DIPARTIMENTO DI MATEMATICA
"Francesco Brioschi"
POLITECNICO DI MILANO

Portfolio Optimization over a Finite Horizon with Fixed and Proportional Transaction Costs and Liquidity Constraints

Baccarin, S.; Marazzina, D.

Collezione dei *Quaderni di Dipartimento*, numero **QDD 144**
Inserito negli *Archivi Digitali di Dipartimento* in data 14-1-2013

Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

Portfolio Optimization over a Finite Horizon with Fixed and Proportional Transaction Costs and Liquidity Constraints

Stefano Baccarin[a,*], Daniele Marazzina[b]

[a] *Università degli Studi di Torino, Dipartimento di Scienze Economico-Sociali e Matematico-Statistiche, Corso Unione Sovietica 218/bis, 10134 Torino, Italy.*
[b] *Politecnico di Milano, Dipartimento di Matematica, Via Bonardi 9, 20133 Milano, Italy.*

Abstract

We investigate a portfolio optimization problem for an agent who invests in two assets, a risk-free and a risky asset modeled by a geometric Brownian motion. The investor faces both fixed and proportional transaction costs and liquidity constraints. His objective is to maximize the expected utility from the portfolio liquidation at a terminal finite horizon. The model is formulated as a parabolic impulse control problem and we characterize the value function as the unique constrained viscosity solution of the associated quasi-variational inequality. We compute numerically the optimal policy by a an iterative finite element discretization technique, presenting extended numerical results in the case of a constant relative risk aversion utility function. Our results show that, even with small transaction costs and distant horizons, the optimal strategy is essentially a buy-and-hold trading strategy where the agent recalibrates his portfolio very few times. This contrasts sharply with the continuous interventions of the Merton's model without transaction costs.

Keywords: Portfolio Optimization, Quasi-variational Inequalities, Transaction Costs, Viscosity Solutions

1. Introduction

Optimal portfolio investment strategies have been widely studied in the literature. In his seminal article, Merton (1969) developed a continuous time model to study the optimal portfolio strategy for an investor managing a portfolio of risky assets, whose prices evolve according to geometric Brownian motions. Since then, research in this area has focused on different aspects, aiming to make the mathematical model closer to the real market, in particular with respect to liquidity issues. It is well known that, in the real economy, investors face nontrivial transaction costs, which influence their trading policies. It is not possible to rebalance a portfolio in a continuous way, as assumed by Merton,

*Corresponding author
Email addresses: stefano.baccarin@unito.it (Stefano Baccarin), daniele.marazzina@polimi.it (Daniele Marazzina)

and solvency constraints and bounds on the amounts of the open short positions are usually present.

In our article we deal with a portfolio optimization problem over a finite horizon with two assets, a risk-free and a risky asset whose value is modeled by a log-normal diffusion. We consider a small agent who does not affect in any significant way the assets prices with his transactions. The investor's objective is to maximize his utility from the liquidation of terminal wealth in the presence of transaction costs and liquidity constraints. More specifically we formulate the classical Merton's problem over a finite horizon, without intermediate consumption, with fixed and proportional transaction costs, a solvency constraint, and bounds on the open short positions. Most of the literature on portfolio optimization with transaction costs considers the problem of maximizing the cumulative expected utility of consumption over a infinite horizon, with proportional transaction costs. See for instance Akian et al. (1996); Davis and Norman (1990); Kumar and Muthuraman (2006); Shreve and Soner (1994). The same infinite horizon problem but with fixed and proportional costs has been studied in Oksendal and Sulem (2002). A second class of articles studies the problem of maximizing the long-term growth rate of portfolio value. See Morton and Pliska (1995) for a problem with transaction costs equal to a fixed fraction of the portfolio value ("portfolio management fee"), (Akian et al., 2001; Assaf et al., 1988; Dumas and Luciano, 1991) for models with proportional transaction costs, and Bielecki and Pliska (2000) in the more general framework of risk-sensitive impulse control. Fewer papers consider a portfolio optimization problem with transaction costs over a finite horizon. Liu and Loewenstein (2002) consider proportional transaction costs and approximate the value function by a sequence of optimal analytical solutions for problems with exponentially distributed horizons. This allows to obtain the optimal solution by a sequence of problems without the time dimension. However for a given terminal date the optimal trading strategy is approximated by a stationary policy. In (Eastham and Hastings, 1988; Korn, 1998) both fixed and variable transaction costs are considered and the model is solved by using impulse control techniques. These last articles, which are the closest to our assumptions, use verification theorems to characterize the value function and the optimal policy and apart from some simple cases only approximate the solution by an asymptotic analysis. In the recent paper by Ly Vath et al. (2007), which has inspired our work, the authors consider a portfolio optimization problem over a finite horizon with a permanent price impact and a fixed transaction cost. The main result in Ly Vath et al. (2007) is a viscosity characterization of the value function, but neither a characterization of the optimal policy nor a numerical solution of the problem is given.

Our portfolio optimization problem is formulated as an impulse control problem, associated by the dynamic programming principle to a Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI), as in Bensoussan and Lions (1984) and all the subsequent literature on impulse control. The features of our stochastic control problem lead to consider a parabolic HJBQVI in two variables and time, and to impose state constraints on the space variables. As the value function of our problem is not necessarily continuous and because of the state constraints, we have considered, as in (Akian et al., 2001; Ly Vath et al., 2007; Oksendal and Sulem, 2002), the very general notion of (possibly) discontinuous constrained viscosity solutions. In fact in Section 3 of this paper, by means of a weak comparison principle, we show that the value function is the unique constrained viscosity solution of the HJBQVI verifying certain boundary conditions, and that it is (almost everywhere) continuous. These results are summarized in Theorem 6.

To simplify the numerical solution of the model, in Section 4 we decompose our impulse control problem into a sequence of iterated optimal stopping problems, as in (Baccarin, 2009; Chancelier et al., 2002). This reduction, first introduced in Bensoussan and Lions (1984), has both a theoretical and computational interest. It allows to represent the value function by the limit of a sequence of solutions of variational inequalities. Moreover it makes it possible to characterize a Markovian quasi-optimal policy which is arbitrarily close to the optimal one. We propose an iterative finite element discretization technique to solve numerically this sequence of variational inequalities, and therefore to compute the value function and the optimal policy.

In Section 5 of the paper we present extended numerical results for our model in the case of a constant relative risk aversion (CRRA) utility function, which is the most commonly used utility function in expected utility maximization problems. See, for instance, Akian et al. (1996); Davis and Norman (1990); Liu and Loewenstein (2002); Ly Vath et al. (2007); Kumar and Muthuraman (2006); Shreve and Soner (1994). We describe the form of the optimal transaction strategy and we investigate how it varies with different values of the model parameters. The article that comes closer to ours in this sense is Liu and Loewenstein (2002), even if authors only considered proportional transaction costs and stationary policies. We analyze the transaction regions, the target portfolios, i.e., the portfolios where it is optimal to move from the transaction regions, and how the agent's non-stationary optimal strategy varies as time goes on and for different horizons. To the best of our knowledge this is the first paper where it is shown explicitly how the transaction regions and the target portfolios change, as time passes, up to the finite horizon. Sensitivity analysis with respect to the market and agent's parameters and a comparison between our optimal strategy and others suggested in literature is also provided. Our numerical results show that the transaction costs have a dramatic impact on the frequency of trading of an optimal policy. This phenomenon has already been noted, in a qualitative way, in Dumas and Luciano (1991); Liu and Loewenstein (2002); Morton and Pliska (1995). We show that the optimal strategy is essentially a buy-and-hold trading strategy where the agent recalibrates his portfolio very few times, in contrast with the continuous interventions of the Merton's model without transaction costs.

2. The model formulation

In this section we give a precise formulation of the model. We consider an investor who holds his wealth in two financial assets: a risky asset, or stock, and a risk-free security, which we call a bank account. We denote by $S(t)$ the value of the stocks held by the investor at time t, and by $B(t)$ his amount of money in the bank account. The initial wealth in $t = 0$ is given by (B_0, S_0). The value $S(t)$ evolves as a geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad S(0) = S_0,$$

where W_t is an adapted Wiener process on the filtered probability space $(\Omega, F, P, \mathbb{F}_t)$, verifying the usual conditions. The bank account grows in a certain way at the fixed rate r

$$dB(t) = rB(t)dt, \quad B(0) = B_0.$$

At any time the investor can buy ($\xi > 0$) or sell ($\xi < 0$) the value $\xi \in \mathbb{R}$ of stocks, reducing (or increasing) correspondingly the bank account. However to make a transaction it is necessary to bear the associated transaction costs $C(\xi)$, which we assume of a fixed plus proportional type

$$C(\xi) = K + c\,|\xi|\,, \quad K > 0,\ 0 \leq c < 1\ .$$

These costs are drawn immediately from the bank account. Therefore if the value ξ of stocks is bought (or sold) the variation in the bank account is given by $-\xi - K - c\,|\xi|$ and the presence of the fixed cost K makes it unprofitable to transact in a continuous way.
A portfolio control policy p is a sequence $\{(\tau_i, \xi_i)\}\,,\ i = 1, 2, \cdots$, of stopping times τ_i and corresponding random variables ξ_i, which represent the value of stocks bought (or sold) in τ_i. We define a policy as feasible if it verifies the following conditions:

$$\begin{cases} \tau_i \text{ is a } \mathbb{F}_t \text{ stopping time} \\ \tau_i \leq \tau_{i+1} \qquad \forall i \\ \lim_{i\to+\infty} \tau_i = +\infty \text{ almost surely} \\ \xi_i \text{ is } F_{\tau_i} \text{ measurable .} \end{cases} \tag{1}$$

Note that condition $\tau_i \to \infty$ a.s. implies that the number of stopping times in any bounded time interval is almost surely finite ($\tau_i = +\infty$ for some $i < \infty$ is possible, it means a policy made of at most $i - 1$ transactions). Starting from the initial amounts (B_0, S_0) of the two assets in $t = 0$, the dynamics of the portfolio $(B^p(t), S^p(t))$, controlled by policy p, is obtained from the following set of stochastic differential equations:

$$\begin{cases} dS^0(t) = \mu S^0(t)dt + \sigma S^0(t)dW(t)\ ,\ S^0(0) = S_0 \\ dB^0(t) = rB^0(t)dt\ ,\ \ B^0(0) = B_0, \end{cases} \quad \text{for any } t \in [0, \tau_1] \tag{2}$$

and, for any $t \in [\tau_i, \tau_{i+1}]$, $i \geq 1$,

$$\begin{cases} dS^i(t) = \mu S^i(t)dt + \sigma S^i(t)dW(t)\ ,\ \ S^i(\tau_i) = S^{i-1}(\tau_i) + \xi_i \\ dB^i(t) = rB^i(t)dt\ ,\ \ B^i(\tau_i) = B^{i-1}(\tau_i) - \xi_i - K - c\,|\xi_i|\ . \end{cases} \tag{3}$$

When $\tau_i < \tau_{i+1}$, we define $(B^p(t), S^p(t)) = (B^i(t), S^i(t))$ for $t \in [\tau_i\ ,\ \tau_{i+1})$. If we have, for example, $\tau_{i-1} < \tau_i = \tau_{i+1} = ... = \tau_{i+n} < \tau_{i+n+1}$, then we set

$$\begin{cases} (B^p(\tau_{i+n}^-), S^p(\tau_{i+n}^-)) = (B^{i-1}(\tau_{i+n}), S^{i-1}(\tau_{i+n})) \\ (B^p(\tau_{i+n}), S^p(\tau_{i+n})) = (B^{i+n}(\tau_{i+n}), S^{i+n}(\tau_{i+n})) \end{cases}$$

where in this case $(B^p(\tau_{i+n}^-), S^p(\tau_{i+n}^-))$ are the left limits in $t = \tau_i = ... = \tau_{i+n}$. The resulting controlled process $(B^p(t), S^p(t))$ is cadlag and adapted to the filtration $\mathbb{F}_t$.

A fundamental notion in our model is that of the liquidation value of the assets. We define the liquidation value $L(B, S)$ of the portfolio (B, S) as

$$L(B, S) = \begin{cases} \max\{S + B - K - c\,|S|\,, B\} & \text{if } S \geq 0 \\ S + B - K - c\,|S| & \text{if } S < 0\ . \end{cases}$$

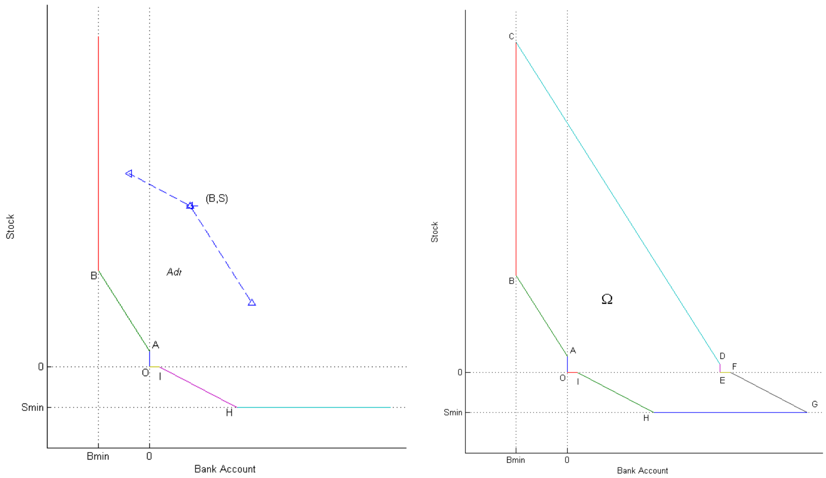

Figure 1: Admissible region (left) and bounded value function domain (right). The points in the figures have coordinates $A = (0, \frac{K}{1-c})$, $B = (B_{\min}, \frac{K-B_{\min}}{1-c})$, $C = \left(B_{\min}, \frac{-B_{\min}+K+L_{\max}}{1-c}\right)$, $D = \left(L_{\max}, \frac{K}{1-c}\right)$, $E = (L_{\max}, 0)$, $F = (L_{\max} + K, 0)$, $G = (-(1+c)S_{\min} + K + L_{\max}, S_{\min})$, $H = (-(1+c)S_{\min} + K, S_{\min})$ and $I = (K, 0)$.

