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A three dimensional Steklov eigenvalue problem with exponential nonlinearity on the boundary

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Abstract

We investigate the existence of pairs (λ, u), with $\lambda > 0$ and u harmonic function in a bounded domain $\Omega \subset \mathbb{R}^3$, such that the nonlinear boundary condition $\partial_\nu u = \lambda\, \mu \sinh u$ holds on $\partial\Omega$, where μ is a non negative weight function. This type of exponential boundary condition arises in corrosion modeling (Butler Volmer condition).

1 Statement of the problem and main results

In the study of mathematical models of corrosion, a common problem is the following: find a harmonic function u in a domain $\Omega \subset \mathbb{R}^N$ satisfying a boundary condition of the form

$$\partial_\nu u(x) = \lambda\mu(x)\big(e^{\beta u(x)} - e^{-(1-\beta)u(x)}\big), \quad x \in \partial\Omega \tag{1.1}$$

for some $\lambda > 0$, where $0 < \beta < 1$ and μ is either identically 1 or the characteristic function of a subset of $\partial\Omega$ [1]. The equation (1.1) is known as Butler-Volmer condition and we refer to [1] (and references therein) for its justification in corrosion modeling.

In dimension $N = 2$, there are suitable variational formulations of the previous problem [1], [2] and of the more general version with a boundary function μ changing sign along $\partial\Omega$ [3], [4].

In [3] it is shown that the problem (1.1) has a solution for λ ranging in some intervals depending on β and on the eigenvalues of the linearized problem.

The symmetric case $\beta = 1/2$ in (1.1) takes on a special interest, both in applications and for theoretical reasons; in [2] the authors prove the existence of infinitely many solutions for any positive λ (assuming $\mu(x) \equiv 1$) by applying variational methods, relying on index theory, which are suitable for even functionals (see [5] chapter 5).

In [4], by exploiting known critical point theorems (for symmetric functionals) based on the topological notions of *index* and *pseudo-index*, [6], [7], existence and multiplicity results are proved for the same problem with an indefinite μ.

Unfortunately, the above mentioned variational methods are no longer applicable to the physical relevant case $N = 3$, since the functional associated to (1.1) does not satisfy the Palais-Smale condition. In fact, existence results for the three dimensional problem seems to be lacking in the literature.

In this paper we discuss the following problem: find nontrivial solutions u to the system

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{in } \Omega \\ \partial_\nu u(x) &= \lambda\, \mu(x) \sinh[u(x)] \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^3$, $\lambda > 0$ and μ is a non negative weight function in $L^\infty(\partial\Omega)$ (with respect to the Hausdorff measure of $\partial\Omega$).

By observing that the above problem (as well as problem (1.1)) has the line of trivial solutions $\{(\lambda, 0) \,|\, \lambda \in \mathbb{R}\}$, it is natural to look for *bifurcation solutions*. To this aim, it is necessary to discuss a related *linear Steklov eigenvalue problem*. This problem is well known in the case of non negative weight functions μ [8]; in section 2.1 we summarize, for reader's convenience, the results on the properties of the eigenvalues and of the eigenfunctions obtained in [3], [9], in the general case $\mu \in L^\infty(\partial\Omega)$. More detailed results on the regularity of the eigenfunctions, which will be necessary for the subsequent discussion, are described in section 2.2.
In section 3 we provide a functional setting for the non linear problem in three dimensions and apply classical results of bifurcation theory [10], [11] to prove that, for every Steklov eigenvalue κ (of the linearized problem) the pair $(\kappa, 0)$ is a bifurcation point for (1.2) (and also for (1.1)). As we will show, the key point is the choice of a suitable Hilbert space formulation of the problem, which allows to apply *variational methods in Bifurcation Theory*.
In the last section, we discuss global existence, regularity and symmetry of the solutions and apply our results to two examples of problem (1.2) respectively in a ball and in a cube, where considerations of symmetry allows to investigate further properties of the solutions. Finally, we also discuss open problems and conjectures on the set of non trivial solutions.

2 The linear eigenvalue problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and consider the following linear Steklov eigenvalue problem in $H^1(\Omega)$:

$$\begin{aligned} \Delta u(x) &= 0 \quad \text{in } \Omega \\ \gamma(\partial_\nu u)(x) &= \lambda\, \mu(x)\gamma(u)(x) \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

where $\lambda \in \mathbb{R}$, $\mu(x) \in L^\infty(\partial\Omega)$ and γ denotes the trace operator on $\partial\Omega$.
We recall that, for a Lipschitz domain Ω, the trace on $\partial\Omega$ of the normal derivative of a $H^1(\Omega)$ function satisfying $\Delta u \in L^2(\Omega)$ (in the weak sense) is well defined as an element of the Sobolev space $H^{-1/2}(\partial\Omega)$. For a general overview of Sobolev spaces and traces of functions see [12].

2.1 Existence and multiplicity of eigenfunctions

It is easily seen that the solutions to (2.1) belong to the subspace $H^1_\mu \subset H^1(\Omega)$ defined as follows:

$$H^1_\mu \equiv \Big\{ u \in H^1(\Omega), \quad \int_{\partial\Omega} \mu\, \gamma(u) = 0 \Big\}. \tag{2.2}$$

Assuming that

$$\int_{\partial\Omega} \mu \neq 0, \tag{2.3}$$

it turns out [3] that the Dirichlet norm $\int_\Omega |\nabla u|^2$ is equivalent to the H^1 norm in H^1_μ and that (2.1) is equivalent to the following variational problem:
Find $u \in H^1_\mu$, $u \neq 0$, such that

$$\int_\Omega \nabla u\, \nabla v = \lambda \int_{\partial\Omega} \mu\gamma(u)\gamma(v) \tag{2.4}$$

holds for every $v \in H^1_\mu$.
Moreover, the expression

$$\|u\|_1^2 = \int_\Omega |\nabla u|^2 + \Big(\int_{\partial\Omega} \mu\gamma(u)\Big)^2, \tag{2.5}$$

defines an equivalent norm in $H^1(\Omega)$. We consider the scalar product in $H^1(\Omega)$ associated to this equivalent norm; then, we have the following result [3]:

Proposition 2.1. *Assume* (2.3). *Then, problem (2.1) has infinitely many eigenvalues λ_n, each of finite multiplicity and such that $|\lambda_n| \to +\infty$. Moreover, the following orthogonal decomposition holds:*

$$H^1 = H_0^1 \oplus c \oplus V_\mu \oplus V_0, \tag{2.6}$$

where c are constants eigenfunctions corresponding to the null eigenvalue $\lambda_0 = 0$, the subspace V_μ is spanned by the eigenfunctions u_n satisfying the variational equations

$$\int_\Omega \nabla u_n \nabla v = \lambda_n \int_{\partial\Omega} \mu\gamma(u_n)\gamma(v), \quad \lambda_n \neq 0, \tag{2.7}$$

for every $v \in H^1$ and V_0 is spanned by (harmonic) functions w such that

$$\int_{\partial\Omega} \mu\gamma(w)\gamma(v) = 0 \tag{2.8}$$

for every $v \in H^1$.

