DIPARTIMENTO DI MATEMATICA
"Francesco Brioschi"
POLITECNICO DI MILANO

Conditions at infinity for the inhomogeneous filtration equation

Grillo, G.; Muratori, M.; Punzo, F.

Collezione dei *Quaderni di Dipartimento*, numero **QDD 133**
Inserito negli *Archivi Digitali di Dipartimento* in data 3-10-2012

Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

Conditions at infinity for the inhomogeneous filtration equation

Gabriele Grillo[*], Matteo Muratori[†] and Fabio Punzo[‡]

Abstract

We investigate existence and uniqueness of solutions to the filtration equation with an inhomogeneous density in $\mathbb{R}^N$, approaching at infinity a given continuous datum of Dirichlet type.

1 Introduction

We provide sufficient conditions for existence and uniqueness of *bounded* solutions to the following nonlinear Cauchy problem (given $T > 0$):

$$\begin{cases} \rho\, \partial_t u = \Delta\big[G(u)\big] & \text{in } I\!R^N \times (0,T] =: S_T \\ u = u_0 & \text{in } I\!R^N \times \{0\}\,. \end{cases} \tag{1.1}$$

Concerning the density ρ, the initial condition u_0 and the nonlinear function G we shall assume the following:

$$(H_0) \qquad \begin{cases} (i) & \rho \in C(I\!R^N)\,,\ \rho > 0\,; \\ (ii) & G \in C^1(I\!R),\ G(0) = 0,\ G'(s) > 0 \text{ for any } s \in I\!R \setminus \{0\}\,, \\ & G' \text{ decreasing in } (-\delta, 0) \text{ and increasing in } (0, \delta), \\ & \text{if } G'(0) = 0\ (\delta > 0); \\ (iii) & u_0 \in L^\infty(I\!R^N) \cap C(I\!R^N)\,. \end{cases}$$

A typical choice for the function G is $G(u) = |u|^{m-1}u$ for some $m \geq 1$. In this case, for $m > 1$, the differential equation in problem (1.1) becomes the *inhomogeneous porous media equation*, which arises in various situations of physical interest. We quote, without any claim of generality, the papers [13], [14], [6], [21], [4], [5], [22], [23], [16]-[20] [11], [8], [9], as references on this topic, and the recent monograph [24] as a general reference on the porous media equation.

As it is well-known, if assumption (H_0) is satisfied, then there exists a bounded solution of problem (1.1) (see, *e.g.*, [14], [7], [21]). Moreover, if $N = 1$ or $N = 2$, and $\rho \in L^\infty(I\!R^N)$, then the solution of problem (1.1) is unique (see [10]).

[*]Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy (gabriele.grillo@polimi.it).

[†]Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy (matteo1.muratori@mail.polimi.it).

[‡]Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", Piazzale Aldo Moro 5, 00185 Roma, Italy (punzo@mat.uniroma1.it).

When $N \geq 3$, we can have uniqueness or nonuniqueness of bounded solutions to problem (1.1), in dependence of the behavior *at infinity* of the density ρ. In fact, given $R > 0$, set $B_R := \{x \in I\!R^N \,\big|\, |x| < R\}$. If

$$(H_1) \qquad \begin{cases} \text{there exist } \widehat{R} > 0 \text{ and } \underline{\rho} \in C([\widehat{R}, \infty)) \text{ such that} \\ (i) \quad \rho(x) \geq \underline{\rho}(|x|) > 0 \text{ for any } x \in I\!R^N \setminus B_{\widehat{R}}, \text{ and} \\ (ii) \quad \int_{\widehat{R}}^{\infty} \eta \underline{\rho}(\eta)\, d\eta \,=\, \infty\,, \end{cases}$$

then problem (1.1) admits at most one bounded solution (see [16], [20]). A natural choice in (H_1) is $\underline{\rho}(\eta) := \eta^{-\alpha}$ $\;(\eta \in [\widehat{R}, \infty))$ for some $\alpha \in (-\infty, 2]$ and $\widehat{R} > 0$.

On the contrary, if ρ satisfies the condition

$$(H_2) \qquad \Gamma * \rho \in L^{\infty}(I\!R^N)\,,$$

where Γ is the fundamental solution of the Laplace equation in $I\!R^N$, then nonuniqueness prevails (see [16], [20]; see also [12] for the case $G(u) = u$). To be specific, for any function $A \in Lip([0, T])$ with $A(0) = 0$ there exists a solution u of problem (1.1) such that

$$\lim_{R \to \infty} \frac{1}{|\partial B_R|} \int_{\partial B_R} \big| U(x, t) - A(t) \big|\, d\sigma \,=\, 0 \tag{1.2}$$

uniformly with respect to $t \in [0, T]$, where

$$U(x, t) := \int_0^t G(u(x, \tau))\, d\tau \quad \big((x, t) \in S_T\big)\,. \tag{1.3}$$

If assumption (H_2) is replaced by the stronger condition

$$(H_3) \qquad \begin{cases} \textit{there exist } \widehat{R} > 0 \textit{ and } \overline{\rho} \in C([\widehat{R}, \infty)) \textit{ such that} \\ (i) \quad \rho(x) \leq \overline{\rho}(|x|) \;\; \textit{for any } x \in I\!R^N \setminus B_{\widehat{R}}, \textit{ and} \\ (ii) \quad \int_{\widehat{R}}^{\infty} \eta \overline{\rho}(\eta)\, d\eta \,<\, \infty\,, \end{cases}$$

then, instead of (1.2), we can impose that

$$\lim_{|x| \to \infty} \big| U(x, t) - A(t) \big| \,=\, 0 \tag{1.4}$$

uniformly with respect to $t \in [0, T]$, with U defined in (1.3).

A natural choice in (H_3) is $\overline{\rho}(\eta) := \eta^{-\alpha}$ $\;(\eta \in [\widehat{R}, \infty))$ for some $\alpha \in (2, \infty]$ and $\widehat{R} > 0$.

Observe that equalities (1.2) and (1.4) can be also regarded as nonhomogeneous Dirichlet conditions at infinity.

It is natural to study whether imposing conditions at infinity restores uniqueness of solutions. In this direction, it was only known that there exists at most one solution $u \in L^{\infty}(S_T)$ to problem (1.1) satisfying condition (1.2) or (1.4) either when $G(u) = u$ (see [12]) or when $u_0 \geq 0$ and $A \equiv 0$ (see [7]). Note that the methods used to obtain such uniqueness results did not work for general G and A.

