phi_t(\omega)|, \quad \phi_t = J\, U_t\big(r \otimes e(f 1_{(0,t)})\big). \tag{38}$$

a. The linear SSE. It is the $(\mathcal{F}_t)$-adapted stochastic process ϕ_t that satisfies the linear SSE and we show now how to get it.

By introducing the Weyl operators $\mathcal{W}_t := \mathcal{W}(f 1_{(0,t)})$ we can write $e(f 1_{(0,t)}) = \mathcal{W}_t e(0)$ and

$$\phi_t = J\, U_t\, \mathcal{W}_t\big(r \otimes e(0)\big). \tag{39}$$

By using the definition of the Weyl operators and the quantum stochastic calculus it is easy to check that $\mathcal{W}_t$ satisfies a quantum SSE (11) with $R_k = f_k(t)$ and $H_0 = 0$. Then, the quantum stochastic differential of $U_t\mathcal{W}_t$ is given by

$$\mathrm{d}\left(U_t\mathcal{W}_t\right) = \big((\mathrm{d}U_t)\,\mathcal{W}_t + U_t\,\mathrm{d}\mathcal{W}_t + (\mathrm{d}U_t)\,\mathrm{d}\mathcal{W}_t\big).$$

This can be computed by using the quantum Itô table (9) and exploiting that operators localized in disjoint time intervals commute and that the differentials $\mathrm{d}B_k^{(\dagger)}(t)$ are localized in $(t, t+\mathrm{d}t)$ with respect to the factorization (4) of the Fock space; in particular this means that $\mathrm{d}B_k(t)$ and U_t commute and the same holds for $\mathrm{d}B_k(t)$ and $\mathcal{W}_t$. The result is

$$\mathrm{d}\left(U_t\mathcal{W}_t\right) = \bigg\{K\mathrm{d}t + \sum_k \Big[\big(R_k + f_k(t)\big)\mathrm{d}B_k^\dagger(t) - \Big(R_k^\dagger + \overline{f_k(t)}\Big)\,\mathrm{d}B_k(t) - \Big(\tfrac{1}{2}\,|f_k(t)|^2 + f_k(t)R_k^\dagger\Big)\,\mathrm{d}t\Big]\bigg\}U_t\,\mathcal{W}_t.$$

Now, we use this result to compute the differential of ϕ_t (39). The key point is that

$$\big(\mathrm{d}B_k(t)\big)U_t\,\mathcal{W}_t\,(r\otimes e(0)) = U_t\,\mathcal{W}_t\,\mathrm{d}B_k(t)\,(r\otimes e(0)) = 0,$$

so that we can change the coefficient of $\mathrm{d}B_k(t)$ as we wish. By this and the fact that $|\ell_k(t)| = 1$, we can write

$$\mathrm{d}\phi_t = J\bigg\{K\mathrm{d}t + \sum_k \Big[\big(R_k + f_k(t)\big)\overline{\ell_k(t)}\,\mathrm{d}Q_k(t) - \Big(\tfrac{1}{2}\,|f_k(t)|^2 + f_k(t)R_k^\dagger\Big)\,\mathrm{d}t\Big]\bigg\}U_t\,\mathcal{W}_t\big(r\otimes e(0)\big).$$

Finally, by Eqs. (10), (26), and (39) we obtain the linear SSE:

$$\mathrm{d}\phi_t = \bigg\{\sum_k\bigg[\overline{\ell_k(t)}\big(R_k + f_k(t)\big)\mathrm{d}W_k(t) - \frac{1}{2}\Big(R_k^\dagger + \overline{f_k(t)}\Big)\,\big(R_k + f_k(t)\big)\mathrm{d}t\bigg] - \mathrm{i}\bigg[H_0 + \frac{\mathrm{i}}{2}\sum_k\Big(\overline{f_k(t)}\,R_k - f_k(t)R_k^\dagger\Big)\bigg]\mathrm{d}t\bigg\}\phi_t. \tag{40}$$

Therefore, the a posteriori evolution of system S under the continuous measurement $\mathcal{I}_t$ is a stochastic evolution mapping pure states into pure states. In particular, the evolution of the non-normalized pure state ϕ_t is given by the linear SSE (40) and it is Markovian and depends on the interaction (11) between S and Γ, on the field initial state $\varrho_\Gamma(f)$ and on the observed quadratures $Q_k^{\mathrm{out}}(s)$.

It is now possible to show that $\|\phi_T(\omega)\|_{\mathcal{H}}^2\,\mathbb{Q}(\mathrm{d}\omega)$ defines a new probability on $(\Omega, \mathcal{F}_T)$ and that, under this new probability, the process $\psi_t := \phi_t/\,\|\phi_t\|_{\mathcal{H}}$, $t\in[0,T]$, satisfies a nonlinear stochastic differential equation. This last equation is the nonlinear SSE, which is the starting point for useful numerical simulation. A key point in the change of probability is the fact that $\|\phi_t\|_{\mathcal{H}}^2$ is a $\mathbb{Q}$-martingale and that the so called Girsanov transformation can be invoked. For the theory of the linear and nonlinear SSE we refer to [16, Sec. 2].

4. The stochastic master equation

By Itô calculus, from the SSE (40) we get the stochastic equation satisfied by the random operator $\sigma_t^{\mathcal{J}}$ (38):

$$\begin{aligned}\mathrm{d}\sigma_t^{\mathcal{J}} =& \mathcal{L}(t)[\sigma_t^{\mathcal{J}}]\,\mathrm{d}t + \sum_k\Big\{\overline{\ell_k(t)}\big(R_k + f_k(t)\big)\sigma_t^{\mathcal{J}} \\ &+ \sigma_t^{\mathcal{J}}\,\ell_k(t)\,\Big(R_k^\dagger + \overline{f_k(t)}\Big)\Big\}\mathrm{d}W_k(t),\end{aligned} \tag{41}$$

where $\mathcal{L}(t)$ is the Liouville operator (15).

We can now get rid of the hypothesis of a pure initial state and prove that Eq. (41) gives the a posteriori evolution for a generic system initial state

$$\rho = \sum_\ell p_\ell|r_\ell\rangle\langle r_\ell|, \quad \|r_\ell\|_{\mathcal{H}} = 1, \quad p_\ell > 0, \quad \sum_\ell p_\ell = 1.$$

Indeed, if we set

$$\phi_t^\ell = J\,U_t\,\mathcal{W}_t\big(r_\ell\otimes e(0)\big),$$

then, by linearity the process

$$\sigma_t^{\mathcal{J}}(\omega) = \sum_\ell p_\ell|\varphi_t^\ell(\omega)\rangle\langle\varphi_t^\ell(\omega)|$$

is adapted, satisfies Eq. (41), and gives the a posteriori state of $\mathcal{J}_t$,

$$\mathcal{J}_t(F)[\rho] = \int_F \sigma_t^{\mathcal{J}}(\omega)\,\mathbb{Q}(\mathrm{d}\omega), \qquad \forall F \in \mathcal{F}_t. \tag{42}$$

Finally, we consider our instrument $\mathcal{I}_t$, which is the restriction of $\mathcal{J}_t$ to $\mathcal{G}_t$, so that Eq. (35), defining the non-normalized a posteriori states, holds with

$$\sigma_t = \mathbb{E}_{\mathbb{Q}}\big[\sigma_t^{\mathcal{J}}\big|\mathcal{G}_t\big]. \tag{43}$$

Let us recal that $\ell_1(t) = \mathrm{e}^{-\mathrm{i}\vartheta}h(t)$, that the $\mathbb{Q}$-mean of any $W_k(s)$ is zero, that $\mathcal{G}_t$ is generated by W_1 and that the other components of the Wiener process are independent from the first one. Then, by applying the conditional expectation with respect to $\mathcal{G}_t$ to (41), we obtain the linear SME for the non-normalized a posteriori states of $\mathcal{I}_t$

$$\begin{aligned}\mathrm{d}\sigma_t =&\mathcal{L}(t)[\sigma_t]\,\mathrm{d}t + \Big\{\mathrm{e}^{\mathrm{i}\vartheta}\overline{h(t)}\big(R_1 + f_1(t)\big)\sigma_t \\ &+ \sigma_t\,\mathrm{e}^{-\mathrm{i}\vartheta}h(t)\left(R_1^\dagger + \overline{f_1(t)}\right)\Big\}\mathrm{d}W_1(t).\end{aligned} \tag{44}$$