Notice that a non trivial w satisfying (2.8) can only exist if $\mu\gamma(w) = c\mu = 0$, i.e. if the function μ vanishes on a subset of positive Hausdorff measure of $\partial\Omega$; otherwise, V_0 is empty.
In the sequel, we will list all the eigenvalues to problem (2.1) as follows

$$...\lambda_{-2} \leq \lambda_{-1} \leq 0 \leq \lambda_1 \leq \lambda_2...$$

The eigenvalue $\lambda_0 = 0$ corresponds to the constant solutions of the homogeneous Neumann problem. By (2.7), we can take all the u_n orthogonal and normalized with respect to the scalar product associated to the Dirichlet norm $\int_\Omega |\nabla u|^2$ and even to the equivalent norm (2.5) by defining $u_0 = \left(\int_{\partial\Omega} \mu\right)^{-1}$; then, we have

$$\int_\Omega \nabla u_n \, \nabla u_m = \int_{\partial\Omega} \mu \, \gamma(u_n)\gamma(u_m) = 0, \tag{2.9}$$

for $n \neq m$.
Note that from the relations

$$\int_\Omega |\nabla u_n|^2 = \lambda_n \int_{\partial\Omega} \mu \, \gamma(u_n)^2, \tag{2.10}$$

we get the inequalities

$$\int_{\partial\Omega} \mu \, \gamma(u_n)^2 > 0, \quad \text{for } n > 0; \qquad \int_{\partial\Omega} \mu \, \gamma(u_n)^2 < 0, \quad \text{for } n < 0. \tag{2.11}$$

Remark 2.2. *If μ has definite sign, $\mu \geq 0$ say, the problem is coercive for $\lambda < 0$ and we have infinitely many positive eigenvalues (this is the case of the classical Steklov problem [8]); an analogous assertion holds if $\mu \leq 0$. But, as soon as μ is positive on some subset of $\partial\Omega$ and negative on some other subset, both subsets being of positive measure, there are infinitely many positive* and *negative eigenvalues (see [3], remark 2.6).*

A relevant question from the point of view of bifurcation theory is the *multiplicity of the eigenvalues.* The results below follow from [9], Theorem 1.2 :

Theorem 2.3. *Let $\mu^\pm = \operatorname{ess\,sup}(0, \pm\mu)$. Then:*

1. *If $\mu^+ > 0$ and $\int_{\partial\Omega} \mu < 0$, the first positive Steklov eigenvalue λ_1 is simple and it is the only nonzero eigenvalue associated to an eigenfunction of definite sign.*
2. *If $\mu^- > 0$ and $\int_{\partial\Omega} \mu > 0$, the first negative Steklov eigenvalue λ_{-1} is simple and it is the only nonzero eigenvalue associated to an eigenfunction of definite sign.*

Remark 2.4. *When $\int_{\partial\Omega} \mu = 0$ (and μ is non trivial) there are still unbounded sequences of positive and negative eigenvalues to problem* (2.1)*; however, the decomposition* (2.6) *does not hold in that case (see [3], remark 2.7) and the occurrence of non zero simple eigenvalues can not be proved. Similarly, if either μ^+ or μ^- vanishes, simple nonzero eigenvales could not exist, as in the case e.g. of the classical Steklov problem ($\mu = 1$) on the sphere.*

2.2 On the regularity of eigenfunctions.

Global regularity of the eigenfunctions of (2.1) depends on the weight μ and on the regularity of the boundary $\partial\Omega$. This issue is also relevant for the subsequent discussion of the nonlinear problem. For, if we want to consider problem (1.2) (or even (1.1)) as an equation for a *continuous operator* (in both the variables λ and u) in a suitable functional space, we should require enough regularity to give meaning to the boundary conditions; on the other hand, non smooth domains are common in corrosion problems and in addition the weight μ may be discontinuous (for example, one often has $\mu = \chi_\Gamma$, the characteristic function of a subset $\Gamma \subset \partial\Omega$).
Recall that any solution of (2.4) belongs to $H^1(\Omega)$; the trace of its normal derivative (which is well defined since u is harmonic in Ω) being proportional to $\mu\gamma(u)$, belongs to $L^2(\partial\Omega)$. Thus, it can be proved that $u \in H^{3/2}(\Omega)$ in a *Lipschitz* domain $\Omega \subset \mathbb{R}^N$ [13].
In case of dimension $N = 2$ this implies (by Sobolev imbedding) $u \in \mathcal{C}(\overline{\Omega})$ without any additional assumption. For $N \geq 3$, again by Sobolev imbedding, we have (in a Lipschitz domain)

$$u \in H^{3/2}(\Omega) \subset W^1_{\frac{2N}{N-1}}(\Omega) \tag{2.12}$$

In particular, for any solution u in a Lipschitz domain $\Omega \subset \mathbb{R}^3$, $u \in W^1_3(\Omega)$.
In order to investigate further regularity depending on μ and on the boundary $\partial\Omega$, we will exploit the following result, which is a special case of theorem 2.4.2.7 in [12]:

Proposition 2.5. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with a $\mathcal{C}^{1,1}$ boundary $\partial\Omega$. Then, for every $f \in L_p(\Omega)$ and every $g \in W_p^{1-\frac{1}{p}}(\partial\Omega)$, there exists a unique $u \in W^2_p(\Omega)$ which is a solution of*

$$\begin{aligned} \Delta u(x) &= f(x) \quad \text{in } \Omega \\ \gamma(\partial_\nu u)(x) &= g(x) \quad \text{on } \partial\Omega \end{aligned} \tag{2.13}$$

Then, we can state

Theorem 2.6. *Assume that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is bounded with $\mathcal{C}^{1,1}$ boundary and let μ be a Lipschitz function on $\partial\Omega$. Then, every solution u to problem 2.1 satisfies $u \in \mathcal{C}(\overline{\Omega})$.*

Proof. By (2.12) and by the regularity of μ, we have

$$\mu\gamma(u) \in W_p^{1-\frac{1}{p}}(\partial\Omega)$$

with $p = \frac{2N}{N-1}$. Then, by applying proposition 2.5 with $f = 0$ and $g = \lambda\mu\gamma(u)$ we get

$$u \in W^2_{\frac{2N}{N-1}}(\Omega)$$

In particular, by Sobolev imbedding, the solutions u are continuous functions up to the boundary for $N \leq 4$. By recalling the inclusion $W^2_{\frac{2N}{N-1}}(\Omega) \subset W^1_{\frac{2N}{N-3}}(\Omega)$, we can repeat the previous arguments with increasing values of p and reach the same conclusion for every N. □

As previously remarked, we would like to treat also the case of a weight μ with *jump discontinuities*, for example when μ is the indicator function of a subset $\Gamma \subset \partial\Omega$.

Proposition 2.7. *Assume that $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, is bounded with $\mathcal{C}^{1,1}$ boundary and let $\mu = \chi_\Gamma$, where $\Gamma \subset \partial\Omega$ has Lipschitz boundary. Then, every solution u to problem 2.1 is continuous up to the boundary.*

Proof. By (2.12) and by the trace theorems, if u solves (2.1) we have $\gamma(u) \in W_p^{1-\frac{1}{p}}(\partial\Omega)$, with $p \leq 4$ if $N = 2$ and $p \leq 3$ if $N = 3$. Then, by applying corollary 1.4.4.5 of [12], it can be shown that

$$\chi_\Gamma \, \gamma(u) \in W_p^{1-\frac{1}{p}}(\partial\Omega) \tag{2.14}$$

whenever $1 - \frac{1}{p} < \frac{1}{p}$ that is for $p < 2$. Then, again by theorem 2.5 and by Sobolev imbedding, u is continuous up to the boundary if $p > N/2$. The two conditions are compatible whenever $N < 2p < 4$, that is for $N = 2$ and $N = 3$. □