It represents the value when the long or short position in stocks is cleared out (eventually by a bin trade if a positive S does not cover the fixed cost K). Note that $L(B,S) < B + S$, except for $S = 0$, and that every transaction cannot increase the liquidation value of the portfolio, that is $L(B,S) \geq L(B - \xi - K - c|\xi|, S + \xi)$, $\forall \xi \in \mathbb{R}$ (the equality holds only if $L(B,S) = S + B - K - c|S|$ and $\xi = -S$). Besides the transaction costs, our investor must face two other kinds of constraints. We assume that there are bounds on the short positions and that the portfolio must satisfy a solvency constraint. Therefore we define the admissible closed region $\overline{Adr} \subset \mathbb{R}^2$,

$$\overline{Adr} = \left\{(B,S) \in \mathbb{R}^2 : (L(B,S) \geq 0) \wedge (B \geq B_{\min}) \wedge (S \geq S_{\min})\right\} ,$$

which represents the set of admissible portfolios. Here $B_{\min} < 0$ and $S_{\min} < 0$ are the bounds in the short position in the bank account and in the risky security, respectively. We assume $B_0 \geq B_{\min}$, $S_0 \geq S_{\min}$ and $L(B_0, S_0) \geq 0$. The admissible region is depicted in Figure 1. The investor's preferences are represented by a continuous, increasing, utility function $U : \mathbb{R}_+ \to \mathbb{R}_+$, with $U(0) = 0$. We assume that U satisfies, for some $C > 0$ and $0 < \gamma < 1$, the upper bound

$$U(L) \leq CL^{\gamma} \ . \tag{4}$$

The objective of our investor is to liquidate his portfolio at a fixed time horizon $T > 0$. This means that the problem is to maximize the expected utility of the portfolio liquidation value at the terminal date T. However we will assume that our investor will be satisfied if his portfolio reaches a threshold liquidation value $L_{\max} > L(B_0, S_0)$, at a time $t < T$. In this case the portfolio will be liquidated in t and $L_{\max}$ will be invested in the bank account up to the finite horizon T. This assumption is not restrictive, if $L_{\max}$ is sufficiently large with respect to $L(B_0, S_0)$ and T. It has the big advantage to let us consider a control problem on a bounded domain in the state variables B and S (if $L_{\max} < \infty$). This makes the numerical solution of the problem more precise, because it avoids the use of artificial boundary conditions when the unbounded domain is localized. See, for instance, Chancelier et al. (2002); Oksendal and Sulem (2002). We define the open control region Cor

$$Cor = \left\{(B,S) \in \mathbb{R}^2 : L(B,S) < L_{\max}\right\}$$

and by $\overline{Cor}$ its closure. Cor is the region where it may be useful to rebalance the portfolio because the threshold value $L_{\max}$ has not been reached yet. Let θ^p be the first exit time of the controlled process from the control region

$$\theta^p = \inf\left\{t : (B^p(t), S^p(t)) \notin Cor\right\}.$$

We set $\vartheta^p \equiv \theta^p \wedge T$, and we will say that a policy p is admissible if the corresponding controlled process verifies $(B^p(t), S^p(t)) \in \overline{Adr}$, $\forall t \in [0, \vartheta^p]$. The payoff functional J^p associated to policy p is then given by

$$J^p = E\left[U(L(B^p(\vartheta^p), S^p(\vartheta^p))\, e^{r(T-\vartheta^p)})\right].$$

Note that the behaviour of $(B^p(t), S^p(t))$ after ϑ^p is irrelevant in our formulation: that is to say $(B^p(t), S^p(t))$ represents the financial position of our investor only up to ϑ^p. If we denote by A the set of admissible policies, the control problem can be formulated as

$$\max_{p\in A} J^p .$$

It is a stochastic impulse control problem over a finite horizon where the system is controlled only in the Cor region, and with the state constraint $(B^p(t), S^p(t)) \in \overline{Adr}$, $\forall t \in [0, \vartheta^p]$. We will solve this problem by using a dynamic programming approach. We consider the compact set $\overline{\Omega} = \overline{Adr} \cap \overline{Cor}$ which is depicted in Figure 1 and we denote by Ω its interior. We define the set $\overline{Q} \equiv [0,T] \times \overline{\Omega}$ and we will denote by Q the subset $[0,T) \times \Omega$. Now we can introduce the value function $V(t,B,S) : \overline{Q} \subset \mathbb{R}^3 \to \mathbb{R}$ defined by

$$V(t,B,S) = \sup_{p\in A(t,B,S)} J^p(t,B,S) .$$

Here $A(t,B,S)$ is the set of admissible policies when the controlled process starts in t with values (B,S) and

$$J^p(t,B,S) = E_{t,B,S}\left[U\left(L(B^p(\vartheta^p), S^p(\vartheta^p))\, e^{r(T-\vartheta^p)}\right)\right].$$

Remark 1. *We have $A(t,B,S) \neq \emptyset$ for any initial condition $(t,B,S) \in \overline{Q}$ because the policy*

$$\begin{cases} \tau_1 = \begin{cases} +\infty & \text{if } S \geq 0 \text{ and } B \geq 0 \\ t & \text{otherwise} \end{cases} \\ \xi_1 = \begin{cases} \text{arbitrary} & \text{if } S \geq 0 \text{ and } B \geq 0 \\ -S & \text{otherwise} \end{cases} \end{cases}, \qquad \begin{cases} \tau_i = +\infty \\ \xi_i \text{ arbitrary} \end{cases} \text{for } i > 1$$

is clearly always admissible. Note that $V(t,0,0) = 0$, $\forall t \in [0,T]$, because the only admissible policy is doing nothing, and $U(0) = 0$ by assumption. Moreover $V(t,B,S) > U(L_{\max}e^{r(T-t)})$ only if $(B,S) \in EF\backslash E$, because the points in $EF\backslash E$ can be reached by an admissible policy only after ϑ^p, if the initial position $(B,S) \notin EF\backslash E$.

The value function V of our problem verifies the following dynamic programming property. See (Fleming and Soner, 1993, Section V.2), or (Ly Vath et al., 2007).

Dynamic Programming Property:
(a) For any $(t, B, S) \in \overline{Q}$*,* $p \in A(t, B, S)$ *and* $\{\mathbb{F}_s\}$*-stopping time* $\alpha \geq t$ *we have*

$$V(t, B, S) \geq E_{t,B,S}\left[V(\vartheta^p \wedge \alpha, B^p(\vartheta^p \wedge \alpha), S^p(\vartheta^p \wedge \alpha))\right]; \tag{5}$$

(b) For any $(t, B, S) \in \overline{Q}$*, and* $\delta > 0$*, there exists* $p'(\delta) \in A(t, B, S)$ *such that for all* $\{\mathbb{F}_s\}$*-stopping time* $\alpha \geq t$ *we have*

$$V(t, B, S) \leq E_{t,B,S}[V(\vartheta^{p'} \wedge \alpha, B^{p'}(\vartheta^{p'} \wedge \alpha), S^{p'}(\vartheta^{p'} \wedge \alpha))] + \delta. \tag{6}$$

Combining (a) and (b) we obtain the following version of the dynamic programming principle, which holds for any $(t, B, S) \in \overline{Q}$ and $\{\mathbb{F}_s\}$-stopping time $\alpha \geq t$

$$V(t, B, S) = \sup_{p \in A(t,B,S)} E_{t,B,S}\left[V(\vartheta^p \wedge \alpha, B^p(\vartheta^p \wedge \alpha), S^p(\vartheta^p \wedge \alpha))\right] .$$

Now, we denote by $F(B, S)$ the set of admissible transactions from $(B, S) \in \overline{\Omega}$

$$F(B, S) = \left\{\xi \in \mathbb{R} : (B - \xi - K - c\,|\xi|\,, S + \xi) \in \overline{\Omega}\right\}$$

and by $\digamma$ the subset of $\overline{\Omega}$ where $F(B, S) \neq \emptyset$.

Remark 2. *The set* $F(B, S)$ *can be empty. For example it is always empty when* $B+S < K$*, but if* $F(B, S) \neq \emptyset$*, then it is a compact subset of* $\mathbb{R}$*. Moreover let* $(B_n, S_n) \in \overline{Adr}$ *be a sequence converging to* $(B', S') \in \overline{Adr}$ *with* $F(B_n, S_n) \neq \emptyset$*. Since the function* L *is upper semicontinuous we have* $L(B', S') \geq 0$ *and* $F(B', S') \neq \emptyset$*. Any sequence* $\xi_n \in F(B_n, S_n)$ *stays bounded and therefore contains a subsequence* ξ'_n *converging to some* $\xi' \in \mathbb{R}$*. As* $L(B_n - \xi'_n - K - c\,|\xi'_n|\,, S_n + \xi'_n) \geq 0$ *and* L *is upper semicontinuous we also obtain that* $\xi' \in F(B', S')$*.*

For any given function $Z : \overline{Q} \to \mathbb{R}$ we define the intervention (non local) operator $\mathcal{M}$ by

$$\mathcal{M}Z\,(t, B, S) = \begin{cases} \sup\limits_{\xi \in F(B,S)} Z(t, B - \xi - K - c\,|\xi|\,, S + \xi) & \text{if } (B, S) \in \digamma \\ -1 & \text{if } (B, S) \notin \digamma. \end{cases} \tag{7}$$

Considering all $p \in A(t, B, S)$ with an immediate transaction in t of arbitrary size $\xi \in F(B, S)$ and setting $\alpha = t$ in (5), we can see as a direct consequence of dynamic programming property that $V(t, B, S) \geq \mathcal{M}V(t, B, S)$, $\forall (t, B, S) \in \overline{Q}$ (this is obvious if $F(B, S) \neq \emptyset$ because V is non-negative).

It is well known that we can associate to the value function of an impulse control problem a Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI) which plays the same role of the HJB equation in continuous optimization. We introduce the second order differential operator $\mathcal{L}$

$$\mathcal{L}V(t,B,S) = rB\frac{\partial V}{\partial B} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}$$

which corresponds to the infinitesimal generator of the uncontrolled process $(B(t), S(t))$. We will show that the value function of our problem is a weak solution of the following HJBQVI in $\overline{Q}$

$$\min\left\{-\frac{\partial V}{\partial t} - \mathcal{L}V, V - \mathcal{M}V\right\} = 0. \tag{8}$$

One cannot hope to show that V is a classical solution of (8). It is easy to see that the value function is not even continuous in some points of ∂Q, such as, for instance, points A, I, in Figure 1, $\forall t \in [0,T]$, or line $S = 0$ in $t = T$. In these points V is only upper-semicontinuous. In the next section we will characterize V as the unique constrained viscosity solution of (8) verifying certain boundary conditions.

3. Boundary properties, bounds, and viscosity characterization of the value function

By $\partial^* Q$ we will denote the subset of ∂Q given by $\partial^* Q \equiv ([0,t)\times\partial\Omega)\cup(T\times\overline{\Omega})$. The boundary $\partial\Omega$ of Ω is divided in two parts:

$$\partial_1\Omega \equiv \{(B,S) \in \partial\Omega : L(B,S) < L_{\max}\}$$

and its complement $\partial_2\Omega \equiv \partial\Omega\backslash\partial_1\Omega$. In the points of $\partial_2\Omega$ the threshold liquidation value $L_{\max}$ has already been reached. It will be also useful to define $\partial_2^* Q \equiv ([0,t) \times \partial_2\Omega) \cup (T \times \overline{\Omega})$, which is the part of $\partial^* Q$ where, or $t = T$ or $L(B,S) \geq L_{\max}$. We will investigate the behavior of V at the boundary $\partial^* Q$.

For $t = T$ we have obviously $V(T,B,S) = U(L(B,S))$ for any $(B,S) \in \overline{\Omega}$. It is always optimal not to intervene in T because any intervention cannot increase the portfolio liquidation value. However one single transaction $\xi = -S$ is also optimal if $S < 0$ or if $S > 0$ and $S + B - K - c\,|S| \geq B$. In this case we have $V = \mathcal{M}V$, otherwise $V > \mathcal{M}V$. Note that V is upper-semicontinuous but not continuous for any point $(T,B,0) \in \overline{Q}$.

For $t \in [0,T)$ the behavior of V depends on which part of $\partial\Omega$ we are considering:

(a) Along the segments OA and OI in Figure 1 it is not possible to intervene because this will bring the process (B,S) outside the admissible region $\overline{Adr}$. Actually in the points A and I there is one admissible transaction which leads us to the origin O, but this is certainly unprofitable. Therefore we have $V > \mathcal{M}V$. Apart from $V(t,0,0) = 0$, the value of V is not known a priori in this part of $\partial^* Q$.

(b) Except for the points A and I, along the segments AB and IH it is necessary to make a transaction, otherwise the process could leave $\overline{Adr}$ with a positive probability. Moreover the only admissible intervention brings the process to O. Consequently it holds $V = \mathcal{M}V = 0$. Note that V is upper-semicontinuous but not continuous in A and I.

(c) In the interior points of the segments BC and HG it is necessary to make a transaction because one of the bounds in the short position is reached. The value of V is not known a priori. We have $V = \mathcal{M}V$.

(d) In the upper part of $\partial^* Q$, that is along the segments CD, DE, EF and FG, the threshold liquidation value $L_{\max}$ has already been reached. The value of V is known. If $(B,0) \in EF$ then $V(t,B,0) = U(B\ e^{r(T-t)})$. If $(B,S) \in CD \cup DE \cup \{FG \backslash F\}$ then $V(t,B,S) = U(L_{\max}\ e^{r(T-t)})$. It is always optimal not to intervene, but we also have $V = \mathcal{M}V$, with $\xi = -S$ in (7), if $S < 0$ or if $S > 0$ and $S + B - K - c\,|S| \geq B$. Note that V is upper-semicontinuous but not continuous in the point F, $\forall t \in [0,T)$.

We give now some bounds on the value function. Since $J^p \geq 0$, $\forall p \in A(t,B,S)$, it is obvious that $V(t,B,S)$ is nonnegative in $[0,T] \times \overline{\Omega}$. By the problem definition we also have $V(t,B,S) \leq U((L_{\max} + K)e^{r(T-t)})$, that is the value function is finite. Moreover, as it holds $U(L(B,S)e^{r(T-t)}) \leq V(t,B,S) \leq U(L_{\max}e^{r(T-t)})$ when $(B,S) \notin EF$, the value function is also continuous in the segments CD, $\{DE \backslash E\}$, $\{FG \backslash F\}$. It is not difficult to show that V is also bounded from above by the value function of the same problem with $U(L) = CL^{\gamma}$ and without transaction costs and liquidity constraints (a Merton problem over a finite horizon without consumption and a CRRA utility function, see Merton (1969)).