As already said at the end of Sec. III C 2, the quantity $\mathrm{Tr}_{\mathcal{H}}\{\sigma_t(\omega)\}$ is the density of the physical probability with respect to $\mathbb{Q}$. Indeed, from Eqs. (35) and (32) we get

$$\mathbb{P}^{\vartheta,h}_{\rho,t}(G) = \int_G \mathrm{Tr}_{\mathcal{H}}\{\sigma_t(\omega)\}\,\mathbb{Q}(\mathrm{d}\omega), \qquad \forall G \in \mathcal{G}_t. \tag{45}$$

By taking the trace of the SME (44), it is possible to show that $\mathrm{Tr}_{\mathcal{H}}\{\sigma_t\}$ is a $\mathbb{Q}$-martingale. In the stochastic formulation it is just this fact that implies the consistency property (30), which we already encountered in the Fock space formulation.

As already seen, if we define $\rho_t = \sigma_t/\mathrm{Tr}_{\mathcal{H}}\{\sigma_t\}$, we get the a posteriori state for the instrument $\mathcal{I}_t$ and the pre-measurement state ρ. By (33), the a posteriori states are related to the system reduced state by

$$\int_\Omega \rho_t(\omega)\,\mathbb{P}^{\vartheta,h}_{\rho,t}(\mathrm{d}\omega) = \int_\Omega \sigma_t(\omega)\,\mathbb{Q}(\mathrm{d}\omega) = \eta_t. \tag{46}$$

It is also possible to prove that, under the physical probability, ρ_t satisfies a nonlinear SME. Moreover, by studying the stochastic differential of $\mathrm{Tr}_{\mathcal{H}}\{\sigma_t\}$ and by using Girsanov theorem, it is possible to prove the following result.

a. The noise. Under the physical probability $\mathbb{P}^{\vartheta,h}_{\rho,T}$ the process

$$\widehat{W}_1(t) := W_1(t) - 2\,\mathrm{Re}\int_0^t \mathrm{e}^{\mathrm{i}\vartheta}h(s)\,\mathrm{Tr}_{\mathcal{H}}\left\{\big(R_1 + f_1(s)\big)\rho_s\right\}\mathrm{d}s$$

is a standard Wiener process for $t \in [0, T]$.

In other terms we can say that, under the physical probability, the instantaneous output $I(t) = \dot{W}_1(t)$ is the sum of a white noise $\mathrm{d}\widehat{W}_1(t)/\mathrm{d}t$ plus a regular signal $2\,\mathrm{Re}\,\mathrm{Tr}_{\mathcal{H}}\left\{\big(R_1 + f_1(t)\big)\rho_t\right\}$. White noise and signal turn out to be correlated in general. Let us stress that this result on the structure of the output is a byproduct of the stochastic representation of the continuous measurements. From this representation and Eq. (46) the mean value of the observed quadrature at time t is

$$\begin{aligned}\mathrm{Tr}\left\{\big(\mathbb{1}_{\mathcal{H}} \otimes Q(t;\vartheta,h)\big)\Sigma_f(t)\right\} &= \int_\Omega W_1(t;\omega)\,\mathbb{P}^{\vartheta,h}_{\rho,t}(\mathrm{d}\omega) \\ &= 2\,\mathrm{Re}\int_0^t \mathrm{e}^{\mathrm{i}\vartheta}h(s)\,\mathrm{Tr}_{\mathcal{H}}\left\{\big(R_1 + f_1(s)\big)\eta_s\right\}\mathrm{d}s.\end{aligned} \tag{47}$$

The full theory of the linear and nonlinear SME's and their relations with the physical probability and the reference probability $\mathbb{Q}$ are presented in [16, Secs. 3 and 5].

D. Characteristic functional and moments

In the study of stochastic processes it is often useful to have explicit formulae for the moments, for instance for the second order moments, which determine the spectrum of the process; see Sec. IV A. In the case of our output, the mean function is given by Eq. (47); to get the higher moments it is useful to introduce the *characteristic functional*, which is the functional Fourier transform of the probability distribution of the process.

Let us denote by $\mathbb{E}^{\vartheta,h}_{\rho,t}$ the expectation with respect to the physical probability $\mathbb{P}^{\vartheta,h}_{\rho,t}$. By recalling that the output is represented by W_1, the characteristic functional up to time $t > 0$ is

$$\Phi_t(k;\vartheta,h) = \mathbb{E}^{\vartheta,h}_{\rho,t}\left[\exp\Big\{\mathrm{i}\int_0^t k(s)\,\mathrm{d}W_1(s)\Big\}\right]; \tag{48}$$

the argument k is any real test function in $L^\infty(\mathbb{R}_+)$.

By functional differentiation with respect to the test function one gets all the moments of the process, as done in Sec. III D 2. Moreover, when the characteristic functional is given, one gets the probabilities by anti-Fourier transform. For instance, the finite-dimensional probability densities of the increments $W_1(t_1) - W_1(t_0)$, $W_1(t_2) - W_1(t_1)$, …, $W_1(t_n) - W_1(t_{n-1})$, with $0 \le t_0 < t_1 < \cdots < t_n \le t$, are given by

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \mathrm{d}\kappa_1\cdots\mathrm{d}\kappa_n\left(\prod_{j=1}^n \mathrm{e}^{-\mathrm{i}\kappa_j\cdot x_j}\right)\Phi_t(k;\vartheta,h),$$

where we have introduced the test function $k(s) = \sum_{j=1}^n 1_{(t_{j-1},t_j)}(s)\,\kappa_j$.

1. Characteristic operators

As our measurement is an indirect observation of S performed by a direct observation of Γ, the characteristic functional (48) can be expressed either in terms of the system S only or in terms of the fields only. First, we want to express it in terms of the quantum observables (21). We introduce the *characteristic operator* $\widehat{\Phi}_t(k;\vartheta,h)$, the Fourier transform of the pvm E^h_ϑ [21, Sec. 11.4.2], [22, Sec. 3.2]:

$$\widehat{\Phi}_t(k;\vartheta,h)=\int_\Omega \exp\left\{\mathrm{i}\left(\int_0^t k(s)\,\mathrm{d}W_1(s)\right)(\omega)\right\}E^h_\vartheta(\mathrm{d}\omega). \tag{49}$$

By using the representation (28) of the pvm, the correspondence between Q and W_1 (26), and the definition of Q (21), we get

$$\begin{aligned}\widehat{\Phi}_t(k;\vartheta,h)&=J^{-1}\exp\left\{\mathrm{i}\int_0^t k(s)\,\mathrm{d}W_1(s)\right\}J\\&=\exp\left\{\mathrm{i}\int_0^t k(s)\,\mathrm{d}Q(s;\vartheta,h)\right\}\\&=\exp\left\{\mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}\int_0^t k(s)h(s)\,\mathrm{d}B_1^\dagger(s)-\text{h.c.}\right\}.\end{aligned} \tag{50}$$

By comparing this expression with Eq. (6), we see that the characteristic operator is the unitary Weyl operator

$$\widehat{\Phi}_t(k;\vartheta,h)=\mathcal{W}(q),\qquad q_j(s)=\delta_{j1}\mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}k(s)h(s)1_{[0,t]}(s).$$

Then, by using the expression (29) of the physical probability, the characteristic functional can be written as

$$\begin{aligned}\Phi_t(k;\vartheta,h)&=\mathrm{Tr}\left\{\widehat{\Phi}_t(k;\vartheta,h)\Sigma_f(t)\right\}\\&=\mathrm{Tr}\left\{\exp\left[\mathrm{i}\int_0^t k(s)\,\mathrm{d}Q^{\mathrm{out}}(s;\vartheta,h)\right]\rho\otimes\varrho_\Gamma(f)\right\}\\&=\mathrm{Tr}_\Gamma\left\{\widehat{\Phi}_t(k;\vartheta,h)\Pi_f(t)\right\}.\end{aligned} \tag{51}$$

The last step is due to the fact that $\widehat{\Phi}_t(k;\vartheta,h)$ depends only on field operators; recall that $\Sigma_f(t)$ is the state of the total system S plus fields, while $\Pi_f(t)$ is the reduced state of the fields (13).