In this paper we shall address existence and uniqueness of bounded solutions to problem (1.1) satisfying at infinity suitable nonhomogeneous Dirichlet conditions. More precisely, at first we shall prove that if assumptions (H_3) and

$$(H_0)^* \qquad \begin{cases} (i) \quad \rho \in C(I\!R^N)\,,\; \rho > 0\,; \\ (ii) \quad G \in C^1(I\!R),\; G(0) = 0,\; G'(s) \geq \alpha_0 > 0 \text{ for any } s \in I\!R\,; \\ (iii) \quad u_0 \in C(I\!R^N),\; \lim_{|x| \to \infty} u_0(x) \text{ exists and is finite} \end{cases}$$

are satisfied, then for any $a \in C([0, T])$ with

$$a(0) = \lim_{|x| \to \infty} u_0(x) \tag{1.5}$$

there exists a bounded solution u to problem (1.1) satisfying

$$\lim_{|x|\to\infty,\, t\to t_0} u(x,t) = a(t_0) \quad \text{for any } \; t_0 \in [0,T] \tag{1.6}$$

(see Theorem 2.2). Observe that hypothesis $(H_0)^*$ is stronger than (H_0), since it amounts to considering nondegenerate nonlinearities G. Furthermore, we can remove assumption $(H_0)^*$ for suitable classes of initial data u_0 and conditions at infinity a. Indeed, if $a_0 := \lim_{|x|\to\infty} u_0(x)$ exists finite, and (H_0) holds true, then there exists a bounded solution to problem (1.1) such that

$$\lim_{|x|\to\infty} u(x,t) = a_0 \quad \text{uniformly for } t \in [0,T] \tag{1.7}$$

(see Theorem 2.3). Note that in this case assumption (H_3) is not required (see Remark 2.4).

Moreover, if $(H_0), (H_3)$ hold true, then there exists a bounded solution u to problem (1.1) satisfying (1.6), for any $a \in C([0,T])$ with $a > 0$ in $[0,T]$, provided u_0 satisfies (1.5) (see Theorem 2.5).

Finally, we shall prove that condition (1.6) implies uniqueness, for general G satisfying $(H_0)(ii)$ and $a \in C([0,T])$ (see Theorem 2.8).

2 Existence and uniqueness results

Solutions, sub- and supersolutions of problem (1.1) are always meant in the following sense.

Definition 2.1 *By a* solution *of problem* (1.1) *we mean a function* $u \in C(S_T) \cap L^\infty(S_T)$ *such that*

$$\begin{aligned}\int_0^\tau \int_{\Omega_1} \{\rho\, u\, \partial_t \psi + G(u)\Delta\psi\}\, dxdt = \int_{\Omega_1} \rho \left[u(x,\tau)\psi(x,\tau) - u_0(x)\psi(x,0)\right] dx + \\ + \int_0^\tau \int_{\partial\Omega_1} G(u)\langle \nabla\psi,\, \nu\rangle d\sigma\, dt\end{aligned} \tag{2.1}$$

for any bounded open set $\Omega_1 \subseteq I\!R^N$ *with smooth boundary* $\partial\Omega_1$, $\tau \in (0,T]$, $\psi \in C^{2,1}(\overline{\Omega_1} \times [0,\tau]), \psi \geq 0, \psi = 0$ *in* $\partial\Omega_1 \times [0,\tau]$*; here* ν *denotes the outer normal to* Ω_1 *and* $\langle\cdot,\cdot\rangle$ *the scalar product in* $I\!R^N$.

Supersolutions (subsolutions) *of* (1.1) *are defined replacing* " $=$ " *by* " $\leq$ " (" $\geq$ ", *respectively) in* (2.1).

These kind of solutions are sometimes referred to as *very weak* solutions. Observe that, according to Definition 2.1, solutions of problem (1.1) we deal with are *bounded* in S_T.

2.1 Existence

In the case of nondegenerate nonlinearities, we have the following result.

Theorem 2.2 *Let* $N \geq 3$. *Let assumptions* $(H_0)^*, (H_3)$ *be satisfied. Let* $a \in C([0,T])$ *and suppose that* (1.5) *holds true. Then there exists a solution to problem* (1.1) *satisfying condition* (1.6).

For appropriate classes of data and possibly degenerate nonlinearities of porous media type, we shall prove the following two results.

Theorem 2.3 *Let* $N \geq 3$. *Let assumption* (H_0) *be satisfied. Suppose that*

$$\lim_{|x|\to\infty} u_0(x) \, = \, a_0 \tag{2.2}$$

for some $a_0 \in I\!R$. *Then there exists a solution to problem* (1.1) *satisfying condition* (1.7).

Remark 2.4 *Let assumptions* $(H_0), (H_1)$ *be satisfied and suppose that* (2.2) *holds. Then by the uniqueness result recalled in the Introduction, and by Theorem 2.3, the* unique *solution to problem* (1.1) *necessarily satisfies condition* (1.7).

Theorem 2.5 *Let* $N \geq 3$. *Let assumptions* $(H_0), (H_3)$ *be satisfied. Let* $a \in C([0,T])$, *with* $\min_{t\in[0,T]} a(t) > 0$. *Suppose that*

$$\lim_{|x|\to\infty} u_0(x) = a(0)\,. \tag{2.3}$$

Then there exists a solution to problem (1.1) *which satisfies condition* (1.6).

Remark 2.6 *Note that the hypotheses made in Theorem 2.5 allow to assume as initial data functions* u_0 *which may be nonpositive in some compact subset* $K \subset I\!R^N$.

Remark 2.7 *As it can be easily seen by the forthcoming proofs, in Theorems 2.2 and 2.5, instead of* (1.6), *we can impose*

$$\lim_{|x|\to\infty} u(x,t) \;=\; a(t) \;\; \textit{uniformly for } t \in [0,T]. \tag{2.4}$$

2.2 Uniqueness

We shall prove the following uniqueness result in the general case of possibly degenerate nonlinearities.

Theorem 2.8 *Let* $N \geq 3$. *Let assumption* (H_0) *be satisfied, and suppose that* $a \in C([0,T])$, $\rho \in L^\infty(I\!R^N)$. *Then there exists at most one solution to problem* (1.1) *satisfying condition* (1.6).

From Theorems 2.3 and 2.8 we deduce the following.

Corollary 2.9 *Let* $N \geq 3$. *Let assumption* (H_0) *be satisfied, and suppose that* $\rho \in L^\infty(I\!R^N)$. *Then there exists a unique solution to problem* (1.1) *satisfying condition* (1.7).

Remark 2.10 *When* (H_0) *and* (H_1) *are fulfilled, then the conclusion of Corollary 2.9 is in agreement with Remark 2.4.*

From Theorems 2.5 and 2.8 we get next.

Corollary 2.11 *Let the assumptions of Theorems 2.5 and 2.8 be satisfied. Then there exists a unique solution to problem* (1.1) *satisfying condition* (2.4).

Finally, in the case of *nondegenerate* nonlinearities, from Theorems 2.2 and 2.8 we also deduce the following.

Corollary 2.12 *Let* $N \geq 3$. *Let assumptions* $(H_0)^*$ *and* (H_3) *be satisfied, and suppose that* $a \in C([0,T])$, $\rho \in L^\infty(I\!R^N)$. *Then there exists a unique solution to problem* (1.1) *satisfying condition* (2.4).

