Classical Dilations à la Quantum Probability of Markov Evolutions in discrete time

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Abstract

We study the Classical Probability analogue of the dilations of a quantum dynamical semigroup in Quantum Probability. Given a (not necessarily homogeneous) Markov chain in discrete time in a finite state space E, we introduce a second system, an environment, and a deterministic invertible time-homogeneous global evolution of the system E with this environment such that the original Markov evolution of E can be realized by a proper choice of the initial random state of the environment. We also compare this dilations with the dilations of a quantum dynamical semigroup in Quantum Probability: given a classical Markov semigroup, we show that it can be extended to a quantum dynamical semigroup for which we can find a quantum dilation to a group of *-automorphisms admitting an invariant abelian subalgebra where this quantum dilation gives just our classical dilation.

AMS Subject Classification: 60J10, 81S25.

1 Introduction

We study the analogue in Classical Probability of the dilations in Quantum Probability of a quantum dynamical semigroup (QDS) in discrete time. A QDS T^t describes the evolution of a quantum system, possibly open, but "Markovian", and homogeneous in time. A dilation of T^t by a quantum stochastic flow j_t allows to represent it by the conditional expectation of a quantum Markov process, analogously to the representation of a classical Markov semigroup (CMS) by a classical Markov process. Anyway, such a dilation in Quantum Probability enjoys a reacher structure which allows to dilate the semigroup T^t at the same time also by a unitary group U^t, thus showing that the system evolution T^t and the flow j_t do not contradict the axioms of Quantum Mechanics, i.e. that they can arise from a unitary time-homogeneous evolution U^t of an isolated bigger system, consisting of the given system and its environment [4, 6, 7, 9–11].

The first aim of this paper is to introduce similar dilations in Classical Probability, choosing a self-contained approach in a completely classical framework. These classical dilations are interesting, not only to better understand the relationship between the two probabilistic theories, but also from a simply classical point of view, to better understand the relationship between Markov processes and deterministic invertible homogeneous dynamics.

The second aim of the paper is to show, by appropriate embeddings, that such dilations in Classical Probability are really analogous to the quantum dilations which inspire them. Even if

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in Classical Probability we shall obtain some extra property which has no analogue in Quantum Probability.

We consider here discrete time Markov evolutions in a finite state space. A well known theorem due to Birkhoff shows that any doubly stochastic matrix is a convex combination of permutation matrices. These describe with matrix terminology one-step evolutions which are both deterministic and invertible. Analogously, a theorem due to A. S. Davis [3] shows that any stochastic matrix, doubly stochastic or not, is a convex combination of deterministic matrices. These describe with matrix terminology one-step evolutions which are deterministic but not necessarily invertible. As a consequence, A. S. Davis shows that each finite state Markov chain can be realized as an automaton with random inputs, thus establishing a connection between Markov chains and automata theory. That is, using a different terminology, that each finite state Markov chain can be realized via an innovation process.

The relationship between Markov chains and deterministic dynamics can be further analyzed investigating if Markov evolutions can be realized as deterministic invertible homogeneous evolutions of the system coupled with a second system. The existence of such representations is theoretically relevant if Markov chains are applied to phenomena, like physical phenomena for example, for which an underlaying theory postulates deterministic invertible homogeneous evolutions in absence of noise and external disturbances. For these phenomena the second system introduced by the dilation models the surrounding world, the environment, the source of the noise, which is given now a dynamical explanation. Of course, here the characterizing property is not simply that the global evolution is deterministic as in innovation theory, but that it is also invertible and homogeneous in time.

More precisely, we consider a system with finite state space E, undergoing a discrete time evolution given by a Markov chain, not necessarily homogeneous. Then, we introduce an environment with its state space $(\Gamma, \mathcal{G})$, a measurable space, together with an invertible one-step global evolution $\alpha : E \times \Gamma \to E \times \Gamma$. Thus, if (i, γ) is the state of the compound system at time 0, then $\alpha^t(i, \gamma)$ is its state at time $t \in \mathbb{N}$, where hence α^t gives a deterministic invertible homogeneous global evolution. Nevertheless, if the environment state is never observed and if initially it is randomly distributed with some law Q on $(\Gamma, \mathcal{G})$, then the evolution of the observed system turns out to be stochastic and, if Γ, $\mathcal{G}$, α and Q are properly built, it is given by the original Markov chain. In this case, we say that $(\Gamma, \mathcal{G}, \alpha, Q)$ is a dilation of the Markov evolution in E.

Actually in this paper, given only the state space E (arbitrary but finite), we build a *universal* dilation $(\Gamma, \mathcal{G}, \alpha, \{Q\})$, where $\{Q\}$ is an entire family of distributions which can produce any Markov chain in E: every Markov chain, homogeneous or not, can be dilated by taking always the same model $(\Gamma, \mathcal{G}, \alpha)$ for the environment and the global evolution, and by choosing every time the proper distribution Q for the initial state of the environment. Moreover, our construction, which relays on A. S. Davis decomposition of stochastic matrices, allows to interpret each Markov chain, not only as the stochastic dynamics resulting from the coupling with an environment, but at the same time also as an automaton with random inputs, which are now dynamically provided by the environment itself.

An unpublished result by Kümmerer [6] provides a dilation of a homogeneous Markov chain with a construction similar to ours, but he looks for faithful environment states and he finds an interaction α which depends on the chain under consideration. Therefore his construction does not exhibit that universality which allows us to dilate also non-homogeneous Markov chains. Our aim is similar also to the aim of Lewis and Maassen [8] when they consider classical mechanics in continuous time and, taken a linear Hamiltonian system modelling a particle and its environment, they describe how Gibbs states of the whole system lead to stationary Gaussian stochastic

processes for the observables pertaining to the particle under consideration. However, we do not look for good global states, but for good states Q of the environment alone which lead to Markov evolutions of the system E, our particle, for every independent choice of its initial state.

The paper also proves that our dilations $(\Gamma, \mathcal{G}, \alpha, Q)$ are restrictions of quantum dilations: every CMS in E admits an extension to a QDS for which we can find a quantum dilation which is itself an extension of the dilation of the CMS. However, we shall not embed the whole universal dilation $(\Gamma, \mathcal{G}, \alpha, \{Q\})$ in the quantum world, as quantum dilations do not exhibit the same universality and they strictly depend on the QDS under consideration, so that it is not enough to change the environment state to get another QDS.

In the sequel, given a complex function f on a domain E, we shall denote with the same symbol f also its extension on a domain $E \times \Gamma$, $f(i, \gamma) = f(i)$. Similarly, given a map $\phi : E \to E$, we shall denote with the same symbol ϕ also its extension, by tensorizing with the identity, on a domain $E \times \Gamma$ to $E \times \Gamma$, $\phi(i, \gamma) = (\phi(i), \gamma)$.

