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Cipriani, F.; Guido, D. Isola, T. Sauvageot J.L.

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Piazza Leonardo da Vinci, 32 - 20133 Milano (Italy)

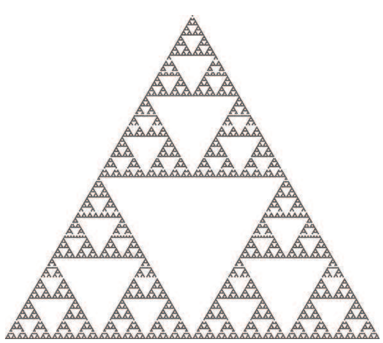
SPECTRAL TRIPLES FOR THE SIERPINSKI GASKET

FABIO CIPRIANI, DANIELE GUIDO, TOMMASO ISOLA, AND JEAN-LUC SAUVAGEOT

Abstract. We construct a 2-parameter family of spectral triples for the Sierpinski Gasket K. We determine their associated Connes' distances in terms of suitable roots of the plane Euclidean metric and their dimensional spectra, and show that the pairing of the associated Fredholm module with (odd) K-theory is non-trivial. We recover the Hausdorff measure of K in terms of the residue of the functional $a \to \mathrm{tr}_\omega(a\,|D|^{-s})$ at the abscissa of convergence d, which coincides with the Hausdorff dimension of the fractal. We recover also the unique, standard Dirichlet form on K, as the residue of the functional $a \to \mathrm{tr}_\omega(|[D,a]|^2\,|D|^{-s})$ at the abscissa of convergence δ, which we call the *energy dimension*. The fact that the volume dimension differs from the energy dimension, $d \neq \delta$, reflects the fact that on K energy and volume are distributed singularly.

1. Introduction

The advent of Noncommutative Geometry allowed to consider from a geometrical and analytical point of view spaces which appear to be singular when analyzed using the classical tools of Differential Calculus and Riemannian Geometry.



In the present paper we approach from a NCG point of view the study of a compact subset K of the plane which is a central example among fractal sets, namely the Sierpinski Gasket. We associate to the gasket a family of spectral triples depending on two parameters. For values of the parameters in suitable ranges, the triple reconstructs the main known features of the gasket, namely its similarity dimension, Hausdorff measure, a distance which is bi-Lipschitz w.r.t. the Euclidean one and the Kigami energy form. Moreover, it pairs non trivially with K-theory and a *energy dimension* appears.

The fundamental topological property of K is its self-similarity, by which it can be reconstructed as a whole from the knowledge of any arbitrary small part of it. More precisely, considering the three contractions F_1, F_2, F_3 of scaling parameter $1/2$ fixing respectively the vertexes p_1, p_2, p_3 of an equilateral triangle, one may characterize K as the only compact set such that

$$K = F_1(K) \cup F_2(K) \cup F_3(K)\,.$$

In particular, K is the fixed point of the map $K \mapsto F_1(K) \cup F_2(K) \cup F_3(K)$ which is a contraction with respect to the Hausdorff distance on compact subsets of the plane. This allows various approximations of K as, for example, the one given by finite graphs.

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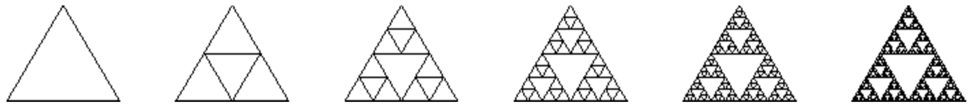

The gasket K was introduced by Sierpińsky for pure topological motivations [39]. Successively, using measure theory, it was noticed that it is a space with a non integer Hausdorff dimension $d = \frac{\ln 3}{\ln 2}$ [28], and later K attracted the attention of Probabilists who constructed a stochastic process X_t with continuous sample paths on K [33]. The process is symmetric w.r.t. the Hausdorff measure μ_H and has a self-adjoint generator Δ in $L^2(K, \mu_H)$ with discrete spectrum. Finally, Kigami [32] associated with the gasket (and other fractals) a so called harmonic structure, which directly produces a Dirichlet form, whose generator in $L^2(K, \mu_H)$ coincides with Δ.

We recall that the main object for the spectral description of the metric aspects of a geometry, introduced by Connes [16], is the so called spectral triple $(\mathcal{A}, \mathcal{H}, D)$. It consists of an algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$, and of a selfadjoint unbounded operator D, the so called Dirac operator. Main requests are the boundedness of the commutators $[D, a]$ of the elements $a \in \mathcal{A}$ with D, and the fact that D has compact resolvent.
With those data, one may reconstruct the basic objects of Riamanniann geometry. The integral $\int a$ w.r.t. the Riemannian volume form is replaced by the functional

$$\mathcal{A} \ni a \mapsto \mathrm{Tr}_\omega(a|D|^{-d}) \,,$$

Tr_ω being the Dixmier logarithmic trace. There is a unique exponent d_D (if any) giving rise to a non-trivial functional, called metric dimension (cf. [23, 17]). The differential 1-form da is replaced by the commutator $[D, a]$, the Lipschitz seminorm $\|da\|_\infty$ is then replaced by $\|[D, a]\|$, and the Riemannian metric can be recovered as a Monge-Kantorovitch distance among states

$$\rho_D(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : \|[D, a]\| = 1\} \,.$$

A simple but fruitful idea is to use the noncommutative analogs considered above for defining the Dirichlet energy:

$$\mathcal{E}[a] = \int |da|^2 \longrightarrow \mathcal{E}[a] = \mathrm{Tr}_\omega([D, a]^2|D|^{-\delta}). \tag{1.1}$$

Similar formulas have been recently considered in [7, 31]. Moreover, spectral triples can reproduce topological invariants. Indeed, a spectral triple gives rise to a class in the K-homology of the algebra $\mathcal{A}$, hence pairs with K-theory. Such pairing may be expressed in terms of the Fredholm module $(\mathcal{A}, \mathcal{H}, F)$ associated with the spectral triple, F being the phase of D. In particular, if the triple is odd, and u is an invertible element of $\mathcal{A}$, the pairing is given by $u \to \mathrm{Ind}(P_+ u P_+)$, where P_+ is the projection on the positive part of the spectrum of F, and $P_+ u P_+$ is a Fredholm operator acting on the space $P_+\mathcal{H}$.

As mentioned above, one of the features of Noncommutative Geometry is that it non only describes noncommutative manifolds, such as the quantum torus or the various quantum spheres, but it is also able to describe some classical (e.g. based on points) but singular geometries. The first example of this fact was given by Connes in [16], where a spectral triple was assigned to the Cantor set. Such a triple could reconstruct dimension, measure and distance of the fractal set, as well as the pairing with K-theory. The peculiar aspects of the construction are the following: the triple is a direct sum of triples associated with elementary building blocks; the building blocks are the lacunas of the fractal, namely the boundaries of the removed intervals in the Cantor set. The first idea can easily be adapted to self-similar fractals, by choosing as building blocks the images of a suitable set via compositions of similarities. This has been exploited in [12] for the Sierpinski gasket (as anticipated in [11]), where the building blocks are the lacunas of the gasket, meant as the boundaries of the removed triangles in K. Again, dimension, measure, distance and pairing with K-theory are reconstructed in spectral terms.