3 Existence: proofs

In view of (H_3) (see [16]) there exists a function $V = V(|x|) \in C^2(I\!R^N \setminus B_{\widehat{R}})$ (let $\widehat{R} > 0$) such that

$$\Delta V \;\leq\; -\rho \quad \text{in} \;\; I\!R^N \setminus \overline{B}_{\widehat{R}}\,, \tag{3.1}$$

$$V(|x|) \;>\; 0 \quad \text{for all} \;\; x \in I\!R^N \setminus B_{\widehat{R}}\,, \tag{3.2}$$

$$|x| \mapsto V(|x|) \;\; \text{is nonincreasing}\,,$$

$$\lim_{|x|\to\infty} V(x) \,=\, 0\,. \tag{3.3}$$

In fact V is a supersolution of $\Delta V = -\rho$ in the whole of $I\!R^N$, but we shall need it only outside a fixed ball.

In some of the following proofs we shall make use of the function G^{-1}, whose domain D need not coincide with $\mathbb{R}$. Since we are dealing with bounded data u_0 (and, by the maximum principle, with bounded solutions), this makes no problem since one can modify the definition of $G(x)$ for $|x|$ large so that such a function is a bijection from $\mathbb{R}$ to itself, without changing the evolution of u_0.

Hereafter, for any $j \in I\!N$, ζ_j will always be a function with the following properties: $\zeta_j \in C^\infty_c(B_j)$ with $0 \le \zeta_j \le 1, \zeta_j \equiv 1$ in $B_{j/2}$.

Proof of Theorem 2.2. Since $a \in C([0,T])$ and $G \in C^1(I\!R)$ is increasing, for any $t_0 \in [0,T], \sigma > 0$ there exists $\delta = \delta(\sigma) > 0$ such that

$$G^{-1}\big[G(a(t_0)) - \sigma\big] \,\le\, a(t) \,\le G^{-1}\big[G(a(t_0)) + \sigma\big] \quad \text{for all } \ t \in [\underline{t}_\delta, \overline{t}_\delta]\,, \tag{3.4}$$

where $\underline{t}_\delta := \max\{t_0 - \delta, 0\}, \overline{t}_\delta := \min\{t_0 + \delta, T\}$. Moreover, in view of $(H_0)^*(iii)$, for any $\sigma > 0$ there exists $R = R(\sigma) > \widehat{R}$ such that

$$G^{-1}\big[G(a(0)) - \sigma\big] \,\le\, u_0(x) \,\le G^{-1}\big[G(a(0)) + \sigma\big] \quad \text{for all } \ x \in I\!R^N \setminus B_R\,. \tag{3.5}$$

For any $j \in I\!N$ let $u_j \in C\big(\overline{B}_j \times [0,T]\big)$ be the unique solution (see, *e.g.*, [15]) to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } B_j \times (0,T] \\ u \,=\, a(t) & \text{in } \partial B_j \times (0,T) \\ u \,=\, u_{0,j} & \text{in } B_j \times \{0\}\,, \end{cases} \tag{3.6}$$

where

$$u_{0,j} := \zeta_j u_0 + (1-\zeta_j)a(0) \quad \text{in } \ \overline{B}_j\,.$$

By comparison principles,

$$|u_j| \le K := \max\{\|u_0\|_\infty, \|a\|_\infty\} \quad \text{in } \ B_j \times (0,T)\,. \tag{3.7}$$

By means of usual compactness arguments (see, *e.g.*, [15]), there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$ which converges, as $k \to \infty$, locally uniformly in $I\!R^N \times (0,T)$ to a solution u of problem (1.1).

Let $t_0 \in [0,T]$. Define

$$\underline{w}(x,t) := G^{-1}\big[-MV(x) - \sigma + G(a(t_0)) - \lambda(t-t_0)^2\big] \ \ \big(x \in I\!R^N \setminus B_{\widehat{R}}, t \in [\underline{t}_\delta, \overline{t}_\delta]\big)\,,$$

where $M > 0$ and $\lambda > 0$ are constants to be chosen later. By $(H_0)^*(ii)$ and (3.1),

$$\begin{aligned} &\rho(x)\partial_t w - \Delta[G(\underline{w})] = -\rho(x)\frac{2\lambda(t-t_0)}{G'(w)} + M\Delta V \le \\ &\le \rho(x)\left(\frac{2\lambda\delta}{\alpha_0} - M\right) \le 0 \quad \text{in } \ [I\!R^N \setminus B_{\widehat{R}}] \times (\underline{t}_\delta, \overline{t}_\delta) \end{aligned} \tag{3.8}$$

provided that

$$M \ge \frac{2\lambda\delta}{\alpha_0}\,. \tag{3.9}$$

For any $j \in \mathbb{N}, j > R$ put, R being as in (3.5),

$$N_{R,j} := B_j \setminus B_R \,.$$

We have

$$\underline{w}(x,t) \leq -K \quad \text{for all } (x,t) \in \partial B_R \times (\underline{t}_\delta, \bar{t}_\delta), \tag{3.10}$$

provided

$$M \geq \frac{G(\|a\|_\infty) - G(-K)}{V(R)} \,. \tag{3.11}$$

Furthermore,

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \sigma\big] \quad \text{for all } (x,t) \in \partial B_j \times (\underline{t}_\delta, \bar{t}_\delta). \tag{3.12}$$

When $\underline{t}_\delta = 0$ there holds

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \sigma\big] \quad \text{for all } (x,t) \in N_{R,j} \times \{\underline{t}_\delta\} \,, \tag{3.13}$$

whereas, when $\underline{t}_\delta > 0$, we have

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \lambda\delta^2\big] \leq -K \quad \text{for all } (x,t) \in N_{R,j} \times \{\underline{t}_\delta\} \,, \tag{3.14}$$

provided

$$\lambda \geq \frac{G(\|a\|_\infty) - G(-K)}{\delta^2} \,. \tag{3.15}$$

Suppose that conditions (3.9), (3.11) and (3.15) are satisfied. Hence, from (3.8), (3.10), (3.12)-(3.14) we can infer that $\underline{w}$ is a subsolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \bar{t}_\delta) \\ u = -K & \text{in } \partial B_R \times (\underline{t}_\delta, \bar{t}_\delta) \\ u = G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{in } \partial B_j \times (\underline{t}_\delta, \bar{t}_\delta) \\ u = -K & \text{in } N_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.16}$$

when $\underline{t}_\delta > 0$, whereas it is a subsolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \bar{t}_\delta) \\ u = -K & \text{in } \partial B_R \times (\underline{t}_\delta, \bar{t}_\delta) \\ u = G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{in } \partial B_j \times (\underline{t}_\delta, \bar{t}_\delta) \\ u = G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{in } N_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.17}$$

when $\underline{t}_\delta = 0$.

On the other hand, (3.4), (3.5), (3.7) show that the boundary data for (3.6) and (3.16),(3.17) are correctly ordered on each part of the parabolic boundary, so that we deduce that u_j is a supersolution to problem (3.16) when $\underline{t}_\delta > 0$, while it is a supersolution to problem (3.17) when $\underline{t}_\delta = 0$.