2 Preliminaries

We consider a system with finite state space $E = \{1, \ldots, N\}$ and power σ-algebra $\mathcal{E}$, fixed for the whole paper.

Markov evolution. We denote by $P = (P_{ij})_{i,j \in E}$ a stochastic matrix on E, so that $P_{ij} \geq 0$ and $\sum_j P_{ij} = 1$ for every i, and we identify the elements of the complex abelian $*$-algebra $\mathcal{L}^\infty(\mathcal{E})$, the system random variables $f : E \to \mathbb{C}$, with the column vectors in $\mathbb{C}^N$, so that every stochastic matrix P in E defines an operator in $\mathcal{L}^\infty(\mathcal{E})$,

$$(Pf)(i) = \sum_{j \in E} P_{ij}\, f(j),$$

describing a one-step probabilistic evolution. Taken a sequence of stochastic matrices $(P(t))_{t \geq 0}$, $P(0) = \mathbb{1}$, the evolution of a system random variable f from time 0 to time $t \geq 0$ is therefore given by

$$f \mapsto P(1) \cdots P(t)\, f, \qquad \forall f \in \mathcal{L}^\infty(\mathcal{E}). \tag{1}$$

We call it Markov evolution and, in the following, we shall denote a sequence $(P(t))_{t \geq 0}$ simply by $\{P\}$. If the sequence is constant, $P(t) = P$ for every t, then the evolution is homogeneous and it is described by the CMS $(P^t)_{t \geq 0}$ in $\mathcal{L}^\infty(\mathcal{E})$.

Decompositions of stochastic matrices. We denote by D a deterministic matrix in E, that is a stochastic matrix with a 1 in each row. Every D describes with matrix terminology a deterministic evolution β, where

$$D = (D_{ij})_{i,j \in E}, \qquad \beta : E \to E, \qquad D_{ij} = \delta_{\beta(i),j}, \tag{2}$$

so that $Df = f \circ \beta$. The invertible (bijective) maps in E correspond to the special cases of permutation matrices. The deterministic matrices are just the extreme points of the convex set of stochastic matrices and every P is a convex combination of deterministic matrices,

$$P = \sum_{\ell \in L} p_\ell\, D_\ell, \qquad p_\ell \geq 0, \quad \sum_{\ell \in L} p_\ell = 1. \tag{3}$$

One can find such a decomposition with N^N terms, with the set L labelling all possible deterministic matrices and weighing each D_ℓ with $p_\ell = P_{1\beta_\ell(1)} \cdots P_{N\beta_\ell(N)}$. Let us remark that

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decomposition (3) is not unique and that, for any given P, no more than $N^2 - N + 1$ terms are needed [3]. Anyway, since we are not going to fix P, we shall employ (3) in the described form: a convex combination of all the deterministic matrices which can produce any P simply by changing the weights $(p_\ell)_{\ell \in L}$.

The Markov chain. For every Markov evolution $\{P\}$, there exists a Markov chain $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E})$ with transition probabilities given by $P(t)$, i.e. a discrete time stochastic process of random variables $X_t : \Omega \to E$, adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$, and a family of probability measures $\mathbb{P}_k$, $k \in E$, such that the starting distribution of the process depends on k, X_0 has Dirac distribution δ_k under $\mathbb{P}_k$, but the process always enjoys the Markov property with transition matrices $P(t)$:

$$\mathbb{P}_k(X_{t+1} = j | \mathcal{F}_t) = \mathbb{P}_k(X_{t+1} = j | X_t) = P(t+1)_{X_t j}, \qquad \forall k, j \in E, \ t \geq 0.$$

Thus, a system random variable $f \in \mathcal{L}^\infty(\mathcal{E})$ has now a stochastic evolution given by the $*$-unital homomorphism

$$j_t : \mathcal{L}^\infty(\mathcal{E}) \to \mathcal{L}^\infty(\mathcal{F}_t), \qquad f \mapsto j_t[f] := f(X_t), \qquad t \geq 0, \tag{4}$$

and the Markov evolution (1) admits the representation

$$\Big(P(1) \cdots P(t) \, f\Big)(k) = \mathbb{E}_k[f(X_t)], \qquad \forall f \in \mathcal{L}^\infty(\mathcal{E}). \tag{5}$$

Markov chains and automata. Let us briefly show a possible realization of the Markov chain by means of the decomposition (3). If $P(t) = \sum_{\ell \in L} p_\ell(t) \, D_\ell$, then one can take

$$\Omega = E \times L^{\mathbb{N}}, \qquad \omega = \big(i, (\ell_n)_{n \in \mathbb{N}}\big),$$
$$X_0(\omega) = i, \qquad X_t(\omega) = \beta_{\ell_t} \circ \cdots \circ \beta_{\ell_1}(i), \qquad Y_t(\omega) = \ell_t, \qquad t \in \mathbb{N},$$
$$\mathcal{F} = \sigma(X_0) \otimes \sigma(Y_n; \ n \in \mathbb{N}), \qquad \mathcal{F}_t = \sigma(X_0, Y_s; \ 1 \leq s \leq t), \tag{6}$$
$$\mathbb{P}_k = \delta_k \otimes \Big(\bigotimes_{t \in \mathbb{N}} p(t)\Big).$$

In this way the Markov chain associated to $\{P\}$ is represented as an automaton with independent random inputs:

$$X_t = \beta_{Y_t}(X_{t-1}),$$

so that at every step the set of all possible mappings $\beta_\ell : E \to E$ is available and the system evolution is determined by the value of the input parameter ℓ which is selected randomly according to $p(t)$ and independently of the previous steps. If the sequence $P(t)$ is constant, then the Markov chain is homogeneous and the input parameters can be chosen identically distributed. To get a dilation, we shall introduce a bigger Ω, namely $E \times G^{\mathbb{Z}}$, with a G bigger than L in order to define an invertible one-shot coupling ϕ, and with $\mathbb{Z}$ instead of $\mathbb{N}$ in order to get a dynamics α^t with group properties.

3 Dilations and universal dilations

Dilation of a Markov evolution. We call dilation of the Markov evolution $\{P\}$ in $\mathcal{L}^\infty(\mathcal{E})$ a term

$$\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E}\Big)$$

such that

- every $Z_t = (X_t, \Upsilon_t)$ is a random variable in $(\Omega, \mathcal{F})$ with values in $(E \times \Gamma, \mathcal{E} \otimes \mathcal{G})$, being $(\Gamma, \mathcal{G})$ a fixed measurable space,

- the term $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_k)_{k \in E})$ is a Markov chain with transition matrices $\{P\}$,

- the random variable (X_0, Υ_0) has distribution $\delta_k \otimes Q$ under $\mathbb{P}_k$, being Q a fixed distribution on $\mathcal{G}$,

- there exists an invertible bimeasurable map $\alpha : E \times \Gamma \to E \times \Gamma$ such that $Z_t = \alpha^t(Z_0)$ for every $t \geq 0$.