Our aim here is to add to this list the Dirichlet form given by the harmonic structure on the gasket, as described by Kigami [32], by making use of formula (1.1). While for regular geometries the energy form may be considered as a derived object, given the other geometric instruments, for fractals this is not the case. Indeed, for self-similar fractals, energy measures are singular w.r.t. the self-similar measures (cf. [26, 27]). This singularity is reflected in the noncommutative picture by the difference between d and δ in the formulas above, quantities which would be equal in the case of regular spaces. Moreover, this provides a new *energy dimension* δ, which, for the standard values of the parameters, is equal to $\frac{\log(12/5)}{\log 2} \approx 1.26$ (smaller than the similarity dimension $d_S := \frac{\log 3}{\log 2} \approx 1.58$), which has no apparent classical analogue. Moreover, we could not prove that the two formulas $\mathrm{tr}_\omega(|[D,a]|^2|D|^{-\delta})$ and $\mathrm{Res}_{s=1}\,\mathrm{tr}(|[D,a]|^2|D|^{-s\delta})$ coincide on elements with finite Kigami energy. Indeed they will, up to a multiplicative constant, only for a sub-algebra which is a core for the Dirichlet form, while the second formula provides a multiple of the Dirichlet form on the whole form domain. Let us mention here that the spectral triple for the gasket proposed in [24] Remark 2.14, whose building blocks were spectral triples associated with the boundary points of the edges of the gasket, could indeed produce the Kigami energy exactly as above. However, being based on a discrete approximation of K, it could not give rise to any pairing with K-theory.

We remark that noncommutative geometry provides also a replacement for the de Rham cohomology in terms of cyclic cohomology, however we do not pursue this direction here. Indeed differential forms have no classical analogue on fractals, and their study in this case is essentially based on [14]. There, the authors associate a bimodule-valued derivation to a Dirichlet form, and define differential 1-forms as the elements of the bimodule. Recent developments in this direction are contained in [29], while in a recent paper of ours [13] concerning the gasket, we give a more concrete description of the 1-forms of [14] in such a way as to define their integrals on paths, and their generalized potentials on suitable coverings.

We now come to a more detailed description of our family of spectral triples. As in [12], our building blocks are associated to the lacunas (boundaries of removed triangles) of the gasket canonically identified with circles. However, we deform the classical spectral triple for the circle $\mathbb{T}$, by replacing the standard Laplacian Δ with its powers Δ^α, $\alpha \in (0,1]$. As proposed in [14], we define a bimodule-valued derivation ∂_α, and define the Dirac operator as

$$D := \begin{pmatrix} 0 & \partial_\alpha \\ \partial_\alpha^* & 0 \end{pmatrix}.$$

While this deformation does not quantize the algebra, which remains $C(\mathbb{T})$, a zest of noncommutativity is nonetheless present, since the left and right action of functions on the Hilbert space do not coincide (functions do not commute with forms). This is related to the fact that, while for $\alpha = 1$ the distributional kernel giving rise to the energy on $\mathbb{T}$ is supported on the diagonal, this is no-longer true for $\alpha < 1$.
A second parameter β deforms the standard metric scaling parameter, which is equal to $1/2$ for the gasket, to $2^{-\beta}$. An unexpected outcome of the construction is that the two

parameters have a quite different role for the gasket as a whole. Indeed, α only plays the role of a threshold parameter. The condition $\alpha \leq \alpha_0 = \frac{\log(10/3)}{\log 4}$ is a necessary condition for formula (1.1) to be finite for finite energy functions, and to reproduce Kigami energy. If, furthermore, $\alpha > \left(2 - \frac{\log(5/3)}{\beta \log 2}\right)^{-1}$, one gets a full-fledged spectral triple, whose features only depend on β, which assumes the role of a deformation parameter.
In fact, for $\alpha_0 < \beta \leq 1$, the metric $\rho_{D,\beta}$ is bi-Lipschitz w.r.t. the Euclidean distance raised to the power β or, equivalently, bi-Lipschitz w.r.t the geodesic metric, induced on K by the Euclidean metric, raised to the power β. Then the metric dimension is given by $d_{D,\beta} = d_{D,1} \cdot \beta^{-1}$, and, as expected, the volume measure $\mu_{D,\beta}$ coincides (up to a multiplicative constant) with the Hausdorff measure for the dimension $d_{D,\beta}$, which in turn coincides with the Hausdorff measure for the dimension d_S. The energy dimension is given by $\delta_{D,\beta} = 2 - \frac{\log(5/3)}{\log 2}\beta^{-1}$, and the corresponding energy form do not even depend on β, apart from a multiplicative constant.

Neither α nor β affect the pairing with K-theory. However we had to tackle another difficulty concerning the Fredholm module associated with the spectral triple. In fact, in order to implement the deformation associated with the parameter α, we had to choose the Hilbert space as the module of differential forms, making the triple (and the Fredholm module) an even one. To recover the pairing with odd K-theory, we have to add a further grading, obtaining a 1-graded Fredholm module, which then has the correct pairing with odd K-theory. We conclude this review of the dependence on α and β by saying that the requests concerning spectral triple properties reflect into independent bounds on the parameters, which finally give rise to a quite small fraction of the (β, α)-plane. The fact that this set is indeed non empty is not at all obvious, and only an analysis of a larger family of fractals and their Dirichlet forms may reveal the reasons of its non-triviality.

We note here that our triples indeed violate one of the requests of a spectral triple as defined in [16], since the kernel of the Dirac operator is infinite dimensional. However, this degeneracy of the kernel does not cause any harm in the construction, when taking the point of view of reading $|D|^{-s}$ as the functional calculus of D with the function $f(t) = 0$ for $t = 0$ and $f(t) = |t|^{-s}$ for $t \neq 0$.
The question is more subtle when the associated Fredholm module is concerned. Indeed, denoting by $P_\pm$ the projection on the positive, resp. negative, part of the spectrum of D, the two formulas for the pairing of the module with the K-theory class of an invertible element u given by $\mathrm{Ind}(P_+\pi(u)P_+)$ and $-\mathrm{Ind}(P_-\pi(u)P_-)$, which are equivalent when the dimension of the kernel of D is finite, may be expected to differ. We call a Fredholm module *tamely degenerate* when such equality holds, hence the kernel of D is irrelevant from the K-theoretical point of view, and check that this condition is satisfied for our triples.

We describe now some technical aspects of our construction. First, in order to construct a Dirac operator for the α-deformed triples on the circle, we had to define a differential square root of Δ^α, or, in other terms, a derivation implementing the corresponding Dirichlet form. This has been done by realizing the corresponding Dirichlet form in terms of an integral operator, whose distributional kernel is written in terms of a special function, the so called Clausen cosine function Ci_s. We show that $-\mathrm{Ci}_{-2\alpha} \geq 0$, for $0 < \alpha < 1$, and describe the Dirac operator in terms of the derivation ∂_α given by $\partial_\alpha f(x,y) = (-2\pi \, \mathrm{Ci}_{-2\alpha}(x-y))^{1/2}(f(x)-f(y))$. By means of some explicit estimates on Ci_α we can show the relation of the Connes' distance for the α-deformed circle and the α-power of the Riemannian distance. In this sense, our deformed circles may be considered as quasi-circles, since the α-power of the Riemannian distance clearly satisfies the so-called reverse triangle inequality [1]. As for the case of the gasket described above, the α-deformation rescales the Hausdorff dimension of the circle and leaves the volume invariant (up to a multiplicative constant).

Second, our study of the noncomutative formula for the Kigami energy produces an interesting situation when Dixmier traces are concerned. Indeed, when used to describe the volume form in noncommutative geometry, the Dixmier trace is computed for elements which belong to the principal ideal generated by $|D|^{-d}$, and the same happens for the computation of the energy form according to formula (1.1) when regular spaces are considered, namely when the metric dimension d_D and the energy dimension δ_D coincide. It is known that the theory of singular traces on principal ideals ([40, 2, 22] etc.) is in a sense simpler than the corresponding theory on symmetrically normed ideals. In the case of fractal spaces however, there is no principal ideal containing all elements of the form $|[D, f]|^2|D|^{-\delta}$, and Dixmier traces on symmetrically normed ideals and the analysis in [9, 10] play a key role.

As for the organization of the paper, it consists of this introduction, three sections, and an appendix. The first section is devoted to some results on (possibly degenerate) spectral triples and Fredholm modules, the second to the description of the α-deformed circles, and the third to the construction and results of the triples on the gasket. The appendix contains some estimates concerning the Clausen functions.

