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Berchio, E.; Ferrero, A.; Grillo, G.

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Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

STABILITY AND QUALITATIVE PROPERTIES OF RADIAL SOLUTIONS OF THE LANE-EMDEN-FOWLER EQUATION ON RIEMANNIAN MODELS

ELVISE BERCHIO, ALBERTO FERRERO, AND GABRIELE GRILLO

Abstract. We study existence, uniqueness and stability of radial solutions of the Lane-Emden-Fowler equation $-\Delta_g u = |u|^{p-1}u$ in a class of Riemannian models (M,g) of dimension $n \geq 3$ which includes the classical hyperbolic space $\mathbb{H}^n$ as well as manifolds with sectional curvatures unbounded below. Sign properties and asymptotic behavior of solutions are influenced by the critical Sobolev exponent while the so-called Joseph-Lundgren exponent is involved in the stability of solutions.

1. Introduction

We study the Lane-Emden-Fowler equation

$$-\Delta_g u = |u|^{p-1}u \quad \text{on } M, \tag{1.1}$$

where $n \geq 3$ and $p > 1$, posed on a *Riemannian model* (M,g) , namely on a manifold admitting a pole o and whose metric is given, in polar or spherical coordinates around o, by

$$ds^2 = dr^2 + (\psi(r))^2 d\Theta^2, \quad r > 0, \Theta \in \mathbb{S}^{n-1} \tag{1.2}$$

for a given function ψ satisfying appropriate conditions. We will denote by g this metric. Here $d\Theta^2$ denotes the canonical metric on the unit sphere $\mathbb{S}^{n-1}$, r is by construction the Riemannian distance between a point whose coordinates are (r,Θ) and o, the function ψ is smooth and positive on $(0,R)$ for some $R \in (0,+\infty]$. In principle R can be finite and in such a case it identifies the *cut locus* of o in M, but hereafter and without further comments we shall assume that $R = +\infty$.

The additional assumptions we shall make later on ψ correspond to considering manifolds which have infinite volume and, at least outside a compact set, have *strictly negative* sectional curvatures. Hence, if such condition holds globally, we are dealing with special classes of Cartan-Hadamard manifolds. The motivating example we have in mind is, therefore, the hyperbolic space $\mathbb{H}^n$, in which some of the problems that we shall study here in greater generality have been recently investigated. In fact, the Riemannian model associated to the choice $\psi(r) = \sinh r$ in (1.2) is a well-known representation of $\mathbb{H}^n$.

In the seminal paper [22] among other results it is shown that, for $p \in \left(1, \frac{n+2}{n-2}\right)$, there is a unique strictly positive radial solution U of (1.1) belonging to the Sobolev space $H^1(\mathbb{H}^n) := \{u \in L^2(V_g);\, \nabla_g u \in L^2(V_g)\}$, where V_g is the Riemannian measure and ∇_g the Riemannian gradient, and U is radial in the sense that it depends only on r. This is in sharp contrast with the Euclidean case, where no such solution exists, and is strongly related to the fact that the L^2 spectrum of $-\Delta_g$ is bounded away from zero, so that an L^2-Poincaré inequality holds. The solution U is rapidly decaying at infinity, but infinitely many other radial positive solutions exist. The precise asymptotics of such slowly decaying solutions was given in [5] also for the case $p \geq \frac{n+2}{n-2}$, together with a classification of

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radial solutions in terms of their sign properties, further investigated in [1]. In fact, sign changing solutions may also exist and are studied in [3, 5]: they can have finite or infinite H^1_r norm, and their asymptotics depend on which of the two cases holds. In [16], the critical case $p = \frac{n+2}{n-2}$ is investigated in further details. See also [9] for other results concerning elliptic problems and [2] for semilinear parabolic problems in $\mathbb{H}^n$.

Our results aim at discussing the cases corresponding to the defining function ψ being everywhere increasing and, moreover, such that $l := \liminf_{r\to+\infty} \frac{\psi'(r)}{\psi(r)} > 0$ (which is Assumption (H_3) in Section 2). While clearly the hyperbolic space satisfies such condition, Riemannian models which are asymptotically hyperbolic satisfy it as well and, more importantly, such a condition allows for *unbounded* negative sectional curvatures: a typical example in which this can hold corresponds to the choice $\psi(r) = e^{r^a}$ for a given $a > 1$ and r large, a case for which (see Section 1.1) sectional curvatures in the radial direction diverge as $-a^2 r^{2(a-1)}$ as $r \to +\infty$. In addition it will be shown later that, under the stated assumption, the L^2 spectrum of $-\Delta_g$ is still bounded away from zero, whereas if $\lim_{r\to+\infty} \frac{\psi'(r)}{\psi(r)} = 0$ then there is no gap in the L^2 spectrum of $-\Delta_g$. Hence, one hardly expects in such situation to be able to construct a positive solution to the equation at hand. It is worth noticing here that if the radial sectional curvature goes to zero as $r \to +\infty$ then necessarily $\lim_{r\to+\infty} \frac{\psi'(r)}{\psi(r)} = 0$ (see Lemma 4.1) and the previous comment applies, whatever the rate of decay of the curvatures is. Hence, in this case no spectral gap is present and the expected picture is of Euclidean type, but we shall not address this issue here.

Under the above mentioned assumptions on ψ, we prove in Theorem 2.2 existence of a finite energy radial solution to (1.1) in the subcritical case $p \in \left(1, \frac{n+2}{n-2}\right)$. Uniqueness of such solution holds under a further technical condition on ψ, see Theorem 2.4. In the supercritical range $p \geq \frac{n+2}{n-2}$, we prove in Theorem 2.7 that if a suitable power of the volume of geodesic balls is *convex* as a function of r, all local radial solutions to (1.1) are everywhere positive and no solution to the Dirichlet problem on geodesic balls exists. In particular, such results hold if ψ itself is convex.

In both subcritical and supercritical cases, we provide an exact description of the asymptotic behavior of positive radial solutions of (1.1). In Theorem 2.6 we show that, in the subcritical case, solutions in the energy space $H^1(M)$ have a fast decay to zero which can be characterized explicitly in terms of the function ψ. An interesting phenomenon occurs for solutions which do not belong to $H^1(M)$: they admit a limit as $r \to +\infty$ which can be strictly positive or equal to zero depending on the integrability at infinity of the function ψ/ψ'.

The same phenomenon occurs in the supercritical case as shown in Theorem 2.9.

The second part of this paper is devoted to stability of solutions. Here by stability we mean the so-called linearized stability. Namely, we say that a solution u of (1.1) is stable if the quadratic form associated with the linearized operator at u is nonnegative definite. Stability of solutions of nonlinear equations in the whole euclidean space is a widely studied problem, especially in the case of the Lane-Emden-Fowler equation and of the Gelfand equation $-\Delta u = e^u$, see e.g. [8, 10, 11, 12, 13, 15] and references therein. See also [6] for results on stability of the Lane-Emden-Fowler and Gelfand equations in bounded domains.

In order to localize the instability of certain solutions we shall also study the stability of solutions outside a compact set, see e.g. [10, 12, 13].

Since the cut locus of the pole o is empty by assumption, any Riemannian model M we are considering is diffeomorphic to $\mathbb{R}^n$, and the main purpose of the present paper is to understand which is the role of the curvature properties of M in determining stability of solutions of (1.1), in particular when sectional curvatures are *negative*. We comment here that the existence of stable solutions to semilinear elliptic equations when Ricci curvature is *positive* has consequences on the structure of the manifold itself (and on the solution as well), as shown recently in [14].

For completeness we first recall what happens when M is the n-dimensional euclidean space. From [12], we know that no nontrivial stable solution (also nonradial) exists if $n \leq 10$ or $n \geq 11$ and $p < p_c(n) = \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}$, where $p_c(n) > \frac{n+2}{n-2}$ is the so-called Joseph-Lundgren exponent, see [20]. On the other hand, for $n \geq 11$ and every $p \geq p_c(n)$ there exists a positive radial stable solution, see [12, 20].

Also we note that when $n \leq 10$ or $n \geq 11$ and $p < p_c(n)$, with $p \neq \frac{n+2}{n-2}$, the euclidean equation admits no nontrivial solution which is stable outside a compact set. On the other hand, if $p = \frac{n+2}{n-2}$ then the euclidean equation admits solutions in $H^1(\mathbb{R}^n)$ which are stable outside a compact set. Among them there are the well-known one-parameter family of solutions of (1.1) which achieve the best Sobolev constant in $\mathbb{R}^n$.

As we said before, under suitable assumptions on ψ, a Poincaré type inequality holds. The validity of this inequality is strictly related to the existence of stable solutions. In Theorems 2.11-2.12 we prove that stable radial solutions of (1.1) always exist in any dimension and for any $p > 1$ provided that their value at the origin is small enough.

This phenomenon is deeply in contrast with the euclidean case where the existence of nontrivial radial stable solutions only depends on n and p but not on the value of the solution at the origin. We also recall that, thanks to rescaling invariance properties of the Lane-Emden-Fowler equation, in the euclidean case all nontrivial radial solutions may be represented as a one-parameter family of rescaled functions. This property explains why there is no dependence of the stability on the value at the origin.

The next step is to understand if radial stable solutions also exist for larger values at the origin. Our main results on stability, Theorem 2.11-2.12, state that independently of the dimension n and of the power p, the set $\mathcal{S}$ of the values at the origin for which the corresponding radial solution of (1.1) is stable, is a closed interval containing 0. One may ask if this interval coincides with $[0, +\infty)$. In Theorem 2.11 we show that, under the same assumptions on n and p for which in the euclidean case we have nonexistence of nontrivial stable solutions, in our Riemannian model the set $\mathcal{S}$ is a bounded closed interval.

This result, which shows instability of radial solutions with a large value at the origin, is based on a blow-up argument which has as a limit problem the Lane-Emden-Fowler equation in the euclidean space. This justifies the relationship between assumptions of Theorem 2.11 and the nonexistence result of stable solutions in the euclidean case.

It is left as an open question to understand if the assumptions of Theorem 2.11 are also necessary for boundedness of the set $\mathcal{S}$.

Stability properties are strictly related to ordering of radial solutions of (1.1). Indeed in Theorem 2.14 we prove that radial solutions of (1.1) corresponding to values at the origin in the set $\mathcal{S}$ are ordered.

Finally, in Theorem 2.15 we show that all radial solutions of (1.1) are stable outside compact sets independently of $n \geq 3$ and $p > 1$, provided that $\psi/\psi' \notin L^{(}0, \infty)$.

This paper is organized as follows: in Section 2 we put the assumptions and the statements of the main results while Sections 3-5 are devoted to the proofs.

1.1. **Notation and preliminaries.** The C^2 smoothness of M around o implies that ψ must be extendible to $r = 0$ with the extension, still denoted by ψ, satisfying $\psi(0) = \psi''(0) = 0$, $\psi'(0) = 1$, the prime indicating right derivative. In greater generality, a power series for ψ near $r = 0$ must contain only odd powers of r should one require additional smoothness at o, see e.g. [25], pp. 179-183, and also [17].

The Riemannian Laplacian of a scalar function f on M is given, in the above coordinates, by

$$\begin{aligned}\Delta_g f(r,\theta_1,\dots,\theta_{n-1}) = & \frac{1}{(\psi(r))^{n-1}}\frac{\partial}{\partial r}\left[(\psi(r))^{n-1}\frac{\partial f}{\partial r}(r,\theta_1,\dots,\theta_{n-1})\right]\\ & + \frac{1}{(\psi(r))^2}\Delta_{\mathbb{S}^{n-1}} f(r,\theta_1,\dots,\theta_{n-1}),\end{aligned}$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Riemannian Laplacian on the unit sphere $\mathbb{S}^{n-1}$. In particular, for *radial* functions, namely functions depending only on r, one has

$$\Delta_g f(r) = \frac{1}{(\psi(r))^{n-1}}\left[(\psi(r))^{n-1}f'(r)\right]' = f''(r) + (n-1)\frac{\psi'(r)}{\psi(r)}f'(r),$$

where from now on a prime will denote, for radial functions, derivative w.r.t. r. Notice that the quantity $(n-1)\frac{\psi'(r)}{\psi(r)}$ has a geometrical meaning, namely it represents mean curvature of the geodesic sphere of radius r in the radial direction. Let ω_n be the volume of the n-dimensional unit sphere. Then

$$S(r) = \omega_n(\psi(r))^{n-1}, \quad V(r) = \int_0^r S(t)\,\mathrm{d}t = \omega_n\int_0^r (\psi(t))^{n-1}\,\mathrm{d}t$$

represent, respectively, the area of the geodesic sphere $\partial B(o,r)$ and the volume of the geodesic ball $B(o,r)$. Moreover (see e.g. [4], [17]) one can show that

$$\frac{1}{n-1}\mathrm{Ric}\,(\partial r,\partial r) = K_\pi(r) = -\frac{\psi''(r)}{\psi(r)},$$

where $\mathrm{Ric}\,(\partial r,\partial r)$ is the Ricci tensor in the radial direction, and $K_\pi(r)$ denotes sectional curvatures w.r.t planes containing ∂r. One shows also that the sectional curvatures w.r.t. planes orthogonal to ∂r is given by $\frac{1-(\psi'(r))^2}{(\psi(r))^2}$. Sectional curvatures equal -1 on the hyperbolic space, whereas they are still negative, but growing in modulus when for example one has, for large r, $\psi(r) = e^{r^a}$ for some $a>1$, a case which can be covered by most of our results.

We consider radial solutions to the Lane-Emden-Fowler equation (1.1). Radial *local* solutions near $r=0$ to (1.1) with $u(0)=\alpha\neq 0$ exist, are unique and satisfy the Cauchy problem

$$\begin{cases} -\dfrac{1}{(\psi(r))^{n-1}}\left[(\psi(r))^{n-1}u'(r)\right]' = |u(r)|^{p-1}u(r) \quad (r>0) \\ \\ u(0)=\alpha \quad u'(0)=0\,. \end{cases} \tag{1.3}$$

For any $r>0$, let us denote by $u_\alpha(r)$ or by $u(\alpha,r)$ the unique solution of the Cauchy problem (1.3).

2. Assumptions and main results

Let ψ be the function defined in the introduction. Let us introduce the following assumptions on ψ:

(H_1) $\psi\in C^2([0,+\infty))$: $\psi(0)=\psi''(0)=0$ and $\psi'(0)=1$;
(H_2) $\psi'(r)\geq 0$ for every $r>0$,
(H_3) $l := \liminf\limits_{r\to+\infty}\dfrac{\psi'(r)}{\psi(r)} > 0$.

Assumption (H_1) is necessary to make the geometric setting outlined in Section 1.1 consistent. Assumptions $(H_2)-(H_3)$ are sufficient conditions to guarantee positivity of bottom of the L^2 spectrum of $-\Delta_g$ in M, see Lemma 4.1. Throughout this paper we denote the bottom of the L^2 spectrum of

$-\Delta_g$ by $\lambda_1(M)$. Under assumptions $(H_1)-(H_3)$ one can show easily that every solution of (1.3) is global.

Proposition 2.1. *Let $p>1$ and assume that ψ satisfies assumptions $(H_1)-(H_3)$. Then, for any $\alpha\neq 0$ the local solution to* (1.3) *may be continued for all $r>0$, $\lim_{r\to+\infty} u'(r)=0$ and $\lim_{r\to+\infty} u(r)$ exists and is finite. In particular* (1.1) *admits infinitely many nontrivial radial solutions.*

Since the proof of Proposition 2.1 can be achieved following the lines of that of [5, Lemma 4.1], we omit it. The same proof does not work if $l=0$ in (H_3). However, if ψ satisfies

$$\exists\,\beta,\beta'>0:\quad \frac{\beta}{r}\le\frac{\psi'(r)}{\psi(r)}\le\beta'\quad \forall r\ge r_0 \tag{2.1}$$

for some $r_0>0$ one may repeat the proof of [24, Theorem 5] to show that $\lim_{r\to+\infty}u(r)=0=\lim_{r\to+\infty}u'(r)$. Clearly, (2.1) includes the euclidean case $\psi(r)=r$ but does not hold if, for instance, $\psi(r)=\log(r)$.

The results concerning existence and qualitative behavior of solutions of (1.1) are strongly influenced by the range in which the power p varies. In the sequel we distinguish the subcritical case $1<p<2^*-1=\frac{n+2}{n-2}$ and the supercritical case $p\ge\frac{n+2}{n-2}$.

• The subcritical case. Let start with the following existence result of a radial $H^1(M)$-solution of (1.1):

Theorem 2.2. *Let $1<p<\frac{n+2}{n-2}$ and ψ satisfy assumptions $(H_1)-(H_3)$. Then* (1.1) *admits a positive radial solution $u\in H^1(M)$.*

One may wonder if (1.1) admits a unique radial solution belonging to $H^1(M)$. This happens in the hyperbolic space, i.e. $\psi(r)=\sinh(r)$, see [22]. In order to guarantee uniqueness of radial $H^1(M)$-solutions, we introduce a supplementary condition on the function ψ. To this purpose we recall from [21] the following definition:

Definition 2.3. *A function $G:(0,+\infty)\to\mathbb{R}$ differentiable, satisfies the $\Lambda-$property if there exists $0\le r_1\le+\infty$ such that $G'\ge 0$ in $(0,r_1)$ and $G'\le 0$ in $(r_1,+\infty)$ with $G'\not\equiv 0$.*

Note that the definition includes the cases in which G is always nondecreasing or nonincreasing in $[0,+\infty)$. We are ready to state the following uniqueness result:

Theorem 2.4. *Let $1<p<\frac{n+2}{n-2}$. Assume that ψ satisfies $(H_1)-(H_2)$ and that there exists*

$$\lim_{r\to+\infty}\frac{\psi'(r)}{\psi(r)}=\lim_{r\to+\infty}\frac{\psi''(r)}{\psi'(r)}=l\in(0,+\infty]\,. \tag{2.2}$$

Furthermore, set $\delta:=\frac{2(n-1)}{p+3}$ and let the function

$$G(r):=\delta\,\psi^{\delta(p-1)-2}(r)\left[(\delta+2-n)\left(\psi'(r)\right)^2-\psi''(r)\psi(r)\right]\quad (r>0)$$

satisfy the $\Lambda-$property.

