

DIPARTIMENTO DI MATEMATICA  
“Francesco Brioschi”  
POLITECNICO DI MILANO

**Optimal impulse control of a portfolio  
with a fixed transaction cost**

Baccarin, S; Marazzina, D.

Collezione dei *Quaderni di Dipartimento*, numero **QDD 138**  
Inserito negli *Archivi Digitali di Dipartimento* in data 11-11-2012



Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

---

# Optimal impulse control of a portfolio with a fixed transaction cost

Stefano Baccarin · Daniele Marazzina

**Abstract** The aim of this work is to investigate a portfolio optimization problem in presence of fixed transaction costs. We consider an economy with two assets, one risky and one risk-free, and an agent fully described by its power utility function. We show how fixed transaction costs influence the agent's behavior, showing when it is optimal to recalibrate his/her portfolio, paying the transaction costs.

**Keywords** Transaction Costs · Portfolio Optimization · Optimal Strategy · Utility Maximization

## 1 The model investment problem

We consider a continuous time economy with a finite horizon  $T$ . We assume that there are two assets: the risk-free asset with a constant instantaneous interest rate  $r$

$$dB(t) = rB(t)dt, \quad B(0) = 1,$$

and a risky asset  $S(t)$ , which evolves as

$$dS(t) = S(t)(bdt + \sigma dZ(t)), \quad S(0) = S_0,$$

---

S. Baccarin  
Università degli Studi di Torino  
Dipartimento di Scienze Economico-Sociali e Matematico-Statistiche  
Corso Unione Sovietica 218/bis, 10134 Torino, Italy  
E-mail: baccarin@econ.unito.it

D. Marazzina  
Politecnico di Milano  
Dipartimento di Matematica 'Francesco Brioschi'  
Via Bonardi 9, 20133 Milano, Italy  
E-mail: daniele.marazzina@polimi.it

where  $b$  is the constant drift of the risky asset price and  $\sigma$  is the (constant) volatility,  $Z(t)$  being a one-dimensional Brownian motion. The agent maximizes the expected utility over the finite horizon. The instantaneous utility at time  $t$  is a function of the wealth, and we assume the following form for the utility function:

$$u(x) = \frac{x^\gamma}{\gamma}, \quad 0 < \gamma < 1. \quad (1)$$

In the considered economy, the agent can invest at any time in the risky asset  $S$  a wealth  $\pi \in \mathbb{R}$ , reducing (or increasing) correspondingly the bank account. However, for each transaction, he/she must face a fixed transaction cost, paying  $K$ . Therefore, if the agent's wealth at a given time  $t$  is  $W$ , and his/her wealth invested in the risky asset is  $\pi_S$ , his/her portfolio is fully described by

$$(\pi_S, \pi_B)$$

with  $\pi_B := W - \pi_S$  being the wealth invested in the risk-free asset (which could also be negative). Buying (or selling) a value  $\xi$  of the risky asset at the same time  $t$ , the portfolio becomes

$$(\pi_S + \xi, \pi_B - K - \xi),$$

i.e.,

$$\pi_S(t) = \pi_S(t^-) + \xi, \quad \pi_B(t) = \pi_B(t^-) - K - \xi.$$

Portfolio optimization in presence of transaction costs has been widely studied in literature, see for example [Davis and Norman 1990, Dumas and Luciano 1991, Korn 1998, Liu and Loewenstein 2002, Oksendal and Sulem 2002] and [Shreve and Soner 1994].

Whenever a transaction is made, the investor must bear a fixed transaction cost of amount  $K$ . A fundamental notion in our model is the liquidation value of the portfolio. We define the liquidation value  $L(\pi_S, \pi_B)$  of the portfolio as

$$L(\pi_S, \pi_B) = \pi_S + \pi_B - K \text{ if } \pi_S \neq 0, \quad L(0, \pi_B) = \pi_B$$

i.e., it is the value when the long or short position in the risky asset is cleared out. Besides the transaction costs, we assume that the agent must face a solvency constraint, requiring that the investor is always solvable, that is a portfolio is admissible only if  $L(\pi_S(t), \pi_B(t)) \geq 0$ ,  $\forall t \in [0, T]$ . We define the region  $\mathcal{P} \subset \mathbb{R}^2$  of admissible portfolios by

$$\mathcal{P} = \{(\pi_S, \pi_B) \in \mathbb{R}^2 : L(\pi_S, \pi_B) \geq 0\}.$$

The investor's preferences are represented by the continuous, increasing, utility function (1). Given the utility function, we can formulate our model as the following optimal impulse control problem

$$\max_{p \in A(0, \pi_S(0), \pi_B(0))} E_{0, \pi_S(0), \pi_B(0)}[u(L(\pi_S(T), \pi_B(T)))]$$

where  $A(0, \pi_S(0), \pi_B(0))$  is the set of admissible policies when the process starts at time 0 with a portfolio  $(\pi_B(0), \pi_S(0))$ . An admissible control policy