Remark 2.8. *More generally, it follows from proposition 2.5 that if μ is such that the product $\mu\gamma(u)$ belongs to the trace space $W_p^{1-\frac{1}{p}}(\partial\Omega)$ (for some $p \leq \frac{2N}{N-1}$) then $u \in W^2_p(\Omega)$. A sufficient condition can be derived from general properties of Sobolev space functions, but it will not be used in the following. By the inclusion $W^2_p(\Omega) \subset W^1_{\frac{Np}{N-p}}(\Omega)$, we may improve (2.12) provided $\frac{Np}{N-p} > \frac{2N}{N-1}$, i.e. for $p > \frac{2N}{N+1}$. By putting $p_1 = \frac{Np}{N-p}$ we get further regularity if μ is such that $\mu\gamma(u) \in W_{p_1}^{1-\frac{1}{p_1}}(\partial\Omega)$. Then, possibly iterating, we achieve $u \in W^2_p(\Omega)$ for some larger p. As previously noticed, we obtain global continuity of the solution if $p > N/2$. In the limit case $p = N/2$ we have*

$$u \in W^2_{N/2}(\Omega) \subset W^1_N(\Omega) \tag{2.15}$$

Remark 2.9. *As we will show below, $W^1_N(\Omega)$ is the largest space where the nonlinear boundary conditions (1.1), (1.2) can be stated in terms of traces at the boundary of functions defined in $\Omega \subset \mathbb{R}^N$. It is remarkable that, in the case $N = 3$, the condition $u \in W^1_3(\Omega)$ holds in a Lipschitz domain for any bounded weight function μ.*

3 Bifurcation solutions of the 3 dimensional problem

We now discuss a possible functional setting for the non linear problem (1.2). For notational simplicity, sometimes we will denote by the same symbol a function defined in Ω and its trace on the boundary $\partial\Omega$.

The previous discussion on the regularity of the solutions of the linear problem suggests to consider the Sobolev space $W^1_N(\Omega)$; actually, if $u \in W^1_N(\Omega)$ it can be proved that the right hand side of (1.1) belongs to $L^p(\partial\Omega)$ for every $p \geq 1$. More precisely, we have

Proposition 3.1. *For every $u \in W^1_N(\Omega)$ and $\alpha \in \mathbb{R}$ one has $e^{\alpha u} \in W^1_p(\Omega)$ for $1 \leq p < N$. Moreover, for $1 < p < N$ the following estimate holds*

$$\int_{\partial\Omega} e^{\alpha u} \leq C(1 + \|u\|^p_{W^1_p}) \, e^{\beta_p \|u\|^N_{W^1_N}} \tag{3.1}$$

Proof. We start from the inequality (see [14], theorem 7.15)

$$\int_\Omega e^{\left(\frac{|v|}{c_1\|\nabla v\|_N}\right)^{N/(N-1)}} \le c_2|\Omega|, \quad \forall\ v \in W_0^{1,N}(\Omega) \tag{3.2}$$

where the costants c_1, c_2 only depend on N. Then, by the inequality

$$\alpha v \le |\alpha v| \le \frac{N-1}{N}\Big(\frac{|v|}{c_1\|\nabla v\|_N}\Big)^{\frac{N}{N-1}} + \frac{1}{N}\Big(|\alpha|\, c_1\|\nabla v\|_N\Big)^N \tag{3.3}$$

one gets

$$\int_\Omega e^{\alpha v} \le C' e^{\beta'\|\nabla v\|_N^N}, \qquad \forall\ v \in W_0^{1,N}(\Omega) \tag{3.4}$$

Let $\Omega \subset\subset \tilde\Omega$ and for any $u \in W_N^1$ take $\tilde u \in W_0^{1,N}(\tilde\Omega)$ such that $\tilde u = u$ on Ω and $\|\nabla\tilde u\|_{L^N(\tilde\Omega)} \le C\|u\|_{W_N^1(\Omega)}$. Then

$$\int_\Omega e^{\alpha u} \le \int_{\tilde\Omega} e^{\alpha\tilde u} \le C e^{\beta\|u\|^N_{W_N^1(\Omega)}}, \qquad \forall\ u \in W_N^1(\Omega) \tag{3.5}$$

By replacing α with α/p in the previous estimate, we find that $e^{\alpha u} \in L_p(\Omega)$ for every p. Moreover, for every $1 \le p < N$, we have

$$\int_\Omega |\nabla\big(e^{\alpha u}\big)|^p = |\alpha|^p\int_\Omega e^{\alpha p u}|\nabla u|^p \le |\alpha|^p\Big(\int_\Omega e^{\alpha\frac{Np}{N-p}u}\Big)^{\frac{N-p}{N}}\Big(\int_\Omega|\nabla u|^N\Big)^{\frac{p}{N}} \le C e^{\tilde\beta\|u\|^N_{W_N^1(\Omega)}}\|\nabla u\|_N^p \tag{3.6}$$

where in the last line we used the estimate (3.5). Then, the first part of the lemma follows.
In order to derive the estimate on the boundary, we fix $1 < p < N$ and consider the bound

$$\int_{\partial\Omega} e^{\alpha u} = \int_{\partial\Omega}|e^{\alpha u/p}|^p \le C\Big(\int_\Omega|\nabla\big(e^{\alpha u/p}\big)|^p + \int_\Omega e^{\alpha u}\Big) \tag{3.7}$$

By the previous estimates (3.5), (3.6) (again by replacing α with α/p) we readily get the bound (3.1). □

Hereafter, we consider the problem (1.2) for $N = 3$. We will assume $\lambda > 0$ and take for simplicity $\mu = 1$; with minor changes, we could also treat the case $\mu \ge 0$.
As we will see below, it is convenient to search three dimensional solutions in the *Hilbert* space $H^{3/2}(\Omega) \subset W_3^1(\Omega)$.
Let $f \in L^2(\partial\Omega)$ satisfies $\int_{\partial\Omega} f = 0$; define the *Neumann to Dirichlet map*

$$\mathcal{G}f = v_0|_{\partial\Omega} \tag{3.8}$$

where v_0 is the *unique* harmonic function in Ω with Neumann datum f and such that $\int_{\partial\Omega} v_0 = 0$.
By the regularity results quoted above, we have $v_0 \in H^{3/2}(\Omega)$ and therefore $\mathcal{G}f \in H^1(\partial\Omega)$.
Let us define the subspace

$$\dot H^1(\partial\Omega) = \Big\{\phi \in H^1(\partial\Omega), \quad \int_{\partial\Omega}\phi = 0\Big\} \tag{3.9}$$

and the operator

$$G(\lambda,\phi) = \lambda\mathcal{G}\Big(\sinh[\phi + s(\phi)]\Big) \tag{3.10}$$

where

$$s(\phi) = -\tanh^{-1}\Big(\frac{\int_{\partial\Omega}\sinh(\phi)}{\int_{\partial\Omega}\cosh(\phi)}\Big) = \frac{1}{2}\log\Big(\frac{\int_{\partial\Omega}e^{-\phi}}{\int_{\partial\Omega}e^{\phi}}\Big) \tag{3.11}$$

By the estimate (3.5) (with $N = 2$) the exponentials $e^{\pm\phi}$ lie in $L^p(\partial\Omega)$ for every $p \geq 1$; moreover, by the definition (3.11), the right hand side of (3.10) has vanishing integral on $\partial\Omega$. Thus, the operator $G(\lambda, \cdot)$ maps $\dot{H}^1(\partial\Omega)$ in itself.
Assume now that ϕ solves the *functional equation*

$$\phi = G(\lambda, \phi) = \lambda \mathcal{G}\Big(\sinh[\phi + s(\phi)]\Big) \tag{3.12}$$