By comparison principles,

$$\underline{w} \leq u_j \quad \text{in } N_{R,j} \times (\underline{t}_\delta, \bar{t}_\delta) \,. \tag{3.18}$$

Define

$$\overline{w}(x,t) := G^{-1}\big[\widetilde{M}V(x) + \sigma + G(a(t_0)) + \widetilde{\lambda}(t-t_0)^2\big] \quad \big(x \in I\!R^N \setminus B_{\widehat{R}}, t \in [\underline{t}_\delta, \overline{t}_\delta]\big)\,,$$

with

$$\widetilde{M} \geq \max\left\{\frac{2\widetilde{\lambda}\delta}{\alpha_0}, \frac{G(K) - G(-\|a\|_\infty)}{V(R)}\right\},$$

and

$$\widetilde{\lambda} \geq \frac{G(K) - G(-\|a\|_\infty)}{\delta^2}\,.$$

By analogous arguments to those used above, we can infer that $\overline{w}$ is a supersolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, K & \text{in } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) + \sigma\big] & \text{in } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,= K & \text{in } N_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.19}$$

when $\underline{t}_\delta > 0$, whereas it is a subsolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, K & \text{in } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) + \sigma\big] & \text{in } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,= G^{-1}\big[G(a(t_0)) + \sigma\big] & \text{in } N_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.20}$$

when $\underline{t}_\delta = 0$.

On the other hand, from (3.4), (3.5), (3.7) we deduce as before that u_j is a subsolution to problem (3.19) when $\underline{t}_\delta > 0$, while it is a subsolution to problem (3.20) when $\underline{t}_\delta = 0$. By comparison principles,

$$u_j \leq \overline{w} \quad \text{in } \; N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.21}$$

From (3.18) and (3.21) with $j = j_k$, sending $k \to \infty$, we obtain:

$$\underline{w} \leq u \leq \overline{w} \quad \text{in } \; B_R^c \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.22}$$

By (3.3), letting $|x| \to \infty$ and $t \to t_0$ in (3.22) we get

$$G^{-1}\big[G(a(t_0)) - \sigma\big] \leq \liminf_{|x|\to\infty, t\to t_0} u(x,t) \leq \limsup_{|x|\to\infty, t\to t_0} u(x,t) \leq G^{-1}\big[G(a(t_0)) + \sigma\big]\,.$$

Letting $\sigma \to 0^+$ we have (1.6): this completes the proof. □

Proof of Theorem 2.3. As in the proof of the previous result note that, in view of (2.2), for any $\sigma > 0$ there exists $R = R(\sigma) > 0$ such that

$$G^{-1}\big[G(a_0) - \sigma\big] \leq u_0(x) \leq G^{-1}\big[G(a_0) + \sigma\big] \quad \text{for all } \; x \in I\!R^N \setminus B_R\,. \tag{3.23}$$

In view of assumption (H_0), by standard results (see, *e.g.*, [1]), for any $j \in I\!N$ there exists a unique solution u_j to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } B_j \times (0,T] \\ u \,=\, a_0 & \text{in } \partial B_j \times (0,T) \\ u \,= u_{0,j} & \text{in } B_j \times \{0\}\,, \end{cases} \tag{3.24}$$

where

$$u_{0,j} := \zeta_j u_0 + (1-\zeta_j)a_0 \quad \text{in } \overline{B}_j \,.$$

Note that, by results in [3], $u_j \in C\big(\overline{B}_j \times [0,T]\big)$. By comparison principles,

$$|u_j| \le K := \max\{\|u_0\|_\infty, |a_0\|\} \quad \text{in } B_j \times (0,T)\,. \tag{3.25}$$

By usual compactness techniques (one can use [2, Lemma 5.2] and a diagonal argument), there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$ which converges, as $k \to \infty$, locally uniformly in $I\!R^N \times (0,T)$ to a solution u of problem (1.1).

Let

$$\Gamma(x) \equiv \Gamma(|x|) := |x|^{2-N}\,, \quad x \in I\!R^N \setminus \{0\}\,.$$

Clearly,

$$\Delta\Gamma \,=\, 0 \quad \text{in } I\!R^N \setminus \{0\}\,, \tag{3.26}$$

$$\Gamma \,>\, 0 \quad \text{in } I\!R^N \setminus \{0\}\,, \tag{3.27}$$

$$\lim_{|x|\to\infty} \Gamma(|x|) \,=\, 0\,. \tag{3.28}$$

Define

$$\overline{W}(x) := G^{-1}\big[M\Gamma(x) + \sigma + G(a_0)\big], \quad x \in I\!R^N \setminus \{0\}\,,$$

where

$$M \ge \frac{G(K) - G(a_0)}{V(R)}\,. \tag{3.29}$$

Then

$$\Delta[G(\overline{W})] \,=\, 0 \quad \text{in } I\!R^N \setminus \{0\}\,. \tag{3.30}$$

In view of (3.29) there holds

$$\overline{W}(x) \ge K \quad \text{for all } x \in \partial B_R\,. \tag{3.31}$$

Furthermore, we have

$$\overline{W}(x) \ge a_0 \quad \text{for all } x \in \partial B_j \tag{3.32}$$

and

$$\overline{W}(x) \ge G^{-1}\big[G(a_0) + \sigma\big] \quad \text{for all } x \in N_{R,j}\,. \tag{3.33}$$

From (3.30)-(3.33) it follows that $\overline{W}$ is a supersolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (0,T) \\ u \,=\, K & \text{in } \partial B_R \times (0,T) \\ u \,=\, a_0 & \text{in } \partial B_j \times (0,T) \\ u \,=\, G^{-1}\big[G(a_0) + \sigma\big] & \text{in } N_{R,j} \times \{0\}\,. \end{cases} \tag{3.34}$$

On the other hand, u_j is a subsolution to problem (3.34). Hence, by comparison principles,

$$u_j \le \overline{W} \quad \text{in } N_{R,j} \times (0,T)\,. \tag{3.35}$$

Define

$$\underline{W}(x) := G^{-1}\big[-\widehat{M}\Gamma(x) - \sigma + G(a_0)\big] \quad x \in I\!R^N \setminus \{0\}\,,$$

where

$$\widehat{M} \ge \frac{G(a_0) - G(-K)}{\Gamma(R)}\,. \tag{3.36}$$

By similar arguments to those used above we can infer that $\underline{W}$ is a subsolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j}\times(0,T) \\ u\;=\;-K & \text{in } \partial B_R\times(0,T) \\ u\;=\;a_0 & \text{in } \partial B_j\times(0,T) \\ u\;=G^{-1}\big[G(a_0)-\sigma\big] & \text{in } N_{R,j}\times\{0\}\,. \end{cases} \tag{3.37}$$

On the other side, u_j is a supersolution to problem (3.37). Hence, by comparison principles,

$$u_j \le \underline{W} \quad \text{in } \; N_{R,j}\times(0,T)\,. \tag{3.38}$$

From (3.35) and (3.38) with $j=j_k$, sending $k\to\infty$, we obtain

$$\underline{W} \le u \le \overline{W} \quad \text{in } \; [I\!R^N\setminus B_R]\times(0,T)\,. \tag{3.39}$$

Letting $|x|\to\infty$ in (3.39), from (3.28) we obtain, for any $t\in[0,T]$,

$$G^{-1}\big[G(a_0)-\sigma\big] \le \liminf_{|x|\to\infty} u(x,t) \le \limsup_{|x|\to\infty} u(x,t) \le G^{-1}\big[G(a_0)+\sigma\big]\,.$$

Letting $\sigma\to0^+$ we get (1.7): the proof is completed. $\square$

In order to prove Theorem 2.5 we need some intermediate results.