Proposition 1. *We have*

$$V(t,B,S) \leq Ce^{\delta(T-t)}\,(B+S)^{\gamma} \tag{9}$$

in $[0,T] \times \overline{\Omega}$, *where*

$$\delta = \gamma \left(r + \frac{(\mu - r)^2}{2\sigma^2(1-\gamma)} \right) .$$

Proof. We set $Z(t,B,S) := Ce^{\delta(T-t)}\,(B+S)^{\gamma}$. The inequality (9) is true in $T \times \overline{\Omega}$ as $V(T,B,S) = U(L(B,S)) \leq C\,L(B,S)^{\gamma} \leq C\,(B+S)^{\gamma}$ and in $[0,T) \times (0,0)$ because here we have $V(t,0,0) = 0$. Moreover, in $\overline{\Omega} \backslash \{\mathbf{0}\}$, Z verifies $Z > \mathcal{M}Z$ and $-\frac{\partial Z}{\partial t} - \mathcal{L}Z \geq 0$. Indeed $\mathcal{M}Z = -1$ if $(B,S) \notin \digamma$, $\mathcal{M}Z \leq Ce^{\delta(T-t)}\,(B+S-K)^{\gamma} < Z$ if $(B,S) \in \digamma$, and, differentiating Z, it is easy to verify that $\frac{\partial Z}{\partial t} + \mathcal{L}Z \leq 0$ in $\overline{\Omega} \backslash \{\mathbf{0}\}$. Now consider an admissible policy $p \in A(t,B,S)$, for the controlled process starting in $t \in [0,T)$ with values $(B,S) \in \overline{\Omega} \backslash \{\mathbf{0}\}$. We define $\tau_0^p = t$ and, almost surely, $n^p(\omega) = \max \{i \geq 0 : \tau_i^p(\omega) \leq \vartheta^p(\omega)\}$. Applying the generalized Itô's formula to the function Z, from t to ϑ^p, we have:

$$Z(\vartheta^p, B^p(\vartheta^p), S^p(\vartheta^p)) = Z(t,B,S) + \int_t^{\vartheta^p} (\tfrac{\partial Z}{\partial t} + \mathcal{L}Z)\,ds + \int_t^{\vartheta^p} \sigma S \tfrac{\partial Z}{\partial S}\,dW(s) +$$
$$+ \sum_{i=1}^{n^p} (Z(\tau_i, B(\tau_i^-) - \xi_i - K - c\,|\xi_i|\,, S(\tau_i^-) + \xi_i) - Z(\tau_i, B(\tau_i^-), S(\tau_i^-)))\ .$$

Since $\frac{\partial Z}{\partial t} + \mathcal{L}Z \leq 0$ and $Z > \mathcal{M}Z$ it follows that, a.s.,

$$Z(\vartheta^p, B^p(\vartheta^p), S^p(\vartheta^p)) < Z(t,B,S) + \int_t^{\vartheta^p} \sigma S \frac{\partial Z}{\partial S}\,dW(s)\ .$$

Taking expectations, the stochastic integral vanishes, since $\frac{\partial Z}{\partial S}$ is bounded, and we obtain

$$Z(t,B,S) > E[Z(\vartheta^p, B^p(\vartheta^p), S^p(\vartheta^p))] \qquad \forall p \in A(t,B,S)\ .$$

Therefore

$$
\begin{aligned}
Z(t,B,S) &\geq \sup_{p\in A(t,B,S)} E[Z(\vartheta^p, B^p(\vartheta^p), S^p(\vartheta^p))] = \\
&= \sup_{p\in A(t,B,S)} E[C(B^p(\vartheta^p)+S^p(\vartheta^p))^\gamma e^{\delta(T-\vartheta^p)}] \geq \\
&\geq \sup_{p\in A(t,B,S)} J^p = V(t,B,S)\ .
\end{aligned}
$$

□

Bound (9) shows in particular that $V(t,B,S)$ is continuous in $(t,0,0)$, where $V(t,0,0)=0$, $\forall t\in[0,T]$. Now we give the precise characterization of the value function as a viscosity solution of (8). Since V is not even continuous at some points in ∂Q it is necessary to consider the notion of discontinuous viscosity solutions. Moreover the state constraint $(B^p(t),S^p(t))\in\overline{Adr}$, $\forall t\in[0,\vartheta^p]$, requires a particular treatment of the lateral boundary conditions when $(t,B,S)\in[0,T)\times\partial_1\Omega$ and the use of constrained viscosity solutions. We recall now the definitions of (possibly discontinuous) constrained viscosity solutions. Let $USC(\overline{Q})$ and $LSC(\overline{Q})$ be respectively the sets of upper-semicontinuous (usc) and lower-semicontinuous (lsc) functions defined on $\overline{Q}$. Given a locally bounded function $u:\overline{Q}\to\mathbb{R}_+$ we will denote by u^* and u_* respectively the usc envelope and the lsc envelope of u

$$
\begin{aligned}
u^*(t,B,S) &= \limsup_{\substack{(t',B',S')\in\overline{Q}\\(t',B',S')\to(t,B,S)}} u(t',B',S') \qquad \forall\,(t,B,S)\in\overline{Q}\\
u_*(t,B,S) &= \liminf_{\substack{(t',B',S')\in\overline{Q}\\(t',B',S')\to(t,B,S)}} u(t',B',S') \qquad \forall\,(t,B,S)\in\overline{Q}\ .
\end{aligned}
$$

We have $u_*\leq u\leq u^*$ and u is usc (lsc) if and only if $u=u^*$ ($u=u_*$). In the following, sometimes we set $x\equiv(B,S)\in\bar{\Omega}$ to simplify the notation.

Definition 1. Given $\mathcal{O}\subset\overline{\Omega}$, a locally bounded function $u:\overline{Q}\to\mathbb{R}_+$ is called a viscosity subsolution (resp. supersolution) of (8) in $[0,T)\times\mathcal{O}$ if for all $(\bar{t},\overline{x})\in[0,T)\times\mathcal{O}$ and $\varphi(t,x)\in C^{1,2}(\overline{Q})$ such that $(u^*-\varphi)(\bar{t},\overline{x})=0$ (resp. $(u_*-\varphi)(\bar{t},\overline{x})=0$) and $(\bar{t},\overline{x})$ is a maximum of $u^*-\varphi$ (resp. a minimum of $u_*-\varphi$) on $[0,T)\times\mathcal{O}$, we have

$$
\min\left\{-\frac{\partial\varphi}{\partial t}(\bar{t},\overline{x})-\mathcal{L}\varphi(\bar{t},\overline{x}),u^*(t,x)-\mathcal{M}u^*(t,x)\right\} \leq 0 \qquad (10)
$$

$$
(\text{resp. } u_* \text{ and } \geq 0) \qquad (11)
$$

On $[0,T)\times\partial_2\Omega$ the value function V verifies the Dirichlet boundary condition $V(t,B,S)=U(L(B,S)e^{r(T-t)})$. To deal properly with the state constraint $(B^p(t),S^p(t))\in\overline{Adr}$, $\forall t\in[0,\vartheta^p]$, it will be necessary to require that V satisfies the subsolution property also on the $[0,T)\times\partial_1\Omega$ part of the lateral boundary $[0,T)\times\partial\Omega$ (see (Crandall et al., 1992, section 7C'), or (Oksendal and Sulem, 2002; Ly Vath et al., 2007)).

Definition 2. We say that a locally bounded function $u:\overline{Q}\to\mathbb{R}_+$ is a $\partial_1\Omega$ constrained viscosity solution of (8) in $Q=[0,T)\times\Omega$ if it is a viscosity supersolution of (8) in Q and a viscosity subsolution of (8) in $[0,T)\times\{\Omega\cup\partial_1\Omega\}$.

We will need the following properties of the non-local operator $\mathcal{M}$.

Lemma 2. *Given a locally bounded function $u : \overline{Q} \to \mathbb{R}_+$ we have:*
(a) if u is lower-semicontinuous (resp. usc) then $\mathcal{M}u$ is lower-semicontinuous (resp. usc)
(b) $\mathcal{M}u_* \leq (\mathcal{M}u)_*$ and $\mathcal{M}u^* \geq (\mathcal{M}u)^*$
(c) if u is upper-semicontinuous then there exists a Borel measurable function $\xi_u^* : \digamma \to \mathbb{R}$ such that for any $(B, S) \in \digamma$

$$\mathcal{M}u(t, B, S) = u(t, B - \xi_u^*(B, S) - K - c\,|\xi_u^*(B, S)|\,, S + \xi_u^*(B, S)). \tag{12}$$

Proof. (a) and (b) can be proven in the same way as in (Ly Vath et al., 2007, Lemma 5.5). As u is upper-semicontinuous and for $(B, S) \in \digamma$ the set $F(B, S)$ is compact the sup in (7) is reached for some values of ξ, $\forall (B, S) \in \digamma$. Moreover, as $\digamma$ is σ-compact, we can select a Borel measurable function $\xi_u^* : \digamma \to \mathbb{R}$ such that (c) holds true (see Fleming and Rishel, 1975, Appendix B, Lemma B). □

We can now state the viscosity property of the value function.

Theorem 3. *The value function $V(t, B, S)$ is a $\partial_1\Omega$ constrained viscosity solution of (8) in Q.*

Proof. Using the dynamic programming property (5-6), and properties (a) and (b) of Lemma 2, the proof can be done in the same way as the proof of (Ly Vath et al., 2007, Theorem 5.3). The only difference is that in our problem it is possible to prove the subsolution property only in Q and in the $[0, T) \times \partial_1\Omega$ part of the lateral boundary. The reason is that an admissible policy can now allow the controlled process to leave $\overline{Q}$ from the subset $[0, T) \times \partial_2\Omega$ of $\partial^* Q$. On $[0, T) \times \partial_2\Omega$ the value function will be determined by the Dirichlet type condition $V(t, B, S) = L(B, S)e^{r(T-t)}$. □

As there can be many viscosity solutions of (8) the next step is to determine the right boundary conditions on $\partial^* Q$ which are sufficient to uniquely determine the value function. The usual way to show uniqueness of viscosity solutions is to prove a comparison theorem between viscosity sub and supersolution. The purpose is to show that a subsolution is lower than a supersolution on the whole domain if it assumes the same or a lower value at the boundary $\partial^* Q$. But in our problem the value of V is not known in some part of $[0, T) \times \partial_1\Omega$, such as the segments BC and HG in Figure 1. Thus on $[0, T) \times \partial_1\Omega$ we will need the viscosity boundary condition given by the subsolution property. Moreover if we look at V^* as a subsolution and at V_* as a supersolution, along the segments $I_T F_T \equiv \left\{(t, B, S) \in \overline{Q} : t = T,\, S = 0, B \geq K\right\}$ and $F_0 F_T \equiv \left\{(t, B, S) \in \overline{Q} : B = L_{\max} + K,\, S = 0\right\}$ the subsolution V^* is greater than the supersolution V_*. Therefore on the rectangular region

$$R \equiv \left\{(t, B, S) \in \overline{Q} : S = 0, B \geq K\right\}$$

we cannot hope to show that $V^* \leq V_*$, and consequently that V is continuous in R, because by definition $V^* \geq V_*$, and thus $V^* = V_*$. All this will induce us to prove only a weaker comparison principle between viscosity sub and supersolutions, which holds on $Q\backslash R$. Thus we will distinguish the cases $S>0$ and $S<0$ denoting by Ω^+, Q^+, $\partial^* Q^+$, Ω^-, $Q^-$the sets

$$\Omega^+ \equiv \{(B,S)\in\Omega : S>0\}, \quad Q^+ \equiv [0,T)\times\Omega^+,$$

and by $\overline{\Omega}^+$, $\overline{Q}^+$, their closures. We also define the boundaries

$$\begin{aligned}
&\partial^* Q^+ \equiv [0,T)\times\partial\Omega^+ \cup T\times\overline{\Omega}^+,\\
&\partial_1\Omega^+ \equiv \left\{(B,S)\in\partial\Omega^+ : L(B,S)<L_{\max}\right\}, \ \partial_2\Omega^+ = \partial\Omega^+\backslash\partial_1\Omega^+,\\
&\partial_2^* Q^+ \equiv [0,T)\times\partial\Omega_2^+ \cup T\times\overline{\Omega}^+.
\end{aligned}$$

The sets Ω^-, Q^-, $\overline{\Omega}^-$, $\overline{Q}^-$, $\partial^* Q^-$, $\partial_2\Omega^-$, $\partial_2^* Q^-$ are defined similarly by setting $S<0$.

Theorem 4 (Weak Comparison Principle)**.** *Assume that $u\in USC(\overline{Q})$ is a viscosity subsolution of (8) in $[0,T)\times\{\Omega\cup\partial_1\Omega\}$ and $v\in LSC(\overline{Q})$ is a viscosity supersolution of (8) in $Q=[0,T)\times\Omega$ Furthermore assume that*

$$\begin{cases}
\limsup\limits_{\substack{(t',B',S')\in Q^+\\(t',B',S')\to(t,B,S)}} u(t',B',S') \le \liminf\limits_{\substack{(t',B',S')\in Q^+\\(t',B',S')\to(t,B,S)}} v(t',B',S') \quad \forall\,(t,B,S)\in\partial_2^* Q^+\\[2ex]
\limsup\limits_{\substack{(t',B',S')\in Q^-\\(t',B',S')\to(t,B,S)}} u(t',B',S') \le \liminf\limits_{\substack{(t',B',S')\in Q^-\\(t',B',S')\to(t,B,S)}} v(t',B',S') \quad \forall\,(t,B,S)\in\partial_2^* Q^-\\[2ex]
\limsup\limits_{\substack{(t',B',S')\in Q\\(t',B',S')\to(t,0,0)}} u(t',0,0) \le \liminf\limits_{\substack{(t',B',S')\in Q\\(t',B',S')\to(t,0,0)}} v(t',0,0) \quad \forall t\in[0,T)\ .
\end{cases} \tag{13}$$

Then $u\le v$ on $Q\backslash R$.

Proof. The proof of this theorem is somewhat long and technical. For the reader's convenience we leave it to the Appendix. □

In order to use the comparison principle to identify the only viscosity solution which represents the value function we need to describe the behavior of V approaching the boundary $\partial_2^* Q$ and taking account of the discontinuity in $\partial_2^* Q\cap R$.

Lemma 5. *The value function V verifies the following limit conditions near the boundary $\partial_2^* Q$:*

$$\begin{cases}
\lim\limits_{\substack{(t',B',S')\in Q\\(t',B',S')\to(t,B,S)}} V(t',B',S') = U(L(B,S)e^{r(T-t)}) \quad \forall(t,B,S)\in\partial_2^* Q\backslash R\\[2ex]
\lim\limits_{\substack{(t',B',S')\in Q^+\\(t',B',S')\to(t,B,0)}} V(t',B',S') = U(B\,e^{r(T-t)}) \qquad \forall(t,B,0)\in\partial_2^* Q\cap R\\[2ex]
\lim\limits_{\substack{(t',B',S')\in Q^-\\(t',B',S')\to(t,B,0)}} V(t',B',S') = U((B-K)e^{r(T-t)}) \quad \forall(t,B,0)\in\partial_2^* Q\cap R
\end{cases} \tag{14}$$

Proof. First consider $(t,B,S) \in \partial_2^* Q \backslash R$. Since $U(L(B,S)e^{r(T-t)})$ is continuous in $(t,B,S) \in \partial_2^* Q \backslash R$ and, by construction, it always holds $V(t',B',S') \geq U(L(B',S')e^{r(T-t')})$, we have for any $(t,B,S) \in \partial_2^* Q \backslash R$

$$V_*(t,B,S) \equiv \liminf_{\substack{(t',B',S') \in Q \\ (t',B',S') \to (t,B,S)}} V(t',B',S') \geq U(L(B,S)e^{r(T-t)}) \ . \tag{15}$$

Now let

$$V^*(t,B,S) = \limsup_{\substack{(t',B',S') \in Q \\ (t',B',S') \to (t,B,S)}} V(t',B',S')$$

and (t_m, B_m, S_m) be a sequence in Q such that

$$\lim_{(t_m,B_m,S_m) \to (t,B,S)} V(t',B',S') = V^*(t,B,S).$$

By (6), for any m there exists a quasi-optimal policy $p^m = \{(\tau_i^m, \xi_i^m)\}$ such that $p^m \in A(t_m, B_m, S_m)$ and $V(t_m, B_m, S_m) \leq J^{p^m}(t_m, B_m, S_m) + \frac{1}{m}$. Denoting the controlled process (B^{p^m}, S^{p^m}) by X^m it follows (here $\vartheta^m = T \wedge \theta^{p^m}$)

$$V(t_m, B_m, S_m) \leq E_{t_m,B_m,S_m} \left[U\left(L(X^m(\vartheta^m))\, e^{r(T-\vartheta^m)}\right) \right] + \frac{1}{m} \ . \tag{16}$$