The results contained in this paper have been described in several conferences, such as Cardiff 2010, Cambridge 2010, Cornell 2011.

2. Spectral Triples and their Fredholm Modules

2.1. **Spectral Triples and their Fredholm Modules.** We define here Spectral Triples $(\mathcal{A}, \mathcal{H}, D)$ and the construction of the associated Fredholm module $(F, \mathcal{H})$, in case the Dirac operator D may have an infinite dimensional kernel. This discussion will be useful later on, when we will construct Spectral Triples on circles and the Sierpinski Gasket, whose Dirac operators have an *infinite dimensional kernel*. There, some extra work will be needed to construct an associated Fredholm module having nontrivial pairing with K theory.

Definition 2.1. (Spectral Triple) A (possibly degenerate, compact) Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$ consists of an involutive unital algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, acting faithfully on a Hilbert space $\mathcal{H}$, and a self-adjoint operator $(D, \mathrm{dom}(D))$ on it, subject to the conditions

(i) the commutators $[D, a]$, initially defined on the domain $\mathrm{dom}(D) \subset \mathcal{H}$ through the sesquilinear forms
$$(\xi, [D, a]\eta) := (D\xi, a\eta) - (a^*\xi, D\eta) \qquad \xi, \eta \in dom(D)\,,$$
extend to bounded operators on $\mathcal{H}$, for all $a \in \mathcal{A}$;

(ii) the operator D^{-1} is compact on $\ker(D)^{\perp}$.

The operator $(D, \mathrm{dom}(D))$ is referred to as the *Dirac operator* of the Spectral Triple.

Notice that if $\ker(D)$ is finite dimensional the condition in (ii) reduces to the compactness of the operators $(I + D^2)^{-1}$. We recover in this way the original definition of a Spectral Triple by Connes [16]. The extended version is required to deal with the Dirac operators we will consider on quasicircles and on the Sierpinski Gasket.
Notice also that the requirement in (ii) amounts to the discreteness of the spectrum of the Dirac operator D.

Example 2.2. As an example of Spectral Triple, where the algebra is commutative, consider the algebra $\mathcal{A} := C^\infty(M)$ of smooth functions on a compact Riemannian manifold M, acting on the Hilbert space $\mathcal{H} := L^2(\Lambda(M))$ of square integrable sections of the exterior bundle over

M, as well the self-adjoint operator $D := d + d^*$, sum of the exterior differential d and its adjoint. Since the square of the Dirac operator is just the Hodge-de Rham operator of M,

$$D^2 = (d + d^*)^2 = dd^* + d^*d = \Delta_{HdR}\,,$$

condition (ii) is verified by the discreteness of the spectrum of Δ_{HdR}, which follows from the compactness of M. The rules of differential calculus on M allow to easily verify that the norm of a commutator coincides with the Lipschitz semi-norm of a function

$$\|[D, a]\| = \sup\{|d_x a|_{T^*_x M} : x \in M\}\,.$$

This also shows that the Riemannian distance on M can be recovered from the Spectral Triple:

$$d_M(x, y) = \sup\{\|[D, a]\| : a \in \mathcal{A}\} \qquad x, y \in M\,.$$

Notice that in this case the kernel of D is just the space of harmonic forms on M.

Definition 2.3. (Fredholm Module) A (possibly degenerate) Fredholm Module $(F, \mathcal{H})$ over an involutive algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, acting faithfully on a Hilbert space $\mathcal{H}$, consists of a bounded operator $F \in \mathcal{B}(\mathcal{H})$ such that

(i) $F^2 - I$ is a compact operator on $\ker(F)^\perp$,
(ii) $F^* - F$ is a compact operator,
(iii) the commutators $[F, a]$ are compact operators, for all $a \in \mathcal{A}$.

It follows from the completeness of the space of compact operators in $\mathcal{B}(\mathcal{H})$ that, if $(F, \mathcal{H})$ is a Fredholm module over $\mathcal{A}$, then it extends automatically to a Fredholm module over the C*-algebra $\overline{\mathcal{A}}$, closure of the algebra $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$.

The classical formulation of Atiyah is recovered when $\ker(F)$ is finite dimensional. Again, the above generalization is required to deal with the Fredholm Modules we will construct on quasicircles and on the Sierpinski Gasket.

Example 2.4. As an example of Fredholm Module, consider the algebra $\mathcal{A} := C(M)$ of continuous functions on a compact Riemannian manifold M, acting on the Hilbert space $\mathcal{H} := L^2(\Lambda(M))$ of square integrable sections of the exterior bundle over M, as well the sign $F := \mathrm{sgn}(D)$ of the Dirac operator $D := d + d^*$ considered in Example 2.2. Here F is self-adjoint, $\ker(F)$ is the finite dimensional space of harmonic forms, so that $F^* = F$, $F^2 - I$ is a finite rank operator, and the first two requirements for a Fredholm Module are fulfilled. The third one follows from the fact that the commutator $[F, a]$ is clearly a pseudo-differential operator of order -1 on M, so that it is a compact operator, by a well known result of Analysis.

A classical result by Baaj and Julg [3] shows that Spectral Triples give rise to Fredholm modules by taking $F = \mathrm{sgn}(D)$ (or any other function which is asymptotic to $\mathrm{sgn}(t)$ for $|t| \to \infty$), whenever $\dim\ker(D) < +\infty$. We need to generalize this result to allow $\dim\ker(D) = +\infty$. The key point is to show that the boundedness of the commutant $[D, a]$ implies the compactness of $[F, a]$, and we simply observe that this remains true even if $\ker(D)$ is not finite dimensional.

Proposition 2.5. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a (possibly degenerate) Spectral Triple over a unital $*$-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. Then, setting $F := \mathrm{sgn}(D)$, $(F, \mathcal{H})$ is a Fredholm module over $\overline{\mathcal{A}}$.*

Proof. Since F is self-adjoint and $F^2 - I$ vanishes on $\ker(F)^\perp = \ker(D)^\perp$ by construction, the first two requirements in the Definition 2.3 of a Fredholm module hold true.

To verify the third, let us first observe that $\frac{1}{\sqrt{x}} = \frac{2}{\pi}\int_0^{+\infty}\frac{dt}{x+t^2}$ for any $x > 0$, from which it follows that $F = \frac{2}{\pi}\int_0^{+\infty}\frac{D}{t^2+D^2}\,dt$, the integral converging in the norm of the Banach space $\mathcal{B}(\mathcal{H})$. Then we have

$$\begin{aligned}
[F,a] &= \frac{2}{\pi}\int_0^{+\infty}\left(\frac{D}{t^2+D^2}\,a - a\,\frac{D}{t^2+D^2}\right)dt\\
&= \frac{2}{\pi}\int_0^{+\infty}\frac{1}{t^2+D^2}\big(Da(t^2+D^2)-(t^2+D^2)aD\big)\frac{1}{t^2+D^2}\,dt\\
&= \frac{2}{\pi}\int_0^{+\infty}\frac{t}{t^2+D^2}\,[D,a]\,\frac{t}{t^2+D^2}\,dt - \frac{2}{\pi}\int_0^{+\infty}\frac{D}{t^2+D^2}\,[D,a]\,\frac{D}{t^2+D^2}\,dt
\end{aligned}$$

and we may try to prove that the two integrals separately converge in $\mathcal{K}(\mathcal{H})$, the subspace of compact operators.