Then, the unique harmonic function $u_0 \in H^1(\Omega)$ such that $u_0|_{\partial\Omega} = \phi$, satisfies the variational equation

$$\int_\Omega \nabla u_0 \nabla v = \lambda \int_{\partial\Omega} \sinh[u_0|_{\partial\Omega} + s(u_0|_{\partial\Omega})]v \tag{3.13}$$

for every v such that $\int_{\partial\Omega} v = 0$.
Finally, by standard regularity results (see section 4) the function $u(x) = u_0(x) + s(u_0|_{\partial\Omega})$ satisfies the boundary value problem (1.2).
We will now write the functional equation (3.12) in an equivalent form which is suitable for the application of variational methods in bifurcation theory. Let us first consider the eigenfunctions v_i of the linear eigenvalue problem normalized according to

$$\int_\Omega \nabla v_i \, \nabla v_j = \delta_{ij}$$

Then, we have on the boundary

$$\int_{\partial\Omega} v_i v_j = \frac{1}{\lambda_i}\delta_{ij} \quad \text{and} \quad \int_{\partial\Omega} v_i = 0 \tag{3.14}$$

where $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_i \leq ...$ are the eigenvalues listed with their multiplicity.
The vectors $\sqrt{\lambda_i} v_i|_{\partial\Omega}$ form an orthonormal system which span the subspace of zero mean functions of $L^2(\partial\Omega)$.
Let us now introduce the Hilbert space E of the sequences of real numbers

$$\mathbf{t} = \{t_i\}_{i=1,2,...} \tag{3.15}$$

such that

$$\|\mathbf{t}\|^2 = \sum_{i=1}^{\infty} \lambda_i \, t_i^2 < \infty \tag{3.16}$$

We now look for ϕ and u_0 in (3.12), (3.13) represented in the form

$$u_0 = u_0(\mathbf{t}) = \sum_{i=1}^{\infty} t_i \, v_i; \quad \phi = \phi(\mathbf{t}) = u_0(\mathbf{t})|_{\partial\Omega} \tag{3.17}$$

By $\partial_\nu v|_{\partial\Omega} = \lambda_i v|_{\partial\Omega}$ and by (3.14) we get

$$\|\partial_\nu u_0(\mathbf{t})\|_{L^2(\partial\Omega)} = \|\mathbf{t}\|$$

so that $u_0 \in H^{3/2}(\Omega)$ and $\phi \in \dot{H}^1(\partial\Omega)$.
By choosing $v = v_i$, $i = 1, 2, ...$ as test functions in (3.13) we obtain

$$t_i = \lambda \int_{\partial\Omega} \sinh[\phi(\mathbf{t}) + s(\phi(\mathbf{t}))] \, v_i \equiv \lambda f_i(\mathbf{t}) \tag{3.18}$$

Note that, by Parseval identity and by the arguments following (3.10),

$$\sum_i \lambda_i f_i^2 = \sum_i \Big(\int_{\partial\Omega} \sinh[\phi + s(\phi)] \sqrt{\lambda_i}\, v_i \Big)^2 = \int_{\partial\Omega} \sinh^2[\phi + s(\phi)] < \infty \tag{3.19}$$

Hence, by defining

$$\mathbf{f}(\mathbf{t}) = \{f_i(\mathbf{t})\}_{i=1,2,...} \tag{3.20}$$

we can write (3.18) in the form

$$\mathbf{t} = \lambda\, \mathbf{f}(\mathbf{t}) \tag{3.21}$$

where both terms of the equation belong to E.
We now introduce the functional $I:\ E \to \mathbb{R}$ defined by

$$I(\mathbf{t}) = \int_{\partial\Omega} \cosh[\phi(\mathbf{t}) + s(\phi(\mathbf{t}))] \tag{3.22}$$

It can be readily checked that I is $\mathcal{C}^2$, even, and satisfies:

$$\partial_{t_i} I(\mathbf{t}) = \int_{\partial\Omega} \sinh[\phi(\mathbf{t}) + s(\phi(\mathbf{t}))] \big(v_i + (s'(\phi(\mathbf{t})), v_i)_{L^2(\partial\Omega)} \big) = f_i(\mathbf{t}) \tag{3.23}$$

where s' denotes the derivative of the functional (3.11) and the last equality follows by the definition (3.18) and by $\int_{\partial\Omega} \sinh[\phi + s(\phi)] = 0$. Hence, we can write

$$I'(\mathbf{t}) = \mathbf{f}(\mathbf{t}) \tag{3.24}$$

where the above equality is meant between elements of the dual space E^*.
For the resolution of (3.21), it is now convenient to define

$$\mathcal{H}(\phi) = \sinh \phi - \phi \tag{3.25}$$

By the orthogonality relations (3.14) we get

$$f_i(\mathbf{t}) = \frac{t_i}{\lambda_i} + \int_{\partial\Omega} \mathcal{H}\big(\phi + s(\phi)\big) v_i = \frac{t_i}{\lambda_i} + H_i(\mathbf{t}), \qquad i = 1, 2, ... \tag{3.26}$$

where the sequence

$$H_i(\mathbf{t}) = \int_{\partial\Omega} \mathcal{H}\big(\phi + s(\phi)\big) v_i, \tag{3.27}$$

satisfies the bound

$$\sum_i \lambda_i H_i(\mathbf{t})^2 = \int_{\partial\Omega} |\mathcal{H}(\phi + s(\phi)|^2 \le C \|\mathbf{t}\|^6 \tag{3.28}$$

for every $\mathbf{t}$ in a bounded set of E (see appendix).
Finally, by defining

$$\mathbf{H}(\mathbf{t}) = \big\{ H_i(\mathbf{t}) \big\}_{i=1,2,...} \tag{3.29}$$

the equations (3.26) are equivalent to

$$\mathbf{f}(\mathbf{t}) = \mathbf{L}\mathbf{t} + \mathbf{H}(\mathbf{t}) \tag{3.30}$$

where the $\mathbf{L}$ at the right hand side denotes the *compact self-adjoint operator*

$$\{\mathbf{L}\mathbf{t}\}_i = \frac{t_i}{\lambda_i}, \qquad i = 1, 2, ... \tag{3.31}$$

Thus, by recalling (3.24) we can write (3.21) in the form

$$\frac{1}{\lambda}\mathbf{t} = I'(\mathbf{t}) = \mathbf{L}\mathbf{t} + \mathbf{H}(\mathbf{t}) \tag{3.32}$$

In order to solve the above equation, we can argue as in theorem 11.4 of [10] and reduce the problem, by the *Lyapunov-Schmidt method,* to searching the critical points of the functional (3.22) *restricted to some submanifold of a finite dimensional manifold* $\mathcal{M} \subset E$. Then, we can state:

Theorem 3.2. *Every eigenvalue* λ_i *of the linear problem* (2.1) *is a bifurcation point for* (3.32). *If the multiplicity of the eigenvalue is* n, *there is an* $r_0 > 0$ *such that for each* $r \in (0, r_0)$ *there exist at least* n *distinct pairs of solutions* $(\lambda_m(r), \pm\mathbf{t}_m(r))$, $1 \le m \le n$, *with* $\|\mathbf{t}_m\| = r$; *moreover* $\lambda_m(r) \to \lambda_i$ *as* $r \to 0$.