Finally, if $l=+\infty$ assume that ψ satisfies the extra condition

$$\frac{\psi'(r)}{\psi(r)}=o(\psi^\delta(r))\,,\qquad \frac{\psi''(r)}{\psi'(r)}=o(\psi^\delta(r))\quad \text{as } r\to+\infty\,. \tag{2.3}$$

Then, problem (1.3) *admits a unique positive solution U belonging to $H^1(M)$. Moreover, every solution to* (1.3) *with $0<\alpha<U(0)$ is of one sign, while any solution to* (1.3) *with $\alpha>U(0)$ is sign-changing.*

Concerning the validity of the $\Lambda-$property for the function G defined in Theorem 2.4 we observe that it is satisfied when $\psi(r) = \sinh(r)$, i.e. $M = \mathbb{H}^n$. For more general function ψ we state the following

Proposition 2.5. *If ψ satisfies assumptions $(H_1) - (H_3)$, and in addition ψ is four times differentiable with $\psi'''(r) > 0$ and $\left(\frac{\psi'(r)}{\psi'''(r)}\right)' \le 0$ for every $r > 0$, then the function G defined in Theorem 2.4 satisfies the $\Lambda-$property for every $\frac{2n+1}{2n-3} \le p < \frac{n+2}{n-2}$.*

By Proposition 2.5, it follows that if $\psi(r) = re^{r^{2\gamma}}$ then the corresponding function G satisfies the $\Lambda-$property for every $\gamma \ge 1$.

Concerning condition (2.3), we observe that it holds, for instance, if $\frac{\psi'(r)}{\psi(r)} = P(r)$ eventually, where P is a nonconstant polynomial.

Finally we state a result dealing with the asymptotic behavior of radial positive solutions of (1.1).

Theorem 2.6. *Let $n \ge 3$ and $1 < p < \frac{n+2}{n-2}$. Suppose that ψ satisfies assumptions $(H_1) - (H_2)$ and*

$$\lim_{r\to+\infty} \frac{\psi'(r)}{\psi(r)} = l \in (0, +\infty] \,. \tag{2.4}$$

Finally in the case $l = +\infty$ assume the supplementary condition

$$\left[\log\left(\frac{\psi'(r)}{\psi(r)}\right)\right]' = O(1) \qquad \text{as } r \to +\infty \,. \tag{2.5}$$

Let u be a radial positive solution of (1.1).

(i) *If $u \in H^1(M)$ then there exists $L \in (-\infty, 0)$ such that*

$$\lim_{r\to+\infty} \psi^{n-1}(r)u'(r) = L \,.$$

Moreover

$$\lim_{r\to+\infty} \psi^{n-1}(r)u(r) = \frac{|L|}{(n-1)l} \qquad \text{if } l < +\infty \,,$$

and

$$\lim_{r\to+\infty} \frac{u(r)}{\int_r^{+\infty} \psi^{1-n}(s)\, ds} = |L| \qquad \text{if } l = +\infty \,.$$

(ii) *If $u \notin H^1(M)$ and $\frac{\psi}{\psi'} \in L^1(0,\infty)$ then*

$$\lim_{r\to+\infty} u(r) \in (0, +\infty) \,.$$

(iii) *If $u \notin H^1(M)$ and $\frac{\psi}{\psi'} \notin L^1(0,\infty)$ then u vanishes at infinity with the following rate*

$$\lim_{r\to+\infty} \left(\int_0^r \frac{\psi(s)}{\psi'(s)}\, ds\right)^{1/(p-1)} u(r) = \left(\frac{n-1}{p-1}\right)^{1/(p-1)} \,.$$

In particular when $l < +\infty$ we have

$$\lim_{r\to+\infty} r^{1/(p-1)}u(r) = \left(\frac{l(n-1)}{p-1}\right)^{1/(p-1)} \,.$$

• The supercritical case. Throughout this paper let us denote by B_R the geodesic ball centered at o of radius R, i.e.

$$B_R := \{(r, \theta_1, \ldots, \theta_{n-1}) : 0 < r < R \text{ and } \theta_1, \ldots, \theta_{n-1} \in \mathbb{S}^{n-1}\} \,.$$

Theorem 2.7. *Let $p \geq \frac{n+2}{n-2}$ and ψ satisfy assumptions $(H_1)-(H_2)$. If $p = \frac{n+2}{n-2}$ assume furthermore that ψ is three times differentiable near 0 with $\psi''(0) = 0$ and $\psi'''(0) > 0$. Finally, let the function*

$$A(r) := \left(\int_0^r (\psi(s))^{n-1}\, ds \right)^{\frac{p-1}{2(p+1)}} = c\, [\operatorname{Vol} B(o,r)]^{\frac{p-1}{2(p+1)}}$$

be convex on $[0,+\infty)$. Then any solution $u(r)$ to (1.3) does not change sign for all $r \in [0,+\infty)$. In particular, the Dirichlet problem

$$\begin{cases} -\Delta_g u = |u|^{p-1}u & \text{in } B_{\overline{r}} \\ u = 0 & \text{on } \partial B_{\overline{r}} \end{cases}$$

with $0 < \overline{r} < +\infty$ has no nontrivial radial solutions.

Concerning the convexity of the function A defined in Theorem 2.7 we state the following

Proposition 2.8. *Assume that ψ satisfies $(H_1) - (H_2)$. Let A be the function defined in Theorem 2.7. Then we have:*

(i) *if ψ is convex, then A is also convex;*
(ii) *if ψ is such that (2.4) holds with $l < +\infty$, then A is eventually convex at $+\infty$;*
(iii) *if ψ is such that (2.4) holds with $l = +\infty$ and (2.5) is satisfied, then A is eventually convex at $+\infty$.*

From Proposition 2.8 it follows that the assumptions of Theorem 2.7 are satisfied either by the hyperbolic model (see [5]) or by models having unbounded negative sectional curvatures such as $\psi(r) = re^{r^{2\gamma}}$ with $\gamma \geq 0$.

Similarly to the subcritical case, for the asymptotic behavior of radial positive solutions of (1.1) we have

Theorem 2.9. *Let $n \geq 3$ and $p \geq \frac{n+2}{n-2}$. Suppose that ψ satisfies assumptions $(H_1)-(H_2)$, (2.4) and that the function $A = A(r)$ defined in Theorem 2.7 is convex. Finally in the case $l = +\infty$ we also assume (2.5). Let u be a radial (positive) solution of (1.1).*

(i) *If $\frac{\psi}{\psi'} \in L^1(0,\infty)$ then*
$$\lim_{r\to+\infty} u(r) \in (0,+\infty)\,.$$
(ii) *If $\frac{\psi}{\psi'} \notin L^1(0,\infty)$ then u vanishes at infinity with the following rate*
$$\lim_{r\to+\infty} \left(\int_0^r \frac{\psi(s)}{\psi'(s)}\, ds \right)^{1/(p-1)} u(r) = \left(\frac{n-1}{p-1} \right)^{1/(p-1)}\,.$$
In particular when $l < +\infty$ we have
$$\lim_{r\to+\infty} r^{1/(p-1)} u(r) = \left(\frac{l(n-1)}{p-1} \right)^{1/(p-1)}\,.$$

• Stability of radial solutions of (1.1). We start by explaining what we mean by stability and stability outside a compact set, see also [15].

Definition 2.10. *A solution $u \in C^2(M)$ to (1.1) is stable if*

$$\int_M |\nabla_g \varphi|_g^2\, dV_g - p \int_M |u|^{p-1}\varphi^2\, dV_g \geq 0 \quad \forall\, \varphi \in C_c^\infty(M). \tag{2.6}$$

A solution $u \in C^2(M)$ to (1.1) *is stable outside the compact set K if*

$$\int_{M\setminus K} |\nabla_g \varphi|_g^2 \, dV_g - p\int_{M\setminus K} |u|^{p-1}\varphi^2 \, dV_g \geq 0 \quad \forall\, \varphi \in C_c^\infty(M\setminus K). \tag{2.7}$$

For any $n \geq 11$, let $p_c(n) = \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}$ be the Joseph-Lundgren exponent. We can now state the first result concerning stability of radial solutions of (1.1).

Theorem 2.11. *Let $3 \leq n \leq 10$ and $p > 1$ or $n \geq 11$ and $1 < p < p_c(n)$. Assume that ψ satisfies $(H_1) - (H_3)$. For any $\alpha \geq 0$ denote by u_α the unique solution of* (1.3)*. There exists $\alpha_0 \in (0, +\infty)$ such that*

(i) *if $\alpha \in [0, \alpha_0]$ then u_α is stable;*
(ii) *if $\alpha > \alpha_0$ then u_α is unstable.*

Furthermore we also have $\alpha_0 \geq \big(p^{-1}\lambda_1(M)\big)^{1/(p-1)}$. The inequality is strict if one of the following alternatives hold

$$\limsup_{r\to+\infty} \frac{\psi'(r)}{\psi(r)} < +\infty \qquad \text{or} \qquad l = +\infty \text{ in } (H_3) \text{ and } \psi \text{ satisfies } (2.5) \text{ and } \psi/\psi' \notin L^1(0,\infty)\,. \tag{2.8}$$

By comparing Theorem 2.11 with the stability result in the euclidean case, one sees that the existence of stable solutions in dimension $n \leq 10$ or in dimension $n \geq 11$ but with $p < p_c(n)$, seems to be strictly related to the validity of the Poincaré inequality (see Table 1 and Table 2 below). Indeed the existence of the positive number α_0 introduced in Theorem 2.11 comes from the the positivity of the bottom of the L^2 spectrum $\lambda_1(M)$ of $-\Delta_g$ in M as one can see from the estimate $\alpha_0 > \big(p^{-1}\lambda_1(M)\big)^{1/(p-1)}$, see also Lemma 4.1. On the contrary, in the euclidean case the Poincaré inequality in $\mathbb{R}^n$ does not hold and, if $n \leq 10$ or $n \geq 11$ but $p < p_c(n)$, all nontrivial solutions of the Lane-Emden-Fowler equation are unstable.

We observe that the assumptions on the dimension n and on the power p in Theorem 2.11 are at least sufficient to show the existence of the switch between stability for small values of α and instability for large values of α but it is not clear if they are also necessary. As a partial result we state the validity of the following alternatives:

Theorem 2.12. *Let $n \geq 11$ and $p \geq p_c(n)$. Assume that ψ satisfies $(H_1) - (H_3)$. For any $\alpha \geq 0$ denote by u_α the unique solution of* (1.3)*. There exists $\alpha_0 \in (0, +\infty]$ such that either $\alpha_0 = +\infty$ and u_α is stable for any $\alpha \geq 0$ or $\alpha_0 < +\infty$ and u_α is stable for any $\alpha \in [0, \alpha_0]$ and unstable for any $\alpha > \alpha_0$.*

Concerning stability of solutions in the energy space, we state the following

Proposition 2.13. *Let $n \geq 3$ and $p > 1$. Assume that ψ satisfies $(H_1) - (H_3)$. Let u be a radial stable solution of* (1.1)*. If $u \in L^2(M)$ then $u \equiv 0$.*

Stability properties of solutions are related to ordering and intersection properties of radial solutions of (1.1):

Theorem 2.14. *Let $n \geq 3$ and $p > 1$. Assume that ψ satisfies $(H_1) - (H_3)$. Let $\alpha, \beta \geq 0$ and let u_α, u_β be the corresponding solutions of* (1.3)*. If u_α and u_β are stable then they do not intersect. In particular stable solutions are strictly positive (or strictly negative) and if $\alpha_0 \in (0, +\infty]$ is as in Theorems 2.11-2.12 then all solutions in the set $\{u_\alpha : \alpha \in [0, \alpha_0)\}$ are ordered.*

We conclude the section by dealing with stability outside a compact set.

Theorem 2.15. *Let $n \geq 3$ and $p > 1$. Assume that ψ satisfies $(H_1)-(H_3)$. Then any radial solution of (1.1) is stable outside a compact set provided that (2.8) holds.*

Differently from the euclidean case, see Table 1 below, Theorem 2.15 states that under assumptions $(H_1) - (H_3)$ and (2.8), all solutions of (1.3) are stable outside a compact independently of the value of the power p. We note that assumption (2.8) assures that solutions of (1.3) vanish as $r \to +\infty$ (see Proposition 2.1, formula (2.1), Theorems 2.6 and 2.9). Hence, the difference from the euclidean case once more comes from the fact that, under assumptions $(H_1) - (H_3)$, the bottom of the L^2 spectrum of $-\Delta_g$ in M is strictly positive.

	$n \leq 10$ or ($n \geq 11$ and $p < p_c(n)$)	$n \geq 11$ and $p \geq p_c(n)$
u_α stable $\forall\ \alpha \neq 0$	NO	YES
u_α unstable $\forall\ \alpha \neq 0$	YES	NO
u_α stable outside a compact $\forall\ \alpha$	NO if $p \neq \frac{n+2}{n-2}$ YES if $p = \frac{n+2}{n-2}$	YES

TABLE 1. Stability of solutions u_α to (1.3) when $\psi(r) = r$ (Euclidean case).

	$n \leq 10$ or ($n \geq 11$ and $p < p_c(n)$)	$n \geq 11$ and $p \geq p_c(n)$
u_α stable $\forall\ 0 < \lvert\alpha\rvert \leq \alpha_0$	YES	YES if $\lvert\alpha\rvert < \alpha_0$
u_α unstable $\forall\ \lvert\alpha\rvert > \alpha_0$	YES	?
u_α stable outside a compact $\forall\ \alpha$	YES if (2.8) holds	YES if (2.8) holds

TABLE 2. Stability of solutions u_α to (1.3) for ψ satisfying $(H_1) - (H_3)$.

3. Proof of the results in the supercritical case

3.1. **Proof of Theorem 2.7.** Let u be a nontrivial solution of (1.3), up to replace u with $-u$, we may assume $\alpha > 0$. For $r \geq 0$, we set

$$P(r) := \left[(p+1)\int_0^r (\psi(s))^{n-1}\,ds\right]\left(\frac{(u'(r))^2}{2} + \frac{|u(r)|^{p+1}}{p+1}\right) + (\psi(r))^{n-1}u(r)u'(r)\,.$$

Then, for u solving (1.3) we get

$$P'(r) = \left[\frac{p+3}{2}(\psi(r))^{n-1} - (n-1)(p+1)\frac{\psi'(r)}{\psi(r)}\int_0^r (\psi(s))^{n-1}\,ds\right](u'(r))^2 := K(r)(u'(r))^2,$$

the latter equality being meant as a definition of $K(r)$. We notice that, as $r \downarrow 0$, the known asymptotics of $\psi(r)$ as $r \to 0$ implies that $K(r) \sim r^{n-1}[(n+2) - (n-2)p]/(2n)$ if $p > \frac{n+2}{n-2}$ and $K(r) \sim r^{n+1}[-2(n-1)/(n^2-4)]\,\psi'''(0)$, if $p = \frac{n+2}{n-2}$, where we exploit the assumptions $\psi''(0) = 0$ and $\psi'''(0) > 0$.

This clearly shows that, in such range of p, $K(r) < 0$ for r sufficiently small, and hence that $P'(r) < 0$ for the same values of r. The strict inequality follows from the fact that $u'(r) \neq 0$ for $r \in (0, \varepsilon)$ for a suitable $\varepsilon > 0$, a fact which holds since u is different from zero in a right neighborhood of zero and by (1.3) we have

$$u'(r) = -\frac{1}{(\psi(r))^{n-1}}\int_0^r (\psi(s))^{n-1}|u(s)|^{p-1}u(s)\,ds\,.$$

Hence, since $P(0) = 0$, we have proven that $P(r) < 0$ in a sufficiently small right neighborhood of zero. We claim that $K(r) \le 0$ for any $r > 0$ which implies P nonincreasing in $(0, +\infty)$; being $P(r) < 0$ for $r > 0$ small enough this yields $P(r) < 0$ for any $r > 0$.

Let us prove the claim. Let $\Psi(r) := \int_0^r (\psi(s))^{n-1}\, ds$. One computes

$$\frac{\Psi''(r)}{\Psi'(r)} = (n-1)\frac{\psi'(r)}{\psi(r)}.$$

Hence, requiring that $K(r) \le 0$ is equivalent to ask that

$$\frac{\Psi''(r)}{\Psi'(r)} \ge \frac{p+3}{2(p+1)}\frac{\Psi'(r)}{\Psi(r)},$$

where we have used the fact that $\Psi(r) > 0$ for all $r \in (0, \infty)$. Recall that, by construction, $\Psi'(r) > 0$ for all $r > 0$. Setting $a_p = \frac{p+3}{2(p+1)}$ we can then rewrite the latter formula as

$$\left[\log\left(\frac{\Psi'(r)}{(\Psi(r))^{a_p}}\right)\right]' \ge 0.$$

or equivalently, setting $c_p = 1 - a_p = \frac{p-1}{2(p+1)}$, as

$$\left[\log\left(((\Psi(r))^{c_p})'\right)\right]' \ge 0.$$

The latter condition is clearly equivalent to $[(\Psi(r))^{c_p}]'' \ge 0$, namely to the fact that $(\Psi(r))^{c_p}$ is convex (recall that ψ is at least C^2). This completes the proof of the claim. Since $u(0) > 0$, if we assume that there exists $\rho > 0$ such that $u(\rho) = 0$ then we have $u'(\rho) < 0$ and hence $P(\rho) > 0$, a contradiction.