As it is always optimal not to intervene in ϑ^m we can assume $\tau_i^m \neq \vartheta^m$, $\forall i$. Defining $\Delta X_s^m \equiv X^m(s) - X^m(s^-)$, where $s \geq t_m$ and $X^m(t_m^-) \equiv (B_m, S_m)$, we have

$$X^m(\vartheta^m) = X^m(t_m^-) + \Delta X_{t_m}^m + \int_{t_m}^{\vartheta^m} \alpha(X^m(s))ds + \int_{t_m}^{\vartheta^m} \beta(X^m(s))dW_s + \sum_{t_m < s < \vartheta^m} \Delta X_s^m \tag{17}$$

where $\alpha(X^m) = [rB^{p^m}, \mu S^{p^m}]$ and $\beta(X^m) = [0, \sigma S^{p^m}]$. Since $(t_m, B_m, S_m) \to (t,B,S) \in \partial_2^* Q$ it follows that $\vartheta^m - t_m$ converges a.s. to zero when $m \to \infty$. Thus the two integrals in (17) vanish because $X^m \in \overline{\Omega}$ is bounded. Moreover the last summation also vanishes because the jump sizes are uniformly bounded and the number of interventions after t_m and before ϑ^m, converges to zero as $t_m \to \vartheta^m$. The first difference $\Delta X_{t_m}^m$, at least for a subsequence, converges to some ΔX_1 when $m \to \infty$. Finally sending m to infinity in (16), by the dominated convergence theorem we obtain

$$V^*(t,B,S) \leq U(L((B,S) + \Delta X_1)e^{r(T-t)}) \leq U(L(B,S)e^{r(T-t)}), \tag{18}$$

and therefore the first condition in (14) is true. If $(t,B,0) \in \partial_2^* Q \cap R$ and $(t',B',S') \in Q^+$ converges to $(t,B,0)$ from above R, we have

$$\lim_{\substack{(t',B',S') \in Q^+ \\ (t',B',S') \to (t,B,0)}} U(L(B',S')e^{r(T-t')}) = U(Be^{r(T-t)}),$$

and we can repeat the same reasoning as for $(t,B,S) \in \partial_2^* Q \backslash R$. But if $(t',B',S') \in Q^-$ converges to $(t,B,0)$ from below R, we have

$$\lim_{\substack{(t',B',S') \in Q^- \\ (t',B',S') \to (t,B,0)}} U(L(B',S')e^{r(T-t')}) = U((B-K)e^{r(T-t)})$$

and, by the same procedure used before to obtain (15) and (18), we get

$$\begin{aligned} V^*_{Q^-}(t,B,0) &\equiv \limsup_{\substack{(t',B',S')\in Q^- \\ (t',B',S')\to(t,B,0)}} V(t',B',S') \le U((B-K)e^{r(T-t)}) \\ &\le V_{*\,Q^-}(t,B,0) \equiv \liminf_{\substack{(t',B',S')\in Q^- \\ (t',B',S')\to(t,B,0)}} V(t',B',S') \ . \end{aligned}$$

□

Now we are able to give the complete viscosity characterization of the value function.

Theorem 6. *The value function $V(t,B,S)$ is continuous in $Q\backslash R$ and it is the unique, in $Q\backslash R$, $\partial_1\Omega$ constrained viscosity solution of (8) which verifies the limit conditions (14) and*

$$\lim_{\substack{(t',B',S')\in\bar{Q} \\ (t',B',S')\to(t,0,0)}} V(t',B',S') = V(t,0,0) = 0 \qquad \forall t \in [0,T] \ . \tag{19}$$

Proof. We apply the comparison principle theorem, using V^* as a subsolution and V_* as a supersolution. In particular the boundary conditions (13) are verified as equalities since (14) and (19) hold true. We derive that $V^* \le V_*$ on $Q\backslash R$ and since by definition $V^* \ge V_*$ we obtain immediately that V is continuous in $Q\backslash R$. Now suppose $\overline{V}$ is another $\partial_1\Omega$ constrained viscosity solution of (8) in Q which verifies the boundary conditions (14), (19). By the comparison principle it follows that $\overline{V}^* \le V_* \le V^* \le \overline{V}_*$ and therefore $\overline{V} = V$ in $Q\backslash R$. □

4. Computation of the value function and the optimal policy by iterated optimal stopping

To simplify the numerical solution of the HJBQVI (8) we reduce our impulse control problem to a sequence of optimal stopping time problems. This reduction, first introduced in Bensoussan and Lions (1984), has the advantage to reduce the solution of a HJBQVI to the solution of an iterative sequence of variational inequalities, where the obstacles are explicit (see Baccarin, 2009; Chancelier et al., 2002; Korn, 1998; Oksendal and Sulem, 2007). We denote by A_n the set of admissible policies with at most $n \ge 1$ interventions, that is

$$A_n(t,B,S) = \{p \in A(t,B,S) : \tau_{n+1} = +\infty\}$$

and by $V_n(t,B,S) : \overline{Q} \subset \mathbb{R}^3 \to \mathbb{R}$ the value function of the corresponding problem with a bounded number of transactions

$$V_n(t,B,S) = \sup_{p\in A_n(t,B,S)} J^p(t,B,S) \ .$$

It is not difficult to show that increasing the number of interventions V_n converges to V.

Theorem 7. *We have $\lim_{n\to\infty} V_n = V$ for all $(t,B,S) \in \overline{Q}$.*

PROOF. As $A_1(t,B,S) \subset A_2(t,B,S) \subset .... \subset A(t,B,S)$, it holds $V_1(t,B,S) \leq V_2(t,B,S) \leq .... \leq V(t,B,S)$ and $\lim_{n\to\infty} V_n \leq V$ for all $(t,B,S) \in \overline{Q}$. To obtain the reverse inequality consider an ϵ-optimal policy $p_\epsilon \in A(t,B,S)$ such that

$$V(t,B,S) \leq J^{p_\epsilon}(t,B,S) + \epsilon \ . \tag{20}$$

Setting $\overline{\tau}_i \equiv \tau_i^{p_\epsilon} \wedge \vartheta^{p_\epsilon}$, by (1) for a.a. ω there exists $n(\omega)$ such that $\overline{\tau}_n(\omega) = \vartheta^{p_\epsilon}(\omega)$. If we define

$$J_n^{\overline{p}}(t,B,S) = E_{t,B,S}\left[U\left(L(B^{p_\epsilon}(\overline{\tau}_n), S^{p_\epsilon}(\overline{\tau}_n)\right) e^{r(T-\overline{\tau}_n)})\right]$$

by the dominated convergence theorem it follows that

$$J^{p_\epsilon}(t,B,S) = \lim_{n\to\infty} J_n^{\overline{p}}(t,B,S),$$

and we can choose $\overline{n}$ such that

$$J^{p_\epsilon}(t,B,S) \leq J_{\overline{n}}^{\overline{p}}(t,B,S) + \epsilon \ . \tag{21}$$

Consider now the policy $\overline{p}_{\overline{n}} = \{(\overline{\tau}_i, \xi_i^{p_\epsilon})\}$, $i = 1, 2, \cdots, \overline{n}$, setting $\overline{\tau}_{\overline{n}+1} = \infty$ a.s.. We have $\overline{p}_{\overline{n}} \in A_{\overline{n}}(t,B,S)$ and combining (20) and (21) we obtain

$$V(t,B,S) \leq J^{\overline{p}_{\overline{n}}}(t,B,S) + 2\epsilon \ .$$

Since ϵ is arbitrary, it follows $V \leq \lim_{n\to\infty} V_n$ for all $(t,B,S) \in \overline{Q}$. □

We consider now the following iterative sequence of optimal stopping problems. Let $(B(s), S(s))$ be the uncontrolled process. We set

$$\theta \equiv \{\inf s \geq t : (B(s), S(s)) \notin Cor\}, \quad \text{and} \quad \vartheta \equiv \theta \wedge T.$$

and we define on $\overline{Q} \cap \mathbb{R}_+^3$

$$P_0(t,B,S) = E_{t,B,S}\left[U\left(L(B(\vartheta), S(\vartheta)\right) e^{r(T-\vartheta)})\right]$$

that is the expected utility without interventions, starting with nonnegative B and S (to be sure the process does not exit from $\overline{Adr}$ before ϑ). Then we define, recursively, for $n \geq 1$

$$\begin{aligned} P_n(t,B,S) \quad = \quad & \sup_{\tau \in A_1(t,B,S)} E_{t,B,S}\Big[\mathcal{M}P_{n-1}(\tau, B(\tau), S(\tau))\chi_{\tau<\vartheta} \\ & + U\left(L(B(\vartheta), S(\vartheta)\right) e^{r(T-\vartheta)})\chi_{\tau\geq\vartheta}\Big] \end{aligned} \tag{22}$$

for all $(t,B,S) \in \overline{Q}$. Here $\mathcal{M}P_{n-1}$ is defined by (7) and it is a given function at step n (note that P_0 is defined in $\overline{Q} \cap \mathbb{R}_+^3$ but all P_n and $(\mathcal{M}P_{n-1})$, $n \geq 1$, are defined in $\overline{Q}$). To the optimal stopping problem (22) it is associated the variational inequality

$$\min\left\{-\frac{\partial P_n}{\partial t} - \mathcal{L}P_n, P_n - \mathcal{M}P_{n-1}\right\} = 0 \ . \tag{23}$$

Using the same techniques of the preceding section it is not difficult to show that P_n is the unique constrained viscosity solution of (23) verifying the same boundary conditions of (8), where $\mathcal{M}P_n$ is replaced by $\mathcal{M}P_{n-1}$. By the following theorem we can reduce the impulse control problem to the sequence (22) of optimal stopping problems.

Theorem 8. *For all* $(t,B,S) \in \overline{Q}$ *and* $n \geq 1$ *it holds* $P_n(t,B,S) = V_n(t,B,S)$. *Moreover for each* $(t,B,S) \in \overline{Q}$ *there exists* $p^* \in A_n(t,B,S)$ *such that*

$$V_n(t,B,S) = J^{p^*}(t,B,S) \ .$$

Proof. We first show that $P_n \geq V_n$, $\forall (t,B,S) \in \overline{Q}$. Let $p \in A_n(t,B,S)$, with $p = \{\tau_i^p, \xi_i^p\}_{i=1,\cdots,n}$, and (B^p, S^p) the corresponding controlled process. Since $\mathcal{M}P_{n-1}$ is given at step n, the function P_n is, for any n, the value function of an optimal stopping problem. By using the dynamic programming principle for the value functions of optimal stopping problems (see Krylov, 1980, Chapter 3, Section 1), it can be shown, as in (Chancelier et al., 2002, Corollary 3.7), that the process

$$Z_n(s) = P_n(s \wedge \vartheta^p, B^p(s \wedge \vartheta^p), S^p(s \wedge \vartheta^p)), \quad s \geq t \tag{24}$$

is a supermartingale, for any n and any given stopping time $\alpha \geq t$. From the optional sampling theorem it follows that if $t \leq \alpha_1 \leq \alpha_2$ are stopping times then we have

$$\begin{aligned} & E_{t,B,S} \left[P_n(\alpha_1 \wedge \vartheta^p, B^p(\alpha_1 \wedge \vartheta^p), S^p(\alpha_1 \wedge \vartheta^p)) \right] \\ \geq \quad & E_{t,B,S} \left[P_n(\alpha_2 \wedge \vartheta^p, B^p(\alpha_2 \wedge \vartheta^p), S^p(\alpha_2 \wedge \vartheta^p)) \right] \ . \end{aligned} \tag{25}$$

Define $\overline{\tau}_0 \equiv 0$, $\overline{\tau}_i \equiv \tau_i^p \wedge \vartheta^p$ and let $(B^p(s), S^p(s)) = (B(s), S(s))$ in any interval $[\overline{\tau}_j, \overline{\tau}_{j+1})$. By (25) and the definitions (7) and (22) we obtain for $j = 0, ..., n-1$

$$\begin{aligned} & E_{t,B,S} \left[P_{n-j}(\overline{\tau}_j, B(\overline{\tau}_j), S(\overline{\tau}_j)) \right] \geq E_{t,B,S} \left[P_{n-j}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-)) \right] \\ & = E_{t,B,S} \left[P_{n-j}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-)) \right]_{\chi_{\tau^p_{j+1i} \leq \vartheta^p}} \\ & + E_{t,B,S} \left[P_{n-j}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-)) \right]_{\chi_{\tau^p_{j+1i} > \vartheta^p}} \\ & \geq E_{t,B,S} \left[\mathcal{M}P_{n-j-1}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-)) \right]_{\chi_{\tau^p_{j+1i} \leq \vartheta^p}} \\ & + E_{t,B,S} \left[P_{n-j-1}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}^-), S(\overline{\tau}_{j+1}^-)) \right]_{\chi_{\tau^p_{j+1i} > \vartheta^p}} \\ & \geq E_{t,B,S} \left[P_{n-j-1}(\overline{\tau}_{j+1}, B(\overline{\tau}_{j+1}), S(\overline{\tau}_{j+1})) \right] \ . \end{aligned} \tag{26}$$

Summing up all these inequalities from $j = 0$ to $j = n-1$ we obtain

$$P_n(t,B,S) \geq E_{t,B,S} \left[P_0(\overline{\tau}_n, B(\overline{\tau}_n), S(\overline{\tau}_n)) \right]. \tag{27}$$

By property (25) we also have

$$\begin{aligned} & E_{t,B,S} \left[P_0(\overline{\tau}_n, B(\overline{\tau}_n), S(\overline{\tau}_n)) \right] \geq E_{t,B,S} \left[P_0(\vartheta^p, B(\vartheta^p), S(\vartheta^p)) \right] \\ & = E_{t,B,S} \left[U\left(L(B(\vartheta^p), S(\vartheta^p)) \, e^{r(T-\vartheta^p)} \right) \right] = J^p(t,B,S) \ . \end{aligned} \tag{28}$$

Thus we have shown that $P_n(t,B,S) \geq J^p(t,B,S)$, $\forall p \in A_n(t,B,S)$ and $P_n \geq V_n$, $\forall (t,B,S) \in \overline{Q}$.