Finally, let us define the *reduced characteristic operator* $\mathcal{G}_t$ as the functional Fourier transform of the instrument (31) [3, 4, 16, 22]:

$$\mathcal{G}_t(k;\vartheta,h)=\int_\Omega \exp\left\{\mathrm{i}\left(\int_0^t k(s)\,\mathrm{d}W_1(s)\right)(\omega)\right\}\mathcal{I}_t(\mathrm{d}\omega). \tag{52}$$

It can be shown that $\mathcal{G}_t$ satisfies a closed differential equation, a kind of modification of the master equation [4]. Then, by the representation (32) of the physical probabilities, we get a further expression of the characteristic functional:

$$\Phi_t(k;\vartheta,h)=\mathrm{Tr}_{\mathcal{H}}\left\{\mathcal{G}_t(k;\vartheta,h)[\rho]\right\}.$$

2. *The output moments*

By functional differentiation of the characteristic functional, we get all the moments of the classical output process. Let us introduce the formal time derivatives $I(t)=\dot{W}_1(t)$ and $\hat{I}(t)=\dot{Q}(t;\vartheta,h)$; from (50) and (51) we obtain immediately the expressions of mean function and autocorrelation function:

$$\begin{aligned}\mathbb{E}^{\vartheta,h}_{\rho,T}[I(t)]&=\mathrm{Tr}\left\{\dot{Q}(t;\vartheta,h)\Sigma_f(T)\right\}=\mathrm{Tr}_\Gamma\left\{\hat{I}(t)\Pi_f(T)\right\}\\&=2\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\vartheta}\,\overline{h(t)}\,\mathrm{Tr}_\Gamma\left\{b_1(t)\,\Pi_f(T)\right\}\right),\end{aligned} \tag{53a}$$

$$\begin{aligned}\mathbb{E}^{\vartheta,h}_{\rho,T}[I(t)I(s)]&=\mathrm{Tr}_\Gamma\left\{\hat{I}(t)\hat{I}(s)\Pi_f(T)\right\}\\&=\delta(t-s)+2\,\mathrm{Re}\Big(\overline{h(s)}\,\mathrm{Tr}_\Gamma\Big\{\Big(h(t)b_1^\dagger(t)\\&\quad+\mathrm{e}^{2\mathrm{i}\vartheta}\,\overline{h(t)}\,b_1(t)\Big)b_1(s)\,\Pi_f(T)\Big\}\Big),\end{aligned} \tag{53b}$$

where $T>t$, $T>s$. Analogous formulae hold for higher moments. Let us note that the order of the operators $\hat{I}(t)$ and $\hat{I}(s)$ in (53b) does not matter, because they commute. Moreover, the moments of the classical process $I(t)$ are expressed in terms of quantum means and quantum correlations of the fields [21, p. 165 and Sec. 11.3.2]: $\mathrm{Tr}_\Gamma\left\{b_1(t)\,\Pi_f(T)\right\}$, $\mathrm{Tr}_\Gamma\left\{b_1^\dagger(t)b_1(s)\,\Pi_f(T)\right\}$, $\mathrm{Tr}_\Gamma\left\{b_1(t)b_1(s)\,\Pi_f(T)\right\}$, and the complex conjugated expressions. The fields are all in normal order because we put in evidence the delta term coming out from a commutator.

By studying the properties of the reduced characteristic operator (52), it is possible to prove that all the moments of our classical output can be expressed by means of quantities concerning only system S [22]. For the mean and autocorrelation functions the final result is [22, Secs. 3.3, 3.5]

$$\mathbb{E}^{\vartheta,h}_{\rho,T}[I(t)]=2\,\mathrm{Re}\left(\mathrm{Tr}_{\mathcal{H}}\left\{Z(t)\eta_t\right\}\right), \tag{54a}$$

$$\begin{aligned}\mathbb{E}^{\vartheta,h}_{\rho,T}[I(t)I(s)]&=\delta(t-s)+2\,\mathrm{Re}\big(\mathrm{Tr}_{\mathcal{H}}\big\{Z(t_2)\\&\quad\times\Upsilon(t_2,t_1)\left[Z(t_1)\eta_{t_1}+\eta_{t_1}Z(t_1)^\dagger\right]\big\}\big),\end{aligned} \tag{54b}$$

where $t_2=t\vee s$, $t_1=t\wedge s$ and

$$Z(t):=\mathrm{e}^{\mathrm{i}\vartheta}\,\overline{h(t)}\,\left(R_1+f_1(t)\right).$$

These expressions are more useful for computations, while the expressions (53) are better suited for theoretical considerations; cf. [21, Sec. 5.4.6].

IV. THE SPECTRUM OF THE OUTPUT

Inside the theory of continuous measurements, the output of the measurement is a classical stochastic process, whose distribution is determined by quantum mechanics; so, the spectrum of the output can be introduced by using the classical definition of spectrum of a stochastic process [42–44].

A. The spectrum of a stationary process

In the classical theory of stochastic processes, the spectrum is related to the Fourier transform of the autocorrelation function [45]. Let Y be a stationary real stochastic process with finite moments; then, the mean is independent of time $\mathbb{E}[Y(t)] = \mathbb{E}[Y(0)] =: m_Y$, $\forall t \in \mathbb{R}$, and the second moment is invariant under time translations: $\forall t, s \in \mathbb{R}$,

$$\mathbb{E}[Y(t)Y(s)] = \mathbb{E}[Y(t-s)Y(0)] =: R_Y(t-s). \qquad (55)$$

The function $R_Y(t)$, $t \in \mathbb{R}$, is called the *autocorrelation function* of the process. Obviously, we have $\operatorname{Cov}[Y(t), Y(s)] = R_Y(t-s) - m_Y^2$.

The *spectrum* of the stationary stochastic process Y is the Fourier transform of its autocorrelation function:

$$S_Y(\mu) := \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\mu t} R_Y(t)\, \mathrm{d}t. \qquad (56)$$

This formula has to be intended in the sense of distributions. If $\operatorname{Cov}[Y(t), Y(0)] \in L^1(\mathbb{R})$, we can write

$$S_Y(\mu) := 2\pi m_Y^2 \delta(\mu) + \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\mu t} \operatorname{Cov}[Y(t), Y(0)]\, \mathrm{d}t. \qquad (57)$$

By the properties of the covariance, the function $\operatorname{Cov}[Y(t), Y(0)]$ is positive definite and, by the properties of positive definite functions, this implies $\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\mu t} \operatorname{Cov}[Y(t), Y(0)]\, \mathrm{d}t \geq 0$; then, also $S_Y(\mu) \geq 0$.

By using the stationarity and some tricks on multiple integrals, one can check that an alternative expression of the spectrum is

$$S_Y(\mu) = \lim_{T\to+\infty} \frac{1}{T}\, \mathbb{E}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu t} Y(t)\, \mathrm{d}t\right|^2\right]. \qquad (58)$$

The advantage now is that the positivity of the spectrum appears explicitly and only positive times are involved [45]. Expression (58) can be generalized also to processes which are stationary only in some asymptotic sense and to singular processes as our $I(t)$.