Observe that $(0,+\infty)\mapsto D(t^2+D^2)^{-1}$ is a $\mathcal{K}(\mathcal{H})$-valued continuous function that can be continuously extended to $[0,+\infty)$ by assigning to it the value $D/D^2\in\mathcal{K}(\mathcal{H})$ at $t=0$. Here we are denoting by D/D^2 the compact operator which is the inverse of D on $\ker(D)^\perp$ and vanishes on $\ker(D)$. Indeed, denoting by λ_1 the first non-zero eigenvalue of $|D|$, we have

$$\left\|\frac{D}{t^2+D^2}-\frac{D}{D^2}\right\| = \left\|\frac{t^2}{(t^2+D^2)}\frac{D}{D^2}\right\| = \frac{t^2}{(t^2+\lambda_1^2)\lambda_1}\to 0 \qquad t\to 0\,.$$

Since moreover $\mathcal{K}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$ and

$$\left\|\frac{D}{t^2+D^2}\,[D,a]\,\frac{D}{t^2+D^2}\right\| = \|[D,a]\|\cdot\left\|\frac{D}{t^2+D^2}\right\|^2 \le \frac{1}{4}\|[D,a]\|\,t^{-2}\in L^1([1,\infty))\,,$$

we have that $\frac{2}{\pi}\int_0^{+\infty}\frac{D}{t^2+D^2}\,[D,a]\,\frac{D}{t^2+D^2}dt$ is a compact operator.

On the other hand, $(0,+\infty)\mapsto t(t^2+D^2)^{-1}\,[D,a]\,t(t^2+D^2)^{-1}$ is a $\mathcal{K}(\mathcal{H})$-valued continuous function which converges to zero as $t\to 0$. Indeed, when restricted to $\ker(D)^\perp$ it appears as a continuous function of $t\in[0,+\infty)$ of products of operators in which at least one factor is compact, and when restricted to $\ker(D)$ it reduces to

$$\frac{t}{t^2+D^2}\,[D,a]\,\frac{t}{t^2+D^2} = \frac{t}{t^2+D^2}\,(Da-aD)\,\frac{t}{t^2+D^2} = \frac{t}{t^2+D^2}\,Da\,\frac{1}{t} = \frac{D}{t^2+D^2}\,a$$

so that it converges to the compact operator $D/D^2\,a$ as $t\to 0$. Since, again, $\mathcal{K}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$ and

$$\left\|\frac{t}{t^2+D^2}\,[D,a]\,\frac{t}{t^2+D^2}\right\| = \|[D,a]\|\cdot\left\|\frac{t}{t^2+D^2}\right\|^2 \le \|[D,a]\|\,t^{-2}\in L^1([1,\infty))\,,$$

we have that $\frac{2}{\pi}\int_0^{+\infty}\frac{t}{t^2+D^2}\,[D,a]\,\frac{t}{t^2+D^2}\,dt$ is a compact operator. This shows that the commutator $[F,a]$ can be written as a sum of compact operators. □

Recall that a *simmetry* of a Hilbert space $\mathcal{H}$ is a bounded operator $\gamma\in\mathcal{B}(\mathcal{H})$ such that

$$\gamma^* = \gamma\,,\qquad \gamma^2 = I\,.$$

Definition 2.6. (Even and odd Spectral Triples and Fredholm Modules)
A Spectral Triple $(\mathcal{A},\mathcal{H},D)$ is called *even* if the exists a simmetry such that

$$D\gamma+\gamma D = 0\,,\qquad a\gamma-\gamma a = 0 \qquad a\in\mathcal{A}\,.$$

A Fredholm Module $(F,\mathcal{H})$ is called *even* if the exists a simmetry such that

$$F\gamma+\gamma F = 0\,,\qquad a\gamma-\gamma a = 0 \qquad a\in\mathcal{A}\,.$$

In other words, the operator D (resp. F) of an even Spectral Triple (resp. Fredholm module) acts as an antidiagonal matrix with respect to the orthogonal demposition of $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ in eigenspaces $\mathcal{H}_\pm$ of the simmetry γ, while the elements of $\mathcal{A}$ act diagonally.

Corollary 2.7. *Let $(\mathcal{A}, \mathcal{H}, D, \gamma)$ be an even Spectral Triple. Then, setting $F := \mathrm{sgn}(D)$, $(F, \mathcal{H}, \gamma)$ is an even Fredholm module over $\overline{\mathcal{A}}$.*

Fredholm modules represent elements of the K-homology groups of the algebra $\mathcal{A}$. These can be paired with elements of the K-theory groups of $\mathcal{A}$. In particular, odd Fredholm modules couple with elements of the group $K_1(\mathcal{A})$, whose elements are represented by invertible or unitary elements of $\mathcal{A}$. Indeed, assume F to be selfadjoint. In this case, for any invertible element $u \in \mathcal{A}$, the operator $P_+\pi(u)P_+$ is Fredholm on $P_+\mathcal{H}$, where P_+ is the projection on the positive part of the spectrum of F, and the pairing is given by

$$\langle F, u\rangle = \mathrm{Ind}(P_+\pi(u)P_+).$$

In the following, we allow F to have a infinite dimensional kernel. The following Proposition justifies in some cases the treatment of such degenerate Fredholm modules.

Proposition 2.8. *Let F be a self-adjoint operator whose spectrum is $\sigma(F) := \{-1, 0+1\}$. Assume $[F, \pi(a)]$ to be compact for any $a \in \mathcal{A}$, and denote by P_ε the spectral projection for the eigenvalue $\varepsilon \in \{-1, 0, 1\}$. Then*

(i) when u is invertible, $P_\varepsilon \pi(u) P_\varepsilon$ is Fredholm, for all $\varepsilon \in \{-1, 0, 1\}$, and

$$\sum_\varepsilon \mathrm{Ind}(P_\varepsilon \pi(u) P_\varepsilon) = 0\,,$$

(ii) $[P_\varepsilon, \pi(a)]$ is compact, for any $a \in \mathcal{A}$, and $\varepsilon \in \{-1, 0+1\}$.

Proof. (i) As $[F, \pi(u)] \in \mathcal{K}(\mathcal{H})$, for all $\varepsilon, \varepsilon' \in \{-1, 0, 1\}$, we have $P_\varepsilon[F, \pi(u)]P_{\varepsilon'} \in \mathcal{K}(\mathcal{H})$,

$$P_\varepsilon[F, \pi(u)]P_{\varepsilon'} = (\varepsilon - \varepsilon')P_\varepsilon \pi(u) P_{\varepsilon'}$$

and, in particular, $P_\varepsilon[F, \pi(u)]P_\varepsilon = 0$. Since

$$\pi(u) = \sum_\varepsilon P_\varepsilon \pi(u) P_\varepsilon + \sum_{\varepsilon \neq \varepsilon'} (\varepsilon - \varepsilon')^{-1} P_\varepsilon [F, \pi(u)] P_{\varepsilon'}$$

and u is invertible, then $\sum_\varepsilon P_\varepsilon \pi(u) P_\varepsilon$ and $P_\varepsilon \pi(u) P_\varepsilon$ are Fredholm operators, for all $\varepsilon \in \{-1, 0, 1\}$, and

$$\sum_\varepsilon \mathrm{Ind}(P_\varepsilon \pi(u) P_\varepsilon) = \mathrm{Ind}(\pi(u)) = 0\,.$$

(ii) Observe that, for all $a \in \mathcal{A}$ and $\{\varepsilon, \varepsilon', \varepsilon''\}$ a permutation of $\{-1, 0, 1\}$, we have

$$[P_\varepsilon, \pi(a)] = \sum_{\varepsilon', \varepsilon''} P_{\varepsilon'}[P_\varepsilon, \pi(a)]P_{\varepsilon''} = P_\varepsilon \pi(a) P_{\varepsilon'} + P_\varepsilon \pi(a) P_{\varepsilon''} - P_{\varepsilon'} \pi(a) P_\varepsilon - P_{\varepsilon''} \pi(a) P_\varepsilon.$$

Since all summands in the last expression are compact by (i), the thesis follows. $\square$

Corollary 2.9. *Let $(\mathcal{A}, (\pi, \mathcal{H}), F)$ be a Fredholm module in the sense of Definition 2.3, with $F^* = F$, and $F^2 = I$ on $(\ker F)^\perp$, and assume that for all invertible $u \in \mathcal{A}$ we have*

$$\mathrm{Ind}(P_0 \pi(u) P_0) = 0\,.$$

Then there exists a Fredholm module $(\mathcal{A}, (\pi, \mathcal{H}), F')$ such that $F'^2 = I$ and

$$\mathrm{Ind}(P_\varepsilon \pi(u) P_\varepsilon) = \mathrm{Ind}(P'_\varepsilon \pi(u) P'_\varepsilon) \qquad \varepsilon = -1\,, +1\,.$$

Here P_ε (resp. P'_ε) denotes the projection of the operator F (resp. F') associated to the eigenvalue $\varepsilon \in \{-1, +1\}$.