3.2. **Proof of Proposition 2.8.** A simple computation yields that $A(r)$ is convex if and only if the function $h(r) := 2(n-1)(p+1)\psi'(r)\int_0^r \psi^{n-1}(s)\, ds - (p+3)\psi^n(r)$ is positive in $(0, +\infty)$. This readily follows if ψ is a convex function too. Indeed, we have $h(0) = 0$ and

$$h'(r) = (p(n-2) - (n+2))\psi'(r)\psi^{n-1}(r) + 2(n-1)(p+1)\psi''(r)\int_0^r (\psi(s))^{n-1}\, ds\,.$$

Hence, statement (i) follows.

Then we turn to the proofs of (ii) and (iii). First we claim that

$$\lim_{r\to+\infty}(n-1)\,\psi'(r)\,\frac{\int_0^r \psi^{n-1}(s)\, ds}{\psi^n(r)} = 1\,. \tag{3.1}$$

By this,

$$\lim_{r\to+\infty} h(r) = \lim_{r\to+\infty}\psi^n(r)\left[2(n-1)(p+1)\psi'(r)\,\frac{\int_0^r \psi^{n-1}(s)\, ds}{\psi^n(r)} - (p+3)\right] = +\infty$$

and we conclude.

Next we prove (3.1). If $\lim_{r\to+\infty}\dfrac{\psi'(r)}{\psi(r)} = l$ for some $0 < l < +\infty$, the claim readily follows by the l'Hôpital rule. Indeed, we have

$$\lim_{r\to+\infty}\frac{\int_0^r \psi^{n-1}(s)\, ds}{\psi^{n-1}(r)} = \lim_{r\to+\infty}\frac{\psi(r)}{(n-1)\psi'(r)} = \frac{1}{(n-1)l}\,.$$

Let now $l = +\infty$. Again, by the l'Hôpital rule we deduce

$$\lim_{r\to+\infty}(n-1)\,\psi'(r)\,\frac{\int_0^r \psi^{n-1}(s)\, ds}{\psi^n(r)} = (n-1)\lim_{r\to+\infty}\frac{\left[\frac{\psi'(r)}{\psi(r)}\int_0^r \psi^{n-1}(s)\, ds\right]'}{[\psi^{n-1}(r)]'}$$

$$= 1 + \lim_{r\to+\infty} \left[\log\left(\frac{\psi'(r)}{\psi(r)}\right)\right]' \frac{\int_0^r \psi^{n-1}(s)\,ds}{\psi^{n-1}(r)}\,.$$

Then, since $\int_0^r \psi^{n-1}(s)\,ds = o(\psi^{n-1}(r))$ as $r \to +\infty$, (3.1) holds for every function ψ such that $\left[\log\left(\frac{\psi'(r)}{\psi(r)}\right)\right]'$ remains bounded.

3.3. **Proof of Theorem 2.9.** We start with the following estimate from below on solutions of (1.3):

Lemma 3.1. *Let the assumptions of Theorem 2.9 hold and u be a positive solution to* (1.3). *There exist no strictly positive constants C, β such that $u(r) \le C(\psi(r))^{-\beta}$ for all $r \ge 0$.*

Proof. Assume by contradiction that there exist C, β such that $u(r) \le C(\psi(r))^{-\beta}$ for all $r \ge 0$. It is not restrictive assuming that $\beta < (n-1)/p$.

After integration in $(0, r)$ we get

$$u'(r) \ge -C^p(\psi(r))^{1-n} \int_0^r (\psi(s))^{n-1-\beta p}\,ds \qquad \text{for any } r > 0\,.$$

Integrating now in $(r, +\infty)$ we obtain

$$u(r) \le C^p \int_r^{+\infty} \left((\psi(s))^{1-n} \int_0^s (\psi(t))^{n-1-\beta p}\,dt \right) ds \qquad \text{for any } r > 0\,.$$

Then, by (2.4) we have

$$u(r) = O((\psi(r))^{-\beta p}) \qquad \text{as } r \to +\infty\,.$$

Iterating this procedure as in the proof of [5, Lemma 5.2] we deduce that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$u(r) \le C_\varepsilon(\psi(r))^{-(n-1-\varepsilon)} \qquad \text{for any } r > 0\,. \tag{3.2}$$

The next purpose is to obtain a lower bound on u in order to reach a contradiction with (3.2).

Let now $P = P(r)$ be the function defined in the proof of Theorem 2.7. Since we are assuming $A = A(r)$ convex, by the proof of Proposition 2.8 we deduce that P is negative and nonincreasing in $(0, +\infty)$.

Therefore

$$\frac{\int_0^r (\psi(s))^{n-1}\,ds}{(\psi(r))^{n-1}} \left(\frac{(u'(r))^2}{2} + \frac{(u(r))^{p+1}}{p+1} \right) + \frac{u(r)u'(r)}{p+1} < 0 \qquad \text{for any } r > 0\,.$$

In particular we obtain

$$u'(r) + \frac{2(\psi(r))^{n-1}}{(p+1)\int_0^r (\psi(s))^{n-1}\,ds}\, u(r) > 0 \qquad \text{for any } r > 0\,. \tag{3.3}$$

By (3.1) we deduce that

$$\frac{(\psi(r))^{n-1}}{\int_0^r (\psi(s))^{n-1}\,ds} \sim (n-1)\frac{\psi'(r)}{\psi(r)} \qquad \text{as } r \to +\infty$$

and hence for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that

$$u'(r) + \frac{2(n-1+\varepsilon)}{p+1}\frac{\psi'(r)}{\psi(r)}\, u(r) > 0 \qquad \text{for any } r > r_\varepsilon$$

and after integration it follows that there exists $\overline{C} > 0$ such that

$$u(r) > \overline{C}(\psi(r))^{-\frac{2(n-1+\varepsilon)}{p+1}} \qquad \text{for any } r > r_\varepsilon\,.$$

Since $p > 1$, this contradicts (3.2) and completes the proof of the lemma. $\square$

Next we prove

Lemma 3.2. *Let the assumptions of Theorem 2.9 hold and u be a positive solution to* (1.3)*. Then*

$$\frac{u'(r)}{u(r)} = o\left(\frac{\psi'(r)}{\psi(r)}\right) \qquad \text{as } r \to +\infty\,.$$

Proof. As a first step, we may exclude the case in which

$$\limsup_{r\to+\infty} \frac{u'(r)}{u(r)}\frac{\psi(r)}{\psi'(r)} < 0 \tag{3.4}$$

since otherwise we would have

$$\frac{u'(r)}{u(r)} < -C_1\frac{\psi'(r)}{\psi(r)} \qquad \text{for any } r > \overline{r}$$

for some $C_1 > 0$ and $\overline{r} > 0$, and after integration it follows

$$u(r) < C_2(\psi(r))^{-C_1} \qquad \text{for any } r > \overline{r}$$

for some constant $C_2 > 0$, in contradiction with Lemma 3.1. Now it is sufficient to prove existence of the limit in (3.4).

Suppose by contradiction that such a limit does not exist. For simplicity, here we consider only the case $l = +\infty$ since the case l finite can be treated exactly as in [5, Lemma 5.3]. Let $r_m \to +\infty$ be the sequence of local maxima and minima points for $\frac{u'(r)}{u(r)}\frac{\psi(r)}{\psi'(r)}$. Then for any m we have

$$u''(r_m)u(r_m) - (u'(r_m))^2 = u(r_m)u'(r_m)\left[\log\left(\frac{\psi'(r_m)}{\psi(r_m)}\right)\right]'$$

By (1.3), (3.1), (3.3) and $p > 1$, it follows that

$$u'(r_m) > -\frac{(u(r_m))^{p+1}}{u(r_m)\left\{(n-1)\frac{\psi'(r_m)}{\psi(r_m)} - \frac{2}{p+1}\frac{(\psi(r_m))^{n-1}}{\int_0^{r_m}(\psi(s))^{n-1}ds} + \left[\log\left(\frac{\psi'(r_m)}{\psi(r_m)}\right)\right]'\right\}}\,.$$

By (3.1), the fact that $l = +\infty$ and that $\left[\log\left(\frac{\psi'(r_m)}{\psi(r_m)}\right)\right]'$ is bounded we obtain

$$u'(r_m) > -\frac{(u(r_m))^p}{\frac{(n-1)(p-1)}{p+1}\frac{\psi'(r_m)}{\psi(r_m)} + o\left(\frac{\psi'(r_m)}{\psi(r_m)}\right)}\,.$$

This shows that $\frac{u'(r_m)}{u(r_m)}\frac{\psi(r_m)}{\psi'(r_m)} \to 0$ as $m \to +\infty$ and by the definition of $\{r_m\}$ we infer

$$\lim_{r\to+\infty}\frac{u'(r)}{u(r)}\frac{\psi(r)}{\psi'(r)} = 0\,,$$

a contradiction. This completes the proof of the lemma. □

Lemma 3.3. *Let the assumptions of Theorem 2.9 hold and u be a positive solution to* (1.3)*. Then*

$$\lim_{r\to+\infty}\frac{u'(r)}{u^p(r)}\frac{\psi'(r)}{\psi(r)} = -\frac{1}{n-1}\,. \tag{3.5}$$

Proof. We omit the proof in the case l finite since it is completely similar to the proof obtained in [5, Section 5]. Let $l = +\infty$. First we prove the existence of the limit in (3.5). Suppose by contradiction that the limit in (3.5) does not exist. Then there exists a sequence $r_m \to +\infty$ of local maxima and minima points for the function $\frac{u'(r)}{u^p(r)}\frac{\psi'(r)}{\psi(r)}$. Then we have

$$u''(r_m)u(r_m) = p(u'(r_m))^2 - u(r_m)u'(r_m)\left[\log\left(\frac{\psi'(r_m)}{\psi(r_m)}\right)\right]' .$$

Inserting this identity in (1.3) multiplied by u we obtain

$$u'(r_m) = -\frac{u^p(r_m)}{(n-1)\frac{\psi'(r_m)}{\psi(r_m)} + p\frac{u'(r_m)}{u(r_m)} - \left[\log\left(\frac{\psi'(r_m)}{\psi(r_m)}\right)\right]'} .$$

Therefore we have

$$\frac{u'(r_m)}{u^p(r_m)}\frac{\psi'(r_m)}{\psi(r_m)} = -\left\{n-1+p\frac{u'(r_m)}{u(r_m)}\frac{\psi(r_m)}{\psi'(r_m)} - \frac{\psi(r_m)}{\psi'(r_m)}\left[\log\left(\frac{\psi'(r_m)}{\psi(r_m)}\right)\right]'\right\}^{-1} .$$

By Lemma 3.2, the fact that $l = +\infty$ and that (2.5) holds true, we obtain

$$\lim_{m\to+\infty}\frac{u'(r_m)}{u^p(r_m)}\frac{\psi'(r_m)}{\psi(r_m)} = -\frac{1}{n-1} .$$

By definition of the sequence $\{r_m\}$, this gives the existence of the limit in (3.5), a contradiction. It remains to compute explicitly the limit in (3.5).

By (1.3) we obtain

$$\frac{u''(r)}{u'(r)}\frac{\psi(r)}{\psi'(r)} + n - 1 + \frac{u^p(r)}{u'(r)}\frac{\psi(r)}{\psi'(r)} = 0$$

and hence there exists the limit

$$\lim_{r\to+\infty}\frac{u''(r)}{u'(r)}\frac{\psi(r)}{\psi'(r)} = 1 - n - \lim_{r\to+\infty}\frac{u^p(r)}{u'(r)}\frac{\psi(r)}{\psi'(r)} . \tag{3.6}$$

On the other hand, by de l'Hôpital rule and Lemma 3.2 we have

$$0 = \lim_{r\to+\infty}\frac{u'(r)}{u(r)}\frac{\psi(r)}{\psi'(r)} = \lim_{r\to+\infty}\frac{[u'(r)\psi(r)(\psi'(r))^{-1}]'}{u'(r)}$$
$$= \lim_{r\to+\infty}\frac{\psi(r)}{\psi'(r)}\left\{\frac{u''(r)}{u'(r)} - \left[\log\left(\frac{\psi'(r)}{\psi(r)}\right)\right]'\right\} = \lim_{r\to+\infty}\frac{\psi(r)}{\psi'(r)}\frac{u''(r)}{u'(r)} .$$

Combining this with (3.6) we arrive to the conclusion of the proof. □

End of the proof of Theorem 2.9. Using (3.5) we have that for any $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that

$$u^{1-p}(r_\varepsilon) + \left(\frac{p-1}{n-1} - \varepsilon\right)\int_{r_\varepsilon}^r \frac{\psi(s)}{\psi'(s)}\,ds < u^{1-p}(r) < u^{1-p}(r_\varepsilon) + \left(\frac{p-1}{n-1} + \varepsilon\right)\int_{r_\varepsilon}^r \frac{\psi(s)}{\psi'(s)}\,ds .$$

If the function $\frac{\psi}{\psi'}$ is integrable in a neighborhood of infinity then $\lim_{r\to+\infty} u(r) > 0$. If $\frac{\psi}{\psi'}$ is not integrable in a neighborhood of infinity then u vanishes at infinity and

$$\lim_{r\to+\infty}\left(\int_{r_\varepsilon}^r \frac{\psi(s)}{\psi'(s)}\,ds\right)^{1/(p-1)} u(r) = \left(\frac{n-1}{p-1}\right)^{1/(p-1)} .$$

This completes the proof of the theorem.

4. Proof of the results in the subcritical case

4.1. **Proof of Theorem 2.2.** By standard arguments we deduce that the bottom of the L^2 spectrum of $-\Delta_g$ in M admits the following variational characterization:

$$\lambda_1(M) := \inf_{\varphi \in C_c^\infty(M)\setminus\{0\}} \frac{\int_M |\nabla_g \varphi|_g^2 \, dV_g}{\int_M \varphi^2 \, dV_g} . \tag{4.1}$$

We start by proving the positivity of $\lambda_1(M)$, and by observing that if instead ψ'/ψ tends to zero, such positivity is false.

Lemma 4.1. *Let $n \geq 3$ and assume that ψ satisfies assumptions $(H_1) - (H_3)$. Then $\lambda_1(M) > 0$. If instead (H_3) does not hold and one has in addition $\psi'(r)/\psi(r) \to 0$ as $r \to +\infty$, then $\lambda_1(M) = 0$. In particular, the latter fact holds if the radial mean curvature at r, or the radial sectional curvature at r, tend to zero as $r \to +\infty$.*

Proof. Let $\lambda_1(B_R)$ be the infimum of the functional in (4.1) with test functions in $C_c^\infty(B_R)$, namely $\lambda_1(B_R)$ is the first eigenvalue of the Laplace-Beltrami operator on B_R under the Dirichlet boundary condition. From [18] we recall the estimate

$$\lambda_1(B_R) \geq \frac{1}{4F(R)} , \tag{4.2}$$

where $F(R) := \sup_{0<r<R} H_R(r)$ for any $R \in (0, +\infty)$ and

$$H_R(r) := \left[\left(\int_0^r (\psi(s))^{n-1} \, ds \right) \left(\int_r^R (\psi(s))^{1-n} \, ds \right) \right] .$$

Since the map $R \mapsto \lambda_1(B_R)$ is decreasing and $\lambda_1(M) = \lim_{R\to+\infty} \lambda_1(B_R)$, one has

$$\lambda_1(M) \geq \lim_{R\to+\infty} \frac{1}{4F(R)} .$$

In particular, the claim can be proved by showing that $F(R)$ stays bounded.