B. The spectrum of the output in a finite time horizon

Let us consider our output $I(t) = \mathrm{d}W_1(t)/\mathrm{d}t$ under the physical probability $\mathbb{P}_{\rho,T}^{\vartheta,h}$. We call "spectrum up to time T" of $I(t)$ the quantity

$$S_T(\mu;\vartheta) = \frac{1}{T}\, \mathbb{E}_{\rho,T}^{\vartheta,h}\left[\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu t}\, \mathrm{d}W_1(t)\right|^2\right]. \qquad (59)$$

Note that the spectrum is an even function of μ: $S_T(\mu;\vartheta) = S_T(-\mu;\vartheta)$. When the limit $T \to +\infty$ exists, we can speak of *spectrum of the output,* but this existence depends on the specific properties of the concrete model.

By writing the second moment defining the spectrum as the square of the mean plus the variance, the spectrum splits in an elastic or coherent part and in an inelastic or incoherent one:

$$S_T(\mu;\vartheta) = S_T^{\mathrm{el}}(\mu;\vartheta) + S_T^{\mathrm{inel}}(\mu;\vartheta), \qquad (60a)$$

$$S_T^{\mathrm{el}}(\mu;\vartheta) = \frac{1}{T}\left|\mathbb{E}_{\rho,T}^{\vartheta,h}\left[\int_0^T \mathrm{e}^{\mathrm{i}\mu t}\, \mathrm{d}W_1(t)\right]\right|^2, \qquad (60b)$$

$$S_T^{\mathrm{inel}}(\mu;\vartheta) = \frac{1}{T}\operatorname{Var}_{\rho,T}^{\vartheta,h}\left[\int_0^T \cos\mu t\, \mathrm{d}W_1(t)\right] + \frac{1}{T}\operatorname{Var}_{\rho,T}^{\vartheta,h}\left[\int_0^T \sin\mu t\, \mathrm{d}W_1(t)\right]. \qquad (60c)$$

Let us note that

$$S_T^{\mathrm{el}}(\mu;\vartheta) = S_T^{\mathrm{el}}(-\mu;\vartheta), \quad S_T^{\mathrm{inel}}(\mu;\vartheta) = S_T^{\mathrm{inel}}(-\mu;\vartheta). \qquad (61)$$

In Sec. III D we have seen two ways of expressing the output moments, by means of field operators or by means of system operators.

By using the expression (53b) for the autocorrelation function of the output, we obtain

$$S_T(\mu;\vartheta) = 1 + \frac{2}{T}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}s\, \mathrm{e}^{\mathrm{i}\mu(t-s)}\, \operatorname{Re}\Big(\overline{h(s)} \times \operatorname{Tr}_\Gamma\Big\{\Big(h(t)b_1^\dagger(t) + \mathrm{e}^{2\mathrm{i}\vartheta}\, \overline{h(t)}\, b_1(t)\Big)\, b_1(s)\, \Pi_f(T)\Big\}\Big). \qquad (62)$$

Let us note that Eq. (62) expresses the spectrum as a Fourier transform (in a finite time interval) of a normal ordered quantum correlation function of the field; cf. [15, Sec. 9.3.2].

By using the expressions (54) for the first two moments we get the spectrum in a form which involves only system operators:

$$S_T^{\mathrm{el}}(\mu;\vartheta) = \frac{1}{T}\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu t}\, \operatorname{Tr}_{\mathcal{H}}\left\{\left(Z(t) + Z(t)^\dagger\right)\eta_t\right\}\mathrm{d}t\right|^2 = \frac{4}{T}\left|\int_0^T \mathrm{e}^{\mathrm{i}\mu t}\, \operatorname{Re}\left[\mathrm{e}^{\mathrm{i}\vartheta}\, \overline{h(t)}\big(\operatorname{Tr}_{\mathcal{H}}\{R_1\eta_t\} + f_1(t)\big)\right]\mathrm{d}t\right|^2, \qquad (63a)$$

$$S_T^{\mathrm{inel}}(\mu;\vartheta) = 1 + \frac{2}{T}\int_0^T \mathrm{d}t\int_0^t \mathrm{d}s\, \cos\mu(t-s) \times \operatorname{Tr}_{\mathcal{H}}\left\{\left(\tilde{Z}(t) + \tilde{Z}(t)^\dagger\right)\Upsilon(t,s)\left[\tilde{Z}(s)\eta_s + \eta_s\tilde{Z}(s)^\dagger\right]\right\}, \qquad (63b)$$

$$\tilde{Z}(t) = Z(t) - \operatorname{Tr}_{\mathcal{H}}\{Z(t)\eta_t\} = \mathrm{e}^{\mathrm{i}\vartheta}\, \overline{h(t)}\big(R_1 - \operatorname{Tr}_{\mathcal{H}}\{R_1\eta_t\}\big).$$

C. Properties of the spectrum and the Heisenberg uncertainty relations

Equations (63) give the spectrum in terms of the reduced description of system S (the fields are traced out); this is useful for concrete computations. But the general properties of the spectrum are more easily obtained by working with the fields; so, here we trace out system S and we start from expression (62).

Let us define the field operators

$$Q_T(\mu;\vartheta) = \frac{1}{\sqrt{T}} \int_0^T \mathrm{e}^{\mathrm{i}\mu t}\, \mathrm{d}Q(t;\vartheta,h), \tag{64a}$$

$$\tilde{Q}_T(\mu;\vartheta) = Q_T(\mu;\vartheta) - \operatorname{Tr}_\Gamma\left\{\Pi_f(T) Q_T(\mu;\vartheta)\right\}; \tag{64b}$$

the local oscillator wave h is fixed. Let us stress that $Q_T(\mu;\vartheta)$ commutes with its adjoint and that $Q_T(\mu;\vartheta)^\dagger = Q_T(-\mu;\vartheta)$. By using Eqs. (51) and (53) and taking first the trace over $\mathcal{H}$, we get

$$S_T(\mu;\vartheta) = \operatorname{Tr}_\Gamma\left\{\Pi_f(T) Q_T(\mu;\vartheta)^\dagger Q_T(\mu;\vartheta)\right\} \geq 0, \tag{65a}$$

$$S_T^{\mathrm{el}}(\mu;\vartheta) = \left|\operatorname{Tr}_\Gamma\left\{\Pi_f(T) Q_T(\mu;\vartheta)\right\}\right|^2 \geq 0, \tag{65b}$$

$$S_T^{\mathrm{inel}}(\mu;\vartheta) = \operatorname{Tr}_\Gamma\left\{\Pi_f(T) \tilde{Q}_T(\mu;\vartheta)^\dagger \tilde{Q}_T(\mu;\vartheta)\right\} \geq 0. \tag{65c}$$

1. *Spectrum and field modes*

To elaborate the previous expressions it is useful to introduce annihilation and creation operators for bosonic temporal modes, as in Sec. II A 3:

$$a_T(\mu) := \frac{1}{\sqrt{T}} \int_0^T \mathrm{e}^{\mathrm{i}\mu t}\overline{h(t)}\, \mathrm{d}B_1(t) \equiv c_1(g_T^\mu), \tag{66}$$

$$g_T^\mu(t) := \frac{\mathrm{e}^{-\mathrm{i}\mu t}}{\sqrt{T}}\, h(t) 1_{[0,T]}(t). \tag{67}$$

The operators $a_T(\mu)$, $a_T^\dagger(\mu)$ are true bosonic modes, as they satisfy the CCR

$$[a_T(\mu), a_T(\mu')] = [a_T^\dagger(\mu), a_T^\dagger(\mu')] = 0, \tag{68a}$$

$$[a_T(\mu), a_T^\dagger(\mu)] = 1. \tag{68b}$$

However, for finite T these modes are only approximately orthogonal, as we get

$$[a_T(\mu), a_T^\dagger(\mu')] = \frac{\mathrm{e}^{\mathrm{i}(\mu-\mu')T} - 1}{\mathrm{i}(\mu-\mu')T} \quad \text{for} \quad \mu' \neq \mu. \tag{69}$$