$p$  is a sequence of stopping times  $\{\tau_i\}$  (with respect to the natural filtration of  $Z$ ) and corresponding random variables  $\{\xi_i\}$  verifying the conditions:

$$\begin{cases} 0 \leq \tau_i \leq \tau_{i+1} & \text{almost surely } \forall i \geq 1 \\ \lim_{i \rightarrow \infty} \tau_i = \infty & \text{almost surely} \\ \xi_i \text{ such that } (\pi_S(\tau_i^-) + \xi_i, \pi_B(\tau_i^-) - \xi_i - K) \in \mathcal{P} \end{cases} .$$

Here  $\xi_i$  represents the value of stocks bought (if  $\xi_i > 0$ ) or sold (if  $\xi_i < 0$ ) at time  $\tau_i$ . We consider a fixed time horizon  $T > 0$ . Notice that  $\lim_{i \rightarrow \infty} \tau_i = \infty$  implies that the number of stopping times  $\tau_i$  which are less or equal to  $T$  is almost surely finite ( $\tau_i = +\infty$  almost surely for some  $i < \infty$  is possible). Starting from the initial condition  $W(0) = \pi_B(0) + \pi_S(0)$  the dynamics of the controlled portfolio can be described by the following set of stochastic differential equations:

$$\begin{cases} d\pi_S = b\pi_S dt + \sigma\pi_S dZ \\ d\pi_B = r\pi_B dt & \text{if } \tau_i < t < \tau_{i+1} \\ \pi_S(\tau_i) = \pi_S(\tau_i^-) + \xi_i \\ \pi_B(\tau_i) = \pi_B(\tau_i^-) - \xi_i - K & \text{if } t = \tau_i . \end{cases}$$

We will solve this problem by using a dynamic programming approach, considering the value function

$$V(t, \pi_S, \pi_B) = \sup_{p \in A(t, \pi_S, \pi_B)} E_{t, \pi_S, \pi_B} [u(L(\pi_S(T), \pi_B(T)))]$$

which is defined in  $[0, T] \times \mathcal{P}$  and where  $A(t, \pi_S, \pi_B)$  is the set of admissible policies when the process starts in  $t$  with a portfolio  $(\pi_S, \pi_B)$ . In the next section we will show heuristically that  $V(t, \pi_S, \pi_B)$  is a solution of a quasi-variational inequality, and that there exists an optimal control of a markovian type for our model.

## 2 The quasi-variational inequality associated to the value function and the optimal control

We can define the following non-local operator  $\mathcal{M}$  for bounded functions in  $[0, T] \times \mathcal{P}$  as

$$\mathcal{M}V := \sup_{\xi \in F(\pi_S, \pi_B)} V(t, \pi_S + \xi, \pi_B - \xi - K),$$

being  $F(\pi_S, \pi_B)$  the set of admissible transactions from  $(\pi_S, \pi_B) \in \mathcal{P}$

$$F(\pi_S, \pi_B) := \{\xi \in \mathbb{R} : (\pi_S + \xi, \pi_B - \xi - K) \in \mathcal{P}\} .$$

Notice that  $\mathcal{M}V$  corresponds to the best transaction the agent can make if he decides to intervene. If  $F(\pi_S, \pi_B) = \emptyset$ , we set  $\mathcal{M}V = -1$ . We also define the second order linear operator  $\mathcal{L}$  by

$$\mathcal{L}V := \frac{\partial V}{\partial t} + r\pi_B \frac{\partial V}{\partial \pi_B} + b\pi_S \frac{\partial V}{\partial \pi_S} + \frac{1}{2} \sigma^2 \pi_S^2 \frac{\partial^2 V}{\partial \pi_S^2}$$

In this section we will show, in a formal way, that the value function  $V$  of our problem is a solution of the following parabolic quasi-variational inequality in  $(0, T) \times \mathcal{P}$

$$V(t, \pi_S, \pi_B) \geq \mathcal{M}V(t, \pi_S, \pi_B) \quad (2)$$

$$\mathcal{L}V(t, \pi_S, \pi_B) \leq 0 \quad (3)$$

$$(V(t, \pi_S, \pi_B) - \mathcal{M}V(t, \pi_S, \pi_B)) \mathcal{L}V(t, \pi_S, \pi_B) = 0 \quad (4)$$

Consider our agent at the time instant  $t$ . He/she can take only one of two possible decisions:

- 1) to let the system evolve freely for the infinitesimal interval  $(t, t + h)$
- 2) to make the best transaction, selling or buying stocks.