We have that $\lim_{r\to R^-} H_R(r) = 0$ and, by applying twice the l'Hôpital rule, that

$$\lim_{r\to 0^+} H_R(r) = \lim_{r\to 0^+} \left(\frac{\int_0^r (\psi(s))^{n-1} \, ds}{(\psi(r))^{n-1}} \right)^2 = \lim_{r\to 0^+} \left(\frac{\psi(r)}{(n-1)\psi'(r)} \right)^2 = 0 .$$

On the other hand, for $0 < r < R$, we have

$$H_R'(r) = (\psi(r))^{n-1} \left(\int_r^R (\psi(s))^{1-n} \, ds \right) - (\psi(r))^{1-n} \left(\int_0^r (\psi(s))^{n-1} \, ds \right) .$$

Since $\lim_{r\to 0^+} H_R(r) = \lim_{r\to R^-} H_R(r) = 0$ and $H_R(r) > 0$ for any $r \in (0, R)$, then H_R admits a local maximum point $r_0 \in (0, R)$. This yields,

$$H_R(r) \leq H_R(r_0) = \left(\frac{\int_0^{r_0} (\psi(s))^{n-1} \, ds}{(\psi(r_0))^{n-1}} \right)^2 \quad \text{for every } r \in (0, R) .$$

Then, condition (H_3) assures the boundedness of the latter quotient and, in turn, proves the claim. To see this, note that $(H_2) - (H_3)$ yield $\lim_{r\to+\infty} \psi(r) = +\infty$. In particular, by the Cauchy Theorem

$$\limsup_{r\to+\infty} \frac{\int_0^r (\psi(s))^{n-1} \, ds}{(\psi(r))^{n-1}} \leq \limsup_{r\to+\infty} \frac{\psi(r)}{(n-1)\psi'(r)} < +\infty .$$

To prove the second part of the statement, we notice that, denoting by $V(r)$ the volume of the geodesic balls of radius r centered at o, we have by l'Hôpital rule, since the last limit below exists:

$$\lim_{r\to+\infty}\frac{V'(r)}{V(r)}=\lim_{r\to+\infty}\frac{(\psi(r))^{n-1}}{\int_0^r(\psi(s))^{n-1}\,ds}=(n-1)\lim_{r\to+\infty}\frac{\psi'(r)}{\psi(r)}=0.$$

Hence for all $\varepsilon>0$ there is r_ε such that $0\le V'(r)/V(r)=(\log V(r))'\le\varepsilon$ for all $r\ge r_\varepsilon$. Integrating between r_ε and r we easily get that

$$\lim_{r\to+\infty}\frac{\log(\psi(r))}{r}=0.$$

By a classical result of Brooks (see [7]) this implies that $\lambda_1(M)=0$. We mention by the sake of completeness that the same conclusion can be obtained by verifying that the necessary and sufficient condition (4.3) for the validity of the Poincaré-Sobolev type inequality below, is not satisfied when $p=1$ under the running assumptions.

The fact that the claim holds when the radial mean curvature at r tends to zero when $r\to+\infty$ is obvious from its expression given in Section 1.1. The radial sectional curvature at r tends to zero when $r\to+\infty$ if and only if $\psi''(r)/\psi(r)\to 0$. This implies that $\psi'(r)/\psi(r)\to 0$ as well. In fact, if $\psi'(r)/\psi(r)$ has a limit, l'Hôpital rule implies that it must be zero, as claimed. Should $\psi'(r)/\psi(r)$ not have a limit, it must have a sequence of stationary points $r_k\to+\infty$ as $k\to+\infty$, so that $\psi''(r_k)/\psi'(r_k)=\psi'(r_k)/\psi(r_k)$, so that in particular $\psi''(r_k)\neq 0$. Hence

$$\frac{\psi'(r_k)}{\psi(r_k)}=\frac{\psi''(r_k)}{\psi(r_k)}\,\frac{\psi'(r_k)}{\psi''(r_k)}=\frac{\psi''(r_k)}{\psi(r_k)}\,\frac{\psi(r_k)}{\psi'(r_k)},\quad\text{or}\quad\left(\frac{\psi'(r_k)}{\psi(r_k)}\right)^2=\frac{\psi''(r_k)}{\psi(r_k)}$$

which tends to zero by assumption if $k\to+\infty$, contradiction. □

Next we show the validity of a Sobolev embedding for the space $H^1_r(M)$ of radial functions in $H^1(M)$.

Lemma 4.2. *Let $n\ge 3$ and assume that ψ satisfies $(H_1)-(H_3)$. If $1<p\le\frac{n+2}{n-2}$ then the embedding $H^1_r(M)\subset L^{p+1}(M)$ is continuous and if $1<p<\frac{n+2}{n-2}$ then the embedding is also compact.*

Proof. Following [23], we define $AC_R(0,+\infty)$ the set of all functions absolutely continuous on every compact subinterval $[a,b]\subset(0,+\infty)$ which tend to zero as $r\to+\infty$. Then, according to [23, Theorem 6.2], the inequality

$$\left(\int_0^{+\infty}|u(r)|^{p+1}\psi^{n-1}(r)\,dr\right)^{\frac{2}{p+1}}\le C_{n,p}\int_0^{+\infty}(u'(r))^2\psi^{n-1}(r)\,dr\quad\text{for all } u\in AC_R(0,+\infty)\,,\tag{4.3}$$

holds for some $C_{n,p}>0$, if and only if

$$\sup_{0<x<+\infty}f_{n,p}(x):=\sup_{0<x<+\infty}\left(\int_0^x\psi^{n-1}(r)\,dr\right)^{\frac{1}{p+1}}\left(\int_x^{+\infty}\psi^{1-n}(r)\,dr\right)^{\frac12}<+\infty\,.$$

The known asymptotics for ψ as $x\to 0^+$ yield

$$f_{n,p}(x)\sim\frac{1}{n^{1/(p+1)}(n-2)^{1/2}}\,x^{\frac{n+2-p(n-2)}{2(p+1)}}\qquad\text{as } x\to 0^+,\tag{4.4}$$

where the integrability of $\psi^{1-n}(r)$ in $(x,+\infty)$ comes from (H_3).

On the other hand, we claim that

$$\limsup_{x\to+\infty}\left(\int_0^x\psi^{n-1}(r)\,dr\right)\left(\int_x^{+\infty}\psi^{1-n}(r)\,dr\right)<+\infty\tag{4.5}$$

from which we easily conclude that for $p>1$

$$\lim_{x\to+\infty} f_{n,p}(x) = \lim_{x\to+\infty} \left(\int_0^x \psi^{n-1}(r)\,dr\right)^{\frac{1}{p+1}-\frac{1}{2}} \left(\int_0^x \psi^{n-1}(r)\,dr\right)^{\frac{1}{2}} \left(\int_x^{+\infty} \psi^{1-n}(r)\,dr\right)^{\frac{1}{2}} = 0\,.$$

To prove (4.5), we first note that $(H_2)-(H_3)$ and the Cauchy's Theorem yield

$$\limsup_{x\to+\infty} \frac{\int_0^x \psi^{n-1}(r)\,dr}{(\psi(x))^{n-1}} \le \limsup_{x\to+\infty} \frac{\psi(x)}{(n-1)\psi'(x)} < +\infty\,.$$

and

$$\limsup_{x\to+\infty} \frac{\int_x^{+\infty} \psi^{1-n}(r)\,dr}{(\psi(x))^{1-n}} \le \limsup_{x\to+\infty} \frac{\psi(x)}{(n-1)\psi'(x)} < +\infty\,.$$

Then,

$$\limsup_{x\to+\infty} \left(\int_0^x \psi^{n-1}(r)\,dr\right)\left(\int_x^{+\infty} \psi^{1-n}(r)\,dr\right)$$
$$= \limsup_{x\to+\infty} \left(\frac{\int_0^x \psi^{n-1}(r)\,dr}{(\psi(x))^{n-1}}\right)\left(\frac{\int_x^{+\infty} \psi^{1-n}(r)\,dr}{(\psi(x))^{1-n}}\right) < +\infty\,.$$

Let us denote by $C^\infty_{c,r}(M)$ the space of radial functions in $C^\infty_c(M)$. By (4.3),(4.4), (4.5) we deduce that if $1<p\le \frac{n+2}{n-2}$ then

$$\|\varphi\|^2_{L^{p+1}(M)} \le C_{n,p} \int_M |\nabla_g \varphi|_g^2\,dV_g\,,$$

for any function $\varphi \in C^\infty_{c,r}(M)$.

Therefore, by density of $C^\infty_{c,r}(M)$ in $H^1_r(M)$ (see [19, Theorem 3.1]) we obtain the continuous embedding $H^1_r(M)\subset L^{p+1}(M)$ for $1<p\le\frac{n+2}{n-2}$. On the other hand [23, Theorem 7.4] yields that the same embedding is compact if and only if $\lim_{x\to0^+} f_{n,p}(x)=0=\lim_{x\to+\infty} f_{n,p}(x)$. This condition is satisfied when $1<p<\frac{n+2}{n-2}$. ☐

End of the proof of Theorem 2.2. The existence of a nonnegative minimizer to

$$\inf_{v\in H^1_r(M)\setminus\{0\}} \frac{\int_M |\nabla_g v|_g^2\,dV_g}{\left(\int_M |v|^{p+1}\,dV_g\right)^{\frac{2}{p+1}}}\,, \tag{4.6}$$

follows in a standard way by Lemmas 4.1-4.2. Up to a constant multiplier a nonnegative minimizer u of (4.6) is actually a radial solution of (1.1) and hence a nonnegative solution of (1.3). Furthermore $u(r)>0$ for any $r>0$ by local uniqueness for a Cauchy problem.

4.2. **Proof of Theorem 2.4.** The proof follows the line of [22, Theorem 1.3] where the case $\psi(r)=\sinh(r)$ is dealt. Hence, in the sequel we will only quote which are the main differences.

First we have uniqueness for Dirichlet problems on bounded domains.

Lemma 4.3. *Let $1<p<\frac{n+2}{n-2}$ and ψ satisfy assumptions $(H_1)-(H_2)$. Furthermore, let G as defined in Theorem 2.4 satisfying the $\Lambda-$property as required there. Then the Dirichlet problem*

$$\begin{cases} -\dfrac{1}{(\psi(r))^{n-1}}\left[(\psi(r))^{n-1}v'(r)\right]' = |v(r)|^{p-1}v(r) \quad r\in(0,R) \\ \\ v'(0)=0 \quad v(R)=0\,. \end{cases} \tag{4.7}$$

has at most one positive solution.

Proof. The proof follows plainly the lines of [22, Proposition 4.4]. The main difference is that the auxiliary energy considered there, here has to be replaced by

$$E_{\widehat{v}}(r) := \frac{1}{2}(\psi(r))^{\delta(p-1)}(\widehat{v}'(r))^2 + \frac{|\widehat{v}(r)|^{p+1}}{p+1} + \frac{1}{2}G(r)(\widehat{v}(r))^2\,,$$

where δ and G are as in the statement of Theorem 2.4 and $\widehat{v}(r) := \psi^{\delta}(r)v(r)$. See also [21] where this substitution was originally introduced. In particular, if v solves (4.7) then $\widehat{v}$ solves

$$\psi^{\delta(p-1)}(r)\,\widehat{v}''(r) + \frac{1}{2}(\psi^{\delta(p-1)}(r))'\,\widehat{v}'(r) + G(r)\,\widehat{v}(r) + \widehat{v}^p(r) = 0 \quad (r>0)$$

and

$$\frac{d}{dr}\,E_{\widehat{v}}(r) = \frac{1}{2}\,G'(r)(\widehat{v}(r))^2\,.$$

We have $G(r) \sim \delta(\delta+2-n)r^{\delta(p-1)-2}$ as $r \to 0^+$, where $\delta+2-n<0$ and $\delta(p-1)-2<0$. Namely, $G(r) \to -\infty$ for $r \to 0^+$. This, combined with the assumptions required on G yields that either $G'(r) \geq 0$ for every $r>0$ or there exists $r_1>0$ such that $G'(r_1)=0$, $G'(r)\geq 0$ for every $r\in(0,r_1)$ and $G'(r)\leq 0$ for every $r>r_1$. Then, all the arguments of [22, Proposition 4.4] work. See also the proof of Lemma 4.8 below. □

Let $u \in H^1(M)$ be a positive radial solution of (1.1) as given in Theorem 2.2 (possibly not unique). The next two lemmas show that every solution v to (1.3) with $0<v(0)<u(0)$, is necessarily of one sign. Furthermore, v intersects u exactly once. First, by exploiting Lemma 4.3, we have

Lemma 4.4. *Let $1<p<\frac{n+2}{n-2}$ and ψ satisfy assumptions $(H_1)-(H_3)$. Furthermore, let G as defined in Theorem 2.4 satisfying the $\Lambda-$property as required there. If u and v are two solutions to* (1.3) *with $u(r)>0$ for every $r\geq 0$ and $0<v(0)<u(0)$, then $v(r)>0$ for every $r\geq 0$.*

The proof of Lemma 4.4 is the same of [22, Lemma 4.1 and Corollary 4.6]. The main tools exploited there are uniqueness for Dirichlet problems on bounded domains and the Poincaré-Sobolev inequality in the hyperbolic space. In our case, they are given, respectively, by Lemma 4.3 and by Lemmas 4.1-4.2.

On the other hand, exactly as in [22, Corollary 4.6], one shows

Lemma 4.5. *Let $1<p<\frac{n+2}{n-2}$ and ψ satisfy assumptions $(H_1)-(H_3)$. Let u and v be two positive solutions to* (1.3) *with $0<v(0)<u(0)$. If $u\in H^1(M)$, then $u-v$ has exactly one zero.*

Next we discuss the asymptotic behavior and uniqueness of radial ground states.

Lemma 4.6. *Assume that ψ satisfies assumptions $(H_1)-(H_2)$ and* (2.2) *holds. Furthermore, let $u\in H^1(M)$ be a positive solution to* (1.3) *with $p>1$. If $l<+\infty$ in* (2.2), *then*

$$\lim_{r\to+\infty}\frac{\log(u(r))}{r} = -(n-1)l = \lim_{r\to+\infty}\frac{\log|u'(r)|}{r} \tag{4.8}$$

and

$$\lim_{r\to+\infty}\frac{u'(r)}{u(r)} = -(n-1)l\,. \tag{4.9}$$

If $l=+\infty$ in (2.2), *we have*

$$\lim_{r\to+\infty}\frac{\log(u(r))}{r} = \lim_{r\to+\infty}\frac{\log|u'(r)|}{r} = -\infty\,, \qquad \lim_{r\to+\infty}\frac{u'(r)}{u(r)} = -\infty\,, \tag{4.10}$$

and

$$\lim_{r\to+\infty}\frac{\log|u'(r)|}{\log(\psi(r))}=-(n-1)\,. \tag{4.11}$$

Proof. We omit the proof in the case $l<+\infty$ since it can be deduced by arguing as in [22, Lemma 3.4]. Suppose now that $l=+\infty$. For every $k>0$ there exists $r_k>0$ such that

$$u''(r)+(n-1)ku'(r)+\frac{1}{k}u(r)\ge 0\quad\text{for all } r\ge r_k\,.$$

Namely,

$$\left(e^{-\lambda_-(k)r}z(r)\right)'\ge 0\quad\text{for all } r\ge r_k\,,$$

where $z:=u'-\lambda_+(k)u$ and $\lambda_\pm(k):=\frac{-(n-1)k\pm\sqrt{(n-1)^2k^2-4/k}}{2}$. Then, two integrations in $[\tau,r]$, with $r_k\le\tau\le r$, yield

$$u(r)\ge B_k(\tau)\,e^{\lambda_+(k)r}-\frac{A_k(\tau)}{\lambda_+(k)-\lambda_-(k)}\,e^{\lambda_-(k)r}\quad\text{for all } r\ge r_k\,,$$

where $A_k(\tau):=e^{-\lambda_-(k)\tau}z(\tau)$ and $B_k(\tau):=u(\tau)e^{-\lambda_+(k)\tau}+\frac{A_k(\tau)}{\lambda_+(k)-\lambda_-(k)}\,e^{-(\lambda_+(k)-\lambda_-(k))\tau}$. We claim that $B_k(\tau)\le 0$ for $\tau\ge r_k$. Otherwise, $B_k(\tau)>0$ eventually. We recall that

$$B_k'(\tau)=\frac{A_k'(\tau)}{\lambda_+(k)-\lambda_-(k)}\,e^{-(\lambda_+(k)-\lambda_-(k))\tau}\ge 0\qquad\text{for any } \tau\ge r_k\,.$$

Here and in the sequel C_k denotes a positive constant sufficiently large which may vary from line to line. Then, $u(r)\ge B_k(\tau)\,e^{\lambda_+(k)r}+o(e^{\lambda_+(k)r})$ as $r\to+\infty$. But this, combined with (2.2), yields $\int_0^{+\infty}\psi^{n-1}(r)u^2(r)\,dr\ge C_k\int_{r_k}^{+\infty}e^{\sqrt{(n-1)^2k^2-4/k}\,r}\,dr$ for some $C_k>0$ and contradicts the fact that $u\in H^1(M)$. Hence, $B_k(\tau)\le 0$ for $\tau\ge r_k$ and we conclude that

$$u'(\tau)\le\lambda_-(k)u(\tau)\quad\text{for all } \tau\ge r_k\,. \tag{4.12}$$

Then,

$$\limsup_{r\to+\infty}\frac{u'(r)}{u(r)}\le\lambda_-(k),\qquad\limsup_{r\to+\infty}\frac{\log(u(r))}{r}\le\lambda_-(k)\quad\text{for every } k>0$$

and the first and the third limit in (4.10) follow since $\lim_{k\to+\infty}\lambda_-(k)=-\infty$. On the other hand, by (1.3), the third limit in (4.10), the fact that $\lim_{r\to+\infty}u(r)=0$ and that $l=+\infty$, we have

$$\lim_{r\to+\infty}\frac{u''(r)}{u'(r)}\frac{\psi(r)}{\psi'(r)}=\lim_{r\to+\infty}\left(-(n-1)-\frac{u^p(r)}{u'(r)}\frac{\psi(r)}{\psi'(r)}\right)=1-n\,.$$

By this, the second limit in (4.10) and (4.11) easily follow from the l'Hôpital rule. □

Lemma 4.7. *Let $n\ge 3$ and $1<p<\frac{n+2}{n-2}$. Assume that ψ satisfies $(H_1)-(H_3)$. Then for any radial positive solution $u\in H^1(M)$ of (1.1), there exists $L\in(-\infty,0)$ such that*

$$\lim_{r\to+\infty}\psi^{n-1}(r)u'(r)=L\,. \tag{4.13}$$

Moreover

$$\lim_{r\to+\infty}\psi^{n-1}(r)u(r)=\frac{|L|}{(n-1)l}\qquad\text{if } l<+\infty\,, \tag{4.14}$$

and

$$\lim_{r\to+\infty}\frac{u(r)}{\int_r^{+\infty}\psi^{1-n}(s)\,ds}=|L|\qquad\text{if } l=+\infty\,. \tag{4.15}$$

Proof. The existence and the negativity of the limit in (4.13) simply follows by (1.3). It remains to prove that $L > -\infty$. If $l < +\infty$ from (4.8) the bound

$$u(r) \le C_\delta e^{-((n-1)l-\delta)r} \quad \text{for all } r \ge 0\,,$$

holds for every $\delta > 0$. By this, (2.2) and (1.3), we deduce that

$$(\psi^{n-1}(r)u'(r))' \ge -C(\varepsilon,\delta)e^{-[((n-1)l-\delta)p-(l+\varepsilon)(n-1)]r} \quad \text{for all } r \ge 0\,,$$

for every $\varepsilon > 0$ and $\delta > 0$, where $C(\varepsilon,\delta) > 0$. Next we fix $\varepsilon = \frac{l(p-1)}{2}$ and we assume $\delta = \delta(\varepsilon)$ to be such that $\delta p < \frac{l(p-1)(n-1)}{2}$. Then, an integration in $[0,r]$ yields

$$\psi^{n-1}(r)u'(r) \ge C[e^{-[\frac{l(p-1)(n-1)}{2}-\delta p]r} - 1]\,.$$

Namely,

$$\psi^{n-1}(r)u'(r) \ge -C \quad \text{for all } r \ge 0$$

and $L > -\infty$.