Then, from Eqs. (21), (64a), (65a) we have easily

$$Q_T(\mu;\vartheta) = \mathrm{e}^{\mathrm{i}\vartheta} a_T(\mu) + \mathrm{e}^{-\mathrm{i}\vartheta} a_T^\dagger(-\mu), \tag{70}$$

$$S_T(\mu;\vartheta) = \operatorname{Tr}_\Gamma\Big\{\Big(\mathrm{e}^{-\mathrm{i}\vartheta} a_T^\dagger(-\mu) + \mathrm{e}^{\mathrm{i}\vartheta} a_T(\mu)\Big) \times \Pi_f(T)\left(\mathrm{e}^{-\mathrm{i}\vartheta} a_T^\dagger(\mu) + \mathrm{e}^{\mathrm{i}\vartheta} a_T(-\mu)\right)\Big\}. \tag{71}$$

Let us stress that only two field modes contribute to the spectrum for $\mu \neq 0$, and only one mode in the case of $\mu = 0$. By using the CCR (68) we get the normal ordered version of (71):

$$S_T(\mu;\vartheta) = 1 + \operatorname{Tr}_\Gamma\Big\{\Pi_f(T)\Big(a_T^\dagger(\mu) a_T(\mu) + a_T^\dagger(-\mu) a_T(-\mu) + \mathrm{e}^{-2\mathrm{i}\vartheta} a_T^\dagger(\mu) a_T^\dagger(-\mu) + \mathrm{e}^{2\mathrm{i}\vartheta} a_T(-\mu) a_T(\mu)\Big)\Big\}. \tag{72}$$

Note that Eq. (69) played no role in the normal ordering operation.

By Eqs. (65b) and (70) we get for the elastic part of the spectrum

$$S_T^{\mathrm{el}}(\mu;\vartheta) = \Big|\operatorname{Tr}_\Gamma\left\{\Pi_f(T) a_T(\mu)\right\} + \mathrm{e}^{-2\mathrm{i}\vartheta}\operatorname{Tr}_\Gamma\left\{\Pi_f(T) a_T^\dagger(-\mu)\right\}\Big|^2. \tag{73}$$

To obtain a similar expression also for the inelastic part, it is convenient to introduce the operators

$$\tilde{a}_T(\mu) := a_T(\mu) - \operatorname{Tr}_\Gamma\left\{\Pi_f(T) a_T(\mu)\right\}, \tag{74}$$

which satisfy the same commutation relations (68), (69) as the operators $a_T(\mu)$ and their adjoint. Then, we get

$$\tilde{Q}_T(\mu;\vartheta) = \mathrm{e}^{\mathrm{i}\vartheta} \tilde{a}_T(\mu) + \mathrm{e}^{-\mathrm{i}\vartheta} \tilde{a}_T^\dagger(-\mu), \tag{75}$$

$$S_T^{\mathrm{inel}}(\mu;\vartheta) = 1 + \operatorname{Tr}_\Gamma\Big\{\Pi_f(T)\Big(\tilde{a}_T^\dagger(\mu)\tilde{a}_T(\mu) + \tilde{a}_T^\dagger(-\mu)\tilde{a}_T(-\mu) + \mathrm{e}^{-2\mathrm{i}\vartheta}\tilde{a}_T^\dagger(\mu)\tilde{a}_T^\dagger(-\mu) + \mathrm{e}^{2\mathrm{i}\vartheta}\tilde{a}_T(-\mu)\tilde{a}_T(\mu)\Big)\Big\}. \tag{76}$$

2. *Spectra of complementary quadratures*

Let us consider two choices of the phase ϑ: ϑ and $\vartheta \pm \pi/2$. From Eq. (23) we get

$$[Q(t;\vartheta,h), Q(s;\vartheta \pm \pi/2,h)] = \mp 2\mathrm{i}\,(t \wedge s),$$

which means that we are considering two incompatible field quadratures, measured by two different setups. For these quadratures we have the important bounds and relations given in the following theorem.

Theorem 1. *For every ϑ and μ we have the following relations:*

$$\frac{1}{2}\big(S_T^{\mathrm{el}}(\mu;\vartheta) + S_T^{\mathrm{el}}(\mu;\vartheta \pm \tfrac{\pi}{2})\big) = \left|\operatorname{Tr}_\Gamma\left\{\Pi_f(T) a_T(\mu)\right\}\right|^2 + \left|\operatorname{Tr}_\Gamma\left\{\Pi_f(T) a_T(-\mu)\right\}\right|^2, \tag{77a}$$

$$\frac{1}{2}\big(S_T^{\mathrm{inel}}(\mu;\vartheta) + S_T^{\mathrm{inel}}(\mu;\vartheta \pm \tfrac{\pi}{2})\big) = 1 + \operatorname{Tr}_\Gamma\Big\{\Pi_f(T)\Big(\tilde{a}_T^\dagger(\mu)\tilde{a}_T(\mu) + \tilde{a}_T^\dagger(-\mu)\tilde{a}_T(-\mu)\Big)\Big\}, \tag{77b}$$

$$\sqrt{S_T^{\rm inel}(\mu;\vartheta)S_T^{\rm inel}(\mu;\vartheta\pm\tfrac{\pi}{2})}\geq 1 + \left|\operatorname{Tr}_\Gamma\left\{\Pi_f(T)\left(\tilde a_T^\dagger(\mu)\tilde a_T(\mu)-\tilde a_T^\dagger(-\mu)\tilde a_T(-\mu)\right)\right\}\right|. \tag{78}$$

Then, independently of the system state ρ, of the field state $\varrho_\Gamma(f)$, of the function h and of the Hudson-Parthasarathy evolution U, the following two important bounds hold:

$$S_T^{\rm inel}(\mu;\vartheta)S_T^{\rm inel}(\mu;\vartheta\pm\tfrac{\pi}{2})\geq 1, \tag{79}$$

$$\frac{1}{2}\left(S_T^{\rm inel}(\mu;\vartheta)+S_T^{\rm inel}(\mu;\vartheta\pm\tfrac{\pi}{2})\right)\geq 1. \tag{80}$$

Proof. First of all from Eqs. (73), (76) we get Eqs. (77). Then, the bound (80) comes immediately from Eq. (77b).

The bound (79) is a trivial consequence of Eq. (78).

To prove the bound (78), we write

$$S_T^{\rm inel}(\mu;\vartheta)=\operatorname{Tr}_\Gamma\Big\{\left(\mathrm{e}^{-\mathrm{i}\vartheta}\tilde a_T^\dagger(-\mu)+\mathrm{e}^{\mathrm{i}\vartheta}\tilde a_T(\mu)\right)\times\Pi_f(T)\left(\mathrm{e}^{-\mathrm{i}\vartheta}\tilde a_T^\dagger(\mu)+\mathrm{e}^{\mathrm{i}\vartheta}\tilde a_T(-\mu)\right)\Big\}.$$

The usual tricks to derive the Heisenberg-Scrödinger-Robertson uncertainty relations can be generalized also to non-selfadjoint operators [8]. For any choice of the state ϱ and of the operators X_1, X_2 (with finite second moments with respect to ϱ) the 2×2 matrix with elements $\operatorname{Tr}\left\{X_i\varrho X_j^\dagger\right\}$ is positive definite and, in particular, its determinant is not negative. Then, we have

$$\operatorname{Tr}\left\{X_1\varrho X_1^\dagger\right\}\operatorname{Tr}\left\{X_2\varrho X_2^\dagger\right\}\geq\left|\operatorname{Tr}\left\{X_1\varrho X_2^\dagger\right\}\right|^2 \geq\left|\operatorname{Im}\operatorname{Tr}\left\{X_1\varrho X_2^\dagger\right\}\right|^2=\frac{1}{4}\left|\operatorname{Tr}\left\{\varrho\left(X_2^\dagger X_1-X_1^\dagger X_2\right)\right\}\right|^2.$$