To obtain the reverse inequality we build an optimal policy $p^* \in A_n(t,B,S)$ such that $J^{p^*}(t,B,S) = P_n(t,B,S)$. First of all, let us define the control sets

$$C_i \equiv \left\{ (t,B,S) \in \overline{Q} : P_i(t,B,S) = \mathcal{M}P_{i-1}(t,B,S) \right\}, \quad i = 1, \cdots, n.$$

Moreover, let I_1 be the set

$$I_1 \equiv \{\vartheta \geq s \geq t : (s, B(s), S(s)) \in C_n\}.$$

We choose τ_1^* such that

$$\tau_1^* = \begin{cases} \inf I_1 & \text{if } I_1 \neq \emptyset \\ +\infty & \text{if } I_1 = \emptyset \end{cases}$$

and ξ_1^* is given by

$$\xi_1^* = \begin{cases} \xi^*_{P_{n-1}}(B(\tau_1^{*\,-}), S(\tau_1^{*\,-})) & \text{if } \tau_1^* < \infty \\ \text{arbitrary if } \tau_1^* = \infty \end{cases}$$

where $\xi^*_{P_{n-1}}(B,S)$ is defined in Lemma 2 (c). If α_1, α_2 are stopping times such that $t \leq \alpha_1 \leq \alpha_2 \leq \tau_1^*$, it follows by the dynamic programming principle that (25) becomes an equality. See (Chancelier et al., 2002, Corollary 3.7b) and (Oksendal and Sulem, 2007, Chapter 7). From this fact and the choice of (τ_1^*, ξ_1^*), all the inequalities in (26) become equalities and, being $\overline{\tau}_1^* \equiv \tau_1^* \wedge \vartheta$, we obtain

$$P_n(t,B,S) = E_{t,B,S}\left[P_{n-1}(\overline{\tau}_1^*, B(\overline{\tau}_1^*), S(\overline{\tau}_1^*))\right]. \quad (29)$$

Now we define the policy p^* recursively by

$$\begin{cases} \tau_i^* = \begin{cases} \inf I_i & \text{if } I_i \neq \emptyset \\ +\infty & \text{if } I_i = \emptyset \end{cases} \\ \xi_i^* = \begin{cases} \xi^*_{P_{n-i}}(B(\tau_i^{*\,-}), S(\tau_i^{*\,-})) & \text{if } \tau_i^* < \infty \\ \text{arbitrary if } \tau_i^* = \infty \end{cases} \end{cases} \quad (30)$$

for $i = 1, ..., n$, with $\tau_0^* \equiv 0$ and where I_i is the random interval

$$I_i \equiv \left\{\vartheta \geq s \geq \tau_{i-1}^* : (\tau_i^*, B^{p^*}(\tau_i^{*\,-}), S^{p^*}(\tau_i^{*\,-})) \in C_{n+1-i}\right\}.$$

By the same argument of (29) we have

$$E_{t,B,S}\left[P_{n-i}(\overline{\tau}_i^*, B^{p^*}(\overline{\tau}_i^*), S^{p^*}(\overline{\tau}_i^*))\right] = E_{t,B,S}\left[P_{n-i-1}(\overline{\tau}_{i+1}^*, B^{p^*}(\overline{\tau}_{i+1}^*), S^{p^*}(\overline{\tau}_{i+1}^*))\right], \quad (31)$$

with $\overline{\tau}_i^* \equiv \tau_i^* \wedge \vartheta^{p^*}$. Considering all the n equalities (31) we end the proof with

$$\begin{aligned} P_n(t,B,S) &= E_{t,B,S}\left[P_0(\overline{\tau}_n^*, B^{p^*}(\overline{\tau}_n^*), S^{p^*}(\overline{\tau}_n^*))\right] \\ &= E_{t,B,S}\left[P_0(\vartheta^{p^*}, B(\vartheta^{p^*}), S(\vartheta^{p^*}))\right] \\ &= E_{t,B,S}\left[U\left(L(B(\vartheta^{p^*}), S(\vartheta^{p^*}))\, e^{r(T-\vartheta^{p^*})}\right)\right] = J^{p^*}(t,B,S)\,. \end{aligned}$$

□

Therefore, as $\lim_{n\to\infty} V_n = \lim_{n\to\infty} P_n = V$, we can compute the value function by solving the sequence (23) of variational inequalities. Each solution $V_n = P_n$ has the meaning of the value function of the same problem with at most n transactions. Moreover the optimal trading strategy p^* described in Theorem 8 gives us, for n large enough, a payoff which is arbitrarily close to the optimal one. In our numerical experiments we have simplified the domain in Figure 1 as in Figure 2, i.e., we have prolonged the segments AB and CD respectively up to the points I and F considering $V(t,B,S)$ continuous at the boundary ∂Q,

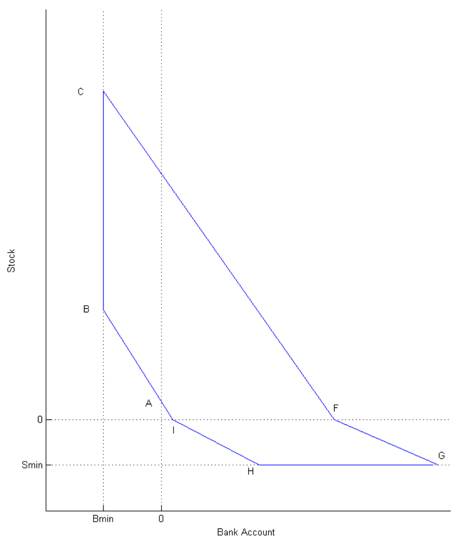

Figure 2: Simplified value function domain for numerical discretization. Coordinates of vertices as in Figure 1.

and therefore everywhere. This corresponds to impose a transaction whenever the investor liquidates his position, i.e., to assume $L(B,S) = S + B - K - c|S|$. We are quite confident that, for the small values of K we used in our numerical experiments, assuming V continuous everywhere is irrelevant for the numerical results. We denote by $\mathcal{D}$ the numerical domain and we set $\overline{Q}' = [0,T] \times \mathcal{D}$. Thus we have slightly modified the boundary conditions stated in the previous section, setting:

- $V(t,B,S) = U(L_{\max} e^{r(T-t)})$, $\forall t \in [0,T]$, along the entire edges CF and FG;
- $V(t,B,S) = 0$ $\forall t \in [0,T]$ along the entire edges BI and HI.

Moreover, we compute the function $P_0(t,B,S)$, that is the expected utility without interventions, solving the PDE

$$-\frac{\partial P_0}{\partial t} - \mathcal{L} P_0 = 0$$

in $\overline{Q}' \cap \mathbb{R}^3_+$ with the additional boundary conditions, when $S = 0$ or $B = 0$:

- $P(t,B,0) = U(Be^{r(T-t)})$;
- $\mu S \frac{\partial P}{\partial S}(t,0,S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2}(t,0,S) + \frac{\partial P}{\partial t}(t,0,S) = 0$.

Thus we are now ready to deal with the numerical discretization of our optimal stopping problem.

4.1. Discretization

Each variational inequality (23) can be solved by a discrete approximation using the finite element method (see (Achdou and Pironneau, 2005; Marazzina et al., 2012) for instance, and Barucci and Marazzina (2012) for an application to a financial optimization problem). Setting $\mathcal{L}_t V = -\frac{\partial V}{\partial t} - \mathcal{L}V$, we discretize the PDE $\mathcal{L}_t V = 0$ with a finite element technique based on polynomial of degree 1, coupled with a Crank-Nicholson scheme. We consider a triangular mesh onto the space $\mathcal{D}$ with N nodes and a equally-spaced time grid $0 = t_0 < t_1 < \cdots < t_W = T$, of W time steps. Denoting by v^i the discrete approximation of V_i, if (B_n, S_n) is a vertex of the mesh, $n = 1, ..., N$, we set $v^i_{j,n} = v^i(t_j, B_n, S_n)$. As proved in Wilmott et al. (1993) with reference to American options, the i-th discrete variational inequality can be solved backward-in-time ($j = W-1, \cdots, 0$) by the following algebraic systems in the unknown vectors $\mathbf{v}^i_j$:

$$\mathbf{v}^i_j \geq \mathbf{M}^i_j, \quad \mathbf{A}\mathbf{v}^i_j \geq \mathbf{b}^i_{j+1}, \quad \left(\mathbf{v}^i_j - \mathbf{M}^i_j\right)\left(\mathbf{A}\mathbf{v}^i_j - \mathbf{b}^i_{j+1}\right) = 0 \tag{32}$$

Here $\mathbf{v}^i_j$ is the N dimensional vector $v^i(t_j, ., .)$, the obstacle $\mathbf{M}_j$, depending on the solution v^{i-1}, is defined by $M^i_{j,n} = Mv^{i-1}(t_j, B_n, S_n)$, $\mathbf{A}$ is the Crank-Nicholson finite element matrix associated to the operator $\mathcal{L}_t$, and the vector $\mathbf{b}^i_{j+1}$ is constructed using vector $\mathbf{v}^i_{j+1}$.

Problem (32) can be solved using a Projected SOR (PSOR) algorithm (see Wilmott et al. (1993)). To compute $\mathbf{v}^i_j$, we used as first guess solution $\mathbf{v}^i_{j+1}$ and we stopped the PSOR iterations when the L^∞ distance between two consecutive solutions falls under a given tolerance (TOL). Similarly, we considered v^i a good approximation of $V(t, B, S)$, the value function of our problem, when the distance between v^i and v^{i-1} falls under another given tolerance (TOL2). In the next section we show, by an example, the convergence of our numerical scheme when we increase the number of mesh nodes and time steps.

5. Numerical results

In this section we present extended numerical results in the case of a CRRA utility function

$$U(L) = \frac{L^\gamma}{\gamma}$$

with $0 < \gamma < 1$. This utility function, which is the most commonly used in the literature, belongs to the class of hyperbolic absolute risk aversion (HARA) utility functions. Using these functions the Merton's portfolio problem without transaction costs admits closed form solutions. Therefore it is possible to compare these exact solutions with the numerical results in the presence of liquidity costs. The main alternative in this class would be to consider the exponential utility which implies a constant absolute risk aversion. However, if we consider our portfolio problem without transaction costs and exponential utility, the optimal strategy would be to maintain constant the discounted amount of money invested in the risky asset, which appears to be a very unrealistic policy (see Korn, 1997, chapter 3). In all the case studies we have set the values $B_{\min} = S_{\min} = -20$, $L_{\min} = 0$, $L_{\max} = 100$, TOL=10^{-5} (the tolerance error in the PSOR algorithm), TOL2=0.001 (the tolerance error to exit from the iterated optimal stopping cycle). In the following, we investigate the form of the

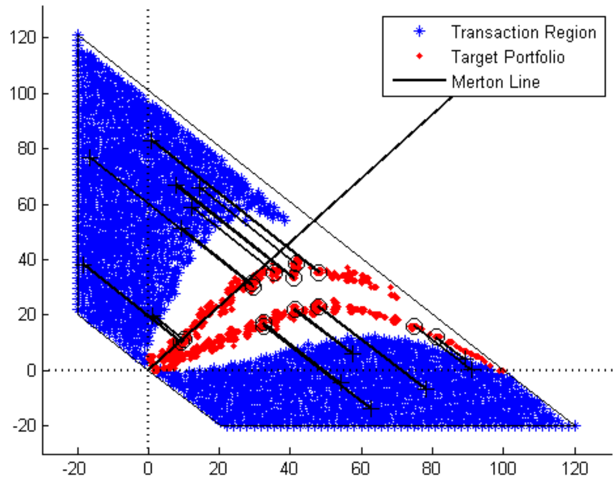

Figure 3: Transaction region in the plane (B, S). Time $t = 0$, $N = 5000$, $W = 250$, $K = 0.1$ and $c = 0.01$.

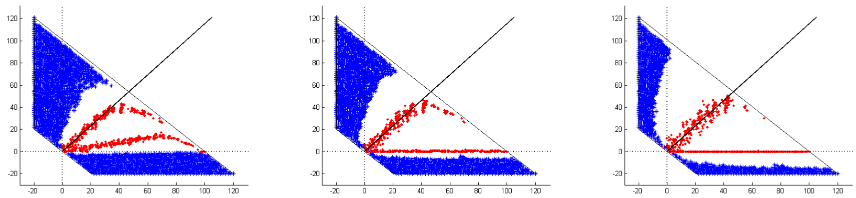

Figure 4: Transaction region in the plane (B, S). Time $t = 0.5$ (left) - $t = 0.75$ (middle) - $t = 0.9$ (right), $N = 5000$, $W = 250$, $K = 0.1$ and $c = 0.01$.

optimal transaction strategy and we describe how it varies with different values of the model parameters. Moreover we show how dramatic is the impact of transaction costs on the frequency of trading of an optimal policy.

In our first numerical experiment, which we use as base case, we set the following values of the model parameters: $K = 0.1$, $c = 0.01$, $r = 0.04$, $\mu = 0.1$, $\sigma = 0.4$, $\gamma = 0.3$ and $T = 1$.

Figures 3 and 4 show the corresponding (blue) transaction regions and (white) continuation regions, at four different time instants: $t = 0$, 0.5, 0.75, 0.9. The red lines represent the re-calibrated portfolios, i.e. the portfolios where it is optimal to move when the investor's position falls in the intervention area. After a possible first transaction, made if the initial portfolio is in the intervention region, the investor will maintain his position inside the white regions re-calibrating his portfolio only if it reaches the boundary of the blue areas. In Figure 3 some optimal transactions at $t = 0$ have been depicted: these lines connect the threshold portfolios in the transaction region to the corresponding target portfolios on the red lines. The target portfolios are always inside the continuation region because the intervention costs make two consecutive transactions unprofitable. Moreover the upper (lower) red line are the target points of the upper (lower) part of the transaction area. Unlikely the infinite horizon case (see Dumas and Luciano, 1991; Davis and Norman, 1990; Shreve and Soner, 1994), the optimal policy is not stationary: the transaction regions, as well as the target portfolios, change as time goes by. As expected the size of the intervention regions decreases as the time increases because, approaching the finite horizon, only a large change in the portfolio composition can compensate the transaction costs. However the evolution of the two parts of the transaction region is not symmetric. The size of the lower part decreases faster than the upper one. This reveals that the finite horizon and the bounded liquidation region induce a bias, as time goes on, in favor of the riskless asset. For example in $t = 0.75$ the lower blue region is already below the axis $S = 0$. This implies that if in $t = 0.75$ the investor has a long position in stock he will never buy again the stock up to $T = 1$. Similarly the lower red line decreases with time towards the axis $S = 0$, and it is already equal to the axis $S = 0$ in $t = 0.75$. The same kind of liquidity preference in case of shorter investment horizons can be noted if we fix $t = 0$ and we consider a variable terminal date T, as we will do in Section 5.2. The more distant is the horizon T, the lower is the no-transaction region and the percentange of cash which is allowed to remain in the portfolio. This result is consistent with the common life-cycle investment advice that a young investor should hold a greater share of stocks in the portfolio than an old investor, see on this point Liu and Loewenstein (2002). In the graphs the Merton straight line is also depicted, which is constant in time and it represents the optimal portfolios for the same problem but without the transaction costs. It is interesting to note that the upper red line remains approximately equal to the Merton line. In both Figures 3-4 the two optimal lines move down approaching to the edges CF and FG of the liquidation region (see Figure 2). This is due to the fact that the portfolio liquidation value is already near to $L_{\max} = 100$, the value considered satisfactory by the investor. Probably he will liquidate his position in short time and before $T = 1$, this induces again a bias in favor of cash. For all t and most of the liquidation domain, the shape of the continuation region closely resembles a cone (enlarging with time) containing two straight lines of optimal portfolios. We conjecture that this would be the exact shape if we considered the same problem with an unbounded liquidation region ($L_{\max} = +\infty$, i.e. the investor is never satisfied before $T = 1$). Figure 5 shows the value and the shape of the value function at time $t = 0$ and the decrease in the optimal expected utility between time 0 and time 0.5.