Next we assume $l = +\infty$. From (4.10) we know that for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\log|u'(r)| \le -((n-1)-\varepsilon)\log\psi(r) \quad \text{for all } r \ge R_\varepsilon\,.$$

Furthermore, from (4.12), for every $k > 0$ there exists $r_k > 0$ such that

$$\log u(r) \le \log|u'(r)| - \log|\lambda_-(k)| \quad \text{for all } r \ge r_k\,,$$

where $\lim_{k\to+\infty} \lambda_-(k) = -\infty$. Fix $\varepsilon = \frac{(p-1)(n-1)}{2p}$ in order to obtain after integration

$$u(r) \le C\psi^{-(n-1)(p+1)/2p}(r) \quad \text{for all } r \ge 0\,,$$

for some $C > 0$. By this and integrating the equation in $[0,r]$, we conclude that

$$\psi^{n-1}(r)u'(r) \ge -C^p \int_0^r \psi^{-(n-1)(p-1)/2}(s)\,ds \ge -K\,,$$

for some finite K and for all $r \ge 0$. Hence, again we infer that $L > -\infty$. □

Lemma 4.8. *Let $1 < p < \frac{n+2}{n-2}$. Assume that ψ satisfies the assumptions of Theorem 2.4. Then* (1.1) *admits a unique radial positive solution $U \in H^1(M)$.*

Proof. We follow the proof of [22, Theorem 1.3]. By contradiction, assume that u and v are two positive solutions to (1.3) such that $u, v \in H^1(M)$ and $v(0) < u(0)$. By Lemma 4.5, u and v intersect exactly once at r_0.

We claim that $\gamma(r) := v(r)/u(r)$ is strictly increasing in $(0,+\infty)$. From the equation we know that

$$[(\psi(r))^{n-1}(v'(r)u(r) - v(r)u'(r))]' = (\psi(r))^{n-1}u(r)v(r)((u(r))^{p-1} - (v(r))^{p-1})\,.$$

Hence,

$$[(\psi(r))^{n-1}(v'(r)u(r) - v(r)u'(r))]'(r_0 - r) > 0 \quad \forall r \ne r_0\,.$$

By (4.13) and the fact that $\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0$, we deduce that

$$\lim_{r\to+\infty} (\psi(r))^{n-1}(v'(r)u(r) - v(r)u'(r)) = 0\,. \tag{4.16}$$

Hence, $v'(r)u(r) - v(r)u'(r) > 0$ for $r > 0$ and $\gamma'(r) > 0$.

Now, we set $\widehat{u}(r) := (\psi(r))^\delta u(r)$ and $\widehat{v}(r) := (\psi(r))^\delta v(r)$, where δ is as in the statement of Theorem 2.4. Then, for $E_{\widehat{v}}$ as in the proof of Lemma 4.3, for any $0 < \varepsilon < R$ and $r \in (0,R)$, we get

$$E_{\widehat{v}}(R) - \gamma^2(r)E_{\widehat{u}}(R) = E_{\widehat{v}}(\varepsilon) - \gamma^2(r)E_{\widehat{u}}(\varepsilon) + \frac{1}{2}\int_\varepsilon^R G'(s)[(\widehat{v}(s))^2 - \gamma^2(r)(\widehat{u}(s))^2]\,ds\,. \tag{4.17}$$

Since $G(r) \to -\infty$ as $r \to 0^+$ (see the proof of Lemma 4.3), by assumption we have that either $G' \geq 0$ in $(0, +\infty)$ or there exists $r_1 > 0$ such that $G'(r_1) = 0$, $G' \geq 0$ in $(0, r_1)$ and $G' \leq 0$ in $(r_1, +\infty)$. We claim that

$$E_{\widehat{v}}(R) \to 0 \quad \text{and} \quad E_{\widehat{u}}(R) \to 0 \quad \text{as } R \to +\infty\,. \tag{4.18}$$

We now show that with the help of (4.18) we arrive to the conclusion of the proof.

If G' does not change sign, take $r = \varepsilon$ in (4.17). Letting $\varepsilon \to 0^+$ we get

$$E_{\widehat{v}}(R) - \gamma^2(0) E_{\widehat{u}}(R) = \frac{1}{2} \int_0^R G'(s)[(\widehat{v}(s))^2 - \gamma^2(0)(\widehat{u}(s))^2]\, ds > 0\,.$$

Letting $R \to +\infty$, (4.18) leads to a contradiction.

If G' changes sign, take $r = r_1$ in (4.17). Letting $\varepsilon \to 0^+$, we get

$$\begin{gathered} E_{\widehat{v}}(R) - \gamma^2(r_1) E_{\widehat{u}}(R) \\ = \frac{1}{2} \int_0^{r_1} G'(s)[(\widehat{v}(s))^2 - \gamma^2(r_1)(\widehat{u}(s))^2]\, ds + \frac{1}{2} \int_{r_1}^R G'(s)[(\widehat{v}(s))^2 - \gamma^2(r_1)(\widehat{u}(s))^2]\, ds < 0\,. \end{gathered}$$

Letting $R \to +\infty$, (4.18) leads again to a contradiction.

It remaind to prove (4.18). First we note that, from (4.14) and (4.15), if $l < +\infty$ we have

$$\psi^\delta(r) v(r) \sim \frac{|L|}{(n-1)l}\, \psi^{-\frac{(p+1)\delta}{2}}(r) \quad \text{as } r \to +\infty$$

and if $l = +\infty$ we have

$$\psi^\delta(r) v(r) \sim |L| \psi^\delta(r) \int_r^{+\infty} \psi^{1-n}(s)\, ds \quad \text{as } r \to +\infty\,.$$

Hence, in both the cases we conclude that $\widehat{v}(r) \to 0$ as $r \to +\infty$. Then we consider

$$G(r)(\widehat{v}(r))^2 = \delta(\delta + 2 - n)\psi^{\delta(p+1)}(r) \left(\frac{\psi'(r)}{\psi(r)}\right)^2 v^2(r) - \delta \psi^{\delta(p+1)}(r) \frac{\psi'(r)}{\psi(r)} \frac{\psi''(r)}{\psi'(r)} v^2(r)\,.$$

If $l < +\infty$ (2.2) and (4.14) give

$$G(r)(\widehat{v}(r))^2 \sim \frac{\delta(\delta + 1 - n)|L|^2}{(n-1)^2}\, \psi^{-2\delta}(r) \quad \text{as } r \to +\infty$$

and $|G(r)|(\widehat{v}(r))^2 \to 0$ as $r \to +\infty$. If $l = +\infty$, (4.12) and (4.13) give

$$\psi^{\delta(p+1)}(r) \left(\frac{\psi'(r)}{\psi(r)}\right)^2 v^2(r) \leq \left[\psi^{\frac{\delta(p+1)}{2}}(r) \frac{\psi'(r)}{\psi(r)} \frac{|v'(r)|}{|\lambda_-(k)|}\right]^2 \sim \left[\frac{|L|\, \psi^{-\delta}(r)}{|\lambda_-(k)|} \frac{\psi'(r)}{\psi(r)}\right]^2,$$

as $r \to +\infty$. Hence, by (2.3), $\psi^{\delta(p+1)}(r) \left(\frac{\psi'(r)}{\psi(r)}\right)^2 v^2(r) \to 0$ as $r \to +\infty$. Similarly, $\psi^{\delta(p+1)}(r) \frac{\psi'(r)}{\psi(r)} \frac{\psi''(r)}{\psi'(r)}\, v^2(r) \to 0$ as $r \to +\infty$ and, in turn, $|G(r)|(\widehat{v}(r))^2 \to 0$ as $r \to +\infty$.

Finally, we compute

$$\begin{gathered} \psi^{\delta(p-1)}(r)(\widehat{v}'(r))^2 \\ = \delta^2 \psi^{\delta(p+1)}(r) \left(\frac{\psi'(r)}{\psi(r)}\right)^2 v^2(r) + 2\delta \psi^{\delta(p+1)}(r) \frac{\psi'(r)}{\psi(r)} v(r) v'(r) + \psi^{\delta(p+1)}(r)(v'(r))^2\,. \end{gathered}$$

If $l < +\infty$ (2.2), (4.13) and (4.14) give

$$\psi^{\delta(p-1)}(r)(\widehat{v}'(r))^2 \sim L^2 \left(\frac{\delta^2}{(n-1)^2} - \frac{2\delta}{n-1} + 1\right) \psi^{-2\delta}(r) \quad \text{as } r \to +\infty\,.$$

Namely, $\psi^{\delta(p-1)}(r)(\widehat{v}'(r))^2 \to 0$ as $r \to +\infty$. When $l = +\infty$, the same conclusion can be reached by exploiting (2.3), (4.12) and (4.13) as shown above. The limits so far proved yield (4.18). $\square$

When α is large, the same proof of [5, Lemma 7.1] gives

Lemma 4.9. *Let ψ satisfy assumptions $(H_1)-(H_2)$. Furthermore, let u be a solution to* (1.3) *with $1<p<\frac{n+2}{n-2}$ and $\alpha>\alpha_0$ sufficiently large. Then u changes sign.*

Finally, following the proofs of [5, Lemmas 7.2, 7.3, 7.4, 7.5], we conclude.

Lemma 4.10. *Let $1<p<\frac{n+2}{n-2}$, ψ satisfy the assumptions of Theorem 2.4 and U be the unique ground state as given in Lemma 4.8. Then, any solution to* (1.3) *with $\alpha>U(0)$ is sign-changing.*

The proof of Theorem 2.4 now follows from Lemmas 4.8-4.10.

4.3. **Proof of Proposition 2.5.** Note that $G'(r)=\delta\psi^{\delta(p-1)-3}(r)h(r)$, where

$$h(r):=(\delta(p-1)-2)(\delta+2-n)(\psi'(r))^3-\psi'''(r)\psi^2(r)+(\delta(3-p)+5-2n)\psi'(r)\psi''(r)\psi(r)\,.$$

Clearly, $h(0)=(\delta(p-1)-2)(\delta+2-n)>0$ for every $1<p<\frac{n+2}{n-2}$. We prove that $h'(\bar r)<0$ for every $\bar r>0$ such that $h(\bar r)=0$, then h admits at most one zero and the $\Lambda-$property follows.

For such $\bar r$, a few computations yield

$$h'(\bar r)=A_{p,n}(\psi'(\bar r))^2\psi''(\bar r)+B_{p,n}\psi(\bar r)\psi'(\bar r)\psi'''(\bar r)+\psi^2(\bar r)\left(\frac{\psi''(\bar r)\psi'''(\bar r)}{\psi'(\bar r)}-\psi^{iv}(\bar r)\right),$$

$$\begin{aligned}A_{p,n}&=2\delta^2(p-1)+\delta((3-2n)p+2n-5)+2n-3\\&=\frac{-(2n-3)^2p^2+6(2n-3)p+4n^2-8n-5}{(p+3)^2}<0\quad\text{for every}\quad p\ge\frac{2n+1}{2n-3}\end{aligned}$$

and $B_{p,n}:=\delta(3-p)+3-2n<0$ for every $p>1$. Note that $\frac{2n+1}{2n-3}\in(1,\frac{n+2}{n-2})$.

Summing up, if ψ satisfies assumptions $(H_1)-(H_3)$, $\psi''(0)=0$, $\psi'''(r)>0$ and $\left(\frac{\psi'(r)}{\psi'''(r)}\right)'\le 0$ for every $r>0$, then G satisfies the $\Lambda-$property for every $\frac{2n+1}{2n-3}\le p<\frac{n+2}{n-2}$.

4.4. **Proof of Theorem 2.6.** The statement of (i) is contained in Lemma 4.7.

Lemma 4.11. *Let the assumptions of Theorem 2.6 hold and let $u\notin H^1(M)$ be a positive solution to* (1.3). *There exist no strictly positive constants C,β such that $u(r)\le C(\psi(r))^{-\beta}$ for all $r\ge 0$.*

Proof. Suppose by contradiction that there exist $C,\beta>0$ such that $u(r)\le C(\psi(r))^{-\beta}$ for all $r>0$. Proceeding exactly as in the proof of Lemma 3.1 we arrive to the estimate (3.2). Integrating (1.3) and exploiting (3.2), we infer $u'(r)\ge -C(\psi(r))^{1-n}$ and any $r>0$, for some constant $C>0$. This shows that $u'\in L^2(M)$. Another integration then yields $u(r)\le C\int_r^{+\infty}(\psi(s))^{1-n}ds$ for any $r>0$ and, in turn, by (H_3) we obtain $u(r)=O((\psi(r))^{1-n})$ as $r\to+\infty$. This implies $u\in L^2(M)$. We have shown that $u\in H^1(M)$, a contradiction. The proof of the lemma is complete. □

Lemma 4.12. *Let the assumptions of Theorem 2.6 hold and let $u\notin H^1(M)$ be a positive solution to* (1.3). *Let $P=P(r)$ be defined as in the proof of Theorem 2.7. Then $P(r)$ admits a limit as $r\to+\infty$.*

Proof. From the proof of Theorem 2.7 we recall that $P'(r):=K(r)(u'(r))^2$. Hence, by (3.1)

$$\lim_{r\to+\infty}K(r)=\lim_{r\to+\infty}(\psi(r))^{n-1}\left[\frac{p+3}{2}-(n-1)(p+1)\psi'(r)\,\frac{\int_0^r(\psi(s))^{n-1}\,ds}{(\psi(r))^n}\right]=-\infty$$

and the statement follows. □

End of the proof of Theorem 2.6. Thanks to Lemma 4.12 we may put $\gamma:=\lim_{r\to+\infty}P(r)$. If $\gamma<0$ then P is obviously eventually negative. In such a case we may proceed exactly as in the proof of Theorem 2.9 and arrive to the estimates (ii) and (iii) of Theorem 2.6.

Suppose now that $\gamma \geq 0$. Since P is eventually nonincreasing then P is eventually nonnegative. Therefore there exists $\overline{r} > 0$ such that

$$(u'(r))^2 + \frac{2}{p+1} \frac{(\psi(r))^{n-1}}{\int_0^r (\psi(s))^{n-1}\, ds} u(r)u'(r) + \frac{2}{p+1}(u(r))^{p+1} \geq 0 \qquad \text{for any } r > \overline{r}\,. \tag{4.19}$$

Suppose now that $\lim_{r\to+\infty} u(r) = 0$. Since $l > 0$ up to enlarging $\overline{r}$, we have that

$$\frac{1}{(p+1)^2}\left(\frac{(\psi(r))^{n-1}}{\int_0^r (\psi(s))^{n-1}\, ds}\right)^2 (u(r))^2 - \frac{2}{p+1}(u(r))^{p+1} > 0 \qquad \text{for any } r > \overline{r}\,.$$

Solving the second order equation in (4.19) with respect to $u'(r)$, we arrive to the following alternatives: either

$$u'(r) \leq -\frac{1}{p+1} \frac{(\psi(r))^{n-1}}{\int_0^r (\psi(s))^{n-1}\, ds} u(r) - \left[\frac{1}{(p+1)^2}\left(\frac{(\psi(r))^{n-1}}{\int_0^r (\psi(s))^{n-1}\, ds}\right)^2 (u(r))^2 - \frac{2}{p+1}(u(r))^{p+1}\right]^{\frac12}$$

or

$$u'(r) \geq -\frac{1}{p+1} \frac{(\psi(r))^{n-1}}{\int_0^r (\psi(s))^{n-1}\, ds} u(r) + \left[\frac{1}{(p+1)^2}\left(\frac{(\psi(r))^{n-1}}{\int_0^r (\psi(s))^{n-1}\, ds}\right)^2 (u(r))^2 - \frac{2}{p+1}(u(r))^{p+1}\right]^{\frac12}.$$

The first alternative may be excluded since otherwise by (3.1) we would have

$$\frac{u'(r)}{u(r)} \leq -\frac{n-1-\varepsilon}{p+1}\frac{\psi'(r)}{\psi(r)} \qquad \text{for any } r > r_\varepsilon$$

for some $\varepsilon \in (0, n-1)$ and $r_\varepsilon > 0$. Integration of this inequality provides a contradiction with Lemma 4.11.