By taking $\varrho=\Pi_f(T)$,

$$X_1=\mathrm{e}^{-\mathrm{i}\vartheta}\tilde a_T^\dagger(-\mu)+\mathrm{e}^{\mathrm{i}\vartheta}\tilde a_T(\mu),$$

$$X_2=\mp\mathrm{i}\mathrm{e}^{-\mathrm{i}\vartheta}\tilde a_T^\dagger(-\mu)\pm\mathrm{i}\mathrm{e}^{\mathrm{i}\vartheta}\tilde a_T(\mu),$$

we get

$$S_T^{\rm inel}(\mu;\vartheta)S_T^{\rm inel}(\mu;\vartheta\pm\tfrac{\pi}{2}) \geq\left|1+\operatorname{Tr}_\Gamma\left\{\Pi_f(T)\left(\tilde a_T^\dagger(-\mu)\tilde a_T(-\mu)-\tilde a_T^\dagger(\mu)\tilde a_T(\mu)\right)\right\}\right|^2.$$

But we can change μ in $-\mu$ and we have also

$$S_T^{\rm inel}(\mu;\vartheta)S_T^{\rm inel}(\mu;\vartheta\pm\tfrac{\pi}{2})=S_T^{\rm inel}(-\mu;\vartheta)S_T^{\rm inel}(-\mu;\vartheta\pm\tfrac{\pi}{2}) \geq\left|1+\operatorname{Tr}_\Gamma\left\{\Pi_f(T)\left(\tilde a_T^\dagger(\mu)\tilde a_T(\mu)-\tilde a_T^\dagger(-\mu)\tilde a_T(-\mu)\right)\right\}\right|^2.$$

The two inequalities together give the final result (78). □

Equations (77) express the independence from ϑ of the arithmetic mean, in both cases of elastic and inelastic spectra. Equation (78) is a bound of Robertson type; such a bound does not depend on ϑ, but it is still dependent on the initial state and on the dynamics. The Heisenberg-type relation (79) and the bound (80) are fully independent of the initial state and of the dynamics.

One speaks of *squeezed field* or of the *spectrum of squeezing* [15, Sec. 9.3.2] if, at least in a region of the μ line, for some ϑ one has $S_T^{\rm inel}(\mu;\vartheta)<1$. If this happens, the bounds (79) and (80) say that necessarily $S_T^{\rm inel}(\mu;\vartheta+\frac{\pi}{2})>1$ in such a way that the product and the arithmetic mean are bigger than one. Note that, with our choice of the environment initial state, any possible squeezing can be imputed to the interaction with S. Indeed, in the case of no interaction, the output W_1 is a Wiener process plus a deterministic drift and $S_T^{\rm inel}(\mu,\vartheta)\equiv 1$.

V. HOMODYNING VERSUS HETERODYNING OF THE FLUORESCENCE LIGHT OF A TWO-LEVEL ATOM

Let us take as system S a two-level atom, which means $\mathcal{H}=\mathbb{C}^2$, $H_0=\frac{\nu_0}{2}\,\sigma_z$; $\nu_0>0$ is the *resonance frequency* of the atom. We denote by σ_- and σ_+ the lowering and rising operators and by $\sigma_x=\sigma_-+\sigma_+$, $\sigma_y=\mathrm{i}(\sigma_--\sigma_+)$, $\sigma_z=\sigma_+\sigma_--\sigma_-\sigma_+$ the Pauli matrices; we set also

$$\sigma_\vartheta=\mathrm{e}^{\mathrm{i}\vartheta}\,\sigma_-+\mathrm{e}^{-\mathrm{i}\vartheta}\,\sigma_+,\qquad P_\pm=\sigma_\pm\sigma_\mp.$$

The atom can absorb and emit light and it is stimulated by a laser; some thermal environment can be present too. The quantum fields Γ model the whole environment.

a. The absorption/emission terms. The electromagnetic field is split in two fields, according to the direction of propagation: one field for the photons in the forward direction ($k=2$), that of the stimulating laser and of the lost light, one field for the photons collected by the detector ($k=1$). In the rotating wave approximation we can take

$$R_1=\sqrt{\gamma p}\,\sigma_-\,,\qquad R_2=\sqrt{\gamma(1-p)}\,\sigma_-\,.$$

The coefficient $\gamma>0$ is the natural *line-width* of the atom, p is the fraction of the detected fluorescence light and $1-p$ is the fraction of the lost light ($0<p<1$) [9, 21, 22, 44].

b. Other dissipation terms. We introduce also the interaction with a thermal bath,

$$R_3=\sqrt{\gamma\overline{n}}\,\sigma_-\,,\qquad R_4=\sqrt{\gamma\overline{n}}\,\sigma_+\,,\qquad \overline{n}\geq 0,$$

and a term responsible of *dephasing* (or decoherence),

$$R_5=\sqrt{\gamma k_d}\,\sigma_z\,,\qquad k_d\geq 0.$$

c. The laser wave. We consider a perfectly coherent monochromatic laser of frequency $\nu > 0$:

$$f_k(t) = \delta_{k2} \frac{\mathrm{i}\Omega}{2\sqrt{\gamma(1-p)}} \mathrm{e}^{-\mathrm{i}\nu t} 1_{[0,T]}(t); \tag{81}$$

T is a time larger than any other time in the theory and the limit $T \to +\infty$ is taken in all the physical quantities. The quantity $\Omega \geq 0$ is called *Rabi frequency* and $\Delta\nu = \nu_0 - \nu$ is called *detuning.*

d. Master equation. With these choices the Liouville operator (15) becomes

$$\begin{aligned}\mathcal{L}(t)[\rho] = &-\frac{\mathrm{i}}{2}\left[\nu_0\sigma_z + \Omega\sigma_{\nu t},\, \rho\right] + \gamma k_d\left(\sigma_z\rho\sigma_z - \rho\right) \\ &+ \gamma\left(\overline{n}+1\right)\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\left\{P_+, \rho\right\}\right) \\ &+ \gamma\overline{n}\left(\sigma_+\rho\sigma_- - \frac{1}{2}\left\{P_-, \rho\right\}\right). \end{aligned} \tag{82}$$

The master equation (14) can be solved by using Bloch equations in the rotating frame [16, Sec. 8.2]. Indeed, we have

$$\Upsilon(t,s)[\rho] = \mathrm{e}^{-\frac{\mathrm{i}}{2}\nu t\sigma_z}\mathrm{e}^{\check{\mathcal{L}}(t-s)}\left[\mathrm{e}^{\frac{\mathrm{i}}{2}\nu s\sigma_z}\rho\mathrm{e}^{-\frac{\mathrm{i}}{2}\nu s\sigma_z}\right]\mathrm{e}^{\frac{\mathrm{i}}{2}\nu t\sigma_z}, \tag{83a}$$

$$\begin{aligned}\check{\mathcal{L}}[\rho] = &-\frac{\mathrm{i}}{2}\left[\nu_0\sigma_z + \Omega\sigma_x,\, \rho\right] + \gamma k_d\left(\sigma_z\rho\sigma_z - \rho\right) \\ &+ \gamma\left(\overline{n}+1\right)\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\left\{P_+, \rho\right\}\right) \\ &+ \gamma\overline{n}\left(\sigma_+\rho\sigma_- - \frac{1}{2}\left\{P_-, \rho\right\}\right). \end{aligned} \tag{83b}$$

The system reduced state turns out to be given by

$$\begin{aligned}\eta_t = \frac{1}{2}\big\{&\mathbb{1} + \left[x(t) + \mathrm{i}y(t)\right]\mathrm{e}^{\mathrm{i}\nu t}\sigma_- \\ &+ \left[x(t) - \mathrm{i}y(t)\right]\mathrm{e}^{-\mathrm{i}\nu t}\sigma_+ + z(t)\sigma_z\big\}, \end{aligned} \tag{84}$$

where

$$\vec{x}(t) = \mathrm{e}^{-At}\vec{x}(0) - \gamma\,\frac{1-\mathrm{e}^{-At}}{A}\begin{pmatrix}0\\0\\1\end{pmatrix},$$

$$A = \begin{pmatrix}\gamma\left(\frac{1}{2}+\overline{n}+2k_\mathrm{d}\right) & \Delta\nu & 0 \\ -\Delta\nu & \gamma\left(\frac{1}{2}+\overline{n}+2k_\mathrm{d}\right) & \Omega \\ 0 & -\Omega & \gamma\left(1+2\overline{n}\right)\end{pmatrix}.$$

A. Homodyning

The squeezing in the fluorescence light is revealed by homodyne detection, which needs to maintain phase coherence between the laser stimulating the atom and the laser in the detection apparatus which determines the observables $Q(t;\vartheta,h)$. To maintain phase coherence for a long time, the stimulating wave f and the local oscillator wave h must be produced by the same physical source and this means to take h proportional to f. So, by including any phase shift in the pase ϑ already present in the definition (21), to describe homodyning we take

$$h(t) = \frac{-\mathrm{i}f_2(t)}{|f_2(t)|}. \tag{85}$$

With the choice (21) for f, we get $h(t) = \mathrm{e}^{-\mathrm{i}\nu t}1_{[0,T]}(t)$.