Proof. Let $\lambda_i = \lambda_{i+1} = ... = \lambda_{i+n-1}$ be of multiplicity n and define $i_j = i + j - 1$, with $j = 1, ..., n$. Let the eigenfunctions $\{v_{i_1}, ..., v_{i_n}\}$ span the relative eigenspace. By projecting (3.32) respectively on the subspace of E spanned by $(t_{i_1}, ..., t_{i_n})$ and on its orthogonal complement and by using the implicit function theorem one can uniquely determine, in a neighborhood $\mathcal{O}$ of the origin of $\mathbb{R}^n$, a sequence of $\mathcal{C}^1$ functions $\chi_k = \chi_k(t_{i_1}, ..., t_{i_n})$, $k \neq i_1, i_2, ..., i_n$, such that every solution $\mathbf{t}$ of sufficiently small norm satisfies

$$t_k = \chi_k(t_{i_1}, ..., t_{i_n}) \qquad k \neq i_1, i_2, ..., i_n \tag{3.33}$$

By definition, $\mathbf{t} \in \mathcal{M}$ *if and only if* $(t_{i_1}, ..., t_{i_n}) \in \mathcal{O}$ *and* (3.33) *holds.*

Furthermore, there is a unique function $\lambda = \lambda(t_{i_1}, ..., t_{i_n})$ (continuously differentiable in a neighborhood of the origin) satisfying

$$\lambda(t_{i_1}, ..., t_{i_n}) \to \lambda_i \quad \text{for} \quad (t_{i_1}, ..., t_{i_n}) \to (0, 0, ..., 0)$$

and such that the following system holds for every $\mathbf{t} \in \mathcal{M}$:

$$\lambda(t_{i_1}, ..., t_{i_n})^{-1}\chi_k(t_{i_1}, ..., t_{i_n}) = \frac{\chi_k(t_{i_1}, ..., t_{i_n})}{\lambda_k} + H_k(\mathbf{t}) \qquad (k \neq i_1, i_2, ..., i_n); \tag{3.34}$$

$$\lambda(t_{i_1}, ..., t_{i_n})^{-1} = \lambda_i^{-1} + \frac{\sum_{j=1}^n H_{i_j}(\mathbf{t})t_{i_j}}{\sum_{j=1}^n t_{i_j}^2} \tag{3.35}$$

(see [10], eq. (11.14)).

Let us denote by $<,>$ the duality product between E and E^*. By (3.24), (3.26), (3.34) and (3.35), we have for every $\mathbf{t} \in \mathcal{M}$:

$$\begin{aligned} < I'(\mathbf{t}), \mathbf{t} > = < f(\mathbf{t}), \mathbf{t} > &= \sum_{j=1}^n f_{i_j}(\mathbf{t})t_{i_j} + \lambda(t_{i_1}, ..., t_{i_n})^{-1} \sum_{k \neq i_1, i_2, ..., i_n} \chi_k^2(t_{i_1}, ..., t_{i_n}) \\ &= \sum_{j=1}^n [\lambda_i^{-1} t_{i_j}^2 + H_{i_j}(\mathbf{t})t_{i_j}] + \lambda(t_{i_1}, ..., t_{i_n})^{-1} \sum_{k \neq i_1, i_2, ..., i_n} \chi_k^2(t_{i_1}, ..., t_{i_n}) \\ &= \lambda(t_{i_1}, ..., t_{i_n})^{-1} \Big[\sum_{j=1}^n t_{i_j}^2 + \sum_{k \neq i_1, i_2, ..., i_n} \chi_k^2(t_{i_1}, ..., t_{i_n}) \Big] \end{aligned} \tag{3.36}$$

The form of the last term suggests to define the *submanifold*

$$\mathcal{D}_\epsilon = \{\mathbf{t} \in \mathcal{M} : \sum_{j=1}^n t_{i_j}^2 + \sum_{k \neq i_1, i_2, ..., i_n} \chi_k^2(t_{i_1}, ..., t_{i_n}) = \epsilon^2\} \tag{3.37}$$

The critical points $\mathbf{t}$ of $I|_{\mathcal{D}_\epsilon}$ satisfy

$$< I'(\mathbf{t}), \mathbf{x} >= 0 \tag{3.38}$$

for all $\mathbf{x}$ in the tangent space $TD_{\epsilon\mathbf{t}} \subset T\mathcal{M}_{\mathbf{t}}$. By (3.33), we get

$$T\mathcal{M}_{\mathbf{t}} = \Big\{ \mathbf{x} = \{x_i\}_{i=1,2,..} : x_k = \sum_{j=1}^{n} x_{i_j} \partial_{i_j} \chi_k(t_{i_1}, ..., t_{i_n}), \, \forall k \neq i_1, ..., i_n \Big\} \tag{3.39}$$

and therefore

$$TD_{\epsilon\mathbf{t}} = \Big\{ \mathbf{x} \in T\mathcal{M}_{\mathbf{t}} : \sum_{j=1}^{n} x_{i_j} t_{i_j} + \sum_{k \neq i_1, i_2, ..., i_n} (\sum_{j=1}^{n} x_{i_j} \partial_{i_j} \chi_k) \chi_k = 0 \Big\} \tag{3.40}$$

Then, by (3.36)-(3.40) every critical point $\mathbf{t}$ of $I|_{\mathcal{D}_\epsilon}$ satisfies

$$0 = \big\langle \mathbf{f}(\mathbf{t}) - \epsilon^{-2} < \mathbf{f}(\mathbf{t}), \mathbf{t} > \mathbf{t} \, , \, \mathbf{y} \big\rangle = \big\langle \mathbf{f}(\mathbf{t}) - \lambda(t_{i_1}, ..., t_{i_n})^{-1} \mathbf{t} \, , \, \mathbf{y} \big\rangle \tag{3.41}$$

for every $\mathbf{y} \in \mathrm{span}\{\mathbf{t}, TD_{\epsilon\mathbf{t}}\}$.

We now prove that the pair $(\lambda, \mathbf{t})$ solves (3.32). By (3.26) and (3.34), we need only to consider in (3.41) the projection

$$P\mathbf{y} \in \ \mathrm{span} \big\{ \{t_{i_j}\}_{j=1,...,n}, \{x_{i_j}\}_{j=1,...,n} \big\}$$

where, by (3.40), the $\{x_{i_j}\}_{j=1,...,n}$ span the $n-1$ dimensional subspace of $\mathbb{R}^n$ of the vectors satisfying

$$\sum_{j=1}^{n} x_{i_j} \big[t_{i_j} + \sum_{k \neq i_1, i_2, ..., i_n} \chi_k \partial_{i_j} \chi_k \big] = 0$$

Thus, we are left to prove that the vector $\{t_{i_j}\}_{j=1,...,n}$ does not belong to the previous subspace. Actually, the equation

$$\sum_{j=1}^{n} t_{i_j}^2 + \sum_{j=1}^{n} t_{i_j} \big[\sum_{k \neq i_1, i_2, ..., i_n} \chi_k \partial_{i_j} \chi_k \big] = 0 \tag{3.42}$$

has no nontrivial solution for small t_{i_j} since the second term at the left hand side is $o(\sum_{j=1}^{n} t_{i_j}^2)$ (see appendix). Thus, every critical point $\mathbf{t}$ of $I|_{\mathcal{D}_\epsilon}$ solves (3.32), with λ given by (3.35). Since I is even, the theorem follows by corollary 11.30 of [10]. □

Now, by recalling (3.17), we may claim that to every solution $\mathbf{t}$ of (3.32), i.e. of (3.21), it corresponds a non trivial solution $u_0(\mathbf{t}) \in H^{3/2}(\Omega)$ to (3.13) or, equivalently, a nontrivial solution $\phi \in H^1(\partial\Omega)$ to (3.12). Thus, we conclude that for λ near to an eigenvalue λ_i of (2.1) of multiplicity n, the nonlinear boundary value problem (1.2) (with a non negative weight μ) has at least n distinct pairs of non trivial solutions.