In Table 1 and 2 we illustrate the convergence of our numerical scheme. We consider the solutions at $t = 0$ increasing the number of sub-intervals of the time-grid (W), and the number of mesh-points (N). In Table 1 we compute the L^2-norm error assuming as exact solution the one computed with $W = 250$ and $N = 5000$. More specifically in the upper part of the table we fix $N = 5000$ and we show the convergence increasing the time grid. Conversely, in the lower part we fix $W = 250$ and we make the space grid more dense. As expected, in both cases the solutions converge. Moreover, in Table 2 we show the convergence when we increase W and N at the same time; we do not assume an exact solution (an analytical solution is not available) but we list the distances, increasing both W and N, between two consecutive solutions in the numerical sequence. We consider both the L^2 and L^∞ relative errors. To show the convergence of the optimal control regions, we have also computed the Hausdorff distances

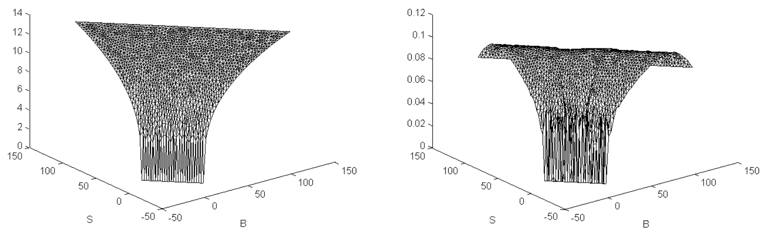

Figure 5: Value function in the plane (B, S). Solution at $t = 0$ (left) - Difference between the solution at $t = 0$ and $t = 0.5$ (right). $N = 5000$, $W = 250$, $K = 0.1$ and $c = 0.01$.

Distance from the $W = 250$ solution, setting $N = 5000$				
$W =$	25	50	100	200
	0.0051	0.0026	0.0012	0.0008

Distance from the $N = 5000$ solution, setting $W = 250$				
$N =$	1000	2000	3000	4000
	0.0032	0.0014	0.0011	0.0005

Table 1: L^2 distance from the solution with $W = 250$ and $N = 5000$ at time $t = 0$, increasing W (above) and N (below).

W	N	L^∞	L^2	HD1	HD2	Iterations	CPU Time (s)
25	1000	-	-	-	-	6	283
50	2000	0.0324	0.0023	0.0734	0.1562	6	1578
75	3000	0.0219	0.0014	0.0565	0.0720	5	3715
100	4000	0.0163	0.0011	0.0350	0.0348	5	8435
250	5000	0.0098	0.0005	0.0291	0.0296	4	23635

Table 2: L^∞ and L^2 errors and Hausdorff distances between the transaction regions (HD1) and the optimal lines (HD2) of two consecutive solutions in the sequence, at time $t = 0$.

(normalized by the length of the domain L) between the transaction regions and between the target portfolios of the consecutive solutions (the Hausdorff distance is the supremum of the distances of the points in one region to the other region, and vice versa). Both Tables 1 and 2 indicate a rapid convergence of the solutions and of the optimal regions. Finally in Table 2 we also list the number of variational inequalities (number of iterations above the obstacle) which were necessary to achieve the TOL2 convergence and the CPU time necessary for the computation. All the computation have been performed in Matlab R2011a and on a personal computer equipped with a Pentium Dual-Core 2.70 GHz and 4 GB RAM.

5.1. Sensitivity analysis

In this section we make a comparative static analysis to investigate how the optimal policy is influenced by the different model parameters. The numerical results have been obtained imposing $W = 100$ and $N = 4000$. Except for the

	$t=0$				$t=0.5$			
$K\backslash c$	0.01	0.005	0.001	0	0.01	0.005	0.001	0
0.5	0.3938	0.4200	0.4397	0.4501	0.2352	0.2533	0.2989	0.3020
0.25	0.4876	0.5288	0.5703	0.5795	0.3764	0.4095	0.4684	0.4735
0.1	0.5841	0.6239	0.6577	0.6607	0.4873	0.5455	0.6138	0.6267
0.05	0.6362	0.6657	0.6804	0.6845	0.5424	0.6667	0.6780	0.6801
0.01	0.6990	0.7307	0.7708	0.7753	0.6185	0.6928	0.7559	0.7667

Table 3: Transaction Region in the plane (B, S) (percentage) for different values of fixed (K) and proportional (c) transaction costs. Other parameters: $r = 0.04$, $\mu = 0.1$, $\sigma = 0.4$, $\gamma = 0.3$ and $T = 1$.

parameters under investigation, the values of the other parameters are the same as in the preceding base case.

5.1.1. Sensitivity with respect to the transaction costs

Naturally enough, increasing the transaction costs, the size of the intervention region decreases. Due to the finite horizon T, if we increase K and c only fewer large transactions can be profitable. In Table 3 we show the percentage of the transaction region on the overall domain decreasing both K and c. Figure 6 depicts the optimal regions of some of the cases considered in the table. Increasing the transaction costs produces a variation in the optimal policy which is similar to that caused by approaching the finite horizon T. The lower part of the intervention region decreases faster than the upper one, indicating a shift towards the riskless asset which is not present with an infinite horizon and an unbounded domain. The lower target portfolios are soon made only of the riskless asset while the upper optimal line stays close to the Merton one. It is interesting to observe how the optimal policy varies when we change the size of the variable cost c with respect to the fixed component K. In Figure 7 we set $K = 0.1$ and we consider different values of c. For vanishing c the lower optimal line converge to the upper one, i.e., the Merton line. In fact, in all our experiments with $c = 0$ we have only one line of target portfolios. Conversely, an increase in c pull the lines apart and closer to the intervention region. Vanishing K the solution tends towards the solution of a singular control problem where the optimal policy is an instantaneous reflection at the boundary of the intervention region (see Davis and Norman, 1990; Shreve and Soner, 1994). This behaviour of the optimal control sets, varying the relative size of the variable and of the fixed part of the intervention costs, has already been noted, for a cash management problem, in Baccarin (2009).

5.1.2. Sensitivity with respect to the other model parameters

When we change the market parameters σ and r or the relative risk aversion coefficient $(1-\gamma)$ the Merton line varies its position and the no transaction area follows it in the same direction. If σ or $(1-\gamma)$ increase the investor will hold more of the riskless asset because he is risk averse, and, if r increases, he will hold more cash since the stock becomes less attractive. Consequently the Merton's line moves down towards the axis $S = 0$. In Table 4 we show the percentage of the transaction area on the overall domain increasing r, σ and $(1-\gamma)$. This percentage grows in all cases, essentially because the upper optimal line follows closely the Merton's one and the upper part of the transaction region becomes

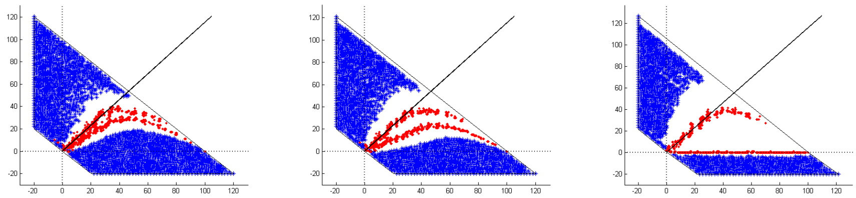

Figure 6: Transaction area in the plane (B, S). Time $t = 0$, $K = 0.05$, $c = 0.005$ (left) - $K = 0.1$, $c = 0.01$ (middle) - $K = 0.25$, $c = 0.05$. Other parameters: $r = 0.04$, $\mu = 0.1$, $\sigma = 0.4$, $\gamma = 0.3$ and $T = 1$.

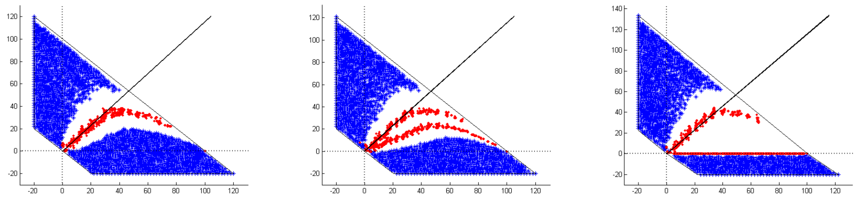

Figure 7: Transaction area in the plane (B, S). Time $t = 0$, $K = 0.1$, $c = 0.001$ (left) - $K = 0.1$, $c = 0.01$ (middle) - $K = 0.1$, $c = 0.1$ (right). Other parameters: $r = 0.04$, $\mu = 0.1$, $\sigma = 0.4$, $\gamma = 0.3$ and $T = 1$.

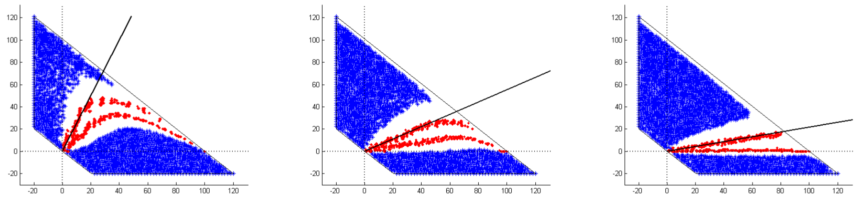

Figure 8: Transaction area in the plane (B, S). Time $t = 0$, $r = 0.02$ (left) - $r = 0.06$ (middle) - $r = 0.08$ (right). Other parameters: $\sigma = 0.4$, $\gamma = 0.3$, $\mu = 0.1$, $K = 0.1$, $c = 0.01$ and $T = 1$

bigger. In Figure 8 we illustrate the optimal regions for some increasing values of r. The qualitative behavior of the optimal policy increasing $(1 - \gamma)$ or σ is similar.

5.2. The impact of transaction costs on the frequency of trading and on the value of the final portfolio

In order to get an estimate of the number of transactions that an investor will make if he follows the optimal policy we have coupled our numerical solution to a Monte Carlo simulation. Precisely we have considered an agent with initial portfolio made only of cash, $B_0 = 20$, $S_0 = 0$, who behaves according to the optimal intervention and continuation regions that we have computed using our numerical procedure. In this section, where we also consider distant horizons, we set the drift parameter of the risky asset $\mu = 0.06$ and the risk-free interest rate $r = 0.02$ (the other model parameters are the same as in our base case). If the investor's position is in the continuation region, which changes dynamically according to our numerical solution, we simulate the evolution of the stock value

	r			
t	0.02	0.04	0.06	0.08
0	0.5713	0.5841	0.6111	0.6746
0.5	0.4565	0.4873	0.5502	0.6185

	σ			
t	0.3	0.4	0.5	0.6
0	0.4748	0.5841	0.6595	0.7122
0.5	0.2880	0.4873	0.6003	0.6763

	γ			
t	0.2	0.3	0.4	0.5
0	0.6085	0.5841	0.5539	0.5109
0.5	0.5243	0.4873	0.4332	0.3748

Table 4: Transaction Region in the plane (B, S) (percentage) for different values of interest rate (r), volatility (σ) and risk aversion coefficient (γ). Other parameters: $\mu = 0.1$, $K = 0.1$, $c = 0.01$ and $T = 1$.

$S(t)$ by a computer generated random walk (the bank account $B(t)$ grows in a deterministic way). Whenever the simulated portfolio falls into the transaction region the agent re-calibrates its portfolio moving to the corresponding (at that time instant) optimal target portfolio and paying the necessary transaction costs. The Monte Carlo simulations were performed with 100 time steps, according to the time grid of the numerical solution computed with $W = 100$ and $N = 4000$. In Table 5 we show the average and the standard deviation of the number of transactions, computed using 500 000 simulations, considering $T = 1$ and different values for the transaction costs. Rather surprisingly, it is clear from this table that the optimal policy is essentially a buy-and-hold trading strategy. In fact, in almost all simulations, or the number of transactions is equal to zero, i.e., the investor does not perform any transaction, or is equal to two, that is the investor transacts at time 0, moving to the lower optimal line, and at time $T = 1$ (or before $T = 1$, if the agent's portfolio exits from the domain). Thus transaction costs result into a strong change in the optimal behavior of the agent: we recall that in the Merton's model, without these costs, the investor transacts continuously at every time instant. It is natural to ask how distant must be the horizon, to have a significant number of interventions. In Tables 6 we show the average number of transactions for increasing values of T, up to forty years. It is surprising to observe that, even with the smallest transaction costs ($K = 0.01$ and $c = 0$), on average more than three years are necessary to have a third transaction and that less than five interventions are made every ten years on the overall period ($T = 40$). Note that if we consider a thousand euros as the unit of measure, $K = 0.01$ means a cost of 10 euros for each transaction, which is a very low fixed cost to rebalance a portfolio of initial value $B_0 = 20$ thousand euros.

It is also interesting to compare some alternative policies with the optimal one. In Table 7 we have considered the following trading strategies:

- the risk-free strategy (RF): the agent only invests his wealth in the risk-free asset

	Average				Standard Deviation			
	c				c			
K	0.01	0.005	0.001	0	0.01	0.005	0.001	0
0.5	0	0	0	0	0	0	0	0
0.1	0	0	2	2.0002	0	0	0	0.0141
0.01	2.0000	2.0028	2.0049	2.0484	0.0045	0.0532	0.0700	0.2168

Table 5: Average (left) and standard deviation (right) of the number of transactions. Parameters: $r = 0.02$, $\mu = 0.06$, $\sigma = 0.4$, $\gamma = 0.3$ and $T = 1$. Initial portfolio: $B_0 = 20$, $S_0 = 0$. Number of simulations equal to 500 000.

		T					
K	c	2	3	5	10	20	40
		Average					
0.1	0.01	2.0007	2.0167	2.0952	2.6626	3.8775	6.1432
0.1	0.005	2.0014	2.0223	2.1233	2.7154	3.9962	6.3354
0.1	0.001	2.0030	2.0283	2.1454	2.7573	4.1726	6.4378
0.1	0	2.0052	2.0322	2.2523	2.8319	4.2174	6.8480
0.01	0.01	2.0414	2.2146	2.9961	4.6006	7.6706	12.5226
0.01	0.005	2.2142	2.5109	3.0501	4.8450	8.0899	13.5611
0.01	0.001	2.2150	2.6285	3.2504	5.5152	8.9542	15.1048
0.01	0	2.2629	2.7840	3.5118	5.6233	9.9649	17.1312
		Standard Deviation					
0.1	0.01	0.0257	0.1361	0.3072	0.7238	1.1943	1.8453
0.1	0.005	0.0379	0.1543	0.3481	0.8113	1.2589	1.8552
0.1	0.001	0.0545	0.1664	0.3761	0.8438	1.2771	1.8743
0.1	0	0.0649	0.1988	0.4733	0.8692	1.4450	1.9174
0.01	0.01	0.2027	0.4427	0.9364	1.6249	2.3478	3.4454
0.01	0.005	0.4256	0.6821	0.9717	1.6285	2.3940	3.5861
0.01	0.001	0.5263	0.7030	1.0277	1.8462	3.0124	5.8325
0.01	0	0.5287	0.8204	1.1833	1.9236	4.9104	8.7569

Table 6: Average (up) and standard deviation (down) of the number of transactions. Parameters: $r = 0.02$, $\mu = 0.06$, $\sigma = 0.4$, and $\gamma = 0.3$. Initial portfolio: $B_0 = 20$, $S_0 = 0$. Number of simulations equal to 500 000.