Therefore the second alternative holds true. Proceeding similarly to the proof of [5, Theorem 2.3] we then obtain

$$u'(r) \geq -2(u(r))^p \frac{\int_0^r \psi^{n-1}(s)\, ds}{\psi^{n-1}(r)}\,.$$

Exploiting (3.1) and (2.4), this implies

$$\lim_{r\to+\infty} \frac{u'(r)}{u(r)} = 0\,.$$

In particular using again (2.4), this gives the validity of Lemma 3.2. Now one can follow exactly all the steps of Theorem 2.9 and arrive to the proof of part (iii) if $\frac{\psi}{\psi'} \notin L^1(0,\infty)$.

Otherwise we arrive to a contradiction with the fact that u vanishes at infinity. This gives the proof of part (ii). We recall that the existence of $\lim_{r\to+\infty} u(r)$ and the fact that it is finite follows from Proposition 2.1.

5. Stability

5.1. **Proof of Theorems 2.11-2.12.** We start with a simpler characterization of stability for radial solutions of (1.1).

Lemma 5.1. *Let ψ satisfy $(H_1)-(H_3)$ and let u be a radial solution of (1.1). Then u is stable if and only if*

$$\int_0^{+\infty} (\chi'(r))^2\, \psi^{n-1}(r)\, dr - p\int_0^{+\infty} |u(r)|^{p-1}\chi^2(r)\, \psi^{n-1}(r)\, dr \geq 0\,, \tag{5.1}$$

for every radial function $\chi \in C_c^\infty(M)$.

Proof. Clearly stability of any solution u of (1.1) is equivalent to

$$\int_{\mathbb{S}^{n-1}}\int_0^{+\infty}[(\varphi_r(r,\Theta))^2+|\nabla_{\mathbb{S}^{n-1}}\varphi(r,\Theta)|^2\,\psi^{-2}(r)]\,\psi^{n-1}(r)\,dr\,d\Theta \tag{5.2}$$
$$-p\int_{\mathbb{S}^{n-1}}\int_0^{+\infty}|u(r,\Theta)|^{p-1}\varphi^2(r,\Theta)\,\psi^{n-1}(r)\,dr\,d\Theta\geq 0\quad\forall\,\varphi\in C_c^\infty(M).$$

In particular if u is radial then (5.1) follows immediately. On the other hand if assume (5.1) we obtain

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}\int_0^{+\infty}&[(\varphi_r(r,\Theta))^2+|\nabla_{\mathbb{S}^{n-1}}\varphi(r,\Theta)|^2\,\psi^{-2}(r)]\,\psi^{n-1}(r)\,dr\,\mathrm{d}\Theta\\ &\geq\int_{\mathbb{S}^{n-1}}\int_0^{+\infty}(\chi_\Theta'(r))^2\,\psi^{n-1}(r)\,dr\,d\Theta\geq\int_{\mathbb{S}^{n-1}}p\int_0^{+\infty}|u(r)|^{p-1}\chi_\Theta^2(r)\,\psi^{n-1}(r)\,dr\,d\Theta\\ &=p\int_{\mathbb{S}^{n-1}}\int_0^{+\infty}|u(r)|^{p-1}\varphi^2(r,\Theta)\,\psi^{n-1}(r)\,dr\,\mathrm{d}\Theta\,,\end{aligned}\tag{5.3}$$

where we have settled $\chi_\Theta(r):=\varphi(r,\Theta)$. □

From the next two lemmas it follows that any solution (1.3) with $\alpha>0$ small enough is stable.

Lemma 5.2. *Let ψ satisfy assumptions $(H_1)-(H_3)$ and let u_α be a solution of* (1.3) *with $\alpha>0$. Then $|u_\alpha(r)|\leq\alpha$ for any $r\in[0,+\infty)$.*

Proof. Let $F_\alpha(r)=\frac{1}{2}|u_\alpha'(r)|^2+\frac{1}{p+1}|u_\alpha(r)|^{p+1}$ be the Lyapunov function corresponding to the solution u_α. From (1.3) one gets that F_α is nonincreasing in $[0,+\infty)$ and hence for any $r>0$

$$\frac{1}{p+1}\alpha^{p+1}=F_\alpha(0)\geq F_\alpha(r)\geq\frac{1}{p+1}|u_\alpha(r)|^{p+1}\,.$$

This completes the proof. □

Lemma 5.3. *Let ψ satisfy assumptions $(H_1)-(H_3)$. Furthermore, let u_α be a solution to* (1.3) *with $|\alpha|\leq\left(\frac{\lambda_1(M)}{p}\right)^{1/(p-1)}$. Then, u_α is stable.*

Proof. For simplicity, let $\alpha>0$. By Lemma 5.2 $|u_\alpha(r)|\leq\alpha$ for every $r\geq0$. The statement follows by combining (4.1) with (2.6). □

Next, under suitable assumptions, we show that stable solutions cannot be sign-changing.

Lemma 5.4. *Let ψ satisfy assumptions $(H_1)-(H_2)$. Then, any stable solution to* (1.3) *has constant sign.*

Proof. By contradiction, let u be a stable solution to (1.3) such that $u(R)=0$ for some $R>0$. Next, we set $v_R(r):=u(r)\chi_{[0,R]}(r)\in H_0^1(B_R)$, where $\chi_{[0,R]}(r)$ denotes the characteristic function of the set $[0,R]$ and B_R is the geodesic ball of center o and radius R. Standard density arguments yield that v_R is a valid test function in (5.2), namely

$$\int_0^{+\infty}(v_R'(r))^2\,\psi^{n-1}(r)\,dr\;-p\int_0^{+\infty}|u(r)|^{p-1}(v_R(r))^2\,\psi^{n-1}(r)\,dr\;\geq0\,. \tag{5.4}$$

On the other hand, multiplying the equation in (1.3) by $v_R(r)\psi^{n-1}(r)$ and integrating, we get

$$\int_0^{+\infty}(v_R'(r))^2\,\psi^{n-1}(r)\,dr\;=\int_0^{+\infty}|u(r)|^{p-1}u\,v_R(r)\,\psi^{n-1}(r)\,dr\,.$$

Recalling the definition of v_R, this yields

$$\int_0^{+\infty} (v_R'(r))^2\, \psi^{n-1}(r)\, dr \; - p\int_0^{+\infty} |u(r)|^{p-1} (v_R(r))^2\, \psi^{n-1}(r)\, dr$$
$$= (1-p)\int_0^{R} |u(r)|^{p+1}\, \psi^{n-1}(r)\, dr < 0\,.$$

The above inequality contradicts (5.4) and concludes the proof. □

Next we exploit well-know results for the euclidean case to deduce the following lemma.

Lemma 5.5. *Let ψ satisfy assumptions $(H_1)-(H_2)$. Let $n \le 10$ and $p>1$ or $n \ge 11$ and $1<p<p_c(n)=\frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}$. Then there exists $\overline{\alpha}>0$ such that for any $\alpha > \overline{\alpha}$, the solution u_α of* (1.3) *is unstable.*

Proof. We argue by contradiction. Let u_λ be a stable solution to (1.3) with $\alpha = \lambda^{2/(p-1)}$. As in [5, Lemma 7.1], we define

$$v_\lambda(s) = \lambda^{-2/(p-1)} u_\lambda\left(\frac{s}{\lambda}\right).$$

Hence, $v_\lambda(0)=1$ and v_λ satisfies

$$v_\lambda''(s) + \frac{n-1}{s}\,\frac{\psi'(s/\lambda)}{\psi(s/\lambda)}\,\frac{s}{\lambda}\, v_\lambda'(s) + |v_\lambda(s)|^{p-1} v_\lambda(s) = 0\,.$$

By (H_1) and Ascoli-Arzelà Theorem we have that $v_\lambda \to \overline{v}$ in $C^1([0,S])$ as $\lambda \to +\infty$, for any $0<S<+\infty$, where $\overline{v}$ solves the equation

$$\overline{v}''(s) + \frac{n-1}{s}\,\overline{v}'(s) + |\overline{v}(s)|^{p-1}\overline{v}(s) = 0\,, \qquad \overline{v}(0)=1\,.$$

On the other hand, by assumption u_λ is stable and from (5.2) we have

$$\int_0^{+\infty} (\chi'(r))^2\, (\psi(r))^{n-1}\, dr - p\,\lambda^2 \int_0^{+\infty} |v_\lambda(\lambda r)|^{p-1}\chi^2(r)\,(\psi(r))^{n-1}\, dr \ge 0\,,$$

for every radial function $\chi \in C_c^\infty(M)$. Next, we set $\eta_\lambda(r) := \eta(r\lambda) \in C_c^\infty(M)$, for some $\eta \in C_c^\infty(M)$ radial. Choosing η_λ as test function in the above inequality and performing the change of variable $s=\lambda r$, we deduce

$$\int_0^{+\infty} (\eta'(s))^2 \left(\psi\left(\frac{s}{\lambda}\right)\right)^{n-1} ds - p\int_0^{+\infty} |v_\lambda(s)|^{p-1}\eta^2(s)\left(\psi\left(\frac{s}{\lambda}\right)\right)^{n-1} ds \ge 0\,,$$

for every radial function $\eta \in C_c^\infty(M)$. Let us fix $S \ge 0$ in such a way that $\operatorname{supp}\eta \subset B_S$. By Lagramge Theorem, for every $s \in [0,S]$ there exist $0<\xi<\frac{s}{\lambda}$ and $0<|\sigma|<\frac{|\psi''(\xi)|}{2}\frac{s}{\lambda}$ such that

$$\left(\psi\left(\frac{s}{\lambda}\right)\right)^{n-1} = \left(\frac{s}{\lambda}\right)^{n-1} + g(\xi,\sigma)\left(\frac{s}{\lambda}\right)^{n} \quad \text{as } \lambda \to +\infty\,, \tag{5.5}$$

where $g(\xi,\sigma) = (n-1)\,(1+\sigma)^{n-2}\,\frac{\psi''(\xi)}{2}$. This yields

$$\int_0^{+\infty} (\eta'(s))^2\, s^{n-1}\, ds + \int_0^{+\infty} (\eta'(s))^2\, \frac{g(\xi,\sigma)}{\lambda}\, s^n\, ds$$
$$-p\int_0^{+\infty} |v_\lambda(s)|^{p-1}\eta^2(s)\, s^{n-1}\, ds - p\int_0^{+\infty} |v_\lambda(s)|^{p-1}\eta^2(s)\,\frac{g(\xi,\sigma)}{\lambda}\, s^n\, ds \ge 0\,.$$

Hence, as $\lambda \to +\infty$, we conclude that

$$\int_0^{+\infty} (\eta'(s))^2\, s^{n-1}\, ds - p\int_0^{+\infty} |\overline{v}(s)|^{p-1}\eta^2(s)\, s^{n-1}\, ds \ge 0\,,$$

for every radial function $\eta \in C_c^\infty(M)$ or, equivalently, for every radial function $\eta \in C_c^\infty(\mathbb{R}^n)$. Namely, $\overline{v}$ is a stable solution to the euclidean equation. Since, by assumption, $n \le 10$ and $p > 1$ or $n \ge 11$ and $1 < p < p_c(n) = \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}$, this contradicts [12, Theorem 1]. $\square$

Let us introduce some notations which will be used in the sequel. For any $r > 0$, let us denote by $v_\alpha(r)$ the derivative with respect to the initial value α, i.e. $v(\alpha, r) := \frac{\partial u}{\partial \alpha}(\alpha, r)$. We will show in Lemma 5.6 that the function v_α is well-defined. For any $\alpha > \beta$ let us define

$$\zeta_{\alpha\beta} := \sup\{r \in (0,\infty) : u_\alpha(s) > u_\beta(s) \text{ for any } s \in (0,r)\} \in (0,+\infty]\,.$$

When $\zeta_{\alpha,\beta} < +\infty$ then $\zeta_{\alpha,\beta}$ is the first zero of $u_\alpha - u_\beta$.

Lemma 5.6. *Let ψ a function satisfying $(H_1)-(H_3)$. Let $a, b, R \in \mathbb{R}$ be such that $b > a > 0$, $R > 0$ and $u_\alpha(r) > 0$ for any $r \in [0,R]$ and $\alpha \in [a,b]$. Then for any $r \in [0,R]$, the map $\alpha \mapsto u(\alpha, r)$ is differentiable in $[a,b]$ and moreover for any $\alpha_0 \in [a,b]$*

$$\lim_{\alpha\to\alpha_0} \sup_{r\in[0,R]} \left|\frac{\partial u}{\partial \alpha}(\alpha,r) - \frac{\partial u}{\partial \alpha}(\alpha_0,r)\right| = 0\,. \tag{5.6}$$

Furthermore for any $\alpha \in [a,b]$ the function $v_\alpha(r) := \frac{\partial u}{\partial \alpha}(\alpha, r)$, $r \in [0,R]$, is a radial solution of the equation

$$-\Delta_g v_\alpha = p|u_\alpha|^{p-1} v_\alpha \qquad \text{in } B_R\,.$$

Proof. For any $r \in [0,R]$ and $\alpha \in [a,b]$ let us define

$$w(r) = \frac{u_\alpha(r) - u_{\alpha_0}(r)}{\alpha - \alpha_0} - v_{\alpha_0}(r) \qquad \text{and} \qquad z(r) = w'(r)$$

where by v_{α_0} we mean the unique solution of the Cauchy problem

$$\begin{cases} v''(r) + (n-1)\dfrac{\psi'(r)}{\psi(r)}\, v'(r) = -p|u_\alpha(r)|^{p-1} v(r) \\ v(0) = 1 \qquad v'(0) = 0 \end{cases} \tag{5.7}$$

corresponding to $\alpha = \alpha_0$. With this notation the following identity holds

$$z'(r) + (n-1)\frac{\psi'(r)}{\psi(r)}\, z(r) = -\left(\frac{|u_\alpha(r)|^{p-1}u_\alpha(r) - |u_{\alpha_0}(r)|^{p-1}u_{\alpha_0}(r)}{\alpha-\alpha_0} - p|u_{\alpha_0}(r)|^{p-1}v_{\alpha_0}(r)\right).$$

By elementary estimates and continuous dependence with respect to α, we deduce that there exist $\overline{\delta} > 0$ and $C > 0$ such that for any $\delta \in (0, \overline{\delta})$, $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta) \cap [a,b]$ and $r \in [0,R]$

$$\begin{aligned} &\left|\frac{|u_\alpha(r)|^{p-1}u_\alpha(r) - |u_{\alpha_0}(r)|^{p-1}u_{\alpha_0}(r)}{\alpha-\alpha_0} - p|u_{\alpha_0}(r)|^{p-1}v_{\alpha_0}(r)\right| \\ &\quad\le \left|\frac{|u_\alpha(r)|^{p-1}u_\alpha(r) - |u_{\alpha_0}(r)|^{p-1}u_{\alpha_0}(r)}{\alpha-\alpha_0} - p|u_{\alpha_0}(r)|^{p-1}\frac{u_\alpha(r)-u_{\alpha_0}(r)}{\alpha-\alpha_0}\right| \\ &\qquad + \left|p|u_{\alpha_0}(r)|^{p-1}\frac{u_\alpha(r)-u_{\alpha_0}(r)}{\alpha-\alpha_0} - p|u_{\alpha_0}(r)|^{p-1}v_{\alpha_0}(r)\right| \\ &\quad\le C\frac{(u_\alpha(r)-u_{\alpha_0}(r))^2}{\alpha-\alpha_0} + p|u_{\alpha_0}(r)|^{p-1}|w(r)|\,. \end{aligned} \tag{5.8}$$

By continuous dependence, for any $\varepsilon > 0$ there exists $\delta \in (0, \overline{\delta})$ such that for any $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta) \cap [a,b]$ and $r \in [0,R]$ we have $\sup_{r\in[0,R]} |u_\alpha(r) - u_{\alpha_0}(r)| < \varepsilon$ and hence by (5.8) and the fact that

$u_{\alpha_0} \le \alpha_0$ and $v_{\alpha_0} \le 1$, we also obtain

$$\left| \frac{|u_\alpha(r)|^{p-1}u_\alpha(r) - |u_{\alpha_0}(r)|^{p-1}u_{\alpha_0}(r)}{\alpha - \alpha_0} - p|u_{\alpha_0}(r)|^{p-1}v_{\alpha_0}(r) \right|$$
$$\le (p\alpha_0^{p-1} + C\varepsilon)|w(r)| + C\varepsilon \qquad \text{for any } r \in [0,R] \text{ and } \alpha \in (\alpha_0 - \delta, \alpha_0 + \delta) \cap [a,b]\,.$$

Since $w(0) = 0$, by the previous inequality we also have

$$\left| \frac{|u_\alpha(r)|^{p-1}u_\alpha(r) - |u_{\alpha_0}(r)|^{p-1}u_{\alpha_0}(r)}{\alpha - \alpha_0} - p|u_{\alpha_0}(r)|^{p-1}v_{\alpha_0}(r) \right|$$
$$\le (p\alpha_0^{p-1} + C\varepsilon)\int_0^r |z(s)|ds + C\varepsilon \qquad \text{for any } r \in [0,R] \text{ and } \alpha \in (\alpha_0 - \delta, \alpha_0 + \delta) \cap [a,b]\,.$$

Simple estimates then yield

$$|z(r)| \le K(p\alpha_0^{p-1} + C\varepsilon)\int_0^r |z(s)|ds + KC\varepsilon$$

for any $r \in [0,R]$ and $\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta) \cap [a,b]$ where $K := \sup_{r \in (0,R]} \frac{\int_0^r \psi^{n-1}(s)ds}{\psi^{n-1}(r)}$. Standard Gronwall-type estimates then yield $\lim_{\alpha \to \alpha_0} \sup_{r \in [0,R]} |z(r)| = 0$ and, in turn,

$$\lim_{\alpha \to \alpha_0} \sup_{r \in [0,R]} |w(r)| = 0\,.$$

This proves the differentiability with respect to α of the map $\alpha \mapsto u(\alpha, r)$ and shows that the derivative with respect to α is a solution of (5.7). The proof of (5.6) is a consequence of a standard continuous dependence result for the Cauchy problem (5.7). □

Lemma 5.7. *Let ψ satisfy $(H_1) - (H_3)$. Let $\alpha_1 > \alpha_2 \ge \alpha_3 > \alpha_4 \ge 0$ be such that $u_{\alpha_1}(r) > 0, u_{\alpha_2}(r) > 0, u_{\alpha_3}(r) > 0, u_{\alpha_4}(r) \ge 0$ for any $r \in [0, R_0)$ for some $0 < R_0 \le +\infty$. If $\zeta_{\alpha_3\alpha_4} \le R_0$ is the first zero of $u_{\alpha_3} - u_{\alpha_4}$ then $\zeta_{\alpha_1\alpha_2}$, the first zero of $u_{\alpha_1} - u_{\alpha_2}$, is finite and it satisfies $\zeta_{\alpha_1\alpha_2} \le \zeta_{\alpha_3\alpha_4}$.*

Proof. The proof can be obtained proceeding exactly as in the proof of [5, Lemma 7.3]. □

We now show that $\lambda_1(B_r)$ diverges as $r \to 0^+$.