Proof. Defining $F' := F + P_0$ we have $F'^* = F'$, $F'^2 = I$ and $\sigma(F') = \{-1, +1\}$. Since $[F', \pi(a)] = [F, \pi(a)] + [P_0, \pi(a)]$, by Proposition 2.8 (ii) we have $[F', \pi(u)] \in \mathcal{K}(\mathcal{H})$, so that $(\mathcal{A}, (\pi, \mathcal{H}), F')$ is a Fredholm module. Finally, since $P'_1 = P_1 + P_0$, and $P_1\pi(u)P_0), P_0\pi(u)P_1$ are compact, by the proof of Proposition 2.8 (i), and since, by assumption, $\mathrm{Ind}(P_0\pi(u)P_0) = 0$, we have

$$\mathrm{Ind}(P'_1\pi(u)P'_1) = \mathrm{Ind}((P_1+P_0)\pi(u)(P_1+P_0)) = \mathrm{Ind}(P_1\pi(u)P_1)+\mathrm{Ind}(P_0\pi(u)P_0) = \mathrm{Ind}(P_1\pi(u)P_1)\,.$$

□

We now recall the definition of p-graded Fredholm module.

Definition 2.10 ([25], Defs 8.1.11 & A.3.1)**.** Let $p \in \{-1, 0, , ...\}$. A p-graded Fredholm module for $\mathcal{A}$ is given by the following data:

(a) a separable Hilbert space $\mathcal{H}$;
(b) $p + 1$ unitary operators $\varepsilon_0, ..., \varepsilon_p$ such that $\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0$ if $i \neq j$, $\varepsilon_i^2 = -1$, for $i \neq 0$, $\varepsilon_0^2 = 1$.
(c) a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $[\varepsilon_i, \pi(a)] = 0$ for any $i = 0, \dots, p$, any $a \in \mathcal{A}$
(d) an operator F on $\mathcal{H}$ such that $\varepsilon_i F - F\varepsilon_i = 0$, $i \neq 0$, $\varepsilon_0 F + F\varepsilon_0 = 0$, and, for all $a \in \mathcal{A}$, $(F^2 - 1)$, $F - F^*$, $[F, \pi(a)]$ are compact.

In particular, odd Fredholm modules are (-1)-graded, and even Fredholm modules are (0)-graded.

It turns out that p-graded Fredholm modules pair with odd K-theory when p is odd, and with even K-theory when p is even. More precisely, order two periodicity is an isomorphism in K-theory (cf. Proposition 8.2.13 in [25]).

In particular, given a 1-graded Fredholm module $\mathcal{F} = (\pi, \mathcal{H}, F, \gamma, \varepsilon)$, and setting $\tilde{F} = i\varepsilon F$, $\tilde{\mathcal{F}} = (\pi, \mathcal{H}, \tilde{F})$ is an odd Fredholm module on $\mathcal{A}$, giving the same pairing with K-theory up to a factor 2. We notice that weakening the condition $[\varepsilon, \pi(a)] = 0$ to $[\varepsilon, \pi(a)] \in \mathcal{K}(\mathcal{H})$ does not alter the previous result.

Definition 2.11. A (possibly degenerate) odd Fredholm module will be called tamely degenerate if the conditions in Corollary 2.9 holds. A (possibly degenerate) 1-graded Fredholm module $(\pi, \mathcal{H}, F, \gamma, \varepsilon)$ is called *tamely degenerate* if $(\pi, \mathcal{H}, \tilde{F})$ is.

Corollary 2.9 proves that a tamely degenerate Fredholm module is equivalent to a (non-degenerate) Fredholm module, as far as their indexes are concerned.

Let us observe that, given an even Fredholm module $(\pi, \mathcal{H}, F, \gamma)$ on $\mathcal{A}$, we can make it a 1-graded Fredholm module $(\pi, \mathcal{H}, F, \gamma, \varepsilon)$ simply by adding a skew-adjoint unitary operator ε which commutes with F, anticommutes with γ, and commutes with $\pi(a)$ (possibly up to compact operators).

3. Spectral triples on quasi-circles

In this section we build a family of spectral triples on the algebra $C(\mathbb{T})$ of continuous functions on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, depending on a parameter $\alpha \in (0, 1]$. For $\alpha = 1$ we get the circle with the standard differential structure, while the triples for $\alpha < 1$ may be considered as deformations of the standard one, the circle being replaced by quasi-circles.

3.1. **Quadratic forms on $\mathbb{T}$.** We will use the following notation. For any $f \in C(\mathbb{T})$, the Fourier coefficients are $f_k := \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-ikt}\,dt$, $k \in \mathbb{Z}$, the convolution between f and $g \in C(\mathbb{T})$ is $f * g(t) := \frac{1}{2\pi}\int_{\mathbb{T}} f(t-\vartheta)g(\vartheta)\,d\vartheta$, and if Ψ is a distribution on $\mathbb{T}$ and $f \in C^\infty(\mathbb{T})$, the pairing is given by $\langle \Psi, f\rangle := \frac{1}{2\pi}\int_{\mathbb{T}} \overline{\Psi(t)} f(t)\,dt$.

For any positive sequence $\{a_k\}$ of polynomial growth on $\mathbb{Z}$ we consider the quadratic form on functions in $\mathcal{C}^\infty(\mathbb{T})$ given by

$$Q[f] = \sum_{k\in\mathbb{Z}} a_k |f_k|^2.$$

In all this section we denote by Φ the distribution given by the Fourier series $\sum_{k\in\mathbb{Z}} a_k e^{ikt}$. Then, a direct computation shows that $\langle \Phi, f\rangle = \sum_{k\in\mathbb{Z}} a_k f_k$, and

$$Q[f] = \langle \Phi, f^* * f\rangle,$$

where $f^*(t) := \overline{f}(-t)$.

Definition 3.1. A sequence $\{a_k \in \mathbb{C} : k \in \mathbb{Z}\}$ is called *positive definite* if

$$\sum_{m,n\in\mathbb{Z}} a_{m-n}\overline{c}_m c_n \geq 0 \tag{3.1}$$

for any finitely supported sequence $\{c_k\}$. A sequence $\{a_k\}$ is called *conditionally positive definite* if

$$\sum_{m,n\in\mathbb{Z}} a_{m-n}(\partial\overline{c})_m(\partial c)_n \geq 0 \tag{3.2}$$

for any finitely supported sequence $\{c_k \in \mathbb{C} : k \in \mathbb{Z}\}$, where $(\partial c)_k = c_k - c_{k-1}$. A sequence is *(conditionally) negative definite* if it is the opposite of a (conditionally) positive definite one.

Theorem 3.2. *Let $\{a_k\}$ be a conditionally positive definite sequence. Then there exist a positive measure μ on $\mathbb{T}$ and a constant b such that*

$$\langle \Phi, f\rangle = \int_{\mathbb{T}} (f(t) - f(0) - f'(0)\sin t)\ d\mu + a_0 f(0) + \frac{1}{2i}(a_1 - a_{-1})f'(0) + bf''(0).$$

Proof. The proof is analogous to that of Thm 1, Chapter II of [21], but we give the details for the convenience of the reader.