The limit $T \to +\infty$ can be taken in Eqs. (63) and it is independent of the atomic initial state [44]. The result is [16, Sec. 9.2.1]

$$\begin{aligned}S^{\mathrm{el}}_{\mathrm{hom}}(\mu;\vartheta) :&= \lim_{T\to+\infty} S^{\mathrm{el}}_T(\mu;\vartheta) \\ &= 2\pi\gamma p\left|\vec{s}(\vartheta)\cdot\vec{x}_{\mathrm{eq}}\right|^2\delta(\mu), \end{aligned} \tag{86}$$

$$\begin{aligned}S^{\mathrm{inel}}_{\mathrm{hom}}(\mu;\vartheta) :&= \lim_{T\to+\infty} S^{\mathrm{inel}}_T(\mu;\vartheta) \\ &= 1 + 2p\gamma\,\vec{s}(\vartheta)\cdot\left(\frac{A}{A^2+\mu^2}\,\vec{t}(\vartheta)\right), \end{aligned} \tag{87}$$

where

$$\vec{t}(\vartheta) = \begin{pmatrix}\left(1+z_{\mathrm{eq}}-x_{\mathrm{eq}}^2\right)\cos\vartheta - x_{\mathrm{eq}}y_{\mathrm{eq}}\sin\vartheta \\ \left(1+z_{\mathrm{eq}}-y_{\mathrm{eq}}^2\right)\sin\vartheta - x_{\mathrm{eq}}y_{\mathrm{eq}}\cos\vartheta \\ -\left(1+z_{\mathrm{eq}}\right)\vec{s}(\vartheta)\cdot\vec{x}_{\mathrm{eq}}\end{pmatrix},$$

$$\vec{s}(\vartheta) = \begin{pmatrix}\cos\vartheta\\ \sin\vartheta\\ 0\end{pmatrix}, \qquad \vec{x}_{\mathrm{eq}} = -\gamma A^{-1}\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Examples of inelastic homodyne spectra are plotted in Figure 1 for $\gamma = 1$, $\overline{n} = k_\mathrm{d} = 0$, $p = 4/5$. The Rabi frequency Ω, the detuning $\Delta\nu$ and the phase ϑ are chosen in order to get the deepest minimum of $S^{\mathrm{inel}}_{\mathrm{hom}}$ in $\mu = 2$. Thus, in this case the analysis of the homodyne spectrum reveals the squeezing of the detected light. Also the complementary spectrum is shown, in order to illustrate the role of the Heisenberg-type uncertainty relation (79). One could also compare the homodyne spectrum with and without $\overline{n}$ and k_d, thus verifying that the squeezing is very sensitive to any small perturbation.

B. Heterodyning

When the local oscillator and the stimulating wave are not produced by the same source, the phase difference cannot be maintained for a long time; in this case we have heterodyne detection. In the case of perfectly monochromatic waves, with a stimulating laser represented by (81), we take as local oscillator

$$h(t) = \mathrm{e}^{-\mathrm{i}\nu_{\mathrm{lo}}t}1_{[0,T]}(t), \qquad \nu_{\mathrm{lo}} \neq \nu. \tag{88}$$

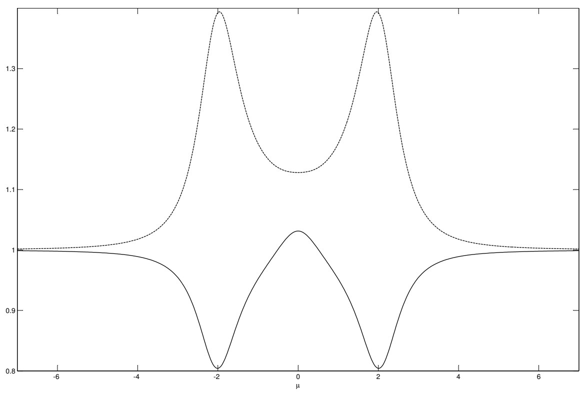

Figure 1. $S^{\rm inel}_{\rm hom}(\mu;\vartheta)$ with $\Delta\nu = 1.4937$, $\Omega = 1.4360$ and $\vartheta = -0.1748$ (solid line), $\vartheta = \frac{\pi}{2} - 0.1748$ (dashed line).

Again the limit $T \to +\infty$ can be taken in Eqs. (63) and it turns out to be independent of the atomic initial state and of ϑ. Let us set

$$v := \nu_{\rm lo} - \nu;$$

then, the final result is: $S_{\rm het}(\mu;\nu_{\rm lo}) = S^{\rm el}_{\rm het}(\mu;\nu_{\rm lo}) + S^{\rm inel}_{\rm het}(\mu;\nu_{\rm lo})$,

$$\begin{aligned} S^{\rm el}_{\rm het}(\mu;\nu_{\rm lo}) &= \lim_{T\to+\infty} S^{\rm el}_T(\mu;\vartheta) \\ &= \frac{\pi}{2}\,\gamma p\left(x_{\rm eq}^2 + y_{\rm eq}^2\right)\left(\delta(\mu - v) + \delta(\mu + v)\right) \\ &= \frac{1}{4}\big[S^{\rm el}_{\rm hom}(\mu - v;0) + S^{\rm el}_{\rm hom}(\mu - v;\pi/2) \\ &\quad + S^{\rm el}_{\rm hom}(\mu + v;0) + S^{\rm el}_{\rm hom}(\mu + v;\pi/2)\big], \end{aligned} \tag{89a}$$

$$\begin{aligned} S^{\rm inel}_{\rm het}(\mu;\nu_{\rm lo}) &= \lim_{T\to+\infty} S^{\rm inel}_T(\mu;\vartheta) = \gamma p\, D(\mu,v) \\ &+ \frac{1}{4}\big[S^{\rm inel}_{\rm hom}(\mu - v;0) + S^{\rm inel}_{\rm hom}(\mu - v;\pi/2) \\ &+ S^{\rm inel}_{\rm hom}(\mu + v;0) + S^{\rm inel}_{\rm hom}(\mu + v;\pi/2)\big], \end{aligned} \tag{89b}$$

$$\begin{aligned} D(\mu,v) &:= \\ &= \vec{s}\left(\tfrac{\pi}{2}\right)\cdot\left(\frac{(\mu+v)/2}{A^2+(\mu+v)^2} - \frac{(\mu-v)/2}{A^2+(\mu-v)^2}\right)\vec{t}(0) \\ &- \vec{s}(0)\cdot\left(\frac{(\mu+v)/2}{A^2+(\mu+v)^2} - \frac{(\mu-v)/2}{A^2+(\mu-v)^2}\right)\vec{t}\left(\tfrac{\pi}{2}\right). \end{aligned} \tag{89c}$$

Recall that $\vec{s}(0) = (1,0,0)$ and $\vec{s}(\pi/2) = (0,1,0)$.