Lemma 5.8. *Let ψ satisfy $(H_1) - (H_2)$. Then*

$$\lim_{r \to 0^+} \lambda_1(B_r) = +\infty\,.$$

Proof. By $(H_1) - (H_2)$, for any $\overline{r}$ there exist $0 < C_1 < C_2$ depending on $\overline{r}$ such that

$$C_1 r \le \psi(r) \le C_2 r \qquad \text{for any } r \in [0, \overline{r}]\,.$$

Fix $\overline{r}$ and for any $r \in [0, \overline{r}]$ let us consider $\varphi \in C_c^\infty(B_r)$ and the quotient

$$\frac{\int_{B_r} |\nabla_g \varphi|_g^2 dV_g}{\int_{B_r} \varphi^2 dV_g} \ge \frac{\min\{C_1^{n-1}, C_1^{n-3}\}}{C_2^{n-1}} \frac{\int_{B_r^E} |\nabla \widetilde{\varphi}(x)|^2 dx}{\int_{B_r^E} \widetilde{\varphi}^2(x)dx} \ge \frac{\min\{C_1^{n-1}, C_1^{n-3}\}}{C_2^{n-1}} \lambda_1(B_r^E)$$

where $B_r^E \subset \mathbb{R}^n$ denotes the euclidean ball of radius r centered at the origin, $\widetilde{\varphi} \in C_c^\infty(B_r^E)$ the function defined by $\widetilde{\varphi}(x) = \varphi(|x|, x/|x|)$ for any $x \in B_r^E$ and $\lambda_1(B_r^E)$ the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions in the euclidean ball B_r^E. Since the previous inequality holds for any $\varphi \in C_c^\infty(B_r)$ then

$$\lambda_1(B_r) \ge \frac{\min\{C_1^{n-1}, C_1^{n-3}\}}{C_2^{n-1}} \lambda_1(B_r^E)\,. \tag{5.9}$$

It is well known that thanks to a rescaling argument one has $\lim_{r\to 0^+} \lambda_1(B_r^E) = +\infty$. Therefore passing to the limit in (5.9) as $r \to 0^+$ we arrive to the conclusion of the proof. $\square$

Lemma 5.9. *Let ψ satisfy $(H_1)-(H_3)$. Let $\alpha > \beta > 0$ and let u_α, u_β be the corresponding solutions of* (1.3)*. If u_β is unstable then u_α is unstable.*

Proof. We assume u_α positive otherwise the statement follows by Lemma 5.4.

First suppose that u_α and u_β have no intersection points. If they are both positive the conclusion is obvious. If u_β changes sign, we reach a contradiction by Lemma 5.7 with $\alpha_1 = \alpha$, $\alpha_2 = \alpha_3 = \beta$ and $\alpha_4 = 0$.

Next we assume that u_α and u_β have at least one intersection point. Let $\zeta_{\alpha\beta}$ be the first zero of the function $u_\alpha - u_\beta$. By (1.3) we deduce that $u'_\alpha(\zeta_{\alpha\beta}) < u'_\beta(\zeta_{\alpha\beta})$ so that there exists $\delta > 0$ such that $u_\alpha(r) < u_\beta(r)$ for any $r \in (\zeta_{\alpha\beta}, \zeta_{\alpha\beta} + \delta)$. By continuous dependence on the initial datum we deduce that there exists $\overline{\alpha} \in (\beta, \alpha)$ such that for any $\gamma \in [\overline{\alpha}, \alpha]$ we have $u_\gamma(r) < u_\beta(r)$ for any $r \in (\zeta_{\alpha\beta} + \delta/2, \zeta_{\alpha\beta} + \delta)$.

By Lemma 5.7 we have that u_α and u_γ admit at least one intersection point and moreover

$$\zeta_{\alpha\gamma} \le \zeta_{\overline{\alpha}\beta} < +\infty \qquad \text{for any } \gamma \in (\overline{\alpha}, \alpha)\,. \tag{5.10}$$

Let us note that as above one can show that for any $\gamma \in (\overline{\alpha}, \alpha)$, $u_\alpha < u_\gamma$ in a arbitrarily right neighborhood of $\zeta_{\alpha\gamma}$.

Let $\{\gamma_k\} \subset [\overline{\alpha}, \alpha)$ be a sequence such that $\gamma_k \uparrow \alpha$.

Then for any k there exists $r_k \in (\zeta_{\alpha\gamma_k}, \zeta_{\overline{\alpha}\beta} + 1)$ such that

$$\frac{u(\alpha, r_k) - u(\gamma_k, r_k)}{\alpha - \gamma_k} < 0$$

and by Lagrange Theorem and Lemma 5.6 we deduce that there exists $\sigma_k \in (\gamma_k, \alpha)$ such that

$$v_{\sigma_k}(r_k) = v(\sigma_k, r_k) = \frac{\partial u}{\partial \alpha}(\sigma_k, r_k) = \frac{u(\alpha, r_k) - u(\gamma_k, r_k)}{\alpha - \gamma_k} < 0\,.$$

On the other hand for any k, $v(\sigma_k, 0) = 1 > 0$ so that there exists $\rho_k \in (0, r_k)$ such that $v(\sigma_k, \rho_k) = 0$. This shows that

$$\begin{cases} -\Delta_g v_{\sigma_k} = p|u_{\sigma_k}|^{p-1} v_{\sigma_k} & \text{in } B_{\rho_k} \\ v_{\sigma_k} = 0 & \text{on } \partial B_{\rho_k}\,. \end{cases} \tag{5.11}$$

By the definitions of r_k and ρ_k we easily deduce that

$$\rho_k \le \zeta_{\overline{\alpha}\beta} + 1 \qquad \text{for any } k \in \mathbb{N}\,.$$

Multiplying both sides of the above equation by v_{σ_k} and integrating by parts we obtain

$$\int_{B_{\rho_k}} |\nabla_g v_{\sigma_k}|_g^2 \, dV_g = \int_{B_{\rho_k}} p|u_{\sigma_k}|^{p-1} v_{\sigma_k}^2 dV_g\,.$$

We want to show that the sequence $\{\rho_k\}$ is also bounded away from zero. Since $v_{\sigma_k} \in H_0^1(B_{\rho_k})$ and $\sigma_k < \alpha$, by Lemma 5.2 we have

$$0 = \int_{B_{\rho_k}} |\nabla_g v_{\sigma_k}|_g^2 \, dV_g - \int_{B_{\rho_k}} p|u_{\sigma_k}|^{p-1} v_{\sigma_k}^2 dV_g \ge (\lambda_1(B_{\rho_k}) - p\alpha^{p-1}) \int_{B_{\rho_k}} v_{\sigma_k}^2 dV_g$$

and hence $\lambda_1(B_{\rho_k}) \le p\alpha^{p-1}$ for any $k \in \mathbb{N}$. Therefore if we assume by contradiction that $\liminf_{k\to+\infty} \rho_k = 0$ then by Lemma 5.8 $\limsup_{k\to+\infty} \lambda_1(B_{\rho_k}) = +\infty$, a contradiction.

Then we may define $\rho_\infty = \liminf_{k\to+\infty} \rho_k \in (0,+\infty)$ and the sequence $\{w_k\} \subset H^1(M)$

$$w_k(x) := \begin{cases} \dfrac{v_{\sigma_k}(x)}{\|v_{\sigma_k}\|_{H^1(M)}} & \text{if } x \in B_{\rho_k} \\ 0 & \text{if } x \in M \setminus B_{\rho_k}\,. \end{cases}$$

Then for any k, w_k satisfies problem (5.11) and

$$\int_M |\nabla_g w_k|_g^2 \, dV_g - \int_M p|u_{\sigma_k}|^{p-1} w_k^2 \, dV_g = 0\,.$$

Moreover $\{w_k\}$ is bounded in $H^1(M)$ and hence up to a subsequence we assume that there exists $w \in H^1(M)$ such that $w_k \rightharpoonup w$ weakly in $H^1(M)$.

Let $\varphi \in C_c^\infty(B_{\rho_\infty})$ such that for any k large enough $\operatorname{supp}\varphi \subset B_{\rho_k}$. Then

$$\int_{B_{\rho_\infty}} \langle \nabla_g w_k, \nabla_g \varphi\rangle_g \, dV_g = p \int_{B_{\rho_\infty}} |u_{\sigma_k}|^{p-1} w_k \varphi \, dV_g\,.$$

Passing to the limit as $k \to +\infty$ and taking into account that by compact embedding $H^1(B_{\rho_\infty}) \subset L^2(B_{\rho_\infty})$ $w_k \to w$ strongly in $L^2(B_{\rho_\infty})$, and that by continuity from the initial data, $u_{\sigma_k} \to u_\alpha$ uniformly on compact sets, we obtain

$$\int_{B_{\rho_\infty}} \langle \nabla_g w, \nabla_g \varphi\rangle_g \, dV_g = p \int_{B_{\rho_\infty}} |u_\alpha|^{p-1} w \varphi \, dV_g \qquad \text{for any } \varphi \in C_c^\infty(B_{\rho_\infty})\,. \tag{5.12}$$

By density, the previous identity holds for any $\varphi \in H_0^1(B_{\rho_\infty})$.

We claim that $w \in H_0^1(B_{\rho_\infty})$. Up to another subsequence we may assume that $w_k \to w$ almost everywhere in M with respect to the volume measure V_g. But up to a subsequence, $\rho_k \to \rho_\infty$ so that for almost every $P \in M \setminus \overline{B}_{\rho_\infty}$, $w_k(P) = 0$ for any k large enough. This proves that $w \equiv 0$ almost everywhere in $M \setminus \overline{B}_{\rho_\infty}$ and since $w \in H^1(M)$ then $w \in H_0^1(B_{\rho_\infty})$.

Since $\operatorname{supp} w_k \subseteq B_{\zeta_{\overline{\alpha}\beta}+1}$ for any k, by compact embedding $H^1(B_{\zeta_{\overline{\alpha}\beta}+1}) \subset L^2(B_{\zeta_{\overline{\alpha}\beta}+1})$, we have that $w_k \to w$ in $L^2(M)$. Together with (5.12) and the fact that $\operatorname{supp} w_k \subset B_{\rho_k}$, $\operatorname{supp} w \subset B_{\rho_\infty}$, this implies

$$\begin{aligned} \lim_{k\to+\infty} \int_M |\nabla_g w_k|_g^2 \, dV_g &= \lim_{k\to+\infty} \int_{B_{\rho_k}} |\nabla_g w_k|_g^2 \, dV_g = \lim_{k\to+\infty} \int_{B_{\rho_k}} p|u_{\sigma_k}|^{p-1} w_k^2 \, dV_g \\ &= \lim_{k\to+\infty} \int_{B_{\zeta_{\overline{\alpha}\beta}+1}} p|u_{\sigma_k}|^{p-1} w_k^2 \, dV_g = \int_{B_{\zeta_{\overline{\alpha}\beta}+1}} p|u_\alpha|^{p-1} w^2 \, dV_g = \int_{B_{\rho_\infty}} p|u_\alpha|^{p-1} w^2 \, dV_g \\ &= \int_{B_{\rho_\infty}} |\nabla_g w|^2 \, dV_g = \int_M |\nabla_g w|^2 \, dV_g\,. \end{aligned}$$

The last identity together with the weak convergence yields $w_k \to w$ strongly in $H^1(M)$. In particular, since $\|w_k\|_{H^1(M)} = 1$ for any $k \in \mathbb{N}$, then $\|w\|_{H^1(M)} = 1$ and hence $w \not\equiv 0$. Summarizing we have found a nontrivial function $w \in H^1(M)$ satisfying

$$\int_M |\nabla_g w|^2 \, dV_g - \int_M p|u_\alpha|^{p-1} w^2 \, dV_g = 0\,.$$

Suppose now by contradiction that u_α is stable. Then

$$\int_M |\nabla_g \varphi|_g^2 \, dV_g - \int_M p|u_\alpha|^{p-1} \varphi^2 \, dV_g \geq 0 \qquad \text{for any } \varphi \in C_c^\infty(M)$$

and by density the previous inequality holds for any $\varphi \in H^1(M)$.

This means that $w \in H^1(M)$ is a minimizer of

$$\inf_{v \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla_g v|_g^2 \, dV_g}{\int_M p|u_\alpha|^{p-1} v^2 \, dV_g} \, .$$

In particular $w \in H^1(M)$ is a solution of the equation

$$-\Delta_g w = p|u_\alpha|^{p-1} w \qquad \text{in } M$$

and by standard regularity theory $w \in C^2(M)$. In particular w is a classical solution of the ordinary differential equation

$$-w''(r) - (n-1)\frac{\psi'(r)}{\psi(r)} w'(r) = p|u_\alpha(r)|^{p-1} w(r) \qquad (r > 0) \, .$$

But $w(r) = 0$ for any $r > \rho_\infty$ and hence by uniqueness of the Cauchy problem we infer $w(r) = 0$ for any $r > 0$ so that $w \equiv 0$ in M, a contradiction. □

Next we define

$$\alpha_0 := \sup\{\alpha \geq 0 : u_\beta \text{ is stable for any } \beta \in (0, \alpha)\} \, .$$

By Lemma 5.3 we know that $\alpha_0 \in (0, +\infty]$ and by Lemma 5.9 we have that u_α is stable for any $\alpha \in [0, \alpha_0)$ and it is unstable for any $\alpha > \alpha_0$ whenever $\alpha_0 < +\infty$. In the next lemma we prove that the set

$$\mathcal{S} := \{\alpha \geq 0 : u_\alpha \text{ is stable}\}$$

is a closed interval.

Lemma 5.10. *Let ψ satisfy $(H_1) - (H_3)$. Then the set $\mathcal{S}$ is a closed interval.*

Proof. We have just shown above that $\mathcal{S}$ is an interval. It remains to show that if $\alpha_0 < +\infty$ then $\alpha_0 \in \mathcal{S}$. We prove that $[0, +\infty) \setminus \mathcal{S}$ is open. Let $\alpha \in [0, +\infty) \setminus \mathcal{S}$ so that u_α is unstable. Hence there exists $\varphi \in C_c^\infty(M)$ such that

$$\int_M |\nabla_g \varphi|_g^2 \, dV_g - \int_M p|u_\alpha|^{p-1} \varphi^2 dV_g < 0 \, . \tag{5.13}$$

We claim that there exists $\delta > 0$ such that

$$\int_M |\nabla_g \varphi|_g^2 \, dV_g - \int_M p|u_\beta|^{p-1} \varphi^2 dV_g < 0$$

for any $\beta \in (\alpha - \delta, \alpha + \delta)$ or in other words $[0, +\infty) \setminus \mathcal{S}$ is open.

Suppose by contradiction that there exists a sequence $\{\alpha_k\} \subset [0, +\infty)$ such that $\alpha_k \to \alpha$ and

$$\int_M |\nabla_g \varphi|_g^2 \, dV_g - \int_M p|u_{\alpha_k}|^{p-1} \varphi^2 dV_g \geq 0 \, . \tag{5.14}$$

Since the $\operatorname{supp} \varphi$ is compact by continuous dependence on the initial data we have that $u_{\alpha_k} \to u_\alpha$ uniformly in any compact set of M. Passing to the limit in (5.14) as $k \to +\infty$ we obtain

$$\int_M |\nabla_g \varphi|_g^2 \, dV_g - \int_M p|u_\alpha|^{p-1} \varphi^2 dV_g \geq 0$$

in contradiction with (5.13). □

The estimate $\alpha_0 \geq \left(p^{-1}\lambda_1(M)\right)^{1/(p-1)}$ follows immediately from Lemma 5.3. It remains to prove that the inequality is strict under some additional assumptions. First we prove the following

Lemma 5.11. *Let ψ satisfy $(H_1)-(H_3)$ and (2.8). Then for any $\alpha > 0$*

$$\Lambda_1(M,\alpha) := \inf_{v\in H^1(M)\setminus\{0\}} \frac{\int_M |\nabla_g v|_g^2\, dV_g}{\int_M p|u_\alpha|^{p-1}v^2\, dV_g}$$

admits a minimizer.