Thus, besides the system E, a second system is introduced, an environment with state space $(\Gamma, \mathcal{G})$. Their states X_t and Υ_t are asked to be random variables on a same measurable space $(\Omega, \mathcal{F})$ such that the global state $Z_t = (X_t, \Upsilon_t)$ undergoes a deterministic invertible homogeneous evolution α^t. Therefore all the X_t and Υ_t are determined by Z_0, so that X_t and Υ_t are measurable with respect to $\sigma(Z_0) = \sigma(X_0, \Upsilon_0) \subseteq \mathcal{F}$ and, depending on the probability chosen on $\mathcal{F}$, they are deterministic if and only if Z_0 is. Nevertheless, a probability $\mathbb{P}_k$ typically fixes only the value of X_0. The space $(\Omega, \mathcal{F})$ is also endowed with a filtration $\mathcal{F}_t$. Note that only the X_t are asked to be adapted to $\mathcal{F}_t$ so that, in particular, Υ_0 does not have to be $\mathcal{F}_0$-measurable. Therefore the X_t are not trivially $\mathcal{F}_0$-measurable, even if their values are completely determined by the values of X_0 and Υ_0, and, neglecting the environment, each $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \mathbb{P}_k)$ can be a non trivial stochastic process. What we ask is that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \mathbb{P}_k)$ actually is a Markov chain starting from k with transition matrices $\{P\}$. At the same time however, this Markov chain is compatible with a deterministic, invertible and homogeneous model for the evolution of E coupled with an environment Γ. In particular, as $X_0 = k$, the whole stochasticity of the process is due only to the randomness of the unobserved initial state Υ_0 of the environment.

A dilation gives another interpretation of every evolution (1), compatible with (5):

$$\left(P(1) \cdots P(t) f \right)(k) = \mathbb{E}_k\big[f(X_t)\big] = \mathbb{E}_k\big[f(Z_t)\big] = \mathbb{E}_k\Big[f(\alpha^t(k, \Upsilon_0))\Big], \qquad \forall f \in \mathcal{L}^\infty(\mathcal{E}).$$

Indeed, the stochastic evolution (4) of a system variable $f \in \mathcal{L}^\infty(\mathcal{E})$ is now described by the $*$-unital homomorphism

$$j_t : \mathcal{L}^\infty(\mathcal{E}) \to \mathcal{L}^\infty(\mathcal{F}_t), \qquad f \mapsto j_t[f] := f(X_t) = f(Z_t) = f \circ \alpha^t(Z_0), \tag{7}$$

which is injective as α^t is invertible. And now we could also consider global random variables $F : E \times \Gamma \to \mathbb{C}$ and their evolution $F \mapsto F(Z_t) = F \circ \alpha^t(Z_0)$.

Universal dilation. Let us denote by $\mathcal{P}$ the set of sequences of stochastic matrices $\{P\}$ in E. We call universal dilation of the Markov evolutions in $\mathcal{L}^\infty(\mathcal{E})$ a term

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (\mathbb{P}_{k, \{P\}})_{k \in E, \{P\} \in \mathcal{P}} \right)$$

such that every $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (\mathbb{P}_{k, \{P\}})_{k \in E})$ is a dilation of the corresponding Markov evolution $\{P\}$. We call universal such a dilation because we ask that the same Ω, $\mathcal{F}$, $\mathcal{F}_t$ and Z_t allow to represent all the Markov evolutions in $\mathcal{L}^\infty(\mathcal{E})$, with the change of the probabilities $\mathbb{P}_{k, \{P\}}$ alone. Therefore, both the environment state space $(\Gamma, \mathcal{G})$ and the global evolution α depend only on the state space E, not on the particular Markov evolution to be dilated.

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Standard dilation and standard universal dilation. In order to show that every state space E admits a universal dilation, now we consider a particular class of dilations and of universal dilations. Let us describe all the special requirements we are interested in for the dilation of a Markov evolution $\{P\}$.

First of all we want the sample space Ω to be just $E \times \Gamma$, the state space of the global system. As we want it to describe all the possible initial global states, we ask the random variable Z_0 to be the identity function and X_0 and Y_0 to be the coordinate variables: if $\omega = (i, \gamma)$, then $Z_0(\omega) = \omega$, $X_0(\omega) = i$ and $\Upsilon_0(\omega) = \gamma$. Thus, for all $t \geq 0$, $Z_t = (X_t, \Upsilon_t) = \alpha^t \circ Z_0 = Z_0 \circ \alpha^t$, $X_t = X_0 \circ \alpha^t$ and $\Upsilon_t = \Upsilon_0 \circ \alpha^t$.

We are interested in an environment $\Gamma = G^{\mathbb{Z}} = \times_{n \in \mathbb{Z}} G_n$ with $G_n = G$ finite set. In this case the environment state $\gamma \in \Gamma$ has infinitely many components $g_n \in G_n$, $n \in \mathbb{Z}$, and we introduce also the coordinate variables $Y_n(\omega) = g_n$, the random n-th components of the environment. Then, endowed G with its power σ-algebra, we want all the functions so far introduced to be measurable and so we ask $\mathcal{G}$ to be the natural σ-algebra $\mathcal{G} = \sigma(\Upsilon_0) = \sigma(Y_n; \ n \in \mathbb{Z})$ on Γ, and $\mathcal{F}$ to be $\mathcal{E} \otimes \mathcal{G} = \sigma(X_0, \Upsilon_0)$ on Ω. Supposing that at time 0 only X_0 is observed and that at each following instant t only the information carried by Y_t is acquired, we want the filtration $\mathcal{F}_0 = \sigma(X_0)$, $\mathcal{F}_t = \sigma(X_0, Y_s; \ 1 \leq s \leq t)$, for $t \geq 1$.

In order to get consistence between these definitions and the global evolution α, we ask that each Y_t is involved in the interaction with the system only once, between time $t-1$ and time t. More precisely, first we ask an invertible map

$$\phi : E \times G \to E \times G, \tag{8}$$

denoted by $\phi(i, g) = \left(\phi^E(i, g), \phi^G(i, g)\right)$, giving the one-step coupling between the system and a single environment component. Secondly, we introduce the left shift

$$\vartheta : \Gamma \to \Gamma, \qquad (g_n)_{n \in \mathbb{Z}} \in \Gamma, \quad g_n \in G_n \quad \mapsto \quad (g_{n+1})_{n \in \mathbb{Z}} \in \Gamma, \quad g_{n+1} \in G_n.$$