Passing to Fourier series, eq. (3.2), which clearly holds also for fast decreasing sequences c_k, may be rephrased as

$$\langle |1 - e^{-it}|^2 \Phi, |f|^2\rangle \geq 0, \tag{3.3}$$

where $f(t) = \sum_{k\in\mathbb{Z}} c_k e^{ikt}$. Since such sums describe all $\mathcal{C}^\infty$ functions, and $|1 - e^{-it}|^2 = 2(1 - \cos t)$, this is equivalent to $\langle (1 - \cos t)\Phi, g\rangle \geq 0$ for any positive function $g \in \mathcal{C}^\infty$, namely $(1 - \cos t)\Phi$ is a positive measure ν. Equivalently, $\langle \Phi, (1 - \cos t)g\rangle = \int g\,d\nu$ for any $g \in \mathcal{C}^\infty$. Since any function h with a zero of order 2 may be written as $h = (1 - \cos t)g$, we get $\langle \Phi, h\rangle = \int h(t)(1 - \cos t)^{-1}\,d\nu$ for any function h with a zero of order 2 in $t = 0$. We then separate the part of ν with support in 0, setting $\nu = b\delta_0 + (1 - \cos t)\mu$, thus getting

$$\langle \Phi, h\rangle = \int_{(0,2\pi)} h(t)\,d\mu + bh''(0)\,.$$

Then, since $f(t) - f(0) - f'(0)\sin t$ has a zero of order 2 for $t = 0$, we get

$$\begin{aligned}\langle \Phi, f\rangle &= \int_{\mathbb{T}} (f(t) - f(0) - f'(0)\sin t)\; d\mu + \langle \Phi, f(0) + f'(0)\sin t\rangle + bf''(0) \\ &= \int_{\mathbb{T}} (f(t) - f(0) - f'(0)\sin t)\; d\mu + a_0 f(0) + \frac{1}{2i}(a_1 - a_{-1})f'(0) + bf''(0).\end{aligned}$$

□

3.2. **Sobolev norms and Clausen functions.** Let $s \in \mathbb{C}$. Then the polylogarithm function of order s is defined as

$$\mathrm{Li}_s(z) := \sum_{k\in\mathbb{N}} \frac{z^k}{k^s}, \quad |z| < 1.$$

It has an analytic continuation on the whole complex plane with the line $[1, +\infty)$ removed, cf. the Appendix. The Clausen cosine function $\mathrm{Ci}_s(t)$ is defined as the sum of the Fourier series

$$\sum_{k\in\mathbb{N}} \frac{\cos kt}{k^s}, \quad \mathrm{Re}\, s > 1.$$

When $\mathrm{Re}\, s \le 1$ it can be defined as the real part of $\mathrm{Li}_s(e^{it})$, hence it is a smooth function for $t \neq 0$.

Some properties of the Clausen function are contained in Lemma A.1 and Proposition A.2.

Proposition 3.3. *Let $\alpha \in (0, 1)$, $a_k = |k|^{2\alpha}$, $k \in \mathbb{Z}$, and Φ the associated distribution as above. Then*

(i) *the sequence a_k is conditionally negative definite,*

(ii) *for any $\mathcal{C}^\infty$ function f,*

$$\langle \Phi, f\rangle = \frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Ci}_{-2\alpha}(t)(f(t) - f(0))\; dt\,.$$

In particular, the Clausen function $\mathrm{Ci}_{-2\alpha}$ is negative.

Proof. (i) It is well known that k^2 is a conditionally negative definite sequence, therefore so is $k^{2\alpha}$, for $\alpha \in (0, 1]$ ([6], page 78).

(ii) Assume $f(0) = 0$. Since Φ is even, the pairing with the odd part of f vanishes, while, by Proposition A.2, the pairing with the even part is given by the integral against $\frac{1}{\pi}\mathrm{Ci}_{-2\alpha}$. According to the results of Theorem 3.2, the measure $d\mu$ (which is now negative) should be replaced by $\frac{1}{\pi}\mathrm{Ci}_{-2\alpha}(t)\, dt$, showing in particular that $\mathrm{Ci}_{-2\alpha}$ is negative. For a general f, again using the parity of Φ, the pairing becomes

$$\langle \Phi, f\rangle = \frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Ci}_{-2\alpha}(t)(f(t) - f(0))\; dt + bf''(0)\,,$$

hence we get the result if we show that $b = 0$. By definition, for any continuous function g, $\langle \Phi, (1-\cos t)g(t)\rangle = b\, g(0) + \int (1-\cos t)g(t)\, d\mu$. In particular, if g has suitably small support, $b\, g(0) = \lim_{\varepsilon\to 0}\langle \Phi, (1 - \cos t)g(t/\varepsilon)\rangle$. Choosing $g(t) = \chi_{[-1,1]}(1 - |t|)$, a direct computation shows that $b = 0$. □

Corollary 3.4. *Let $\alpha \in (0, 1]$, $a_k = |k|^{2\alpha}$, $k \in \mathbb{Z}$, and denote by $\mathcal{E}_\alpha$ the corresponding quadratic form. Then*

(i) *$\|f\|_2^2 + \mathcal{E}_\alpha[f]$ is the square of the Sobolev norm for the space $H^\alpha(\mathbb{T})$,* *(ii) the quadratic form $\mathcal{E}_\alpha$ is given by*

$$\mathcal{E}_\alpha[f] = -\frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(x-y)|f(x)-f(y)|^2\, dxdy - b\|f'\|_2^2,$$

where $b=0$, for $\alpha<1$, while $\mathrm{Ci}_{-2}=0$ and $b=-1$, for $\alpha=1$.

Proof. (i) Obvious.
(ii) We have

$$\mathcal{E}_\alpha[f] = \langle \Phi, f^* * f\rangle = \frac{1}{\pi}\int_{\mathbb{T}} \mathrm{Ci}_{-2\alpha}(t)((f^* * f)(t) - (f^* * f)(0))\, dt + b(f^* * f)''(0).$$

Since $\mathrm{Ci}_{-2\alpha}(x-y) = \mathrm{Ci}_{-2\alpha}(y-x)$, we have

$$\begin{aligned}
2\int_{\mathbb{T}} &\mathrm{Ci}_{-2\alpha}(t)((f^* * f)(t) - (f^* * f)(0))\, dt \\
&= 2\int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(t)\big(\overline{f}(x-t)f(x) - \overline{f}(x)f(x)\big)\, dt\, dx \\
&= 2\int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(x-y)\big(\overline{f}(y)f(x) - \overline{f}(x)f(x)\big)\, dy\, dx \\
&= \int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(x-y)\big(\overline{f}(y)f(x) - \overline{f}(x)f(x) + \overline{f}(x)f(y) - \overline{f}(y)f(y)\big)\, dy\, dx \\
&= -\int_{\mathbb{T}\times\mathbb{T}} \mathrm{Ci}_{-2\alpha}(x-y)\big|f(x)-f(y)\big|^2\, dy\, dx\,.
\end{aligned}$$

As for the second summand,

$$(f^* * f)''(0) = -\big((f')^* * f'\big)(0) = -\|f'\|_2^2\,,$$

which proves the equation. Since the quadratic form gives rise to the Sobolev norm, the last summand should be absent, when $\alpha<1$, while, for $\alpha=1$, the Clausen function vanishes by a direct computation, and $\mathcal{E}_\alpha[f] = \|f'\|_2^2$, giving $b=-1$. ☐

3.3. **The construction of the triple.** Let us consider, for each fixed $0<\alpha\le 1$, the Dirichlet form $\mathcal{E}_\alpha$ on $L^2(\mathbb{T})$, with domain $\mathcal{F}_\alpha := \{f\in L^2(\mathbb{T}) : \mathcal{E}_\alpha[f] < +\infty\}$.
As shown in Corollary 3.4, the Sobolev space $H^\alpha(\mathbb{T})$ coincides with $\mathcal{F}_\alpha$ and has norm

$$\|f\|_\alpha^2 = \|f\|_{L^2(\mathbb{T})}^2 + \mathcal{E}_\alpha[f]\,.$$

In this section we construct a Spectral Triple associated to the above Dirichlet space for each value of the parameter $0<\alpha\le 1$. The construction is based on the differential calculus, given in terms of a closable derivation with values in a suitable bimodule, underlying any regular Dirichlet form (see [14], [15]).
We summarize below, the main known properties of the Dirichlet spaces on the circle, we are considering. Proofs may be found in [20].