Proof. Let $\{v_k\}\subset H^1(M)$ be a minimizing sequence for $\Lambda_1(M,\alpha)$ such that

$$\int_M p|u_\alpha|^{p-1}v_k^2\, dV_g = 1\,.$$

Then $\{v_k\}$ is bounded in $H^1(M)$ and hence up to a subsequence there exists $v\in H^1(M)$ such that $v_k \rightharpoonup v$ weakly in $H^1(M)$. By compact embedding $H^1(B_r)\subset L^2(B_r)$ we have that $v_k\to v$ strongly in $L^2(B_r)$ for any $r>0$.

By the assumptions of this lemma, combined with Proposition 2.1, formula (2.1), Theorems 2.6 and 2.9, we have that $u_\alpha(r)\to 0$ as $r\to+\infty$. Hence for any $\varepsilon>0$ we may choose $R_\varepsilon>0$ such that $p|u_\alpha(r)|^{p-1}<\varepsilon$ for any $r>R_\varepsilon$. Hence we obtain

$$\begin{aligned}
&\left|\int_M p|u_\alpha|^{p-1}v_k^2\, dV_g - \int_M p|u_\alpha|^{p-1}v^2\, dV_g\right| \le \left|\int_{B_{R_\varepsilon}} p|u_\alpha|^{p-1}v_k^2\, dV_g - \int_{B_{R_\varepsilon}} p|u_\alpha|^{p-1}v^2\, dV_g\right| \\
&\qquad + \left|\int_{M\setminus B_{R_\varepsilon}} p|u_\alpha|^{p-1}v_k^2\, dV_g - \int_{M\setminus B_{R_\varepsilon}} p|u_\alpha|^{p-1}v^2\, dV_g\right| \\
&\le \left|\int_{B_{R_\varepsilon}} p|u_\alpha|^{p-1}v_k^2\, dV_g - \int_{B_{R_\varepsilon}} p|u_\alpha|^{p-1}v^2\, dV_g\right| + \frac{\varepsilon}{\lambda_1(M)}\left(\|v_k\|_{H^1(M)}^2 + \|v\|_{H^1(M)}^2\right).
\end{aligned}$$

Passing to the limit as $k\to+\infty$ we obtain

$$\limsup_{k\to+\infty}\left|\int_M p|u_\alpha|^{p-1}v_k^2\, dV_g - \int_M p|u_\alpha|^{p-1}v^2\, dV_g\right| \le \frac{2\Lambda_1(M,\alpha)}{\lambda_1(M)}\,\varepsilon \qquad \text{for any } \varepsilon>0\,.$$

Hence,

$$\lim_{k\to+\infty}\int_M p|u_\alpha|^{p-1}v_k^2\, dV_g = \int_M p|u_\alpha|^{p-1}v^2\, dV_g\,.$$

This shows that $v\neq 0$ and that, by the lower semicontinuity of the $H^1(M)$-norm, v is a minimizer for $\Lambda_1(M,\alpha)$. ☐

Lemma 5.12. *Let ψ satisfy $(H_1)-(H_3)$ and (2.8). Then $\alpha_0 > \left(\frac{\lambda_1(M)}{p}\right)^{\frac{1}{p-1}}$.*

Proof. Define $\overline{\alpha} := \left(\frac{\lambda_1(M)}{p}\right)^{\frac{1}{p-1}}$. We claim that $\Lambda(M,\overline{\alpha})>1$. To see this, by Lemma 5.11 we introduce a minimizer $w\in H^1(M)$ of $\Lambda(M,\overline{\alpha})$. By Poincaré inequality and the fact that, by Lemma 5.2, $u_{\overline{\alpha}}\le\overline{\alpha}$, we have

$$\Lambda_1(M,\overline{\alpha}) = \frac{\int_M |\nabla_g w|_g^2\, dV_g}{\int_M p|u_{\overline{\alpha}}|^{p-1}w^2\, dV_g} \ge \frac{\int_M |\nabla_g w|_g^2\, dV_g}{\lambda_1(M)\int_M w^2\, dV_g} \ge 1\,.$$

If assume by contradiction that $\Lambda_1(M,\overline{\alpha})=1$ then the inequalities above are equalities and hence $w\in H^1(M)$ is a minimizer for $\lambda_1(M)$. Hence, it solves the equation

$$-\Delta_g w = \lambda_1(M) w \qquad \text{in } M$$

and this contradicts the fact that w solves

$$-\Delta_g w = \Lambda_1(M,\overline{\alpha})|u_{\overline{\alpha}}|^{p-1}w \qquad \text{in } M\,.$$

This completes the proof of the claim. Let us consider a sequence $\{\alpha_k\}$ such that $\alpha_k \downarrow \overline{\alpha}$. We prove that for any large k, $\Lambda_1(M,\alpha_k) > 1$.

If we proceed by contradiction, we may assume that $\Lambda_1(M,\alpha_k) \le 1$ for any large k.

Let $\{w_k\} \subset H^1(M)$ be a sequence of minimizers for $\Lambda_1(M,\alpha_k)$ such that $\int_M p|u_{\alpha_k}|^{p-1} w_k^2\, dV_g = 1$.

Then $\{w_k\}$ is bounded in $H^1(M)$ and up to a subsequence we may assume that there exists $\overline{w} \in H^1(M)$ such that $w_k \rightharpoonup \overline{w}$ weakly in $H^1(M)$.

For any $\alpha > 0$, consider the Lyapunov function

$$F_\alpha(r) := \frac{1}{2}|u_\alpha'(r)|^2 + \frac{1}{p+1}|u_\alpha(r)|^{p+1} \qquad \text{for any } r > 0\,.$$

For any $\varepsilon > 0$ let $R_\varepsilon > 0$ be such that

$$F_{\overline{\alpha}}(R_\varepsilon) < \varepsilon\,.$$

We recall that as in the proof of Lemma 5.11 we have $\lim\limits_{r\to+\infty} u_\alpha(r) = \lim\limits_{r\to+\infty} u_\alpha'(r) = 0$, for any $\alpha > 0$.

Since $u_{\alpha_k}(r) \to u_{\overline{\alpha}}(r)$ and $u_{\alpha_k}'(r) \to u_{\overline{\alpha}}'(r)$ for any $r > 0$, there exists $\overline{k}$ such that

$$F_{\alpha_k}(R_\varepsilon) < \varepsilon \qquad \text{for any } k > \overline{k}\,.$$

But we know that for any $\alpha > 0$ the function F_α is nonincreasing and hence

$$F_{\alpha_k}(r) < \varepsilon \qquad \text{for any } r \ge R_\varepsilon\,, \text{for any } k > \overline{k}\,,$$

so that

$$p|u_{\alpha_k}(r)|^{p-1} \le p[(p+1)\varepsilon]^{\frac{p-1}{p+1}} \qquad \text{for any } r \ge R_\varepsilon\,, \text{for any } k > \overline{k}\,.$$

Therefore

$$\begin{aligned}
\left|\int_M p|u_{\alpha_k}|^{p-1} w_k^2\, dV_g - \int_M p|u_{\overline{\alpha}}|^{p-1}\overline{w}^2\, dV_g\right| &\le \left|\int_{B_{R_\varepsilon}} p|u_{\alpha_k}|^{p-1} w_k^2\, dV_g - \int_{B_{R_\varepsilon}} p|u_{\overline{\alpha}}|^{p-1} w_k^2\, dV_g\right| \\
&+ \left|\int_{B_{R_\varepsilon}} p|u_{\overline{\alpha}}|^{p-1} w_k^2\, dV_g - \int_{B_{R_\varepsilon}} p|u_{\overline{\alpha}}|^{p-1}\overline{w}^2\, dV_g\right| \\
&+ \left|\int_{M\setminus B_{R_\varepsilon}} p|u_{\alpha_k}|^{p-1} w_k^2\, dV_g - \int_{M\setminus B_{R_\varepsilon}} p|u_{\overline{\alpha}}|^{p-1}\overline{w}^2\, dV_g\right| \\
\le \sup_{B_{R_\varepsilon}} |p|u_{\alpha_k}|^{p-1} - p|u_{\overline{\alpha}}|^{p-1}| \int_{B_{R_\varepsilon}} w_k^2\, dV_g &+ \left|\int_{B_{R_\varepsilon}} p|u_{\overline{\alpha}}|^{p-1} w_k^2\, dV_g - \int_{B_{R_\varepsilon}} p|u_{\overline{\alpha}}|^{p-1}\overline{w}^2\, dV_g\right| \\
&+ p[(p+1)\varepsilon]^{\frac{p-1}{p+1}} \int_{M\setminus B_{R_\varepsilon}} w_k^2\, dV_g + p[(p+1)\varepsilon]^{\frac{p-1}{p+1}} \int_{M\setminus B_{R_\varepsilon}} \overline{w}^2\, dV_g\,.
\end{aligned}$$

By strong convergence $w_k \to \overline{w}$ in $L^2(B_{R_\varepsilon})$, uniform convergence $u_{\alpha_k} \to u_\alpha$ in B_{R_ε}, Poincaré inequality, weak lower semicontinuity of the $H^1(M)$-norm and the fact that $\Lambda(M,\alpha_k) \le 1$, we obtain

$$\limsup_{k\to+\infty} \left|\int_M p|u_{\alpha_k}|^{p-1} w_k^2\, dV_g - \int_M p|u_{\overline{\alpha}}|^{p-1}\overline{w}^2\, dV_g\right| \le \frac{2}{\lambda_1(M)}\, p[(p+1)\varepsilon]^{\frac{p-1}{p+1}} \qquad \text{for any } \varepsilon > 0\,.$$

This proves that

$$\lim_{k\to+\infty} \int_M p|u_{\alpha_k}|^{p-1} w_k^2\, dV_g = \int_M p|u_{\overline{\alpha}}|^{p-1}\overline{w}^2\, dV_g\,.$$

Therefore using again the weak lower semicontinuity of the $H^1(M)$-norm, we obtain

$$1 < \Lambda_1(M,\overline{\alpha}) \le \frac{\int_M |\nabla_g \overline{w}|_g^2\, dV_g}{\int_M p|u_{\overline{\alpha}}|^{p-1}\overline{w}^2\, dV_g} \le \liminf_{k\to+\infty} \frac{\int_M |\nabla_g w_k|_g^2\, dV_g}{\int_M p|u_{\alpha_k}|^{p-1} w_k^2\, dV_g} = \liminf_{k\to+\infty} \Lambda_1(M,\alpha_k)\,,$$

a contradiction. This proves that $\Lambda_1(M,\alpha_k) > 1$ for any large k.

In particular for any large k and any $\varphi \in C_c^\infty(M)$ we have

$$\int_M |\nabla_g \varphi|_g^2 \, dV_g \geq \Lambda_1(M, \alpha_k) \int_M p|u_{\alpha_k}|^{p-1} \varphi^2 \, dV_g \geq \int_M p|u_{\alpha_k}|^{p-1} \varphi^2 \, dV_g \, .$$

We found a sequence of values $\alpha_k > \overline{\alpha}$ such that u_{α_k} is stable. This completes the proof of the lemma. $\square$

textitEnd of the proof of Theorems 2.11-2.12. The proof of Theorem 2.11 simply follows by combining Lemma 5.10 with Lemma 5.3 and Lemma 5.5. The estimate from below on α_0 follows from Lemma 5.12.

5.2. **Proof of Theorem 2.14.** Let $\alpha, \beta \in \mathcal{S}$ with $\alpha > \beta$. We want to prove that $u_\alpha(r) > u_\beta(r) > 0$ for any $r > 0$. Suppose by contradiction that there exists $\overline{r} > 0$ such that $u_\alpha(\overline{r}) < u_\beta(\overline{r})$. By Lagrange Theorem and Lemma 5.6 we deduce that there exists $\sigma \in (\beta, \alpha)$ such that

$$v_\sigma(\overline{r}) = v(\sigma, \overline{r}) = \frac{\partial u}{\partial \alpha}(\sigma, \overline{r}) = \frac{u(\alpha, \overline{r}) - u(\beta, \overline{r})}{\alpha - \beta} < 0$$

and proceeding as in the proof of Lemma 5.9 we find $\rho \in (0, \overline{r})$ such that

$$\begin{cases} -\Delta_g v_\sigma = p|u_\sigma|^{p-1} v_\sigma & \text{in } B_\rho \\ v_\sigma = 0 & \text{on } \partial B_\rho \, . \end{cases}$$

Testing the above problem with $v_\sigma \in H_0^1(B_\rho)$, we obtain

$$\int_{B_\rho} |\nabla_g v_\sigma|_g^2 \, dV_g - \int_{B_\rho} p|u_\sigma|^{p-1} v_\sigma^2 \, dV_g = 0 \, .$$

Next we define $w_\sigma \in H^1(M)$ as the trivial extension of v_σ outside B_ρ in such a way that

$$\int_M |\nabla_g w_\sigma|_g^2 \, dV_g - \int_M p|u_\sigma|^{p-1} w_\sigma^2 \, dV_g = 0 \, .$$

But $\sigma \in [0, \alpha_0]$ and hence by Lemma 5.10 u_σ is stable. Therefore w_σ is a minimizer of the problem

$$\inf_{v \in H^1(M) \setminus \{0\}} \frac{\int_M |\nabla_g v|_g^2 \, dV_g}{\int_M p|u_\sigma|^{p-1} v^2 \, dV_g}$$

and proceeding as in the proof of Lemma 5.9 we arrive to a contradiction.

5.3. **Proof of Theorem 2.15.** By Lemma 4.1 we have $\lambda_1(M) > 0$. By Proposition 2.1, (2.1), Theorem 2.6 and Theorem 2.9 we get the existence of $R > 0$ such that $p|u(r)|^{p-1} \leq \lambda_1(M)$ for every $r > R$. Let now B_R be the geodesic ball of radius R centered at o. From what just remarked and (4.1), inequality (5.2) holds for every $\psi \in C_c^\infty(M \setminus K)$ and for every compact K such that $B_R \subset K$. In particular, u is stable outside a compact set.

5.4. **Proof of Proposition 2.13.** Since u is stable, from (5.2) we have

$$\int_0^{+\infty} (\chi'(r))^2 \, \psi^{n-1}(r) dr - p \int_0^{+\infty} |u(r)|^{p-1} \chi^2(r) \, \psi^{n-1}(r) \, dr \geq 0 \, , \tag{5.15}$$

for every radial function $\chi \in C_c^\infty(M)$.

Inequality (5.15) holds for every $\chi \in H^1 \cap L^\infty(M)$ with compact support in M. Next, we choose $\chi(r) = u(r)\eta(r)$ with $\eta \in C_c^1(0, +\infty)$ in (5.15) and we get

$$\int_0^{+\infty} (u'(r))^2 (\eta(r))^2 \, \psi^{n-1}(r) \, dr + \int_0^{+\infty} (u(r))^2 (\eta'(r))^2 \, \psi^{n-1}(r) \, dr$$

$$+\int_0^{+\infty} u'(r)u(r)(\eta^2(r))'\,\psi^{n-1}(r)\,dr \geq p\int_0^{+\infty} |u(r)|^{p+1}\eta^2(r)\,\psi^{n-1}(r)\,dr\,.$$

An integration by parts and (1.3) yield

$$\int_0^{+\infty} (u(r))^2(\eta'(r))^2\,\psi^{n-1}(r)\,dr \geq (p-1)\int_0^{+\infty} |u(r)|^{p+1}\eta^2(r)\,\psi^{n-1}(r)\,dr\,, \tag{5.16}$$

for every radial function $\eta \in C_c^\infty(M)$.

For $R>0$, let now $\eta_R(r)=\eta(r/R)$, where $\eta(r)\in C^1([0,+\infty))$ is such that $\eta(r)=1$ for $0\leq r<1$ and $\eta(r)=0$ for $r\geq 2$. Taking η_R as test function in (5.16), we get

$$\begin{aligned}\frac{\|\eta'\|_{L^\infty(1,2)}}{R^2}\int_R^{2R}(u(r))^2\,\psi^{n-1}(r)\,dr &\geq (p-1)\int_0^{2R}|u(r)|^{p+1}\eta^2(r/R)\,\psi^{n-1}(r)\,dr\\ &\geq (p-1)\int_0^R |u(r)|^{p+1}\,\psi^{n-1}(r)\,dr\,.\end{aligned}$$

As $R\to+\infty$, recalling that $\int_0^{+\infty}(u(r))^2\,\psi^{n-1}(r)\,dr<+\infty$, we finally conclude that

$$\int_0^{+\infty}|u(r)|^{p+1}\,\psi^{n-1}(r)\,dr\;=0\,.$$

Hence, $u\equiv 0$ in $(0,+\infty)$.

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Politecnico di Milano,
Dipartimento di Matematica “Francesco Brioschi”,
Piazza Leonardo da Vinci 32, 20133 Milano, Italy.

E-mail address: elvise.berchio@polimi.it

Università degli Studi del Piemonte Orientale,
Dipartimento di Scienze e Innovazione Tecnologica,
Viale Teresa Michel 11, 15121 Alessandria, Italy.

E-mail addresses: alberto.ferrero@mfn.unipmn.it

Politecnico di Milano,
Dipartimento di Matematica “Francesco Brioschi”,
Piazza Leonardo da Vinci 32, 20133 Milano, Italy.

E-mail address: gabriele.grillo@polimi.it