Lemma 3.1 *Let $N\ge3$. For any $\alpha,R,M>0$ there exists a subsolution $\overline{u}_0$ to the equation $\Delta[G(u)]=0$ in $I\!R^N$ which is bounded, continuous, radial, nondecreasing as a function of $|x|$, satisfies $\lim_{|x|\to+\infty}\overline{u}_0(x)=\alpha$ and is equal to $-M$ in B_R.*

Proof. Define

$$\widetilde{U}_0(x) := G(\alpha)-\frac{\beta}{|x|} \quad (x\in I\!R^N\setminus B_\varepsilon)\,,$$

with $0<\varepsilon<\gamma:=\frac{\beta}{G(\alpha)-G(-M)}$. It is easily seen that

$$\Delta\widetilde{U}_0(x)\ge0 \quad \text{for all } \; x\in I\!R^N\setminus\overline{B}_\varepsilon\,.$$

Then $\widetilde{u}_0 := G^{-1}(\widetilde{U}_0)$ is a subsolution to $\Delta[G(u)]=0$ in $I\!R^N\setminus\overline{B}_\varepsilon$. Define

$$\widehat{u}_0 := \sup_{I\!R^N\setminus B_\varepsilon}\big\{\widetilde{u}_0,-M\big\} \quad \text{in } \; I\!R^N\setminus\overline{B}_\varepsilon\,.$$

Since $v\equiv-M$ solves $\Delta[G(u)]=0$ in $I\!R^N$, from Kato's inequality $\widehat{u}_0$ is a subsolution to $\Delta[G(u)]=0$ in $I\!R^N\setminus\overline{B}_\varepsilon$. Now, since $\widehat{u}_0=-M$ in $B_\gamma\setminus B_\varepsilon$, the function

$$\overline{u}_0 := \begin{cases} \widehat{u}_0 & \text{in } I\!R^N\setminus B_\varepsilon \\ -M & \text{in } \; B_\varepsilon \end{cases}$$

is a subsolution to $\Delta[G(u)]=0$ in $I\!R^N$. The fact that $\overline{u}_0$ is bounded, continuous, radial, nondecreasing as a function of $|x|$ and satisfies the limit property at infinity is clear by construction. The constant condition in B_R is achieved by choosing $\beta=R\,(G(\alpha)-G(-M))$. $\square$

Lemma 3.2 *Suppose that, besides the assumptions of Theorem 2.5, there exists a function $\overline{u}_0$ having the properties stated in Lemma 3.1 and such that, for a suitable $\varepsilon > 0$ small enough,*

$$u_0(x) \geq \overline{u}_0(x) \quad \textit{for all } x \in I\!R^N , \tag{3.40}$$

$$\lim_{|x|\to\infty} \overline{u}_0(|x|) = \min_{t\in[0,T]} a(t) - \varepsilon > 0 . \tag{3.41}$$

Moreover assume that, for the same ε given above,

$$2\,G\left(\min_{t\in[0,T]} a(t) - \varepsilon\right) > G(\|a\|_\infty) . \tag{3.42}$$

Then there exists a solution to problem (1.1) *satisfying condition* (1.6).

Proof. We can repeat the proof of Theorem 2.2 up to the construction of the sequence $\{u_j\}$, keeping the same notation. Note that, as in Theorem 2.3, when we allow for a degenerate nonlinearity G, in view of hypothesis (H_0) existence of solutions to problem (3.6) is due to standard results (see, *e.g.*, [1]). Again by results in [3], $u_j \in C\big(\overline{B}_j \times [0,T]\big)$.

First of all, by the assumptions on $\overline{u}_0$, (3.41), (3.42) and (H_0) we can find $\beta > 0$ and $\widetilde{R} > \widehat{R}$ such that for all $R \geq \widetilde{R}$

$$\begin{gathered} \beta < \overline{u}_0(R) , \\ 2G\big(\overline{u}_0(R)\big) - G(\beta) - G(\|a\|_\infty) > 0 . \end{gathered} \tag{3.43}$$

Still from the assumptions on $\overline{u}_0$ we deduce that it is a subsolution to problem (3.6). By comparison principles we have, K being as in (3.7),

$$\overline{u}_0(|x|) \leq u_j(x,t) \leq K \quad \text{for all } B_j \times (0,T) . \tag{3.44}$$

Hence, by the monotonicity of $\overline{u}_0$,

$$\overline{u}_0(R) \leq u_j(x,t) \leq K \quad \text{for all } (x,t) \in N_{R,j} \times (0,T) . \tag{3.45}$$

Put

$$\gamma := \min_{[\beta,K]} G' . \tag{3.46}$$

Given $\sigma > 0$, in view of (3.3) we can choose $R = R(\sigma) > \widetilde{R}$ in (3.5) great enough so that in (3.4) we are allowed to set

$$\delta = \frac{2}{\gamma} V(R) . \tag{3.47}$$

Note that β and γ are independent of R and δ. Let $t_0 \in [0,T]$. Define

$$\lambda := \frac{G(a(t_0)) - G(\overline{u}_0(R))}{\delta^2}, \quad M := \frac{2\lambda\delta}{\gamma} . \tag{3.48}$$

From (3.43), (3.47), (3.48) it follows that

$$M = \frac{G(a(t_0)) - G(\overline{u}_0(R))}{V(R)} , \tag{3.49}$$

$$-MV(R) - \sigma + G(a(t_0)) - \lambda\delta^2 > G(\beta) \tag{3.50}$$

for $\sigma > 0$ small enough.