Since there is no other alternative it is likely that the following version of Bellman's optimality principle holds true:

$$V(t, \pi_S, \pi_B) = \max \{ E_{t, \pi_S, \pi_B} [V(t + h, \pi_S(t + h), \pi_B(t + h))] , \mathcal{M}V(t, \pi_S, \pi_B) \}.$$

Therefore we obtain immediately  $V \geq \mathcal{M}V$ , which is condition (2). Now, suppose the value function is regular enough to apply the Dynkin's formula in the interval  $(t, t + h)$ . We obtain

$$\begin{aligned} & E_{t, \pi_S, \pi_B} [V(t + h, \pi_S(t + h), \pi_B(t + h))] \\ &= V(t, \pi_S, \pi_B) + E_{t, \pi_S, \pi_B} \left[ \int_t^{t+h} \frac{\partial V}{\partial s} + r\pi_B \frac{\partial V}{\partial \pi_B} + b\pi_S \frac{\partial V}{\partial \pi_S} + \frac{1}{2} \sigma^2 \pi_S^2 \frac{\partial V}{\partial \pi_S^2} ds \right]. \end{aligned}$$

But, from the Bellman' principle, it holds

$$V(t, \pi_S, \pi_B) \geq E_{t, \pi_S, \pi_B} [V(t + h, \pi_S(t + h), \pi_B(t + h))]$$

and consequently we have

$$E_{t, \pi_S, \pi_B} \left[ \int_t^{t+h} \frac{\partial V}{\partial s} + r\pi_B \frac{\partial V}{\partial \pi_B} + b\pi_S \frac{\partial V}{\partial \pi_S} + \frac{1}{2} \sigma^2 \pi_S^2 \frac{\partial V}{\partial \pi_S^2} ds \right] \leq 0.$$

Letting  $h \rightarrow 0^+$  and using the integral version of the mean value theorem we obtain the inequality (3), that is  $\mathcal{L}V \leq 0$ . Since no other alternative is possible, the third equality (4),  $(V - \mathcal{M}V) \mathcal{L}V = 0$ , is also verified. To uniquely characterize  $V$  as a solution of (2-4) in  $[0, T] \times \mathcal{P}$  we must consider the behavior of the value function at the boundary of  $(0, T) \times \mathcal{P}$ . At the terminal date  $T$  it holds, obviously,  $V(T, \pi_S, \pi_B) = u(L(\pi_S, \pi_B))$ ,  $\forall (\pi_S, \pi_B) \in \mathcal{P}$ . Along the straight line  $\pi_S + \pi_B = K$ , we have

$$V(t, \pi_S, \pi_B - K) = u(0) \quad \forall t \in [0, T]$$

because a policy must prescribe an intervention to be sure to stay in the solvency region, and the only admissible transaction leads the process to zero. Moreover the value function is also determined by the fact that it is upper

bounded, because it is certainly lower than the value function of a corresponding Merton's problem without transaction costs.

We also can show heuristically that an optimal control of a markovian type always exists for our model. We divide the  $[0, T] \times \mathcal{P}$  domain into two regions, the transaction region

$$A \equiv \{(t, \pi_S, \pi_B) \in [0, T] \times \mathcal{P} : V = \mathcal{M}V\}$$

and the complementary continuation region

$$C \equiv \{(t, \pi_S, \pi_B) \in [0, T] \times \mathcal{P} : V > \mathcal{M}V\} .$$

Setting  $\tau_0^* \equiv t$ , the optimal policy  $p^*(t, \pi_S, \pi_B)$  for the process starting in  $(t, \pi_S, \pi_B)$  is given by:

$$p^* = \begin{cases} \tau_i^* = \begin{cases} \inf \{I_i \equiv \{T \geq t \geq \tau_{i-1}^* : (t, \pi_S(t^-), \pi_B(t^-)) \in A\}\} & \text{if } I_i \neq \emptyset \\ +\infty & \text{if } I_i = \emptyset \end{cases} \\ \xi_i^* = \begin{cases} \arg \max_{\xi \in F(\pi_S(\tau_i^-), \pi_B(\tau_i^-))} V(\tau_i, \pi_S(\tau_i^{*-}) + \xi, \pi_B(\tau_i^{*-}) - \xi - K) & \text{if } \tau_i^* < \infty \\ \text{arbitrary} & \text{if } \tau_i^* = +\infty \end{cases} \end{cases}$$