Proposition 3.5. *The Dirichlet space $(\mathcal{E}_\alpha, \mathcal{F}_\alpha)$ on $L^2(\mathbb{T})$ is regular in the sense that the Dirichlet algebra $\mathcal{F}_\alpha\cap C(\mathbb{T})$ is dense both in $C(\mathbb{T})$ with respect to the uniform norm and in $\mathcal{F}_\alpha$ with respect to the graph norm. In particular, the algebra $C^\gamma(\mathbb{T})$ of Hölder continuous functions of order $\gamma\in(\alpha,1]$ is a form core contained in the Dirichlet algebra. We observe that $\mathcal{F}_\alpha\subset C(\mathbb{T})$, for $\alpha>\frac{1}{2}$.*

When $\alpha = 1$ we set $\partial_\alpha f = df$, and we get $\mathcal{E}_\alpha[f] = \|\partial_\alpha f\|^2_{L^2(\Omega^1(\mathbb{T}))}$.

Let us now choose $\alpha < 1$. By Corollary 3.4,

$$\mathcal{E}_\alpha[f] = \frac{1}{4\pi^2}\int_{\mathbb{T}}\int_{\mathbb{T}} \varphi_\alpha(z-w)|f(z)-f(w)|^2\,dzdw,$$

where we set $\varphi_\alpha = -2\pi\,\mathrm{Ci}_{-2\alpha}$.

The linear map defined as

$$\partial_\alpha : \mathcal{F}_\alpha \to L^2(\mathbb{T}\times\mathbb{T}) \qquad \partial_\alpha(f)(z,w) = \varphi_\alpha(z-w)^{1/2}(f(z)-f(w))$$

is a closed operator acting on $L^2(\mathbb{T})$, since $\mathcal{E}_\alpha[f] = \|\partial_\alpha f\|^2_{L^2(\mathbb{T}\times\mathbb{T})}$.

Endowing the Hilbert space $L^2(\mathbb{T}\times\mathbb{T})$ with the $C(\mathbb{T})$-bimodule structure defined by the left and right actions of $C(\mathbb{T})$ given by

$$(f\xi)(z,w) := f(z)\xi(z,w)\,, \qquad (\xi g)(z,w) := \xi(z,w)g(w)\,, \qquad z,w\in\mathbb{T}\,,$$

and by the anti-linear involution

$$(\mathcal{J}\xi)(z,w) := \overline{\xi(w,z)}, \qquad z,w\in\mathbb{T}\,,$$

for $f,g\in C(\mathbb{T})$ and $\xi\in L^2(\mathbb{T}\times\mathbb{T})$, it is easy to see that the map ∂_α is a *derivation* on the Dirichlet algebra $\mathcal{F}_\alpha\cap C(\mathbb{T})$, since it is *symmetric*

$$\mathcal{J}(\partial_\alpha(f)) = \partial_\alpha(\overline{f}), \qquad f\in C^\gamma(\mathbb{T}),$$

and satisfies the *Leibniz rule*

$$\partial_\alpha(fg) = (\partial_\alpha f)g + f(\partial_\alpha g), \qquad f,g\in C^\gamma(\mathbb{T})\,.$$

Moreover, the map ∂_α is a *differential square root* of the self-adjoint operator Δ^α on $L^2(\mathbb{T})$ having $(\mathcal{E}_\alpha,\mathcal{F}_\alpha)$ as closed quadratic form, because of the identities

$$\mathcal{E}_\alpha[f] := \|\Delta^{\alpha/2}f\|^2_{L^2(\mathbb{T})} = \|\partial_\alpha f\|^2_{L^2(\mathbb{T}\times\mathbb{T})}, \qquad f\in\mathcal{F}_\alpha\,.$$

We accommodate in the following Lemma some technical results which will be useful later.

Lemma 3.6. *Let us denote by $\{e_k : k\in\mathbb{Z}\}$ the orthonormal basis of eigenfunctions of the standard Laplacian Δ:*

$$e_k(t) := e^{ikt}, \qquad \Delta e_k = k^2 e_k.$$

(1) $\mathcal{E}_\alpha(e_k,e_j) = (e_k,\partial_\alpha^*\partial_\alpha e_j) = |k|^{2\alpha}\delta_{kj}$.
(2) Let $e'_n = |n|^{-\alpha}\partial_\alpha e_n$, $n\in\mathbb{Z}\setminus\{0\}$. Then, the family $\{e'_n\}_{n\in\mathbb{Z}\setminus\{0\}}$ is an orthonormal basis for the range of ∂_α.
(3) The following equation holds:

$$\partial_\alpha^*((\partial_\alpha e_p)e_n) = \frac{1}{2}(|p|^{2\alpha}+|n+p|^{2\alpha}-|n|^{2\alpha})e_{n+p}. \tag{3.4}$$

(4) For any $k,p\in\mathbb{Z}$, $|p|^{2\alpha}+|k|^{2\alpha}-|k-p|^{2\alpha}\le 2|k|^\alpha|p|^\alpha$.
(5) Let us consider the map $S_f : C(\mathbb{T})\to L^2(\mathbb{T}\times\mathbb{T})$ defined, for a fixed $f\in C(\mathbb{T})$, as $S_f g := (\partial_\alpha f)g$, with $g\in C(\mathbb{T})$. Then, for $s>\alpha^{-1}$,

$$tr(S_f S_f^*(\partial_\alpha\partial_\alpha^*)^{-s/2}) \le 2\zeta(\alpha s)\mathcal{E}[f] = tr(S_f^* S_f(\partial_\alpha^*\partial_\alpha)^{-s/2}). \tag{3.5}$$

Proof. The equality $\partial_\alpha^*\partial_\alpha = \Delta^\alpha$ gives (1), while (2) follows from a direct computation, and (3) amounts to verify that $(\partial_\alpha e_k,(\partial_\alpha e_p)e_n) = \frac{1}{2}(p^{2\alpha}+(n+p)^{2\alpha}-n^{2\alpha})\delta_{k,n+p}$. We now show (4). We observe that it certainly holds for $p=0$ or $k=0$. Observe that, $|p|^{2\alpha}+|k|^{2\alpha}-|k-p|^{2\alpha}\le|p|^{2\alpha}+|k|^{2\alpha}-\big||k|-|p|\big|^{2\alpha}$, hence, setting $e^{2t}=|p|/|k|$, it is enough to show that $2\sinh(\alpha t)\le(2\sinh t)^\alpha$ or, equivalently, $\log(2\sinh(\alpha t))-\alpha\log(2\sinh t)\le 0$, for $t>0$. The

latter function tends to 0 for $t \to \infty$, and its derivative $\alpha(\coth(\alpha t) - \coth t)$ is positive for $\alpha \in [0,1]$ since coth is decreasing, proving the inequality.
As for the inequality in (5), we have