Remark 3.3. *In the general case of the boundary condition* (1.1) *(with $\mu \geq 0$) we can repeat with obvious modifications the discussion leading to theorem 3.2. The main difference is that the functional I is no more symmetric, so that its restriction to the submanifold $\mathcal{D}_\epsilon$ has (at least) two different critical points corresponding to the maximum and minimum. As a consequence, there is an $r_0 > 0$ such that for each $r \in (0, r_0)$ there exist at least 2 distinct solutions $(\lambda_k(r), \mathbf{t}_k(r))$, $k = 1, 2$, with $\|\mathbf{t}_k\| = r$; moreover $\lambda_k(r) \to \lambda_i$ as $r \to 0$.*

4 Properties of the solutions and applications

Let us first investigate the regularity of the solutions to (1.2) found in the previous section; recall that, for λ near to the bifurcation values and assuming $\mu \geq 0$, such solutions lie in the Sobolev space $H^{3/2}(\Omega)$ and satisfy (1.2). As in the linear problem, more regularity on the data allows more regularity of the solutions.

Theorem 4.1. *Assume that $\Omega \subset \mathbb{R}^3$ is bounded with $\mathcal{C}^{1,1}$ boundary and let μ be a non negative Lipschitz function on $\partial\Omega$. Then, every solution $u \in H^{3/2}(\Omega)$ to problem (1.2) belongs to $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ for every $\alpha < 1$.*

Proof. By the imbedding $u \in H^{3/2}(\Omega) \subset W^1_3(\Omega)$, by proposition 3.1 and by the regularity of μ, we have

$$\mu \sinh u \in W_p^{1-\frac{1}{p}}(\partial\Omega)$$

for $1 \leq p < 3$. Hence, by theorem 2.5 we get $u \in W^2_p(\Omega)$ for the same values of p. By the Sobolev immersion $W^2_p(\Omega) \subset \mathcal{C}^{0,\alpha}(\bar{\Omega})$, $\alpha = 2 - \frac{3}{p} < 1$, we further obtain

$$\mu \sinh u \in \mathcal{C}^{0,\alpha}(\partial\Omega)$$

Then, the theorem follows by standard Hölder regularity results [14]. □

Remark 4.2. *By iteration of the above proof, if μ is smooth (e.g. $\mu = 1$) on a smooth boundary (e.g. a sphere) we get solutions $u \in \mathcal{C}^\infty(\bar{\Omega})$. If on the contrary μ has a jump discontinuity (e.g. μ is the indicator function of a subset of $\partial\Omega$) by the same arguments as in the proof of proposition 2.7 we still have $u \in \mathcal{C}^0(\bar{\Omega})$.*

We now consider global existence of the bifurcation solutions. It can be checked that the operator G in equation (3.12) is continuous in $\dot{H}^1(\partial\Omega)$ and that maps bounded sets into relatively compact sets. The latter property follows by the bound (3.5), by the previous regularity results and by Sobolev imbedding; continuity can be proved by using arguments similar to those of lemma 2.2 in [2]. Then, in the case of bifurcation from eigenvalues of odd multiplicity, a global result holds (see [11], theorem 1.10). By denoting with $\mathcal{S} \subset \mathbb{R} \times \dot{H}^1(\partial\Omega)$ the closure of the set of the non trivial solutions (λ, ϕ) to (3.12), we have

Proposition 4.3. *Let κ be an eigenvalue of odd multiplicity of the linear problem (2.1) and let $\mathcal{C}$ be the component (i.e. a closed connected subset maximal with respect to inclusion) of $\mathcal{S}$ to which $(\kappa, 0)$ belongs. Then $\mathcal{C}$ is either unbounded or contains $(\bar{\kappa}, 0)$, where $\bar{\kappa} \neq \kappa$.*

Example 4.4. *Let us consider the problem (1.2) with Ω the unit ball of $\mathbb{R}^3$ and $\mu = 1$. It is well known that the eigenfunctions of the corresponding linear Steklov problem are the* homogenous harmonic polynomials *of degree n and that the Steklov eigenvalues are precisely n, $n = 0, 1, 2, \ldots$ Moreover, the dimension of each eigenspace is $2n + 1$. Hence, proposition 4.3 applies to the component of $\mathcal{S}$ containing $(n, 0)$ for every $n = 1, 2, \ldots$.* □

In a spherical domain it is natural to look for solutions with an *axial symmetry* with respect to a diameter (note that there are no nontrivial *radially* symmetric solutions to (1.2) in the ball). By suitably choosing the coordinate system, we may consider solutions symmetric with respect to the z axis, i.e. solutions which are constant along the parallel lines of the sphere; in spherical coordinates, they will only depend on the distance $r = \sqrt{x^2 + y^2 + z^2}$ from the origin, and on the polar angle θ. Let us denote by $H^{3/2}_{ax}(\Omega)$ the subspace of the functions $v \in H^{3/2}(\Omega)$ with the above axial symmetry; the boundary traces $v|_{\partial\Omega}$ with vanishing integral on the sphere will belong to a subspace of (3.9) denoted by $\dot{H}^1_{ax}(\partial\Omega)$. Now, by rotational invariance of the Laplacian and by uniqueness of the solution of the Neumann problem, one can check that *the operator* $G(\lambda,\cdot)$ *defined by* (3.10) *maps* $\dot{H}^1_{ax}(\partial\Omega)$ *in itself*. Moreover, the (non constant) axially symmetric eigenfunctions of the Steklov problem in the ball are those harmonic polynomials which (in polar coordinates) are independent of the azimuthal angle, that is $r^n P_n(\cos\theta)$, $n = 1, 2, ...$ where the P_n are the *Legendre polynomials*. The restrictions of these eigenfunctions to the spherical surface span the subspace of axially symmetric, zero mean functions of $L^2(\partial\Omega)$.

We now define $u_0 \in H^{3/2}_{ax}(\Omega)$ and $\phi \in \dot{H}^1_{ax}(\partial\Omega)$ by restricting (3.17) to the above symmetric eigenfunctions; then, by the same arguments as in section 3 we find nontrivial solutions (λ, u) of (1.2) bifurcating from $(n, 0)$, $n = 1, 2, ...$ and such that $u \in H^{3/2}_{ax}(\Omega)$. We stress that there is a *unique* (normalized) axially symmetric eigenfunction for every eigenvalue n, so that all the eigenvalues of the linear problem in $H^{3/2}_{ax}(\Omega)$ are *simple*. Thus, we get

Proposition 4.5. *Let Ω be the unit ball and let $\mu = 1$. Then, for any $n = 1, 2, ...$ there is a component $\mathcal{C}_n \subset \mathbb{R} \times \dot{H}^1_{ax}(\partial\Omega)$ of $\mathcal{S}$ which meets the point $(n, 0)$; each $\mathcal{C}_n$ is either unbounded or meets $(m, 0)$, with $m \neq n$.*

It would be interesting to establish which of the alternatives of the previous proposition actually holds. We can partially answer to this question by further restricting to the subspace of the axially symmetric functions u in the ball which are *odd* with respect to z; by writing, in spherical coordinates, $u = u(r, \cos\theta)$ we have

$$u(r, \cos\theta) = -u(r, \cos(\pi - \theta)) \qquad \forall\, r \geq 0, \quad 0 \leq \theta \leq \pi \tag{4.1}$$

By putting $\cos\theta = t$, $-1 \leq t \leq 1$, we get $u = u(r, t)$, with $u(r, -t) = -u(r, t)$. The following lemma is required to prove our results:

Lemma 4.6. *Let Ω be the unit ball and let $u \in \mathcal{C}^1(\bar{\Omega})$ an axially symmetric solution to problem* (1.2) *(with $\mu = 1$) which satisfies* (4.1)*. Let us write $u = u(r, t)$ and assume that one of the following cases occurs:*

i) *The function $u(1, \cdot)$ has double zeros, but no simple zeros in $(-1, 0) \cup (0, 1)$;*

ii) *$u(1, \cdot)$ has no zeros in $(-1, 0) \cup (0, 1)$ and it vanishes at $t = 0, \pm 1$;*

iii) *$u(1, \cdot)$ has a unique double zero at $t = 0$.*

Then, $u = 0$.