Define

$$\underline{w}(x,t) := G^{-1}\big[- MV(x) - \sigma + G(a(t_0)) - \lambda(t-t_0)^2\big] \quad \big(x \in I\!R^N \setminus B_{\widehat{R}}, t \in [\underline{t}_\delta, \overline{t}_\delta]\big) .$$

Since $|x| \mapsto V(|x|)$ is nonincreasing, by (3.50)

$$\underline{w}(x,t) \geq \beta \quad \text{for all } \ (x,t) \in N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.51}$$

By $(H_0)(ii)$, (3.1), (3.51), (3.46) and (3.48),

$$\begin{aligned} &\rho(x)\partial_t \underline{w} - \Delta[G(\underline{w})] = -\rho(x)\frac{2\lambda(t-t_0)}{G'(\underline{w})} + M\Delta V \leq \\ &\leq \rho(x)\left(\frac{2\lambda\delta}{\gamma} - M\right) = 0 \quad \text{in } \ [I\!R^N \setminus B_{\widehat{R}}] \times (\underline{t}_\delta, \overline{t}_\delta)\,. \end{aligned} \tag{3.52}$$

By (3.49),

$$\underline{w}(x,t) \leq \overline{u}_0(R) \quad \text{for all } \ (x,t) \in \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.53}$$

Furthermore,

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \sigma\big] \quad \text{for all } \ (x,t) \in \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta). \tag{3.54}$$

When $\underline{t}_\delta = 0$ there holds

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \sigma\big] \quad \text{for all } \ (x,t) \in N_{R,j} \times \{\underline{t}_\delta\} \tag{3.55}$$

whereas, when $\underline{t}_\delta > 0$, we have

$$\underline{w}(x,t) \leq G^{-1}\big[G(a(t_0)) - \lambda\delta^2\big] = \overline{u}_0(R) \quad \text{for all } \ (x,t) \in N_{R,j} \times \{\underline{t}_\delta\}\,; \tag{3.56}$$

here (3.48) has been used.

From (3.52), (3.53), (3.54)-(3.56) we can infer that $\underline{w}$ is a subsolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, \overline{u}_0(R) & \text{in } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{in } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, \overline{u}_0(R) & \text{in } N_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.57}$$

when $\underline{t}_\delta > 0$, whereas it is a subsolution to problem

$$\begin{cases} \rho\,\partial_t u = \Delta\big[G(u)\big] & \text{in } N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, \overline{u}_0(R) & \text{in } \partial B_R \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{in } \partial B_j \times (\underline{t}_\delta, \overline{t}_\delta) \\ u \,=\, G^{-1}\big[G(a(t_0)) - \sigma\big] & \text{in } N_{R,j} \times \{\underline{t}_\delta\} \end{cases} \tag{3.58}$$

when $\underline{t}_\delta = 0$.

On the other hand, from (3.4), (3.5) (which, recall, holds true as a consequence of (2.3)) (3.45) we easily deduce that u_j is a supersolution to problem (3.57) when $\underline{t}_\delta > 0$, while it is a supersolution to problem (3.58) when $\underline{t}_\delta = 0$.

By comparison principles,

$$\underline{w} \leq u_j \quad \text{in } \ N_{R,j} \times (\underline{t}_\delta, \overline{t}_\delta)\,. \tag{3.59}$$

Define

$$\overline{w}(x,t) := G^{-1}\big[\widetilde{M}V(x) + \sigma + G(a(t_0)) + \widetilde{\lambda}(t-t_0)^2\big] \ \ \big(x \in I\!R^N \setminus B_{\widehat{R}}, t \in [\underline{t}_\delta, \overline{t}_\delta]\big)\,;$$

we have

$$\overline{w} \geq \min_{t\in[0,T]} a(t) \quad \text{in} \quad \left(I\!\!R^N \setminus B_{\widehat{R}}\right) \times [\underline{t}_\delta, \overline{t}_\delta]\,.$$

Choose

$$\widetilde{M} \geq \max\left\{ \frac{2\widetilde{\lambda}\delta}{G'(\min_{t\in[0,T]} a(t))}, \frac{G(K) - G(-\|a\|_\infty)}{V(R)} \right\},$$

and

$$\widetilde{\lambda} \geq \frac{G(K) - G(-\|a\|_\infty)}{\delta^2}\,.$$

By analogous arguments to those used above, we can infer that $\overline{w}$ is a supersolution to problem (3.19) when $\underline{t}_\delta > 0$, whereas it is a subsolution to problem (3.20) when $\underline{t}_\delta = 0$.

On the other hand, from (3.4), (3.5), (3.7) we easily deduce that u_j is a subsolution to problem (3.19) when $\underline{t}_\delta > 0$, while it is a subsolution to problem (3.20) when $\underline{t}_\delta = 0$. As in the proof of Theorem 2.3, by a compactness argument which makes use of [2, Lemma 5.2] and by a diagonal procedure, there exists a subsequence $\{u_{j_k}\} \subseteq \{u_j\}$ which converges, as $k \to \infty$, locally uniformly in $I\!\!R^N \times (0,T)$ to a solution u of problem (1.1). We conclude arguing as in the final part of the proof of Theorem 2.2. □

Proof of Theorem 2.5 . First consider a datum $a(t)$ at infinity such that, for some $\varepsilon > 0$, (3.42) holds and $\min_{t\in[0,T]} a(t) - \varepsilon > 0$. Consider then the function $\overline{u}_0$ given in Lemma 3.1 with the choices $\alpha = \min_{t\in[0,T]} a(t) - \varepsilon$, R great enough so that $u_0(x) \geq \min_{t\in[0,T]} a(t) - \varepsilon$ for all $x \in B_R^c$ and $M = \max(0, -\inf_{x\in I\!\!R^N} u_0(x))$. Clearly, under these assumptions, $u_0(x) \geq \overline{u}_0(x)$ for all $x \in I\!\!R^N$. Therefore the assertion of Lemma 3.2 holds true.

If there exists no $\varepsilon > 0$ such that $a(t)$ fulfils (3.42) in the time interval $[0,T]$, we can always find $\varepsilon, \tau > 0$ small enough such that

$$2\,G\left(\min_{s\in[t,(t+\tau)\wedge T]} a(s) - \varepsilon\right) > G\left(\max_{s\in[t,(t+\tau)\wedge T]} a(s)\right) \quad \forall t \in [0,T)\,. \tag{3.60}$$

This is a consequence of the uniform continuity of $G(a(t))$ and of its strict positivity in $[0,T]$. Hence we get existence in the time interval $[0,\tau]$. Repeating this procedure starting from $t = \tau$ we get existence in the time interval $[\tau, 2\tau \wedge T]$ with initial datum $u(\tau)$ and hence, by Definition 2.1, existence in the time interval $[0, 2\tau \wedge T]$. A finite number of iterations yields the claim.

□

4 Uniqueness: proofs

Let u_1, u_2 be any two solutions to problem (1.1). Define

$$q(x,t) := \begin{cases} \frac{G(u_1)-G(u_2)}{u_1-u_2} & \text{if } u_1 \neq u_2 \\ \\ 0 & \text{if } u_1 = u_2\,. \end{cases}$$

Observe that, in view of $(H_0) - (ii)$, $q \geq 0$ in S_T and $q \in L^\infty(S_T)$. Consider a sequence $\{q_n\} \subseteq C^\infty(S_T)$ such that for every $n \in I\!\!N$ there hold:

$$\frac{1}{n^2} \leq q_n \leq \|q\|_{L^\infty(S_T)} + \frac{1}{n^2} \quad \text{in} \quad Q_{n,\tau} := B_n \times (0,\tau) \tag{4.1}$$

and

$$\left\| \frac{(q_n - q)}{\sqrt{q_n}} \right\|_{L^2(Q_{n,\tau})} \to 0 \quad \text{as } n \to \infty\,. \tag{4.2}$$

For any $n \in I\!N$ let $\psi_n \in C^2(\overline{Q}_{n,\tau})$ be the unique solution to the backward parabolic problem

$$\begin{cases} \rho\,\partial_t\psi_n + q_n\Delta\psi_n = 0 & \text{in } Q_{n,\tau} \\ \psi_n = 0 & \text{in } \partial Q_{n,\tau} \\ \psi_n = \chi(x) & \text{in } B_n \times \{\tau\}\,, \end{cases} \tag{4.3}$$

where $\chi \in C^\infty(I\!R^N)$, $0 \le \chi \le 1$ and *supp* $\chi \subseteq B_{n_0}$, for some $n_0 \in I\!N$.