Indeed, if we apply the Dynkin's formula, separately in the intervals  $(\tau_{i-1}^* \wedge T, \tau_i^* \wedge T)$  to the process  $(\pi_S^*, \pi_B^*)$  controlled by policy  $p^*$ , and we take account of the jumps  $\xi_i^*$ , we have ( $i = 1, \dots, m$ ) :

$$\begin{aligned} & E_{t, \pi_S, \pi_B} [V(\tau_m^* \wedge T, \pi_S(\tau_m^{*-} \wedge T), \pi_B(\tau_m^{*-} \wedge T))] = V(t, \pi_S, \pi_B) \\ & + E_{t, \pi_S, \pi_B} [\sum_{i=0}^{m-1} \int_{\tau_i^* \wedge T}^{\tau_{i+1}^* \wedge T} (\frac{\partial V}{\partial s} + r\pi_B \frac{\partial V}{\partial \pi_B} + b\pi_S \frac{\partial V}{\partial \pi_S} + \frac{1}{2}\sigma^2\pi_S^2 \frac{\partial V}{\partial \pi_S^2}) ds] \\ & + E_{t, \pi_S, \pi_B} [\sum_{i=1}^{m-1} (V(\tau_i^*, \pi_S(\tau_i^{*-}) + \xi_i^*, \pi_B(\tau_i^{*-}) - \xi_i^* - K) \\ & \quad - V(\tau_i, \pi_S(\tau_i^{*-}), \pi_B(\tau_i^{*-}))\chi_{\tau_i^* < \infty}] . \end{aligned}$$

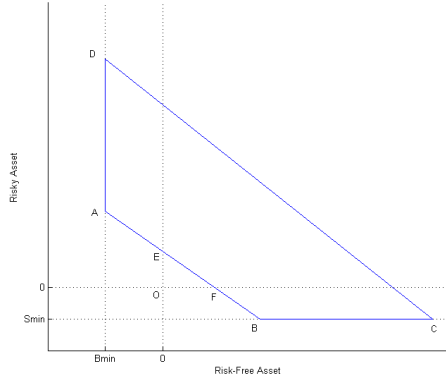
Since  $V$  verifies  $\mathcal{L}V = 0$  when  $V > \mathcal{M}V$ , and by construction  $(s, \pi_S^*, \pi_B^*) \in C$  in the intervals  $(\tau_i^* \wedge T, \tau_{i+1}^* \wedge T)$  when  $\tau_i^* \wedge T < \tau_{i+1}^* \wedge T$ , all the terms in the first expectation vanish. Similarly, as  $V$  verifies  $V = \mathcal{M}V$  in  $A$ , and by construction  $(\tau_i^*, \pi_S(\tau_i^{*-}), \pi_B(\tau_i^{*-})) \in A$  if  $\tau_i^* < \infty$ , we have  $V(\tau_i^*, \pi_S(\tau_i^{*-}) + \xi_i^*, \pi_B(\tau_i^{*-}) - \xi_i^* - K)_{\tau_i^* < \infty} = V(\tau_i^*, \pi_S(\tau_i^{*-}), \pi_B(\tau_i^{*-}))_{\tau_i^* < \infty}$ , and also the second expectation vanishes. Therefore we obtain

$$V(t, \pi_S, \pi_B) = E_{t, \pi_S, \pi_B} [V(\tau_m^* \wedge T, \pi_S(\tau_m^{*-} \wedge T), \pi_B(\tau_m^{*-} \wedge T))] .$$

By taking the limit for  $m \rightarrow \infty$ , as  $\tau_m^* \rightarrow \infty$  almost surely because  $p^*$  is admissible, we have

$$V(t, \pi_S, \pi_B) = E_{t, \pi_S, \pi_B} [V(T, \pi_S^*(T), \pi_B^*(T))] = E_{t, \pi_S, \pi_B} [U(L(\pi_S^*(T), \pi_B^*(T)))] \quad (5)$$

and by the definition of the value function the policy  $p^*$  is optimal.



**Fig. 1** Admissible Region ( $\mathcal{P}_h$ )

### 3 A numerical solution

To solve numerically our model problem, it is necessary to deal with a finite domain. Since the region  $\mathcal{P}$  is unbounded, besides the transaction costs, we assume that the agent must face other kinds of constraints, which have a natural economic meaning. More precisely, we define the bounded region  $\mathcal{P}_h \subset \mathcal{P}$  of admissible portfolios by

$$\mathcal{P}_h = \{(\pi_S, \pi_B) \in \mathbb{R}^2 : (0 \leq L(\pi_S, \pi_B) \leq L_{\max}) \cap (\pi_B \geq B_{\min}) \cap (\pi_S \geq S_{\min})\}.$$

Thus we introduce bounds  $B_{\min} < 0$  and  $S_{\min} < 0$  in the short position in the bank account and in the risky security, respectively. Moreover, we assume that our agent is fully satisfied if his/her portfolio reaches the threshold liquidation value  $L_{\max}$ , at a time  $t < T$ . In this case the portfolio will be liquidated in  $t$  and  $L_{\max}$  will be invested in the bank account up to the finite horizon  $T$ . The bounded region  $\mathcal{P}_h$  is depicted in Figure 1:  $\mathcal{P}_h$  consists of the trapezoidal domain ABCD and the segment OF. For computational purposes, we only consider the bounded domain ABCD. The length of segments OE and OF is  $K$ .