- the Merton strategy (Mer): it is the optimal strategy without transaction costs. The expected utility of the final position is given by the closed formula

$$E\left[U(L(B(T),S(T)))\right] = \frac{B_0^\gamma}{\gamma}\exp\left(\gamma(r+\frac{(\mu-r)^2}{2\sigma^2(1-\gamma)})T\right)$$

- the optimal strategy (Opt): in this case we have $E\ [U(L(B(T),S(T)))] = V(B_0,0,T)$. To obtain the average number of transactions we couple the Monte Carlo simulation with the numerical solution, as described above
- the Merton strategy with transaction costs: the agent recalibrates his portfolio moving to the Merton's line when the distance between his portfolio and the line itself is bigger than 5% (MTC(5%)) or 10% (MTC(10%)) of his wealth

c	T	RF	Mer	Opt	MTC(5%)	MTC(10%)	Bar(1%)
		Certainty Equivalent					
0.01	3	21.236	21.696	21.482	21.393	21.473	21.445
0.01	5	22.103	22.906	22.747	22.539	22.651	22.520
0.01	10	24.427	26.236	26.014	25.687	25.889	25.481
0.005	3	21.236	21.696	21.531	21.480	21.511	21.435
0.005	5	22.103	22.906	22.789	22.626	22.722	22.506
0.005	10	24.427	26.236	26.058	25.778	25.979	25.470
0	3	21.236	21.696	21.629	21.553	21.624	21.431
0	5	22.103	22.906	22.816	22.685	22.801	22.487
0	10	24.427	26.236	26.099	25.885	26.081	25.439
		Average Number of transactions					
0.01	3	0	∞	2.21	13.42	5.44	5.23
0.01	5	0	∞	3.00	18.87	7.44	6.14
0.01	10	0	∞	4.60	29.14	11.76	7.78
0.005	3	0	∞	2.51	13.43	5.44	5.33
0.005	5	0	∞	3.05	18.87	7.45	6.25
0.005	10	0	∞	4.84	29.14	11.77	7.83
0	3	0	∞	2.78	13.43	5.45	5.44
0	5	0	∞	3.51	18.87	7.46	6.37
0	10	0	∞	5.62	29.15	11.81	8.13

Table 7: Comparison of different strategies: certainty equivalent and average number of transactions. Parameters: $K = 0.01$, $r = 0.02$, $\mu = 0.06$, $\sigma = 0.4$, and $\gamma = 0.3$. Initial portfolio: $B_0 = 20$, $S_0 = 0$. Number of simulations equal to 500 000.

- the barrier strategy (Bar(1%)): here we assume that the no-transaction region is a time-independent cone delimited by two barriers passing through the origin. The agent recalibrates his portfolio only when his position touches one of the two barriers and he makes the minimal transactions necessary to stay inside the cone. We have fixed the barriers as follows: they define the biggest cone which remains included at time $t = 0$ in the optimal transaction region that we have computed numerically. To avoid unbounded transaction costs, due the fixed component K, we assumed that the portfolio is recalibrated towards the lower/upper barrier only if it falls below/above the barrier by more than the 1% of the agent's wealth.

For each of the last three strategies we have simulated 500 000 possible scenarios, and thus 500 000 possible values of $B(T)$ and $S(T)$, computing the mean value of $U(L(B(T), S(T))$ and the average number of transactions. To make a more readable comparison among the different policies, in Table 7, besides the average number of transactions, we have shown the certainty equivalent of the utility of the final positions, that is $U^{-1}(E\ [U(L(B(T), S(T)))])$. As expected, if we do not consider the Merton (Mer) strategy, without transaction costs, the optimal strategy is the best one, i.e., the one with the highest certainty equivalent. It is also the policy with the lowest average number of interventions. We also notice that the optimal strategy and the MTC(10%) one are close, while the Bar(1%) strategy is the worst one, despite a low number of transactions. Notice that the Bar(1%) strategy is similar to the trading strategy which has been proven optimal for portfolio optimization problems with only

γ	$c = 0.1$				$c = 0.001$			
	TR (%)	Av	Std	CE	TR (%)	Av	Std	CE
	$T = 1$							
0.1	0.704	0	0	20.404	0.820	2.022	0.068	20.434
0.2	0.702	0	0	20.404	0.810	2.016	0.056	20.454
0.3	0.670	0	0	20.404	0.806	2.004	0.070	20.476
0.4	0.639	0	0	20.404	0.789	2.002	0.039	20.506
0.5	0.586	0	0	20.404	0.758	2.002	0.047	20.543
0.6	0.517	0	0	20.404	0.728	2.001	0.037	20.590
0.7	0.434	0	0	20.404	0.685	2.000	0.006	20.677
	$T = 5$							
0.1	0.715	2.003	0.037	22.182	0.837	4.923	4.292	22.590
0.2	0.708	2.003	0.034	22.191	0.832	3.342	1.056	22.665
0.3	0.687	2.002	0.046	22.194	0.822	3.250	1.027	22.791
0.4	0.658	2.003	0.042	22.195	0.811	3.141	1.007	22.924
0.5	0.611	2.002	0.048	22.195	0.794	3.038	1.000	23.116
0.6	0.565	2.002	0.055	22.200	0.770	2.945	0.926	23.404
0.7	0.499	2.002	0.059	22.203	0.745	2.366	0.942	23.892
	$T = 10$							
0.1	0.727	4.059	1.073	24.643	0.839	6.578	4.191	25.638
0.2	0.723	3.705	1.124	24.664	0.836	5.980	2.003	25.775
0.3	0.705	3.025	0.957	24.683	0.826	5.515	1.846	26.074
0.4	0.673	2.553	0.986	24.718	0.815	5.012	1.998	26.345
0.5	0.632	2.519	0.837	24.790	0.803	4.596	1.790	26.774
0.6	0.591	2.444	0.774	24.932	0.785	4.211	1.674	27.400
0.7	0.544	2.207	0.854	25.139	0.764	4.023	1.948	28.335

Table 8: Sensitivity with respect to γ considering a fixed transaction cost $K = 0.01$: transaction region at $t = 0$ (TR), average (Av) and standard deviation (Std) of the number of transactions, and certainty equivalent for the optimal strategy with transaction costs (CE). Other parameters: $r = 0.02$, $\mu = 0.06$, and $\sigma = 0.4$. Initial portfolio: $B_0 = 20$, $S_0 = 0$. Number of simulations equal to 500 000.

proportional transaction costs (see Davis and Norman, 1990; Dumas and Luciano, 1991; Fleming and Soner, 1993; Liu and Loewenstein, 2002). Thus trying to use this kind of policy in the presence of a fixed cost K different from zero clearly becomes unprofitable (and the results are even worse if we decrease the 1% level). We also notice that this strategy results in a lower utility when the proportional cost c approaches to zero. This rather surprising effect depends on the increased number of transactions due to a smaller no-transaction region (cone), and thus on the increased fixed transaction costs.

Finally, in order to understand how the risk aversion index $1 - \gamma$ influences the agent's behavior, in Table 8 we report the average number of transactions for agents who use our optimal policy with different values of γ. We have considered $K = 0.01$, and $c = 0.1$ or $c = 0.001$ (the other parameters are the same considered in this section). We notice that, increasing γ, i.e., considering less risk-averse investors, both the percentage of the transaction region and the average number of transactions decrease. This is due to the fact that a more risk-averse agent prefers to pay higher transaction costs to maintain his portfolio into a less risky position.

References

Achdou, Y., Pironneau, O., 2005. Computational Methods for Option Pricing. SIAM Frontiers in Applied Mathematics.

Akian, M., Menaldi, J.L., Sulem, A., 1996. On an investment-consumption model with transaction costs. SIAM Journal of Control and Optimization 34-1, 329-364.

Akian, M., Sulem, A., Taksar, M., 2001. Dynamic optimization of long-term growth rate for a portfolio with transaction costs and logarithmic utility. Mathematical Finance 11-2, 153-188.

Assaf, D., Klass, M., Taksar, M., 1988. A diffusion model for optimal porfolio selection in the presence of brokerage fees. Mathematics of Operations Research 13, 277-294.

Baccarin, S., 2009. Optimal impulse control for a multidimensional cash management system with generalized cost functions. European Journal of Operational Research 196-1, 198-206.

Barles, G., 1994. Solutions de Viscosité des équations de Hamilton-Jacobi. Collection "Mathématiques et Applications" de la SMAI, no.17. Springer.

Barucci, E., Marazzina, D., 2012. Optimal investment, stochastic labor income and retirement. Applied Mathematics and Computation 218-9, 5588-5604.

Bensoussan, A., Lions, J.L., 1984. Impulse Control and Quasi-Variational Inequalities. Gauthiers-Villars, Paris.

Bielecki, T., Pliska, S.R., 2000. Risk sensitive asset management with transaction costs. Finance and Stochastics 4, 1-33.

Crandall, M.G., Ishii, H., Lions, P.L., 1992. User's guide to viscosity solutions of second order partial differential equations. Bulletin of the American Mathematical Society 27, 1-67.

Chancelier, J., Oksendal, B., Sulem, A., 2002. Combined stochastic control and optimal stopping, and application to numerical approximation of combined stochastic and impulse control. Proceedings of the Steklov Institute of Mathematics 237, 140-163.

Davis, M., Norman, A., 1990. Portfolio selection with transaction costs. Mathematics of Operations Research 15, 676-713.

Dumas, B., Luciano, E., 1991. An exact solution to a dynamic portfolio choice problem. The Journal of Finance 46, 577-595.

Eastham, J.E., Hastings, K.J., 1988. Optimal impulse control of portfolios. Mathematics of Operations Research 13, 588-605.

Fleming, W.H., Rishel, R.W., 1975. Deterministic and Stochastic Optimal Control. Springer-Verlag.

Fleming, W.H., Soner, M., 1993. Controlled Markov processes and viscosity solutions. Springer.

Korn, R., 1997. Optimal Portfolios. World Scientific.

Korn, R., 1998. Portfolio optimization with strictly positive transaction costs and impulse control. Finance and Stochastics 2, 85-114.

Krylov, N.V., 1980. Controlled Diffusion Processes. Springer.

Liu, H., Loewenstein, M., 2002. Optimal portfolio selection with transaction costs and finite horizon. The Review of Financial Studies 15-3, 805-835.

Ly Vath, V., Mnif, M., Pham H., 2007. A model of optimal portfolio selection under liquidity risk and price impact. Finance and Stochastics 11, 51-90.

Marazzina, D., Reichmann, O., Schwab, Ch., 2012. hp-DGFEM for Kolmogorov-Fokker-Planck equations of multivariate Lévy processes. M3AS: Mathematical Models and Methods in Applied Sciences 22-1, 1150005.1-37.

Merton, R.C., 1969. Lifetime portfolio selection under uncertainty: the continuous time case. The Review of Economics and Statistics 51, 247-257.

Kumar, S., Muthuraman, K., 2006. Multidimensional portfolio optimization with proportional transaction costs. Mathematical Finance 16-2, 301-335.

Morton, A.J., Pliska, S.R., 1995. Optimal portfolio management with fixed transaction costs. Mathematical Finance 5-4, 337-356.

Oksendal, B., Sulem, A., 2002. Optimal consumption and portfolio with both fixed and proportional costs. SIAM Journal of Control and Optimization 40-6, 1765-1790.

Oksendal, B., Sulem, A., 2007. Applied Stochastic Control of Jump Diffusions. Collection "Universitext", Springer.

Shreve, S., Soner, M., 1994. Optimal investment and consumption with transaction costs. Annals of Applied Probability 4, 609-692.

Wilmott, P., Howison, S., Dewynne, J., 1993. Option Pricing: Mathematical Models and Computation. Oxford Financial Press, Oxford.

Appendix A. Proof of Theorem 4

In this appendix we prove the weak comparison principle, adapting to our problem the techniques in (Akian et al., 2001; Barles, 1994; Ly Vath et al., 2007; Oksendal and Sulem, 2002) and giving all the necessary preliminary definitions and results. To prove comparison results for second-order equations is useful to give equivalent definitions of viscosity solutions in terms of parabolic second order super and subdifferentials (see Crandall et al., 1992). We will denote by $\mathcal{S}^2$ the set of all 2×2 symmetric matrices and, when it is convenient, by x the couple $(B, S) \in \overline{\Omega}$.

Definition 3. 1) The set of parabolic second order superdifferentials of a function $u : \overline{Q} \to \mathbb{R}$ at the point $(t,x) \in \overline{Q}$ is defined by

$$\begin{aligned} D^{+(1,2)}u(t,x) &= \Big\{(q,p,A) \in \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}^2 : \\ &\left.\limsup_{\substack{(h,y)\to 0 \\ (t+h,x+y)\in\overline{Q}}} \frac{u(t+h,x+y) - u(t,x) - qh - py - \frac{1}{2}Ay\cdot y}{|h| + |y|^2} \le 0 \right\} \end{aligned} \tag{A.1}$$

2) A triplet $(q,p,A) \in \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}^2$ belongs to $\overline{D}^{+(1,2)}u(t,x)$, the closure of $D^{+(1,2)}u(t,x)$, if there exists a sequence (t_m, x_m) converging to (t,x), and another sequence

$$(q_m, p_m, A_m) \in D^{+(1,2)}u(t_m, x_m)$$

converging to (q,p,A) as m tends to infinity.

The set $D^{-(1,2)}u(t,x)$ of parabolic second order subdifferentials of $u : \overline{Q} \to \mathbb{R}$ at $(t,x) \in \overline{Q}$ is defined in a symmetric way using the $\liminf$ and the $\ge$ inequality in (A.1) and the definition of its closure $\overline{D}^{-(1,2)}u(t,x)$ is analogous to the definition of $\overline{D}^{+(1,2)}u(t,x)$.

Definition 4. Given $\mathcal{O} \subset \overline{\Omega}$, a locally bounded function $u : \overline{Q} \to \mathbb{R}_+$ is called a viscosity subsolution (resp. supersolution) of (8) in $[0,T) \times \mathcal{O}$ if

$$\begin{aligned} \min\left\{-q - rBp_1 - \mu Sp_2 - \frac{1}{2}\sigma^2 S^2 A_{22}, u^*(t,x) - \mathcal{M}u^*(t,x)\right\} &\le 0 \\ \text{(resp. } u_* \text{ and } &\ge 0) \end{aligned}$$

for all $(t,x) \in [0,T) \times \mathcal{O}$, $(q,p,A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) \in \overline{D}^{+(1,2)}u^*(t,x)$ (resp. $\overline{D}^{-(1,2)}u_*(t,x)$).

In order to prove the weak comparison principle it is useful to obtain strict viscosity supersolutions of (8) in $Q = [0,T) \times \Omega$.

Lemma 9. *Fix $\delta' > \delta = \gamma\left(r + \frac{(\mu-r)^2}{2\sigma^2(1-\gamma)}\right)$ and consider the smooth perturbation function $g(t,B,S) = e^{\delta'(T-t)}(B+S)^\gamma$. Let $v \in LSC(\overline{Q})$ be a viscosity supersolution of (8) in Q. Then for any $\varepsilon > 0$ the lsc function $v_\varepsilon = v + \epsilon g$ is a strict viscosity supersolution of (8) in any compact set $G \subset Q$. This means that for any compact $G \subset Q$ there exists a constant $\rho > 0$, depending on G, such that*

$$\min\left\{-q - rBp_1 - \mu Sp_2 - \frac{1}{2}\sigma^2 S^2 A_{22}, v_\varepsilon - \mathcal{M}v_\varepsilon\right\} \ge \varepsilon\rho$$

for all $(t,B,S) \in G$, $\epsilon > 0$ and $(q,p,A) \in \overline{D}^{-(1,2)}v_\epsilon(t,B,S)$.