The following lemma will play a central role in the proof of Theorem 2.8.

Lemma 4.1 *For every $n \in I\!N$ let $\psi_n \in C^2(\overline{Q}_{n,\tau})$ be the unique solution to problem* (4.3). *Then, for every $n \in I\!N$,*

$$0 \le \psi_n \le 1 \quad in \;\; Q_{n,\tau}\,. \tag{4.4}$$

Furthermore, there exists a constant $C > 0$ such that for every $n > n_0$

$$-\frac{C}{R^{N-1}} \le \langle \nabla\psi_n, \nu_n\rangle \le 0 \quad in \;\; \partial B_n \times (0,\tau)\,, \tag{4.5}$$

where ν_n is the outer normal at ∂B_n.

Proof. Since $\underline{\psi} \equiv 0$ is a subsolution, while $\overline{\psi}$ is a supersolution to problem (4.3), by comparison principle we get (4.4). Now, since

$$\psi_n = 0 \quad \text{in } \;\partial B_n \times (0,\tau)\,,$$

for all $n \in I\!N$, from (4.4) we deduce that

$$\langle \nabla\psi_n, \nu_n\rangle \le 0 \quad \text{in } \;\partial B_n \times (0,T)\,. \tag{4.6}$$

for all $n \in I\!N$. For every $n > n_0$ set

$$E_n := B_n \setminus B_{n_0}\,.$$

From (4.4), and the fact that $\operatorname{supp}\chi \subset B_{n_0}$, we can infer that, for all $n > n_0$, the function ψ_n is a subsolution to problem

$$\begin{cases} \rho\,\partial_t\psi_n + q_n\Delta\psi_n = 0 & \text{in } E_n \times (0,T) \\ \psi = 1 & \text{in } \partial B_{n_0} \times (0,T) \\ \psi = 0 & \text{in } \partial Q_{n,\tau} \\ \psi = 0 & \text{in } B_n \times \{\tau\}\,, \end{cases} \tag{4.7}$$

For every $n > n_0$ define

$$z(x) := \widehat{C}\,\frac{|x|^{2-N} - n^{2-N}}{1 - n^{2-N}} \quad \big(x \in E_n\big)\,.$$

It is easily seen that, for $\widehat{C} = \widehat{C}_{n_0}$ sufficiently large, the function z is a supersolution to problem (4.7) for all $n > n_0$. Furthermore,

$$\psi_n = z = 0 \quad \text{in } \;\partial B_n \times (0,\tau),$$

hence

$$\langle \psi_n, \nu_n\rangle \ge \langle z, \nu_n\rangle = (2-N)\widehat{C}n^{1-N} \quad \text{in } \;\partial B_n \times (0,\tau) \tag{4.8}$$

for all $n > n_0$. From (4.6) and (4.8) it follows (4.5) with $C := (N-2)\widehat{C} > 0$. This completes the proof. $\square$

Proof of Theorem 2.8. Let u_1, u_2 be two bounded solutions of problem (1.1) satisfying

$$\lim_{|x|\to\infty, t\to t_0} u_1(x,t) = \lim_{|x|\to\infty, t\to t_0} u_2(x,t) = a(t_0) \quad \text{for any } \ t_0 \in [0,T]\,.$$

Clearly, this implies that for any $\tau \in (0,T)$

$$\lim_{R\to\infty} \frac{1}{R^{N-1}} \int_0^\tau \int_{\partial B_R} \big|G\big(u_1(x,t)\big) - G\big(u_2(x,t)\big)\big| d\sigma dt = 0\,. \tag{4.9}$$

Put $w := u_1 - u_2$. By Definition 2.1,

$$\begin{aligned}
&\int_{\Omega_1} \rho w(x,\tau)\psi(x,\tau)dx \leq \\
&= \int_0^\tau \int_{\Omega_1} \big\{\rho w \psi_t + [G(u_1) - G(u_2)]\Delta\psi\big\} dxdt - \\
&- \int_0^\tau \int_{\partial\Omega_1} [G(u_1) - G(u_2)]\langle \nabla\psi, \nu\rangle d\sigma dt
\end{aligned} \tag{4.10}$$

for any τ, Ω_1 and ψ as in Definition 2.1.

Moreover, multiplying the first equation in (4.3) by $\frac{\Delta\psi_n}{\rho}$ and integrating by parts, since $\rho \in L^\infty(I\!R^N)$, we obtain for any $n \in I\!N$:

$$\int_0^\tau \int_{B_n} q_n (\Delta\psi_n)^2 dxdt \leq \widetilde{C}\,. \tag{4.11}$$

for some constant $\widetilde{C} > 0$.

Taking $\Omega_1 = B_n$ and $\psi = \psi_n$ in (4.10), we get for any $n \in I\!N$

$$\begin{aligned}
\int_0^\tau \int_{B_n} \rho\chi w(x,\tau)dx = &\int_0^\tau \int_{B_n} (q - q_n)\Delta\psi_n dxdt - \\
&- \int_0^\tau \int_{\partial B_n} qw\langle \nabla\psi, \nu_n\rangle d\sigma dt.
\end{aligned} \tag{4.12}$$

We shall prove that both integrals in the right-hand side of inequality (4.12) tend to 0 as $n \to \infty$.

In fact, from (4.2) and (4.11) we have:

$$\begin{aligned}
&\left(\int_0^\tau \int_{B_n} (q - q_n)\Delta\psi_n dxdt\right)^2 \leq \\
&\leq \overline{C} \int_0^\tau \int_{B_n} \left|\frac{q - q_n}{\sqrt{q_n}}\right|^2 dxdt \int_0^\tau \int_{B_n} q_n |\Delta\psi_n|^2 dxdt \to 0 \quad \text{as } \ n \to \infty\,,
\end{aligned} \tag{4.13}$$

where $\overline{C} := (\|u_1\|_\infty + \|u_2\|_\infty)^2$.