Proof. By (4.1), $u(r, 0) = 0$ for every $r \in [0, 1]$. Hence, the plane $z = 0$ is a nodal surface for u. Assume that $u(1, \cdot)$ vanishes at $\pm t \in (-1, 0) \cup (0, 1)$; then, there are two nodal circles on the spherical boundary, symmetric with respect to the equatorial plane. If we consider any longitudinal section of the sphere, we have $u = 0$ on the diameter of a disk and at two pairs of symmetric points on its circular boundary, one pair on the upper half, the other obtained by reflection on the lower half. By symmetry, it is enough to consider the zeros of u in the (closed) upper semi disk; we denote by C such domain and by ∂C_+ the part of its boundary not lying on the diameter. Note that every symmetric pair of zeros on ∂C_+ corresponds to a nodal circle on the upper spherical surface and every nodal line in C corresponds to a nodal rotational surface in the upper half of the ball.

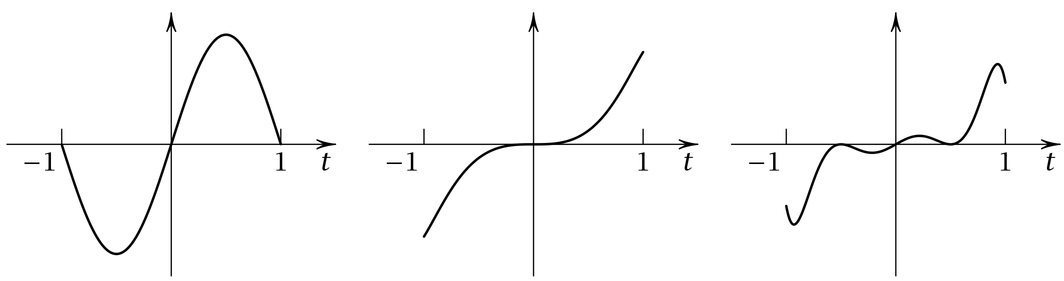
If u does not vanish identically, by the Hopf principle the zeros on ∂C_+ are not extremum points since the boundary condition implies $\partial_r u(1, t) = 0$; moreover, if $u(1, \cdot)$ has a *double zero* at t, we have $\nabla u(1, t) = 0$. Hence, we may assume that from each of these points originate (at least) *two* distinct

nodal lines of u. In fact, should a single nodal line arise, we get in the three dimensional space a nodal surface of double zeros of u; since u is harmonic, this implies $u = 0$.
Thus, we have two symmetric pairs of nodal lines beginning at symmetric points on ∂C_+; assuming that the only other zeros of u are the points of the diameter, these nodal lines will necessarily form, possibly with part of the diameter itself, a closed path in C. In the three dimensional space, we get a closed nodal surface for u, so that $u = 0$ by the maximum principle. The same conclusion holds if there are arbitrary pairs of double zeros in $(-1, 0) \cup (0, 1)$, but no simple zeros.
Assume now that *ii*) holds. Hence, u vanishes at the top of C; if there exist at least two nodal lines originating from this point, we can repeat the previous arguments to conclude that $u = 0$. On the other hand, if there is only one nodal line outgoing from the top, this is necessarily the perpendicular to the diameter, so that $u = 0$ again by the maximum principle.
Finally, should u have a double zero at $t = 0$, either the diameter is a line of double zeros, or it is part of a closed nodal line in C. In both cases, we get $u = 0$. $\square$

Let us now denote by $V \subset \dot{H}^1_{ax}(\partial\Omega)$ the subspace of the functions ϕ such that $\phi(-t) = -\phi(t)$; by the invariance of the Laplace operator with respect to the reflection $z \mapsto -z$ and by the symmetry of the Neumann condition on the sphere, it follows that the (axially symmetric) solutions of the Neumann problem in the ball, with boundary data in V and having zero mean on the surface, are *odd* functions of $t = \cos\theta$. Hence, we can further restrict the functional formulation of the nonlinear equation (3.12) to the subspace V. The representation (3.17) will now contain only the eigenfunctions with odd indices $r^{2k+1}P_{2k+1}(\cos\theta)$, $k = 0, 1, 2, ...$ Then, we can state

Theorem 4.7. *Let Ω be the unit ball and let $\mu = 1$. Then, for any $k = 0, 1, 2, ...$ there is a component $\mathcal{D}_k \subset \mathbb{R} \times V$ of $\mathcal{S}$ which meets the point $(2k + 1, 0)$; each $\mathcal{D}_k$ is either unbounded or meets $(2j + 1, 0)$ for some $j \neq k$. The component $\mathcal{D}_0$ is unbounded.*

Proof. The first part of the theorem follows by the previous discussion and by proposition 4.3. We now show that the $\mathcal{D}_0$ is actually unbounded. Recall that, by regularity, every solution u to problem (1.2) belongs to $\mathcal{C}^1(\bar{\Omega})$. Let us consider, in the subspace of the *odd* functions $\phi(t) \in \mathcal{C}^1([-1, 1])$, the subset S_k of the functions ϕ having exactly $2k+1$ simple zeros in $(-1, 1)$ and such that $\phi(1) > 0$. The set S_k is open and $P_{2k+1}(t) \in S_k$. Then, by the same arguments as in Lemma 2.7 of [11], there is a neighborhood $\mathcal{N}_k \subset \mathbb{R} \times V$ of $(2k+1, 0)$ such that if $(\lambda, \phi) \in \mathcal{N}_k$ is a non trivial solution, then $\phi \in S_k$. In particular, we have $\mathcal{D}_k \cap \mathcal{N}_k \subset (\mathbb{R} \times S_k) \cup (2k+1, 0)$ for every k. We claim that $\mathcal{D}_0 \subset (\mathbb{R} \times S_0) \cup (1, 0)$; then, since $S_j \cap S_k = \emptyset$ for $j \neq k$, the theorem follows.
In order to prove the claim, assume that $\mathcal{C}_0 \not\subset \mathbb{R} \times S_0 \cup (1, 0)$; then, there is $(\lambda, \phi) \in \mathcal{C}_0 \cap (\mathbb{R} \times \partial S_0)$, $(\lambda, \phi) \neq (1, 0)$, which is a limit point of a sequence $(\lambda_n, \phi_n) \in \mathbb{R} \times S_0$; but, as shown in the examples below, a function in ∂S_0 either vanishes at $t = \pm 1$ or it has a double zero at the origin or symmetric double zeros in $(-1, 0) \cup (0, 1)$.