Then, denoted by ϕ_1 the map ϕ in $E \times G_1$, extended ϕ_1 and ϑ in Ω by tensorizing with identities, we ask

$$\alpha = \vartheta \circ \phi_1.$$

Roughly speaking, when α is applied for the first time between time 0 and time 1, the map ϕ_1 couples the system state X_0 with Y_1, giving the new system state $X_1 = X_0 \circ \alpha = X_0 \circ \phi_1 = \phi^E(X_0, Y_1)$, and then the shift ϑ prepares Y_2 for the following interaction with X_1. So the state of the system at a positive time t is $X_t = X_0 \circ \alpha^t = \phi^E(X_{t-1}, Y_t)$, which is automatically adapted to $\mathcal{F}_t$. Thus ϑ could be interpreted as a free evolution of the environment. If we explicitly introduce also the random variables $Y_n^{(t)} = Y_n \circ \alpha^t$, the n-th environment components at time $t \geq 1$, then

$$Y_n^{(t)} = Y_n \circ \alpha^t = \begin{cases} Y_{n+1}^{(t-1)}, & \text{if } n \neq 0, \\ \phi^G(X_{t-1}, Y_1^{(t-1)}), & \text{if } n = 0, \end{cases} = \begin{cases} Y_{n+t}, & \text{if } n \leq -t, \ n \geq 1, \\ \phi^G(X_{t-1+n}, Y_{n+t}), & \text{if } -t+1 \leq n \leq 0, \end{cases}$$

and $X_t = \phi^E(X_{t-1}, Y_1^{(t-1)})$.

At long last, we consider the probabilities $\mathbb{P}_k$. Of course, they have to be factorized as $\delta_k \otimes Q$ on $\mathcal{F} = \mathcal{E} \otimes \mathcal{G}$. We also require Q to be factorized as $Q_0 \otimes \left(\bigotimes_{n \in \mathbb{N}} q(n) \right)$ on the σ-algebra $\sigma(Y_n; \ n \leq 0) \otimes \sigma(Y_n; \ n \geq 1)$, so that $(Y_n)_{n \leq 0} \sim Q_0$ while $Y_n \sim q(n)$ for every $n \geq 1$ and they are all independent. This guarantees the Markov property for the process X_t with respect to $\mathcal{F}_t$ with transition probabilities

$$P(t)_{ij} = \sum_{g \in G} q_g(t) \, \delta_{\phi^E(i,g),j}, \qquad \forall i, j \in E, \ t \geq 1, \tag{9}$$

so that the only point is to check if the transition probabilities (9) are the desired ones.

A dilation like this will be called standard in the following. Summarizing, a dilation is standard if

- $\Gamma = G^{\mathbb{Z}} = \times_{n \in \mathbb{Z}} G_n, \; G_n = G$ finite set with power σ-algebra, $\mathcal{G} =$ cylindric σ-algebra,

- $\Omega = E \times \Gamma, \qquad \omega = (i, \gamma) = (i, (g_n)_{n \in \mathbb{Z}}) \in \Omega, \quad i \in E, \quad \gamma \in \Gamma, \quad g_n \in G,$

- $X_0(\omega) = i, \quad \Upsilon_0 = (Y_n)_{n \in \mathbb{Z}}, \quad Y_n(\omega) = g_n, \quad \Upsilon_0(\omega) = \gamma, \quad Z_0(\omega) = \omega,$

- $\mathcal{F} = \mathcal{E} \otimes \mathcal{G} = \sigma(X_0) \otimes \sigma(Y_n; \, n \leq 0) \otimes \sigma(Y_n; \, n \geq 1), \qquad \mathcal{F}_t = \sigma(X_0, Y_s; \, 1 \leq s \leq t), \quad t \geq 0,$

- $\mathbb{P}_k = \delta_k \otimes Q_0 \otimes \big(\bigotimes_{n \in \mathbb{N}} q(n) \big),$

- $\alpha = \vartheta \circ \phi_1$, with an invertible $\phi_1 = \phi : E \times G_1 \to E \times G_1$, and with the left shift ϑ on Γ.

Then, for every $t \geq 1$ we have

- $Z_t = \alpha^t \circ Z_0 = Z_0 \circ \alpha^t = (X_t, \Upsilon_t),$

- $\Upsilon_t = \Upsilon_0 \circ \alpha^t = (Y_n^{(t)})_{n \in \mathbb{Z}},$

- $X_t = X_0 \circ \alpha^t = \phi^E(X_{t-1}, Y_t) = \phi^E(X_{t-1}, Y_1^{(t-1)}).$

A standard dilation is therefore specified by the term $\big(G, \phi, Q_0 \otimes (\bigotimes_{n \in \mathbb{N}} q(n)) \big)$.

With a standard dilation the evolution of every global random variable is described by the group of $*$-automorphisms J^t, $t \in \mathbb{Z}$, where

$$J : \mathcal{L}^{\infty}(\mathcal{F}) \to \mathcal{L}^{\infty}(\mathcal{F}), \qquad J(F) = F \circ \alpha.$$

Note that a standard dilation always allows to see every Markov chain also as an automaton with independent random inputs Y_t, which now are provided by the environment via the dynamics α. Given a standard dilation $\big(G, \phi, Q_0 \otimes (\bigotimes_{n \in \mathbb{N}} q(n)) \big)$ for a CMS P^t, the random inputs Y_t can always be choosen independent and identically distributed by taking $q = q(1)$ and replacing $Q_0 \otimes (\bigotimes_{n \in \mathbb{N}} q(n))$ with $q^{\otimes \mathbb{Z}}$.

A universal dilation will be called standard if it is given by a family of standard dilations $(G, \phi, Q_{\{P\}})$, $\{P\} \in \mathcal{P}$, all of them with the same G and ϕ. Thus we fix the model for the environment and for the global evolution with G and ϕ, and then we require the existence of a family of initial distributions $Q_{\{P\}}$ for the environment state, each one giving rise to a different Markov evolution for the system. A standard universal dilation is therefore specified by the term $(G, \phi, (Q_{\{P\}})_{\{P\} \in \mathcal{P}})$. A standard universal dilation is not uniquely determined by the state space E, but it always exists.

Theorem 1. *For every finite state space E, there exists a standard universal dilation $(G, \phi, (Q_{\{P\}})_{\{P\} \in \mathcal{P}})$ of the Markov evolutions in $\mathcal{L}^{\infty}(\mathcal{E})$.*

Proof. We only have to exhibit a proper set G, together with the coupling ϕ and the probability measures $Q_{\{P\}}$ on $\mathcal{G}$.

Given $E = \{1, \ldots, N\}$ and the set L labelling the all possible maps $\beta : E \to E$, we set

$$G = E \times L, \qquad i, j, k \in E, \qquad \ell \in L, \qquad g = (j, \ell) \in G.$$

Arbitrarily fixed $j = 1$, we focus on points $(1, \ell)$ in G. Thus, taken two points $(i, (1, \ell)) \neq (i', (1, \ell'))$ in $E \times G$, we get $(\beta_\ell(i), (i, \ell)) \neq (\beta_{\ell'}(i'), (i', \ell'))$ and so we can find an invertible map

$$\phi : E \times G \to E \times G, \qquad \phi(i, (j, \ell)) = \begin{cases} (\beta_\ell(i), (i, \ell)), & \text{if } j = 1, \\ \dots, & \text{if } j \neq 1. \end{cases} \tag{10}$$

We choose an arbitrary ϕ satisfying (10).