We consider a backward-in-time problem: as specified in the previous section, at the final time  $T$  it holds

$$V(T, \pi_S, \pi_B) = u(L(\pi_S, \pi_B)). \quad (6)$$

The value function under fixed ( $K$ ) transaction costs is fully described by the terminal condition problem (2-4), i.e.,

$$\begin{aligned} V(t, \pi_S, \pi_B) &\geq \mathcal{M}V(t, \pi_S, \pi_B) \\ \mathcal{L}V(t, \pi_S, \pi_B) &\leq 0 \\ (V(t, \pi_S, \pi_B) - \mathcal{M}V(t, \pi_S, \pi_B)) L_t V(t, \pi_S, \pi_B) &= 0 \end{aligned} \quad (7)$$

for any  $t \in [0, T)$  and  $(\pi_S, \pi_B) \in \mathcal{P}_h$ . The boundary conditions on the domain depicted in Figure 1 are chosen in order to solve this approximation of the problem stated in the previous section. We set:

- on edge CD:  $V(t, \pi_S, \pi_B) = u(L(\pi_S, \pi_B)e^{r(T-t)});$
- on edge BC and AD:  $V(t, \pi_S, \pi_B) = \mathcal{M}V(t, \pi_S, \pi_B);$
- on edge AB:  $V(t, \pi_S, \pi_B) = 0.$

The first boundary condition is due to the assumption on the threshold liquidation value  $L_{\max}$ : if the agent reaches this lever, he/she is satisfied and thus recalibrate its portfolio investing only in the risk-free asset. The second boundary condition is due to the  $B_{\min}$  and  $S_{\min}$  bounds, where the agent is obliged to transact. Finally, condition on AB is due to the bankruptcy that the agent faces if its liquidation value reach the zero level.

The above problem can be solved with a projected SOR method coupled with an iterative procedure *above the obstacle* (7). Beginning with a guest solution  $V_0(t, \pi_S, \pi_B)$ , one defines  $V_i(t, \pi_S, \pi_B)$ ,  $i \geq 1$ , as the solution of

$$\begin{aligned} V_i(t, \pi_S, \pi_B) &\geq \mathcal{M}V_{i-1}(t, \pi_S, \pi_B), \\ \mathcal{L}V_i(t, \pi_S, \pi_B) &\leq 0, \\ (V_i(t, \pi_S, \pi_B) - \mathcal{M}V_{i-1}(t, \pi_S, \pi_B)) \mathcal{L}V_i(t, \pi_S, \pi_B) &= 0 \end{aligned} \tag{8}$$

with boundary condition  $V_i(t, \pi_S, \pi_B) = \mathcal{M}V_{i-1}(t, \pi_S, \pi_B)$  on edges BC and AD. We consider as guest solution the function  $V_0(t, \pi_S, \pi_B)$  such that

$$\mathcal{L}V_0(t, \pi_S, \pi_B) = 0$$

for any  $T \in [0, T)$  and  $(\pi_S, \pi_B) \in \mathcal{P}_h$ , with boundary conditions:

- on edges BC, CD, AD:  $V(t, \pi_S, \pi_B) = u(L(\pi_S, \pi_B)e^{r(T-t)});$
- on edge AB:  $V(t, \pi_S, \pi_B) = 0.$

This is the solution when no transactions are permitted.

As shown, for example, in [Chancelier et al. 2002, Eastham and Hastings 1988], the solution  $V_i(t, \pi_S, \pi_B)$  can be interpreted as the solution when at most  $i$  transaction can be considered. Thus it holds

$$V_0(t, \pi_S, \pi_B) \leq V_1(t, \pi_S, \pi_B) \leq V_2(t, \pi_S, \pi_B) \leq \dots \leq V(t, \pi_S, \pi_B),$$

and, due to the finiteness of the number of transactions in the transaction cost framework, this implies that the sequence  $V_i$  converge. See Appendix A for the interpretation of the increasing sequence of variational inequalities.