Proof. From the definition (7) we have, for $\epsilon > 0$,

$$\mathcal{M}v + \epsilon\mathcal{M}g \ge \mathcal{M}v_\varepsilon$$

and thus

$$v_\varepsilon - \mathcal{M}v_\varepsilon \geq v - \mathcal{M}v + \epsilon(g - \mathcal{M}g) \ . \tag{A.2}$$

Since v is a supersolution it holds $v - \mathcal{M}v \geq 0$. Moreover from (7) and the definition of g it follows

$$g(t,B,S) - \mathcal{M}g\,(t,B,S) \geq \begin{cases} e^{\delta'(T-t)}\left[(B+S)^\gamma - (B+S-k)^\gamma\right] & \text{if } (B,S) \in F \\ 1 & \text{if } (B,S) \notin F \ . \end{cases}$$

Hence for any compact $G \subset Q$ there exists $\rho_1 > 0$ such that $g - \mathcal{M}g \geq \rho_1$ for $(t,B,S) \in G$. Combining this with (A.2) we obtain $v_\varepsilon - \mathcal{M}v_\varepsilon \geq \varepsilon\rho_1$ in G. We consider now $-\frac{\partial g}{\partial t} - \mathcal{L}g$. We have

$$-\frac{\partial g}{\partial t} - \mathcal{L}g = e^{\delta'(T-t)}(B+S)^\gamma\left[\delta' - \gamma\frac{rB+\mu S}{B+S} - \frac{1}{2}\gamma(\gamma-1)\sigma^2\frac{S^2}{(B+S)^2}\right] \tag{A.3}$$

and, setting $\frac{S}{B+S} = \alpha$, $\frac{B}{B+S} = (1-\alpha)$, it is not difficult to see that $\delta' > \gamma\left(r + \frac{(\mu-r)^2}{2\sigma^2(1-\gamma)}\right)$ is sufficient to get $-\frac{\partial g}{\partial t} - \mathcal{L}g > 0$, when $B+S > 0$. Therefore for any compact $G \subset Q$ there exists $\rho_2 > 0$ such that $-\frac{\partial g}{\partial t} - \mathcal{L}g \geq \rho_2$ for all $(t,B,S) \in G$. Since v is already a supersolution of (8), we obtain that

$$-q - rBp_1 - \mu Sp_2 - \frac{1}{2}\sigma^2 S^2 A_{22} \geq \epsilon\rho_2$$

for all $(t,B,S) \in G$ and $(q,p,A) \in \overline{D}^{-(1,2)}v_\epsilon(t,B,S)$. Therefore v_ϵ is a strict viscosity supersolution of (8) in any compact set $G \subset Q$. □

Now it is sufficient to prove the weak comparison principle between a viscosity subsolution u and a strict viscosity supersolution $v_\varepsilon = v + \epsilon\, f$, for all $\epsilon > 0$, because $u \leq v$ in $Q \backslash R$ will follow in the limit $\epsilon \downarrow 0$. We show the result first reasoning in $\overline{Q}^+$. Let u and v be as in theorem 4. We redefine the supersolution v on $\partial^* Q^+$ by

$$v(t,B,S) = \liminf_{\substack{(t',B',S') \in Q^+ \\ (t',B',S') \to (t,B,S)}} v(t',B',S') \qquad \forall\,(t,B,S) \in \partial^* Q^+, \tag{A.4}$$

and we still denote v this function. Now we consider the difference $u - v_\epsilon$ in $\overline{Q}^+$, and we argue by contradiction supposing that

$$m \equiv \sup_{(t,B,S) \in \overline{Q}^+} u - v_\epsilon > 0 \ . \tag{A.5}$$

Since $u - v_\epsilon$ is u.s.c, $\overline{Q}^+$ is compact and the boundary conditions (13) hold true, the maximum m is attained in some point $(t_0,x_0) \in \{[0,T) \times \{\Omega^+ \cup \partial_1\Omega^+\}\} \setminus \{\mathbf{0}\}$. To obtain a contradiction we apply the Ishii's technique redoubling the variables and penalizing this doubling (see Barles, 1994; Crandall et al., 1992). First suppose $(t_0,x_0) \in Q^+$ and consider the test functions for $i \geq 1$

$$\Phi_i(t,x,x') = u(t,x) - v_\epsilon(t,x') - \varphi_i(t,x,x'),$$

where

$$\varphi_i(t,x,x') = |t-t_0|^2 + |x-x_0|^4 + \frac{i}{2}|x-x'| \, .$$

As $\Phi_i(t,x,x')$ is usc in $\overline{Q}^+$, there exists $(\hat{t}_i, \hat{x}_i, \hat{x}'_i) \in \overline{Q}^+$ such that

$$m_i = \sup_{(t,x,x')\in[0,T]\times\overline{\Omega}^+\times\overline{\Omega}^+} \Phi_i(t,x,x') = \Phi_i(\hat{t}_i, \hat{x}_i, \hat{x}'_i), \tag{A.6}$$

and, at least for a subsequence, $(\hat{t}_i, \hat{x}_i, \hat{x}'_i)$ converges to some $(\hat{t}_0, \hat{x}_0, \hat{x}'_0) \in \overline{Q}^+$. By definition we have

$$m \le m_i \le u(\hat{t}_i, \hat{x}_i) - v_\epsilon(\hat{t}_i, \hat{x}'_i),$$

and it is not difficult to show that, sending i to infinity, we obtain

$$\begin{cases} \hat{t}_0 = t_0, \ x_0 = \hat{x}_0 = \hat{x}'_0 \\ m_i \longrightarrow m \\ \frac{i}{2}\left|\hat{x}_i - \hat{x}'_i\right| \to 0. \end{cases} \tag{A.7}$$

Therefore we can apply Ishii's lemma to the interior maximum $(\hat{t}_i, \hat{x}_i, \hat{x}'_i) \in [0,T) \times \Omega^+ \times \Omega^+$ of Φ_i (see Crandall et al., 1992, Theorem 8.3). There exist $q, q' \in \mathbb{R}$, $p, p' \in \mathbb{R}^2$ and $A, A' \in \mathcal{S}^2$ such that

$$(q,p,A) \in \overline{D}^{+(1,2)} u(\hat{t}_i, \hat{x}_i) \quad \text{and} \quad (q',p',A') \in \overline{D}^{-(1,2)} v_\epsilon(\hat{t}_i, \hat{x}_i),$$

where

$$\begin{cases} q - q' = \frac{\partial \varphi_i}{\partial t}(\hat{t}_i, \hat{x}_i, \hat{x}'_i) = 2(\hat{t}_i - t_0) \\ p = \frac{\partial \varphi_i}{\partial x}(\hat{t}_i, \hat{x}_i, \hat{x}'_i) = 4(\hat{x}_i - x_0)\left|\hat{x}_i - x_0\right|^2 + i(\hat{x}_i - \hat{x}'_i) \\ p' = -\frac{\partial \varphi_i}{\partial x'}(\hat{t}_i, \hat{x}_i, \hat{x}'_i) = i(\hat{x}_i - \hat{x}'_i), \end{cases} \tag{A.8}$$

and A, A' are such that

$$\begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix} \le \frac{\partial^2 \varphi_i}{\partial x \partial x'}(\hat{t}_i, \hat{x}_i, \hat{x}'_i) + \frac{1}{i}\left(\frac{\partial^2 \varphi_i}{\partial x \partial x'}(\hat{t}_i, \hat{x}_i, \hat{x}'_i)\right)^2 . \tag{A.9}$$

The subsolution property of u in $(\hat{t}_i, \hat{x}_i)$ and the strict supersolution property of v_ϵ in $(\hat{t}_i, \hat{x}'_i)$, imply that

$$\min\left\{-q - r\hat{B}_i p_1 - \mu \hat{S}_i p_2 - \frac{1}{2}\sigma^2 \hat{S}_i^2 A_{22},\ u(\hat{t}_i, \hat{x}_i) - \mathcal{M}u(\hat{t}_i, \hat{x}_i)\right\} \le 0 \tag{A.10}$$

$$\min\left\{-q' - r\hat{B}'_i p'_1 - \mu \hat{S}'_i p'_2 - \frac{1}{2}\sigma^2 \hat{S}_i'^2 A'_{22},\ z_\varepsilon(\hat{t}_i, \hat{x}'_i) - \mathcal{M}z_\varepsilon(\hat{t}_i, \hat{x}'_i)\right\} \ge \varepsilon\rho. \tag{A.11}$$

If $u(\hat{t}_i, \hat{x}_i) - \mathcal{M}u(\hat{t}_i, \hat{x}_i) \le 0$ in (A.10), then, combining with $z_\varepsilon(\hat{t}_i, \hat{x}'_i) - \mathcal{M}z_\varepsilon(\hat{t}_i, \hat{x}'_i) \ge \varepsilon\rho$ due to (A.11), we obtain

$$m_i \le u(\hat{t}_i, \hat{x}_i) - v_\epsilon(\hat{t}_i, \hat{x}'_i) \le \mathcal{M}u(\hat{t}_i, \hat{x}_i) - \mathcal{M}z_\varepsilon(\hat{t}_i, \hat{x}'_i) - \varepsilon\rho .$$

Using Lemma 2 and (A.7), when i goes to infinity, we have

$$m \le \mathcal{M}u(t_0, x_0) - \mathcal{M}z_\varepsilon(t_0, x_0) - \varepsilon\rho .$$

Since by Remark 1, $F(x_0)$ is compact, if it is not empty, and u is usc, then there exists x'_0 such that $\mathcal{M}u(t_0, x_0) = u(t_0, x'_0)$ and we obtain a contradiction using the definitions of m and $\mathcal{M}$

$$m \le u(t_0, x'_0) - z_\varepsilon(t_0, x'_0) - \varepsilon\rho \le m - \varepsilon\rho .$$

Therefore it must be $-q - r\hat{B}_i p_1 - \mu \hat{S}_i p_2 - \frac{1}{2}\sigma^2 \hat{S}_i^2 A_{22} \leq 0$ in (A.10), and, combining with $-q' - r\hat{B}'_i p'_1 - \mu \hat{S}'_i p'_2 - \frac{1}{2}\sigma^2 \hat{S}'^2_i A'_{22} \geq \varepsilon\rho$ of (A.11), we obtain

$$-(q - q') - r(\hat{B}_i p_1 - \hat{B}'_i p'_1) - \mu(\hat{S}_i p_2 - \hat{S}'_i p'_2) - \frac{1}{2}\sigma^2(\hat{S}_i^2 A_{22} - \hat{S}'^2_i A'_{22}) \leq -\varepsilon\rho \ . \tag{A.12}$$

By (A.7) and (A.8), as i goes to infinity, $(q - q')$, $(\hat{B}_i p_1 - \hat{B}'_i p'_1)$, $(\hat{S}_i p_2 - \hat{S}'_i p'_2)$ converge to zero. Moreover by (A.9) it follows

$$(\hat{S}_i^2 A_{22} - \hat{S}'^2_i A'_{22}) \leq \beta_i, \tag{A.13}$$

where

$$\beta_i = s_i \left[\frac{\partial^2 \varphi_i}{\partial x \partial x'}(\hat{t}_i, \hat{x}_i, \hat{x}'_i) + \frac{1}{i}\left(\frac{\partial^2 \varphi_i}{\partial x \partial x'}(\hat{t}_i, \hat{x}_i, \hat{x}'_i) \right)^2 \right] s_i^T, \tag{A.14}$$

with $s_i = [0, \hat{S}_i, 0, \hat{S}'_i]$. We have

$$\frac{\partial^2 \varphi_i}{\partial x \partial x'}(\hat{t}_i, \hat{x}_i, \hat{x}'_i) = \begin{bmatrix} iI_2 + Q_i & -iI_2 \\ -iI_2 & iI_2 \end{bmatrix}, \tag{A.15}$$

where $Q_i = 8\,|\hat{x}_i - x_0|^2\, I_2 + 8(\hat{x}_i - x_0)(\hat{x}_i - x_0)^T$ and I_2 is the (2×2) identity matrix. Substituting (A.15) into (A.14), after some computations we obtain

$$\beta_i = 3i(\hat{S}_i - \hat{S}'_i)^2 + s_i \left(\begin{bmatrix} 3Q_i & -Q_i \\ -Q_i & 0 \end{bmatrix} + \frac{1}{i}\begin{bmatrix} Q_i^2 & 0 \\ 0 & 0 \end{bmatrix} \right) s_i^T \ . \tag{A.16}$$

By (A.7) and (A.16), β_i also converges to zero as i goes to infinity and therefore (A.12) and (A.13) lead to another contradiction when $i \to \infty$. Therefore we have shown that the maximizer (t_0, x_0) of (A.5) cannot belong to Q^+. The more difficult case ,when we suppose the maximizer (t_0, x_0) is on the border $\{[0,T) \times \partial_1\Omega^+\} \setminus \{\mathbf{0}\}$, can be faced as in (Ly Vath et al., 2007; Oksendal and Sulem, 2002) using a technique proposed in Barles (1994) which assumes some regularity of the boundary. Specifically if we denote by $d(x)$ the distance from x to $\partial\Omega^+$, this distance must be twice continuously differentiable in a neighborhood of x_0. It can be shown as in Ly Vath et al. (2007) that this regularity is satisfied on the border $\{[0,T) \times \partial_1\Omega^+\} \setminus \{\mathbf{0}\}$. By (A.4) there exists a sequence (t_i, x_i) in Q^+ converging to (t_0, x_0). Define $\alpha_i = |t_i - t_0|$, $\gamma_i = |x_i - x_0|$ and consider, as in Ly Vath et al. (2007), the test functions for $i \geq 1$

$$\Phi_i(t, t', x, x') = u(t, x) - v_\epsilon(t', x') - \varphi_i(t, t', x, x'), \tag{A.17}$$

where

$$\varphi_i(t, t', x, x') = |t - t_0|^2 + |x - x_0|^4 + \frac{|t - t'|^2}{2\alpha_i} + \frac{|x - x'|^2}{2\gamma_i} + \left(\frac{d(x')}{d(x_i)} - 1 \right)^4 .$$

It is not difficult to show that in the maximizer $(\hat{t}_i, \hat{t}'_i, \hat{x}_i, \hat{x}'_i)$ of Φ_i, the point $\hat{x}'_i$ always verifies $d(\hat{x}'_i) > 0$. Therefore we can still use the strict supersolution property of v_ϵ in $(\hat{t}'_i, \hat{x}'_i)$. Applying Ishii's lemma to the point $(\hat{t}_i, \hat{t}'_i, \hat{x}_i, \hat{x}'_i)$ and repeating the preceding arguments with the test functions (A.17) we obtain again, by contradiction, that it must be $m \geq 0$. Finally to get $u \leq v$ also in $Q^- \backslash R$ it is sufficient to redefine the subsolution u on $\partial^* Q^-$ by

$$v(t,B,S) = \limsup_{\substack{(t',B',S') \in Q^- \\ (t',B',S') \to (t,B,S)}} v(t',B',S') \qquad \forall\, (t,B,S) \in \partial^* Q^- \tag{A.18}$$

and to repeat the same proof of $\overline{Q}^+$ in $\overline{Q}^-$.