The inelastic heterodyne spectrum (89c) can also be written as

$$S^{\rm inel}_{\rm het}(\mu;\nu_{\rm lo}) = 1 + 2\pi p\left[\Sigma_{\rm inel}(\mu + v) + \Sigma_{\rm inel}(\mu - v)\right], \tag{90}$$

$$\Sigma_{\rm inel}(\mu) = \frac{\gamma}{4\pi}\,{\rm Re}\left((1,{\rm i},0)\cdot\frac{1}{A+{\rm i}\mu}\left[\vec{t}(0) - {\rm i}\vec{t}(\pi/2)\right]\right). \tag{91}$$

1. Properties of the heterodyne spectrum

By explicit computations, it is possible to prove [16, Proposition 9.3 and Remark 9.4 in Sec. 9.1.2] that

$$\Delta\nu = 0 \quad\Rightarrow\quad D(\mu,v) = 0$$

and that

$$\overline{n} = 0 \ \text{ and } \ k_d = 0 \quad\Rightarrow\quad D(\mu,v) = 0.$$

In these cases the heterodyne spectrum reduces to a linear combination of different homodyne contributions.

a. The lower bound of the heterodyne spectrum. Being $S^{\rm inel}_{\rm het}(\mu;\nu_{\rm lo})$ independent of ϑ, any one of the two bounds in Theorem 1 implies

$$S^{\rm inel}_{\rm het}(\mu;\nu_{\rm lo}) \geq 1, \qquad \forall\mu,\quad \forall\nu_{\rm lo}. \tag{92}$$

This means that it is impossible to see squeezing by heterodyning. This is true not only in the model of this section, but in any physical set up for which the dependence on ϑ is lost.

2. The power spectrum

Let us consider now the heterodyne spectrum as a function of the frequency of the local oscillator in the case $\mu = 0$. By particularizing the expressions (89), we get

$$S^{\rm el}_{\rm het}(0;\nu_{\rm lo}) = \pi\gamma p\left(x_{\rm eq}^2 + y_{\rm eq}^2\right)\delta(v),$$

$$\begin{aligned} S^{\rm inel}_{\rm het}(0;\nu_{\rm lo}) &= \frac{1}{2}\left(S^{\rm inel}_{\rm hom}(v;0) + S^{\rm inel}_{\rm hom}\left(v;\tfrac{\pi}{2}\right)\right) + \gamma p\, D(0;v) \\ &= 1 + 4\pi p\Sigma_{\rm inel}(v), \end{aligned} \tag{93}$$

$$D(0,v) = \vec{s}\left(\tfrac{\pi}{2}\right)\cdot\left(\frac{v}{A^2+v^2}\,\vec{t}(0)\right) - \vec{s}(0)\cdot\left(\frac{v}{A^2+v^2}\,\vec{t}\left(\tfrac{\pi}{2}\right)\right).$$

It can be shown that $S_{\rm het}(0;\nu_{\rm lo})$ is proportional to the mean of the power of the heterodyne current [16, Sec. 9.1.1]; so, as a function of $\nu_{\rm lo}$, it represents the *power spectrum*. This interpretation can be strengthened by expressing $S_{\rm het}(0;\nu_{\rm lo})$ in terms of the fields. Let us consider now the mode operators (66) in the case $h(t) = 1$:

$$a_T(\mu)\big|_{h=1} \equiv \frac{1}{\sqrt{T}}\int_0^T {\rm e}^{{\rm i}\mu t}\,{\rm d}B_1(t) =: \hat{a}_T(\mu).$$

These operators, together with their adjoint, satisfy the bosonic commutation relations (68). By taking into account that the ϑ-dependent terms vanish in the limit $T \to \infty$, from (76) we get

$$S_{\rm het}(0;\nu_{\rm lo}) = 1 + 2\lim_{T\to+\infty}{\rm Tr}_\Gamma\left\{\Pi_f(T)\hat{a}^\dagger_T(\nu_{\rm lo})\hat{a}_T(\nu_{\rm lo})\right\}. \tag{94}$$

So, the mean observed power spectrum is composed by the flat spectrum of the shot noise, conventionally set equal to 1, plus a term proportional to the mean number of photons in the temporal mode of frequency approximately equal to $\nu_{\rm lo}$.

a. The fluorescence spectrum. The quantity

$$\frac{S_{\mathrm{het}}(0;\nu_{\mathrm{lo}})-1}{4\pi p}=\frac{\gamma\left(x_{\mathrm{eq}}^2+y_{\mathrm{eq}}^2\right)}{4}\,\delta(v)+\Sigma_{\mathrm{inel}}(v)$$

is interpreted as the *fluorescence spectrum* of the atom [16, Sec. 9.1.2]; the normalization is chosen in order to have its integral over v equal to the rate of emission of photons in the equilibrium state. For $\overline{n}=k_d=0$ this quantity coincides with the original Mollow spectrum [16, Sec. 9.1.2.2], [17, pp. 178–181], [21, p. 288]. In Figure 2 we give the plot of the inelastic part of the fluorescence spectrum in an asymmetric case ($\overline{n}\neq 0$, $k_{\mathrm{d}}\neq 0$) in which the Mollow triplet is well visible (Ω large).

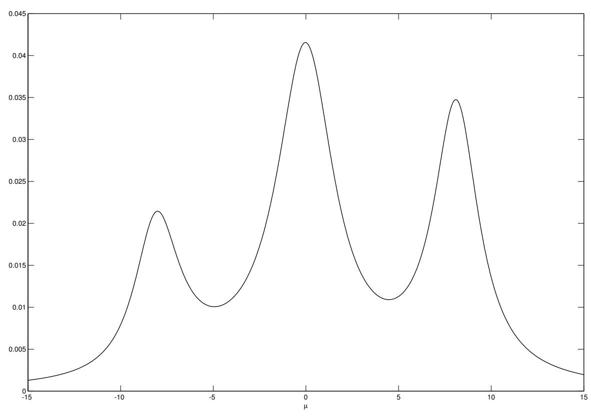

Figure 2. $\Sigma_{\mathrm{inel}}(\mu)$ with $\Delta\nu=1.5$, $\Omega=8.0$, $\overline{n}=0.01$, $k_{\mathrm{d}}=0.7$.

VI. DISCUSSION AND CONCLUSIONS

In this paper we have presented two formulations of the quantum theory of measurements in continuous time. The first one is based on the Hudson-Parthasarathy equation, on quantum stochastic calculus and on the observation of commuting field observables. The other one is based on the SSE, the stochastic master equation and the notion of conditional states. We have also shown how to connect the two formulations. Then, we have studied the properties of the observed output and in particular the output spectrum and the uncertainty relations on the spectra of incompatible quadratures.

Being the two formulations of quantum continuous measurements equivalent, the output spectra can be deduced from the classical SSE or from the quantum one, and the bounds (of Heisenberg type) on the spectra hold independently of the formulation. This point is conceptually relevant, because this equivalence and the presence of uncertainty relations in both formulations show that the "classical" SSE is not a semiclassical approximation to some "quantum" theory, but it is itself fully "quantum". However, let us stress that the proof of the bounds is based on the quantum field formulation and on the uncertainty relations for *incompatible quadratures* of the fields. On the other side we have shown that the most complete results on the probabilistic structure of the output can be obtained only in the stochastic formulation.

The final part of the paper is devoted to a relatively simple example which allows to show squeezing in the homodyne spectrum and to illustrate the differences between homodyning and heterodyning.

Various generalizations of the theory presented in this paper are possible, first of all by introducing direct detection [2, 4, 7, 9, 15, 21–25, 30] and Markovian feedback [10, 11, 16, 42, 44, 46–49]. The quantum trajectory approach can be generalized to introduce also feedback control with delay, coloured noises, various non-Markovian effects [50–55]. The simplest non-Markovian contribution is to take the laser wave f to be random [22, 25]. This means to consider as initial state of the field a classical mixture of coherent states, the mean of $\varrho_\Gamma(f)$. Even the local oscillator wave h can be random, which again means to take a mixture of coherent vectors as state of the local oscillator [22, 55]. This randomness can be used to introduce more realistic models of laser light, not only perfectly coherent monochromatic waves, but also waves exhibiting some coherence time. This allows for a better analysis of the differences between homodyning and heterodyning [55].

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