Moreover, for every $n > n_0$, by (4.5) and (4.9),

$$\begin{aligned}
&\Big|\int_0^\tau \int_{\partial B_n} qw\langle \nabla\psi_n, \nu_n\rangle d\sigma dt\Big| = \\
&= \Big|\int_0^\tau \int_{\partial B_n} \big[G(u_1) - G(u_2)\big]\langle \nabla\psi_n, \nu_n\rangle d\sigma dt\Big| \leq \\
&\leq \max_{\partial B_n} |\langle \nabla\psi_n, \nu_n\rangle| \int_0^\tau \int_{\partial B_n} \big|G(u_1) - G(u_2)\big| d\sigma dt \leq \\
&\leq \frac{C}{n^{N-1}} \int_0^\tau \int_{\partial B_n} \big|G(u_1) - G(u_2)\big| d\sigma dt \to 0
\end{aligned} \tag{4.14}$$

as $n \to \infty$.

Sending $n \to \infty$ in (4.12), from (4.13) and (4.14) it follows that

$$\int_{I\!R^N} \rho(x)\chi(x)w(x,\tau)dx = 0 \tag{4.15}$$

for any $\tau \in (0,T)$ and $\chi \in C_c^\infty(I\!R^N)$ with $0 \le \chi \le 1$.

Fix any compact subset $K \subset I\!R^N$ and $t \in (0,T)$. Define

$$\zeta(x,t) := \begin{cases} 1 & \text{if } x \in K, t \in (0,T), w(x) > 0 \\ \\ 0 & \text{if } [x \in K, t \in (0,T), w(x) < 0\,] \text{ or } [x \in I\!R^N \setminus K, t \in (0,T)\,]\,. \end{cases}$$

Now, choose a sequence $\{\chi_n\} \subseteq C_c^\infty(I\!R^N)$, with $0 \le \chi_n \le 1$ for any $n \in I\!N$, such that $\chi_n(x) \to \zeta(x)$ as $n \to \infty$ for any $x \in I\!R^N$. In view of (4.15) we deduce that for any $n \in I\!N$

$$\int_{I\!R^N} \rho(x)\chi_n(x)w(x,\tau)d\mu = 0. \tag{4.16}$$

Letting $n \to \infty$ in (4.16), by the dominated convergence theorem we get

$$\int_K \rho(x)w(x,\tau)dx = 0\,. \tag{4.17}$$

Hence $w(x,\tau) \equiv 0$ for any $x \in K$. Since the compact subset $K \subset I\!R^N$ and $\tau \in (0,T)$ were arbitrary, we get

$$w \equiv 0 \quad \text{in } \; I\!R^N \times (0,T),$$

so

$$u_1 \equiv u_2 \quad \text{in } \; I\!R^N \times (0,T).$$

This completes the proof. □

References

[1] D. Aronson, M.C. Crandall, L.A. Peletier, *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlin. Anal. TMA **6** (1982), 1001-1022 .

[2] E. Di Benedetto, *Continuity of Weak Solutions to a General Porous Medium Equation,* Indiana Univ. Math. J. **32** (1983), 83–118 .

[3] E. Di Benedetto, *A Boundary Modulus of Continuity for a Class of Singular Parabolic Equations,* J. Diff. Eq. **63** (1986), 418–447 .

[4] J. Dolbeault, I. Gentil, A. Guillin, F.-Y. Wang, *L^q-functional inequalities and weighted porous media equations*, Potential Anal. 28 (2008), 35–59.

[5] J. Dolbeault, B. Nazaret, G. Savaré, *On the Bakry-Emery criterion for linear diffusions and weighted porous media equations*, Commun. Math. Sci. 6 (2008), 477–494.

[6] D. Eidus, *The Cauchy problem for the nonlinear filtration equation in an inhomogeneous medium*, J. Differential Equations 84 (1990), 309–318.

[7] D. Eidus, S. Kamin, *The filtration equation in a class of functions decreasing at infinity*, Proc. Amer. Math. Soc. **120** (1994), 825–830.

[8] G. Grillo, M. Muratori, *Sharp short and long time L^∞ bounds for solutions to porous media equations with Neumann boundary conditions*, preprint 2012, arXiv:1206.5332.

[9] G. Grillo, M. Muratori, M. M. Porzio, *Porous media equations with two weights: existence, uniqueness, smoothing and decay properties of energy solutions via Poincaré inequalities*, preprint 2012, arXiv:1204.6159.

[10] M. Guedda, D. Hilhorst, M.A. Peletier, *Disappearing interfaces in nonlinear diffusion*, Adv. Math. Sc. Appl. **7** (1997), 695–710.

[11] S. Kamin, G. Reyes, J. L. Vázquez, *Long time behavior for the inhomogeneous PME in a medium with rapidly decaying density*, Discrete Contin. Dyn. Syst. 26 (2010), 521–549.

[12] S. Kamin, M.A. Pozio, A. Tesei, *Admissible conditions for parabolic equations degenerating at infinity*, St. Petersburg Math. J. **19** (2008), 239–251.

[13] S. Kamin, P. Rosenau, *Propagation of thermal waves in an inhomogeneous medium*, Comm. Pure Appl. Math. 34 (1981), 831–852.

[14] S. Kamin, P. Rosenau, *Non-linear diffusion in a finite mass medium*, Comm. Pure Appl. Math. **35** (1982), 113–127.

[15] O.A. Ladyzhenskaya, V.A. Solonnikov, N.A. Uraltseva, "Linear and Quasilinear Equations of Parabolic Type", Nauka, Mocow (1967) (English translation: series Transl. Math. Monographs, **23** AMS, Providence, 1968) .

[16] F. Punzo, *On the Cauchy problem for nonlinear parabolic equations with variable density*, J. Evol. Equations **9** (2009), 429-447.

[17] F. Punzo, *Uniqueness and nonuniqueness of bounded solutions to singular nonlinear parabolic equations,* Nonlin. Anal. TMA **70** (2009), 3020–3029.

[18] F. Punzo, *On support of solutions to singular nonlinear parabolic equations in bounded domains*, Interf. Free Bound. **13** (2011), 397–410.

[19] F. Punzo, *Uniqueness and support properties of solutions to singular quasilinear parabolic equations on surfaces of revolution*, Ann. Mat. Pura Appl. **191** (2012), 311–338.

[20] F. Punzo, *Uniqueness and Nonuniqueness of Solutions to Quasilinear Parabolic Equations with a Singular Coefficient on Weighted Riemannian Manifolds*, Asympt. Anal. (to appear)

[21] G. Reyes, J. L. Vazquez, *The Cauchy problem for the inhomogeneous porous medium equation*, Netw. Heterog. Media **2** (2006), 337–351.

[22] G. Reyes, J. L. Vázquez, *The inhomogeneous PME in several space dimensions. Existence and uniqueness of finite energy solutions*, Commun. Pure Appl. Anal. 7 (2008), 1275–1294.

[23] G. Reyes, J. L. Vázquez, *Long time behavior for the inhomogeneous PME in a medium with slowly decaying density*, Commun. Pure Appl. Anal. 8 (2009), 493–508.

[24] J. L. Vázquez, *The porous medium equation. Mathematical theory*, The Clarendon Press, Oxford University Press, Oxford, 2007.