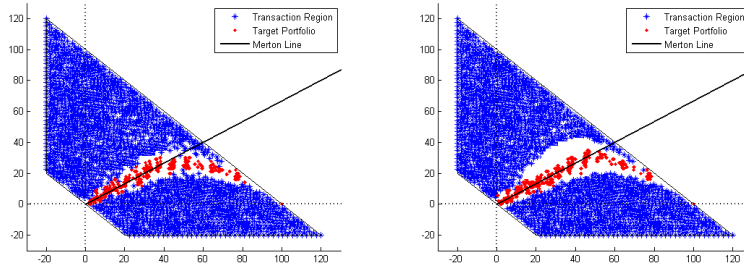
Each variational inequality (8) can be solved with a projected SOR method. We discretize the PDE  $\mathcal{L}V = 0$  considering a finite element technique with polynomial of degree 1, and a Crank-Nicholson scheme. For details on the implementation of the PSOR algorithm see, for example, [Wilmott et al. 1995].



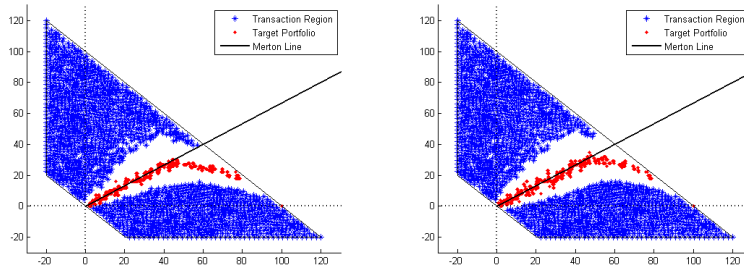
## 4 Numerical results

In this section we deal with the optimal investment strategy of an agent fully described by the utility function (1) with  $\gamma = 0.5$ . Moreover, we assume an interest rate  $r = 3\%$ , and a risky asset with drift  $\mu = 0.08$  and volatility  $\sigma = 0.5$ . The finite horizon is one year ( $T = 1$ ) and the bounds of the domain (see Figure 1) are  $B_{\min} = S_{\min} = -20$  and  $L_{\max} = 100$ . The finite element discretization is done with a mesh of 3000 points (approximately 5000 triangles) and a time grid of 50 steps. The numerical experiments are performed with Matlab R2011a.

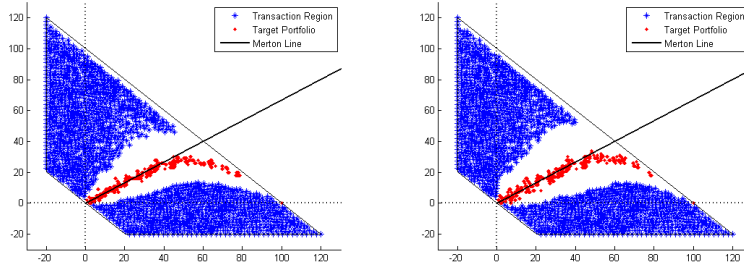
To analyze the influence of transaction costs, in Figures 2-5 we show the transaction region (in blue) as well as the target portfolios (in red), i.e., the portfolio where it is optimal to move when the agent portfolio falls into the transaction region. It is well known that, without transaction costs, the optimal policy is to transact continuously, moving to the Merton's line. Thus, if transaction costs are not faced the target portfolios belongs to the Merton's line, and the transaction region is the whole domain, with the exclusion of the Merton's line.



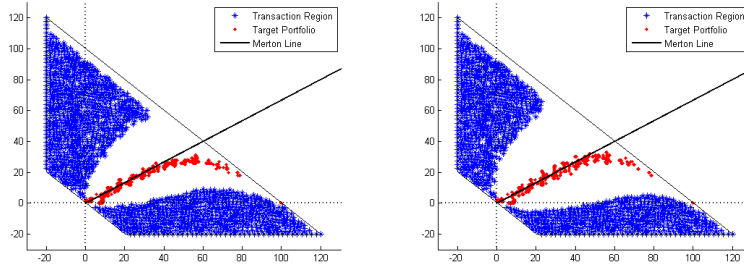
**Fig. 2** Transaction area in the plane  $(\pi_B, \pi_S)$ . Time  $t = 0$  (left) and  $t = 0.5$  (right).  $K = 0.01$ .



**Fig. 3** Transaction area in the plane  $(\pi_B, \pi_S)$ . Time  $t = 0$  (left) and  $t = 0.5$  (right).  $K = 0.05$ .



**Fig. 4** Transaction area in the plane  $(\pi_B, \pi_S)$ . Time  $t = 0$  (left) and  $t = 0.5$  (right).  $K = 0.1$ .



**Fig. 5** Transaction area in the plane  $(\pi_B, \pi_S)$ . Time  $t = 0$  (left) and  $t = 0.5$  (right).  $K = 0.25$ .