Given $\{P\}$, we fix a decomposition (3) for every $P(t)$, thus obtaining the distributions $p(t)$ on L; for every distribution $p(t)$ on L we define on $G = E \times L$ the distribution

$$q(t) = \delta_1 \otimes p(t).$$

Thus $P(t)_{ij} = \sum_{\ell \in L} p_\ell(t)\, \delta_{\beta_\ell(i), j} = \sum_{g \in G} q_g(t)\, \delta_{\phi^E(i, g), j}$ for every $t \in \mathbb{N}$. Chosen an arbitrary distribution Q_0 on $\sigma(Y_n; n \leq 0)$, if we define $Q_{\{P\}} = Q_0 \otimes \left(\bigotimes_{t \in \mathbb{N}} q(t) \right)$, then every stochastic process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_{k, \{P\}})_{k \in E})$ is a Markov chain with transition matrices $\{P\}$, independently of Q_0 and of the definition of $\phi(i, (j, \ell))$ for $j \neq 1$. Therefore $(G, \phi, (Q_{\{P\}})_{\{P\} \in \mathcal{P}})$ is a standard universal dilation of the Markov evolutions in $\mathcal{L}^\infty(\mathcal{E})$. $\qquad \square$

In this construction, even if each $g \in G$ has two components, $g = (j, \ell)$, the probability is always concentrated only on those g of the kind $g = (1, \ell)$, but we need the first component j to define an invertible ϕ. Analogously, we are considering the evolution only for positive times so that all the components g_n, $n \leq 0$, are never involved in the interaction with the system, but they are needed to define an invertible shift ϑ.

Since the decomposition (3) is not unique, just as the choice of Q_0, we could find other probabilities $\mathbb{P}$ on $\mathcal{F}$ inducing the same Markov evolutions for the system. On the other hand, considering non-factorized Q or even non-factorized $\mathbb{P}$, we would obtain new stochastic processes $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, \mathbb{P})$, without any Markov property guaranteed, which now would depend on the definition of ϕ for $j \neq 1$.

Let us remark also that, just because of the universality of the construction, a dilation $(G, \phi, Q_{\{P\}})$ is usually non minimal for a particular evolution $\{P\}$.

The cocycle approach to standard dilations. Given a Markov evolution $\{P\}$ with a standard dilation $(G, \phi, Q_0 \otimes (\bigotimes_{t \in \mathbb{N}} q(t)))$, where G, ϕ, $q(t)$ can be defined as in the proof of Theorem 1 or not, as long as we consider only system random variables neglecting the environment, we can avoid the shift ϑ and define a deterministic, invertible, but inhomogeneous global evolution φ_t which never involves the environment components g_n for $n \leq 0$, but which generates the same Markov chain and the same Markov evolution for the system. Consider indeed the invertible maps in Ω

$$\phi_t := \vartheta^{-(t-1)} \circ \phi_1 \circ \vartheta^{t-1}, \qquad \varphi_t := \phi_t \circ \dots \circ \phi_1 = \vartheta^{-t} \circ \alpha^t, \qquad t \geq 1,$$

where ϕ_t actually acts only in $E \times G_t$, where it is just the coupling (10), while φ_t actually acts only in $E \times G_1 \times \dots \times G_t$. Then, set $\varphi_0 = \mathrm{Id}$, we get $X_t = X_0 \circ \alpha^t = X_0 \circ \varphi_t$ for every $t \geq 0$. Because of the group properties of ϑ^t and α^t, the evolution φ_t satisfies the cocycle property

$$\varphi_{t+s} = \vartheta^{-t} \circ \varphi_s \circ \vartheta^t \circ \varphi_t, \qquad \forall t, s \geq 1,$$

and, of course, its knowledge is equivalent to the knowledge of α^t.

In particular the $*$-unital injective homomorphism (7) can be written as

$$j_t : \mathcal{L}^\infty(\mathcal{E}) \to \mathcal{L}^\infty(\mathcal{F}_t), \qquad f \mapsto j_t[f] := f(X_t) = f \circ \varphi_t, \qquad t \geq 0. \tag{11}$$

If we denote by $\mathbb{E}_g[f \circ \phi]$ the system random variable in $\mathcal{L}^\infty(\mathcal{E})$ defined by $i \mapsto f \circ \phi(i, g)$, then the stochastic evolution (11) satisfies

$$j_0[f] = f(X_0), \qquad j_t[f] = \sum_{g \in G} j_{t-1}\Big[\mathbb{E}_g[f \circ \phi]\Big] I_{(Y_t = g)}, \qquad \forall f \in \mathcal{L}^\infty(\mathcal{E}), \ t \geq 1, \tag{12}$$

where $j_t\Big[\mathbb{E}_g[f \circ \phi]\Big]$ are $\mathcal{F}_t$-adapted processes, while the random variables $I_{(Y_t = g)}$, indicators of the events $(Y_t = g)$, are the increments of the $\mathcal{F}_t$-adapted processes $N_t^g = \sum_{s=1}^t I_{(Y_s = g)}$, $t \geq 0$. Hence Eq. (12) can be read as a stochastic equation with respect to the noises N_t^g.

We can even reduce the sample space Ω from $E \times G^\mathbb{Z}$ to $E \times G^\mathbb{N}$, restrict here $\mathcal{F}$, $\mathcal{F}_t$ and $\mathbb{P}_k$, and define φ_t in $E \times G^\mathbb{N}$ by (8) and (12). Anyway, thanks to the cocycle properties of φ_t, it is always possible to introduce later $\times_{n \leq 0} G_n$ and the shift ϑ, in order to recover the whole environment state space $(G^\mathbb{Z}, \mathcal{G})$, the evolution α and the initial environment distribution $Q_0 \otimes (\bigotimes_{t \in \mathbb{N}} q(t))$, so that the two constructions are equivalent and can be considered different descriptions of the same situation.

Choosing the cocycle approach, a standard dilation of a Markov evolution $\{P\}$ is a Markov chain $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}_{k,\{P\}})_{k \in E})$, where

$$\Omega = E \times G^\mathbb{N}, \qquad \omega = (i, (g_n)_{n \in \mathbb{N}}),$$
$$X_0(\omega) = i, \qquad Y_t(\omega) = g_t, \qquad X_t = \phi^E(X_{t-1}, Y_t), \qquad t \in \mathbb{N},$$
$$\mathcal{F} = \sigma(X_0) \otimes \sigma(Y_t; \ t \in \mathbb{N}), \qquad \mathcal{F}_t = \sigma(X_0, Y_s, 1 \leq s \leq t), \tag{13}$$
$$\mathbb{P}_k = \delta_k \otimes \Big(\bigotimes_{t \in \mathbb{N}} q(t)\Big).$$

This chain is specified by the term $(G, \phi, \bigotimes_{t \in \mathbb{N}} q(t))$.