In any case, since $\phi(t) = u(1, t)$ for some solution u of (1.2), it follows by lemma 4.6 that $\phi = 0$. Hence, $\lambda = 2j + 1$ for some $j > 0$; but this implies that (λ_n, ϕ_n) is definitively in $\mathbb{R} \times S_j$, a contradiction. $\square$

Remark 4.8. *The results obtained by variational methods for the analogous two-dimensional problem in a disk seems to indicate that in the* $(\lambda, \|\phi\|)$ *plane the branches of solutions outgoing from* $(n, 0)$ *become asymptotic to the* $\lambda = 0$ *axis.*

Finally, as an example of different type, we consider a problem in a cube, where μ is the indicator function of a single face.

Example 4.9. *Let* Ω *be the unit cube* $[0,1]^3$ *and let* $\mu = 1$ *on* $[0,1] \times [0,1] \times \{1\}$ *and* $\mu = 0$ *on the remaining part of the boundary. By separation of variables, the eigenfunctions of the linear problem* (2.1) *are*

$$u_{j,k}(x, y, z) = \cos(\pi jx)\cos(\pi ky)\cosh(p_{j,k}z), \qquad j, k \,=\, 0, 1, 2, ...$$

where $p_{j,k} = \pi\sqrt{j^2 + k^2}$*. The corresponding Steklov eigenvalues are*

$$\lambda_{j,k} = p_{j,k} \tanh p_{j,k}$$

Then, by theorem 3.2, if the eigenvalue $\lambda_{j,k}$ *has multiplicity* n *there are at least* n *distinct pairs of non trivial solutions to the non linear problem* (1.2) *bifurcating from* $(\lambda_{j,k}, 0)$*. Since* $p_{j,k} = p_{k,j}$*, if* $j \neq k$ *each eigenvalue has at least multiplicity two, but we can still reduce the multiplicity by considerations of symmetry. In fact, by still focusing on the symmetry properties of the solutions to the Neumann problem, one can prove that the subspaces of the functions* u *which are respectively even and odd for the reflection* $u(x, y) \mapsto u(y, x)$ *are both invariant for the action of the operator* $G(\lambda, \cdot)$ *defined by* (3.10)*. Hence, we can find non trivial solutions of the non linear problem in these subspaces by the same arguments which led to proposition 4.7; in each subspace, the eigenvalue* $\lambda_{j,k}$ *is simple provided that the integer* $(p_{j,k}/\pi)^2$ *can be expressed in a unique way as a sum of two squares, ignoring order and signs. Nevertheless, it is known that there are positive integers which can be represented in more than one way as a sum of two squares, depending on their factorization [15]. For example, we have* $25 = 0^2 + 5^2 = 3^2 + 4^2$*, or* $65 = 1^2 + 8^2 = 4^2 + 7^2$*; hence, even in a subspace of symmetric functions, the eigenvalues* $5\pi \tanh(5\pi)$ *and* $\sqrt{65}\pi \tanh(\sqrt{65}\pi)$ *have multiplicity two.* □

We conclude by observing that if a bifurcation branch contains non trivial solutions ϕ of arbitrarily large norm (in $H^1(\partial\Omega)$) and if λ is bounded along the same branch, then also the sup norm $\|\phi\|_{L^\infty(\Omega)}$ becomes arbitrarily large; if not, the Neumann datum $\lambda \sinh(\phi + s(\phi))$ would be bounded in $L^2(\partial\Omega)$, and the same holds for the $H^{3/2}(\Omega)$ norm of the solution u to (1.2). But since $u = u_0 + s(\phi)$, with $u_0|_{\partial\Omega} = \phi$, we would obtain that also $\|\phi\|_{H^1(\partial\Omega)}$ is bounded, a contradiction.

5 Appendix

We first prove (3.28), i.e.

$$\sum_i \lambda_i H_i(\mathbf{t})^2 = \int_{\partial\Omega} |\mathcal{H}(\phi + s(\phi)|^2 \leq C\|\mathbf{t}\|^6 \tag{5.1}$$

The first equality follows readily by definition (3.27) and by Parseval identity. Let us now estimate the integral. By Taylor expansion of the exponentials in (3.11) and recalling that $\int_{\partial\Omega} \phi = 0$, we can write

$$s(\phi) = \log \frac{1 + \frac{1}{|\Omega|}\int_{\partial\Omega} \phi^2 g_-(\phi)}{1 + \frac{1}{|\Omega|}\int_{\partial\Omega} \phi^2 g_+(\phi)}$$

where the function $g_+(\phi)$ $[g_-(\phi)]$ is bounded for $\phi \leq 0$ $[\phi \leq 0]$ and such that $g_+(\phi) \leq e^\phi/2$ for $\phi > 0$ $[g_-(\phi) \leq e^{-\phi}/2$ for $\phi < 0]$. Hence, by the estimates of section 3 and by Hölder inequality we get

$$|s(\phi)| \leq C_1\|\phi\|^2_{L^4(\partial\Omega)} \leq C_2\|\phi\|^2_{H^1(\partial\Omega)} \leq C_3\|\mathbf{t}\|^2 \tag{5.2}$$

for $\mathbf{t}$ in a bounded set of E.
Now, by (3.25) we have

$$\mathcal{H}\big(\phi + s(\phi)\big) = \big(\phi + s(\phi)\big)^3 \tilde{\mathcal{H}}\big(\phi + s\phi)\big)$$

where $0 < \tilde{\mathcal{H}}(\phi) < \cosh(\phi)/6$. Then, for $\mathbf{t}$ in a bounded set,

$$\int_{\partial\Omega} \mathcal{H}\big(\phi + s(\phi)\big)^2 \le K_1 \|\phi + s(\phi)\|^6_{L^{12}(\partial\Omega)} \le K_2 \|\phi + s(\phi)\|^6_{H^1(\partial\Omega)} \le K_3 \|\mathbf{t}\|^6 \tag{5.3}$$

where in the last inequality we used (5.2).
We prove now the claim following (3.42). We need an estimate of the term

$$\sum_{k \neq i_1, i_2, \ldots, i_n} \chi_k(t_{i_1}, ..., t_{i_n}) \partial_{i_j} \chi_k(t_{i_1}, ..., t_{i_n}) \tag{5.4}$$

Let us write (3.34) in the form

$$\chi_k(t_{i_1}, ..., t_{i_n}) = \big(\lambda(t_{i_1}, ..., t_{i_n})^{-1} - \lambda_k^{-1}\big)^{-1} H_k(\mathbf{t}) \tag{5.5}$$

where the components of $\mathbf{t}$ satisfy $t_k = \chi_k(t_{i_1}, ..., t_{i_n})$ and $k \neq t_{i_1}, ..., t_{i_n}$. By recalling that, for small t_{i_j}, $\lambda(t_{i_1}, ..., t_{i_n}) \approx \lambda_i \neq \lambda_k$ and by the bound (5.1), we obtain

$$\sum_{k \neq i_1, i_2, \ldots, i_n} \lambda_k \chi_k^2(t_{i_1}, ..., t_{i_n}) \le C\big(\sum_{j=1}^{n} t_{i_j}^2\big)^3 \tag{5.6}$$

for $\{t_{i_j}\}_{j=1,\ldots,n}$ in a neighborhood of the origin.
Moreover, by differentiating (3.34) and (3.35) with respect to t_{i_1}, $j = 1, 2, ...n$, and by the same calculations as in [10], equations (1.16)-(1.19), one can prove (we omit the details)

$$\sum_k \lambda_k |\partial_{i_j} \chi_k|^2 \le C\big(\sum_{l=1}^{n} t_{i_l}^2\big)^2 \tag{5.7}$$

Finally, by estimating (5.4) taking account of (5.6), (5.7), the claim follows.

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