More precisely, in Figures 2-5 we consider different transaction costs ( $K = 0.01, 0.05, 0.1$  and  $0.25$ ) and we show transaction areas at time  $t = 0$  and  $t = 0.5$ . From these numerical experiments we notice that the target portfolios belong to the Merton's line, with few exceptions near the edge CD (see Figure 1), due to the boundedness of the domain and the boundary condition considered. The transaction region consists of two parts, and it seems nearly symmetric with respect to the Merton's line. The shape of the continuation region (white) is similar to a cone, enlarging as time increases. We conjecture that this will be the exact shape if we considered the same problem with an unbounded liquidation region.

Moreover, as expected, transaction costs strongly influence the optimal strategies. The transaction region, in fact, decreases as the transaction cost  $K$  increases. Moreover, it also decreases as time increases: this happens because, as the time to maturity  $T - t$  decreases, only a large change in the portfolio composition can compensate the transaction costs.

## References

- [Chancelier et al. 2002] Chancelier J, Oksendal B., Sulem A. (2002) Combined Stochastic Control and Optimal Stopping, and Application to Numerical Approximation of Combined Stochastic and Impulse Control, Proceedings of the Steklov Institute of Mathematics, 237: 140-163.
- [Davis and Norman 1990] Davis M., Norman A. (1990) Portfolio selection with transaction costs, Mathem Operations Research 15: 676-713.
- [Dumas and Luciano 1991] Dumas B, Luciano E (1991) An exact solution to a dynamic portfolio choice problem, J Finance, 46: 577-595.
- [Eastham and Hastings 1988] Eastham JE, Hastings KJ (1988) Optimal impulse control of portfolios, Mathem Operations Research 13, 588-605.
- [Korn 1998] Korn R. (1998) Portfolio optimization with strictly positive transaction costs and impulse control. Financ Stoch 2: 85-114
- [Liu and Loewenstein 2002] Liu H, Loewenstein M (2002) Optimal Portfolio Selection with Transaction Costs and Finite Horizon, Rev Financ Stud, 15-3: 805-835:
- [Oksendal and Sulem 2002] Oksendal B, Sulem A (2002) Optimal Consumption and Portfolio with both Fixed and Proportional Costs, SIAM J Control Optim 40-6: 1765-1790.
- [Shreve and Soner 1994] Shreve S, Soner H (1994) Optimal investment and consumption with transaction costs, Ann Appl Probab, 4: 609-692.
- [Wilmott et al. 1995] Wilmott P, Howison S, Dewynne J (1995) The Mathematics of Financial Derivatives: A Student Introduction, Cambridge University Press.

## A Interpretation of the increasing sequence of variational inequalities

We denote by  $A^n(t, \pi_S, \pi_B)$  the set of admissible policies when at most  $n$  interventions are admitted before the final time  $T$ . A policy  $p = (\tau_i, \xi_i) \in A^n(t, \pi_S, \pi_B)$  if  $p \in A(t, \pi_S, \pi_B)$  and  $\tau_{n+1} = +\infty$  almost surely. We introduce the value function  $V^n$  when the set of admissible policies is restricted to  $A^n(t, \pi_S, \pi_B)$ :

$$V^n(t, \pi_S, \pi_B) := \sup_{p \in A^n(t, \pi_S, \pi_B)} E_{t, \pi_S, \pi_B} [V(\pi_S^p(T), \pi_B^p(T))] = \sup_{p \in A^n(t, \pi_S, \pi_B)} J(p)$$

where  $(\pi_S^p(t), \pi_B^p(t))$  is the process controlled by policy  $p$ , and  $J(p)$  the corresponding objective value.

We define  $V_0$  as the solution of the partial differential equation (with the other appropriate boundary conditions):

$$\begin{cases} \mathcal{L}V_0(t, \pi_S, \pi_B) = 0 \\ V_0(T, \pi_S, \pi_B) = u(L(\pi_S, \pi_B)) \end{cases} .$$

Consider now the sequence of variational inequalities ( $j = 1, \dots, n$ ):

$$\begin{cases} \mathcal{L}V_j \leq 0 \\ V_j \geq \mathcal{M}V_{j-1} \\ (V_j - \mathcal{M}V_{j-1})\mathcal{L}V_j = 0 \\ V_j(T, \pi_S, \pi_B) = u(L(\pi_S, \pi_B)) \end{cases}$$

starting from  $j = 1$ . We show in a heuristic way that  $V_n = V^n$ , that is the solution of the  $n$ -th variational inequality is the value function when at most  $n$  interventions are admitted.