Let us underline that the Markov chain (13) is similar to (6), as also this one is an automaton with independent random inputs. Nevertheless, the invertibility of ϕ endows this chain with a reacher structure because it implicitly introduces also the deterministic invertible homogeneous evolution α^t.

4   Dilations of classical Markov semigroups and of quantum dynamical semigroups

We want to compare a standard dilation with the dilation of a QDS in Quantum Probability.

Given a Hilbert space $\mathcal{H}$, always complex separable in the paper, let us denote its vectors by h, or $|h\rangle$ using Dirac's notation, so that $\langle h'|h \rangle$ denotes the scalar product (linear in h) and $|h'\rangle\langle h|$ denotes the operator $h'' \mapsto \langle h|h'' \rangle h'$. Let $\mathcal{B}(\mathcal{H})$ be the complex $*$-algebra of bounded operators in $\mathcal{H}$. Given a Hilbert space $L^2(\mu)$, with a probability measure μ on some measurable space, and given a measurable complex function f on the same measurable space, let m_f denote the multiplication operator

$$m_f : \mathrm{Dom}(m_f) \to L^2(\mu), \qquad \mathrm{Dom}(m_f) = \{h \in L^2(\mu) \ : \ fh \in L^2(\mu)\}, \qquad m_f h = fh,$$

which is bounded if and only if $f \in L^\infty(\mu)$. Let $\mathcal{D}(L^2(\mu))$ be the abelian complex $*$-algebra of bounded multiplication operators in $L^2(\mu)$. Given a finite space S with its power σ-algebra $\mathcal{S}$,

let μ_S denote the uniform probability on $(S, \mathcal{S})$ and let $\{|i\rangle\}_{i \in S}$ denote the canonical basis of $L^2(\mu_S)$. Moreover, let us denote $\mathcal{L}^\infty(\mathcal{S}) = L^\infty(\mu_S)$ also by $\mathcal{L}^\infty(S)$. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a trace operator τ in $\mathcal{K}$, let $\mathbb{E}_\tau : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ denote the conditional expectation with respect to τ. Given a vector $\kappa \in \mathcal{K}$, let us denote the conditional expectation with respect to $|\kappa\rangle\langle\kappa|$ simply by $\mathbb{E}_\kappa$. In the sequel, given a bounded operator a in $\mathcal{H}$, we shall identify it with its extension $a \otimes \mathbb{1}_\mathcal{K}$ in $\mathcal{H} \otimes \mathcal{K}$.

Quantum extension of a CMS. A discrete-time QDS in $\mathcal{B}(\mathcal{H})$ is a semigroup $(T^t)_{t \geq 0}$ with $T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ stochastic map, that is a linear, bounded, completely positive, normal and identity preserving operator.

In order to extend a CMS in $\mathcal{L}^\infty(\mathcal{E})$ by a QDS in some $\mathcal{B}(\mathcal{H})$, we take $\mathcal{H} = L^2(\mu_E)$ and we embed $\mathcal{L}^\infty(\mathcal{E})$ in $\mathcal{B}(\mathcal{H})$ by the $*$-isomorphism $f \mapsto m_f$ between $\mathcal{L}^\infty(\mathcal{E})$ and $\mathcal{D}(\mathcal{H})$.

We say that a QDS T^t in $\mathcal{B}(\mathcal{H})$ extends a CMS P^t in $\mathcal{L}^\infty(\mathcal{E})$ if

$$T m_f = m_{Pf}, \qquad \forall f \in \mathcal{L}^\infty(\mathcal{E}).$$

Such extension always exists. For example, given a stochastic matrix P, using a representation (3) and notations (2), P is extended by the stochastic map

$$T a = \sum_{\substack{\ell \in L \\ i \in E}} p_\ell \, |i\rangle\langle\beta_\ell(i)| \, a \, |\beta_\ell(i)\rangle\langle i|, \qquad \forall a \in \mathcal{B}(\mathcal{H}). \tag{14}$$

Note that this extension T^t maps $\mathcal{B}(\mathcal{H})$ to $\mathcal{D}(\mathcal{H})$ in only one step. The extension of a CMS is not unique at all: for example every permutation P can be extended also by a $*$-automorphism $Ta = u^* a\, u$, provided that, for every j in E, $u\,|j\rangle = P^* \,|j\rangle$ up to a phase factor (necessary and sufficient condition).

QP-dilation of a QDS. Given a QDS T^t in $\mathcal{B}(\mathcal{H})$, a typical Quantum Probability construction employs a Hilbert space $\mathfrak{Z}$, a unitary operator V in $\mathcal{H} \otimes \mathfrak{Z}$ and a unit vector v in $\mathfrak{Z}$, to dilate T^t at the same time by a quantum stochastic flow and by a group of $*$-automorphisms.

Taken infinitely many copies $\mathfrak{Z}_n$ of $\mathfrak{Z}$, the quantum stochastic flow

$$j_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}\Big(\bigotimes_{n=1}^{t} \mathfrak{Z}_n \Big), \qquad t \geq 0,$$

is the solution of the quantum stochastic equation

$$j_0 = \mathrm{Id}_{\mathcal{B}(\mathcal{H})}, \qquad j_t(a) = \sum_{zz'} j_{t-1}\Big(\mathbb{E}_{|z\rangle\langle z'|}\big[V^* \cdot a \otimes \mathbb{1}_\mathfrak{Z} \cdot V \big] \Big) \otimes |z'\rangle\langle z|, \quad t \geq 1,$$

where $\{|z\rangle\}$ is a given basis in $\mathfrak{Z}$.

Denoted by $\mathcal{K}$ the infinite tensor product $\bigotimes_{n \in \mathbb{Z}} \mathfrak{Z}_n$ with respect to the stabilizing sequence of unit vectors $\psi_n \equiv v$, denoted by Θ the left shift operator in $\mathcal{K}$, denoted by V_1 the operator V in $\mathcal{H} \otimes \mathfrak{Z}_1$, extended in $\mathcal{H} \otimes \mathcal{K}$ every operator, consider the unitary operator in $\mathcal{H} \otimes \mathcal{K}$

$$U = \Theta \, V_1.$$

The group of $*$-automorphisms is J^t, $t \in \mathbb{Z}$, where

$$J : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}), \qquad J(A) = U^* A \, U.$$

Then $J^t(a) = j_t(a)$ for every $a \in \mathcal{B}(\mathcal{H})$ and $t \geq 0$.