Let  $p \in A^n(t, \pi_S, \pi_B)$  and define  $\hat{\tau}_i^p = \tau_i^p \wedge T$ . By Ito's formula applied to  $V_n$  on the interval  $(t, \hat{\tau}_1)$ , taking expectations and recalling that  $\mathcal{L}V_n \leq 0$  (and assuming that the expectation of the stochastic integral vanishes), we obtain

$$V_n(t, \pi_S, \pi_B) \geq E_{t, \pi_S, \pi_B} [V_n(\hat{\tau}_1, \pi_S(\hat{\tau}_1^-), \pi_B(\hat{\tau}_1^-))] .$$

Moreover since  $V_n \geq \mathcal{M}V_{n-1}$  we also have

$$\begin{aligned} & V_n(t, \pi_S, \pi_B) \\ & \geq E_{t, \pi_S, \pi_B} [V_n(\hat{\tau}_1, \pi_S(\hat{\tau}_1^-), \pi_B(\hat{\tau}_1^-))]_{\mathcal{X}_{T < \tau_1^p}} + E_{t, \pi_S, \pi_B} [V_n(\hat{\tau}_1, \pi_S(\hat{\tau}_1^-), \pi_B(\hat{\tau}_1^-))]_{\mathcal{X}_{T \geq \tau_1^p}} \\ & \geq E_{t, \pi_S, \pi_B} [V_n(\hat{\tau}_1, \pi_S(\hat{\tau}_1^-), \pi_B(\hat{\tau}_1^-))]_{\mathcal{X}_{T < \tau_1^p}} \\ & \quad + E_{t, \pi_S, \pi_B} [V_{n-1}(\hat{\tau}_1, \pi_S(\hat{\tau}_1^-) + \xi_1, \pi_B(\hat{\tau}_1^-) - \xi_1 - K)]_{\mathcal{X}_{T \geq \tau_1^p}} \\ & = E_{t, \pi_S, \pi_B} [V_{n-1}(\hat{\tau}_1, \pi_S(\hat{\tau}_1), \pi_B(\hat{\tau}_1))] . \end{aligned}$$

Repeating the same reasoning we obtain ( $j = 1, \dots, n-1$ )

$$E_{t, \pi_S, \pi_B} [V_{n-j}(\hat{\tau}_j, \pi_S(\hat{\tau}_j), \pi_B(\hat{\tau}_j))] \geq E_{t, \pi_S, \pi_B} [V_{n-j-1}(\hat{\tau}_{j-1}, \pi_S(\hat{\tau}_{j-1}), \pi_B(\hat{\tau}_{j-1}))] .$$

Summing up these inequalities we end with

$$V_n(t, \pi_S, \pi_B) \geq E_{t, \pi_S, \pi_B} [V_0(\hat{\tau}_n, \pi_S(\hat{\tau}_n), \pi_B(\hat{\tau}_n))] .$$

Furthermore we have

$$\begin{aligned} E_{t, \pi_S, \pi_B} [V_0(\hat{\tau}_n, \pi_S(\hat{\tau}_n), \pi_B(\hat{\tau}_n))] & \geq E_{t, \pi_S, \pi_B} [V_0(T, \pi_S^p(T), \pi_B^p(T))] \\ & = E_{t, \pi_S, \pi_B} [u(L(\pi_S^p(T), \pi_B^p(T)))] = J(p) . \end{aligned}$$

Therefore we have shown that

$$V_n(t, \pi_S, \pi_B) \geq J(p), \quad \forall p \in A^n(t, \pi_S, \pi_B) .$$

Now we show that there exists  $p^* \in A^n(t, \pi_S, \pi_B)$  such that  $V_n(t, \pi_S, \pi_B) = J(p^*)$  and  $V_n \equiv V^n$ , the value function with at most  $n$  interventions. By  $A^i$ ,  $i = 1, \dots, n$ , we define the set

$$A^i := \{(t, \pi_S, \pi_B) : V_{n+1-i} = \mathcal{M}V^{n-i}\} .$$

We consider the policy  $p^*$  given recursively by ( $\tau_0^* \equiv t$ ,  $i = 1, \dots, n$ )

$$p^* = \begin{cases} \tau_i^* = \begin{cases} \inf \left\{ I_i \equiv \left\{ T \geq t \geq \tau_{i-1}^* : V_{n+1-i}(\tau_i^*, \pi_S(\tau_i^{*-}), \pi_B(\tau_i^{*-})) \in A^i \right\} \right\} & \text{if } I_i \neq \emptyset \\ +\infty & \text{if } I_i = \emptyset \end{cases} \\ \xi_i^* = \begin{cases} \arg \max_{\xi \in \mathbb{R}} V_{n-i}(\tau_i^*, \pi_S(\tau_i^{*-}) + \xi, \pi_B(\tau_i^{*-}) - \xi - K) & \text{if } \tau_i^* < +\infty \\ \text{arbitrary} & \text{if } \tau_i^* = +\infty \end{cases} \end{cases} .$$

It is not difficult to see that using this policy the above inequalities become equalities and we have

$$V^n(t, \pi_S, \pi_B) = J(p^*) = V_n(t, \pi_S, \pi_B) .$$