A term $(\mathfrak{Z}, V, \upsilon)$ defines a QP-dilation of the QDS T^t in $\mathcal{B}(\mathcal{H})$ if

$$T^t a = \mathbb{E}_{\upsilon^{\otimes t}}[j_t(a)] = \mathbb{E}_{\upsilon^{\otimes \mathbb{Z}}}[J^t(a)], \qquad \forall a \in \mathcal{B}(\mathcal{H}), \ t \geq 0. \tag{15}$$

The equalities hold for $t \geq 0$ if they hold for $t = 1$, that is if $Ta = \mathbb{E}_\upsilon[V^* a V]$ for all a in $\mathcal{B}(\mathcal{H})$.

Given $\mathfrak{Z}$, the spatial tensor product $\otimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n)$ is a C^*-algebra which can be naturally embedded in $\mathcal{B}(\mathcal{K})$ for every stabilizing sequence ψ_n used in the definition of $\mathcal{K}$. Given V and υ, the C^*-algebra $\mathcal{B}(\mathcal{H}) \otimes \left(\otimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n) \right) \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ is invariant for J. Indeed, denoted by $\widehat{\Theta}$ the right shift in $\otimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n)$, the restriction of J gives

$$J : \mathcal{B}(\mathcal{H}) \otimes \left(\bigotimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n) \right) \to \mathcal{B}(\mathcal{H}) \otimes \left(\bigotimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n) \right), \qquad J(A) = V_1^* \, \widehat{\Theta}(A) \, V_1, \tag{16}$$

which is a $*$-automorphism, is independent of υ, and generates a group J^t satisfying (15) with $\upsilon^{\otimes \mathbb{Z}}$ pure and locally normal state. Thus, at the C^*-algebras level, a QP-dilation $(\mathfrak{Z}, V, \upsilon)$ of T^t consists of a C^*-algebra $\otimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n)$ depending on $\mathfrak{Z}$, of a quantum stochastic flow j_t and of a group of $*$-automorphisms J^t depending on V, and of a pure locally normal state $\upsilon^{\otimes \mathbb{Z}}$ depending on υ, such that representations (15) hold.

Quantum extension of a standard dilation. Given a stochastic matrix P in E and a standard dilation $(G, \phi, q^{\otimes \mathbb{Z}})$ of the CMS P^t, we show that P^t can be extended to a QDS T^t for which we can find a QP-dilation which is itself an extension of $(G, \phi, q^{\otimes \mathbb{Z}})$. This shows that a standard dilation is a classical analogue of a QP-dilation.

The first step to study this relationship is to embed the standard dilation $(G, \phi, q^{\otimes \mathbb{Z}})$ in the quantum world. In order to get this embedding at the Hilbert space level, we should introduce a proper measure on $(G^{\mathbb{Z}}, \mathcal{G})$. Taken μ_E on E, the proper measure on $(G^{\mathbb{Z}}, \mathcal{G})$ should give a product measure on $(E \times G^{\mathbb{Z}}, \mathcal{E} \otimes \mathcal{G})$ invariant for the deterministic invertible evolutions α, ϑ, ϕ_1. Thus the natural choice would be the probability measure $\mu_G^{\otimes \mathbb{Z}}$. Nevertheless, it would be singular with respect to the initial environment distribution $q^{\otimes \mathbb{Z}}$, so that this latter could not be obtained from a state in $L^2(\mu_G^{\otimes \mathbb{Z}})$. Therefore now we work at C^*-algebras level.

We embed in the quantum world the spatial tensor product of C^*-algebras

$$\mathcal{L}_{\mathbb{Z}}^{\infty} = \mathcal{L}^{\infty}(\mathcal{E}) \otimes \left(\bigotimes_{n \in \mathbb{Z}} \mathcal{L}^{\infty}(G_n) \right).$$

It is the sub C^*-algebra of $\mathcal{L}^{\infty}(\mathcal{F})$ which consists of all continuous functions in the compact set $E \times G^{\mathbb{Z}}$, and it is invariant for $\circ \phi_1$, $\circ \vartheta$ and $\circ \alpha$.

Taken again $\mathcal{H} = L^2(\mu_E)$, we introduce the sequence of Hilbert spaces $\mathfrak{Z}_n = L^2(\mu_G)$, $n \in \mathbb{Z}$, the algebras of multiplication operators $\mathcal{D}(\mathfrak{Z}_n)$, and the spatial tensor product

$$\mathcal{D}_{\mathbb{Z}} = \mathcal{D}(\mathcal{H}) \otimes \left(\bigotimes_{n \in \mathbb{Z}} \mathcal{D}(\mathfrak{Z}_n) \right),$$

which is an abelian sub C^*-algebra of $\mathcal{B}(\mathcal{H}) \otimes \left(\otimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n) \right)$.

We embed $\mathcal{L}_{\mathbb{Z}}^{\infty}$ in $\mathcal{B}(\mathcal{H}) \otimes \left(\otimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n) \right)$ by the $*$-isomorphism $F \mapsto m_F$ between $\mathcal{L}_{\mathbb{Z}}^{\infty}$ and $\mathcal{D}_{\mathbb{Z}}$ defined by mapping each F belonging to

$$\mathcal{L}_{[-t,t]}^{\infty} := \mathcal{L}^{\infty}(\mathcal{E}) \otimes \left(\bigotimes_{n=-t}^{t} \mathcal{L}^{\infty}(G_n) \right), \qquad t \in \mathbb{N},$$

to the multiplication operator m_F belonging to $\mathcal{D}(\mathcal{H}) \otimes \left(\bigotimes_{n=-t}^{t} \mathcal{D}(\mathfrak{Z}_n) \right) = \mathcal{D}\left(L^2(\mu_E \otimes \mu_G^{\otimes(2t+1)}) \right)$, abelian subalgebra of $\mathcal{B}(\mathcal{H}) \otimes \left(\bigotimes_{n=-t}^{t} \mathcal{B}(\mathfrak{Z}_n) \right) = \mathcal{B}\left(L^2(\mu_E \otimes \mu_G^{\otimes(2t+1)}) \right)$.

The invertible map ϕ in $E \times G$ defines the unitary operator in $\mathcal{H} \otimes \mathfrak{Z} = L^2(\mu_E \otimes \mu_G)$

$$V = \sum_{\substack{i \in E \\ g \in G}} |\phi(i,g)\rangle\langle i,g|, \tag{17}$$

which is a quantum extension of the deterministic invertible evolution ϕ as

$$V^* m_F V = m_{F \circ \phi}, \qquad \forall F \in \mathcal{L}^\infty(E) \otimes \mathcal{L}^\infty(G).$$

With respect to the canonical basis of $\mathfrak{Z}$, we have $V = \sum_{g,g' \in G} V_{gg'} \otimes |g\rangle\langle g'|$ with

$$V_{gg'} = \sum_{i,j \in E} |i\rangle\langle i, g|\phi(j,g')\rangle\langle j|.$$

According to Section 3, in the following let $\mathbb{E}_k$ denote the expectation of a random variable on $(E \times G^{\mathbb{Z}}, \mathcal{E} \otimes \mathcal{G})$ with respect to the probability measure $\delta_k \otimes q^{\otimes \mathbb{Z}}$.

Theorem 2. *Let P^t be a classical Markov semigroup in a finite state space E and let $(G, \phi, q^{\otimes \mathbb{Z}})$ be a standard dilation. Let $\mathcal{H} = L^2(\mu_E)$, let $\mathfrak{Z} = L^2(\mu_G)$, let V be the unitary operator (17) in $\mathcal{H} \otimes \mathfrak{Z}$, and let J be the $*$-automorphism (16) in $\mathcal{B}(\mathcal{H}) \otimes \left(\bigotimes_{n \in \mathbb{Z}} \mathcal{B}(\mathfrak{Z}_n) \right)$. Then*

(1) for every global random variable F belonging to $\mathcal{L}_{\mathbb{Z}}^\infty$, it holds

$$J(m_F) = m_{F \circ \alpha}; \tag{18}$$

(2) taken $\upsilon = \sum_{g \in G} \sqrt{q_g} |g\rangle \in \mathfrak{Z}$,

 — for every $k \in E$, every $F_1, \ldots, F_n \in \mathcal{L}_{\mathbb{Z}}^\infty$ and every bounded and continuous function $\eta : \mathbb{C}^n \to \mathbb{C}$, it holds

$$\mathbb{E}_k[\eta(F_1, \ldots, F_n)] = \mathrm{tr}\left[\eta(m_{F_1}, \ldots, m_{F_n}) |k\rangle\langle k| \otimes |\upsilon^{\otimes \mathbb{Z}}\rangle\langle \upsilon^{\otimes \mathbb{Z}}| \right], \tag{19}$$

 — $(\mathfrak{Z}, V, \upsilon)$ is a QP-dilation of the quantum dynamical semigroup T^t,

$$Ta = \sum_{g \in G} \left(\sum_{g' \in G} \sqrt{q_{g'}} V_{gg'}^* \right) a \left(\sum_{g'' \in G} \sqrt{q_{g''}} V_{gg''} \right), \qquad a \in \mathcal{B}(\mathcal{H}), \tag{20}$$

 which extends the classical Markov semigroup P^t.

Proof. (1) For every F belonging to an algebra $\mathcal{L}_{[-t,t]}^\infty$,

$$\widehat{\Theta}(m_F) = m_{F \circ \vartheta}, \qquad V_1^* m_F V_1 = m_{F \circ \phi_1}, \qquad J(m_F) = m_{F \circ \alpha},$$

so that, by continuity, Eq. (18) is proved for all $F \in \mathcal{L}_{\mathbb{Z}}^\infty$.

(2) For every $t \in \mathbb{N}$, the pure locally normal state $|\upsilon^{\otimes \mathbb{Z}}\rangle\langle \upsilon^{\otimes \mathbb{Z}}|$ restricted to $\bigotimes_{n=-t}^{t} \mathcal{B}(\mathfrak{Z}_n)$ gives the state $|\upsilon^{\otimes(2t+1)}\rangle\langle \upsilon^{\otimes(2t+1)}|$, where the unit vector $\upsilon^{\otimes(2t+1)}$ belongs to $\bigotimes_{n=-t}^{t} \mathfrak{Z}_n = L^2(\mu_G^{\otimes(2t+1)})$ and has $|\upsilon^{\otimes(2t+1)}|^2 = \mathrm{d}q^{\otimes(2t+1)}/\mathrm{d}\mu_G^{\otimes(2t+1)}$. Thus

$$\mathbb{E}_k[F] = \mathrm{tr}\left[m_F \cdot |k\rangle\langle k| \otimes |\upsilon^{\otimes \mathbb{Z}}\rangle\langle \upsilon^{\otimes \mathbb{Z}}| \right]$$

for all $F \in \mathcal{L}^{\infty}_{[-t,t]}$ and hence, by continuity, for all $F \in \mathcal{L}^{\infty}_{\mathbb{Z}}$. Then Eq. (19) is proved, as $\eta(F_1, \ldots, F_n)$ belongs to $\mathcal{L}^{\infty}_{\mathbb{Z}}$ whenever $F_1, \ldots, F_n \in \mathcal{L}^{\infty}_{\mathbb{Z}}$ and η is bounded and continuous.

Moreover, for every $a \in \mathcal{B}(\mathcal{H})$,

$$\mathbb{E}_v[V^* a V] = \sum_{j', j'' \in E} |j'\rangle \langle j'| \otimes \langle v| V^* a V |j''\rangle \otimes |v\rangle \langle j''|$$

$$= \sum_{\substack{j', j'' \in E \\ g', g'' \in G}} |j'\rangle \sqrt{q_{g'}} \langle j', g'| V^* a V |j'', g''\rangle \sqrt{q_{g''}} \langle j''| = \sum_{\substack{j', j'' \in E \\ g', g'' \in G}} \sqrt{q_{g'} q_{g''}} |j'\rangle\langle \phi(j', g')| a |\phi(j'', g''))\rangle\langle j''|$$

$$= \sum_{g, g', g'' \in G} \sqrt{q_{g'} q_{g''}} V^*_{gg'} a V_{gg''} = Ta.$$

Then

$$Tm_f = \sum_{\substack{g \in G \\ i \in E}} q_g |i\rangle\langle \phi^E(i,g)| m_f |\phi^E(i,g)\rangle\langle i| = m_{Pf}, \qquad \forall f \in \mathcal{L}^{\infty}(\mathcal{E}).$$

$\square$

Equation (18) shows that the group of $*$-automorphisms J^t defined by $(\mathbf{3}, V)$ is a quantum extension of the deterministic invertible homogeneous evolution α^t defined by (G, ϕ). Then, automatically, also the quantum stochastic flow j_t extends the classical j_t. Moreover, Equation (19) shows that the whole standard dilation $(G, \phi, q^{\otimes \mathbb{Z}})$ of P^t is extended by the QP-dilation $(\mathbf{3}, V, v)$ of T^t as the states $q^{\otimes \mathbb{Z}}$ and $v^{\otimes \mathbb{Z}}$ of the environments, together with the dynamics $\circ \alpha$ and J, give rise to the same joint distribution for the trajectories of the global random variables F belonging to $\mathcal{L}^{\infty}_{\mathbb{Z}}$ and for the trajectories of the corresponding normal operators m_F belonging to $\mathcal{B}(\mathcal{H}) \otimes \left(\bigotimes_{n \in \mathbb{Z}} \mathcal{B}(\mathbf{3}_n) \right)$. This happens for every possible starting state k of the classical Markov system and for the corresponding state $|k\rangle\langle k|$ of its quantum counterpart.

Let us note also that if $(G, \phi, q^{\otimes \mathbb{Z}})$ is built as in Theorem 1, then

$$V_{(i,\ell)(1,\ell')} = \delta_{\ell\ell'} |\beta_\ell(i)\rangle\langle i|,$$

and the stochastic map (20) becomes just the stochastic map (14).

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