$$
\begin{aligned}
\|S_f^*\partial_\alpha e_k\|^2 &= \sum_n |(e_n, S_f^*\partial_\alpha e_k)|^2 = \sum_n |(\partial_\alpha f, ((\partial_\alpha e_k)e_{-n})|^2 \\
&= \frac{1}{4}\sum_n (|k|^{2\alpha} + |n-k|^{2\alpha} - |n|^{2\alpha})^2 |(f, e_{n+k})|^2 \\
&= \frac{1}{4}\sum_p (|k|^{2\alpha} + |p|^{2\alpha} - |p-k|^{2\alpha})^2 |(f, e_p)|^2 \\
&\leq |k|^{2\alpha}\sum_p |p|^{2\alpha}|(f,e_p)|^2 \; = |k|^{2\alpha}\mathcal{E}_\alpha[f]\,,
\end{aligned}
\tag{3.6}
$$

where the inequality in the last row follows by (4). Then

$$
\begin{aligned}
tr(S_f S_f^*(\partial_\alpha\partial_\alpha^*)^{-s/2}) &= \sum_k (e_k', S_f S_f^*(\partial_\alpha\partial_\alpha^*)^{-s/2}e_k') = \sum_k |k|^{-(s+2)\alpha}\|S_f^*\partial_\alpha e_k\|^2 \\
&\leq \sum_k |k|^{-s\alpha}\|\partial_\alpha f\|^2_{L^2(\mathbb{T}\times\mathbb{T})} = 2\zeta(\alpha s)\mathcal{E}_\alpha[f]\,.
\end{aligned}
\tag{3.7}
$$

Concerning the equality in (5) we have

$$
\begin{aligned}
tr(S_f^* S_f(\partial_\alpha^*\partial_\alpha)^{-s/2}) &= \sum_k (e_k, S_f^* S_f(\partial_\alpha^*\partial_\alpha)^{-s/2}e_k) = \sum_k |k|^{-s\alpha}\|S_f e_k\|^2 \\
&= \sum_k |k|^{-s\alpha}\|(\partial_\alpha f)e_k)\|^2 = 2\zeta(\alpha s)\mathcal{E}_\alpha[f]\,.
\end{aligned}
$$

□

We now construct the promised family of spectral triples. For $\alpha = 1$, the Hilbert space is simply $L^2(\Omega^*(\mathbb{T})) = L^2(\Omega^1(\mathbb{T}))\oplus L^2(\Omega^0(\mathbb{T}))$. For $\alpha < 1$, we interpret ∂_α as a deformed external derivation, and set $\mathcal{K}_\alpha := L^2(\Omega^*_\alpha(\mathbb{T})) = L^2(\Omega^1_\alpha(\mathbb{T}))\oplus L^2(\Omega^0(\mathbb{T}))$, with $L^2(\Omega^1_\alpha(\mathbb{T})) = L^2(\mathbb{T}\times\mathbb{T})$. The Dirac operator $(D_\alpha, \mathrm{dom}(D_\alpha))$ on $\mathcal{K}_\alpha$ is defined as

$$
D_\alpha := \begin{pmatrix} 0 & \partial_\alpha \\ \partial_\alpha^* & 0 \end{pmatrix}, \quad \text{so that} \quad D_\alpha \begin{pmatrix} \xi \\ f \end{pmatrix} = \begin{pmatrix} \partial_\alpha f \\ \partial_\alpha^*\xi \end{pmatrix},
$$

on the domain $\mathrm{dom}(D_\alpha) := \mathrm{dom}(\partial_\alpha^*) \oplus \mathcal{F}_\alpha$. The $*$-algebra is $\mathcal{A}_\alpha = \{f \in C(\mathbb{T}) : \|[D, L_f]\| < \infty\}$, where, if $f \in C(\mathbb{T})$, L_f denotes its left action on $\mathcal{K}_\alpha$ resulting from the direct sum of those on $L^2(\mathbb{T}\times\mathbb{T})$ and on $L^2(\mathbb{T})$. We also consider the seminorm p_α given by $p_\alpha(f) = \|f'\|_\infty$, for $\alpha = 1$, or

$$
p_\alpha(f)^2 = \frac{1}{2\pi}\sup_{x\in\mathbb{T}}\int_{\mathbb{T}} \varphi_\alpha(x-y)|f(x)-f(y)|^2 dy < +\infty,
$$

for $\alpha < 1$. Some estimates for p_α are contained in Proposition A.3.

Theorem 3.7. *Let us consider the regular Dirichlet form $(\mathcal{E}_\alpha, \mathcal{F}_\alpha)$ on $L^2(\mathbb{T})$, the associated derivation $(\partial_\alpha, \mathcal{F}_\alpha)$ and the triple $(\mathcal{A}_\alpha, \mathcal{K}_\alpha, D_\alpha)$ described above. Then,*

- *(i) D_α^{-1} is a compact operator on $\mathcal{K}_\alpha$ on the orthogonal complement of* $\ker D_\alpha$,
- *(ii) $\mathcal{A}_\alpha = \{f \in C(\mathbb{T}) : p_\alpha(f) < \infty\}$, and analogous results hold true upon replacing the left module structure of $\mathcal{K}_\alpha$ by the right one,*
- *(iii) $\mathcal{A}_\alpha$ is a uniformly dense subalgebra of $C(\mathbb{T})$. In particular, $\mathcal{A}_\alpha = C^{0,1}(\mathbb{T})$, for $\alpha = 1$, while, for $\alpha < 1$, $C^{0,\alpha+\varepsilon}(\mathbb{T}) \subset \mathcal{A}_\alpha$. Finally, if $\alpha \geq \frac{1}{2}$, $\mathcal{A}_\alpha \subset C^{0,\alpha}(\mathbb{T})$.*

As a consequence, $(\mathcal{A}_\alpha, \mathcal{K}_\alpha, D_\alpha)$ is a densely defined Spectral Triple on the algebra $C(\mathbb{T})$, in the sense of A. Connes.

Proof. The proof for $\alpha = 1$ is classical, so we consider the case $\alpha < 1$.
(i) Notice first that, since the self-adjoint operators $\partial_\alpha^*\partial_\alpha$ and $\partial_\alpha\partial_\alpha^*$ are unitarily equivalent on the orthogonal complement of their kernels, it suffices to prove that $\partial_\alpha^*\partial_\alpha$ has discrete spectrum on $L^2(\mathbb{T})$. Indeed, Lemma 3.6 (1) shows that the spectrum of the self-adjoint operator $\partial_\alpha^*\partial_\alpha$ is discrete and coincides with $\{k^{2\alpha} : k \in \mathbb{N}\}$.
(ii) Let us consider first the map $S_f : C(\mathbb{T}) \to L^2(\mathbb{T}\times\mathbb{T})$ defined, for a fixed $f \in C(\mathbb{T})$, as follows

$$S_f g := (\partial_\alpha f)g \qquad g \in C(\mathbb{T})\,.$$

This map extends to a bounded map on $L^2(\mathbb{T})$, provided $f \in \mathcal{A}_\alpha$, because

$$\begin{aligned}
\|S_f g\|^2_{L^2(\mathbb{T}\times\mathbb{T})} &= \frac{1}{4\pi^2}\int_{\mathbb{T}\times\mathbb{T}} |(\partial_\alpha f)(z,w)g(w)|^2\,dzdw \\
&= \frac{1}{4\pi^2}\int_{\mathbb{T}} |g(w)|^2 \int_{\mathbb{T}} \varphi_\alpha(z-w)|f(z)-f(w)|^2\,dzdw \\
&\le \frac{1}{2\pi}\|g\|^2_{L^2(\mathbb{T})}\sup_{w\in\mathbb{T}}\int_{\mathbb{T}} \varphi_\alpha(z-w)|f(z)-f(w)|^2\,dz, \qquad g \in L^2(\mathbb{T}),
\end{aligned}$$

so that

$$\|S_f\|^2 = \frac{1}{2\pi}\sup_{w\in\mathbb{T}}\int_{\mathbb{T}} \varphi_\alpha(z-w)|f(z)-f(w)|^2\,dz = p_\alpha(f)^2\,. \tag{3.8}$$

Let us compute now, using the Leibniz rule for the derivation ∂_α, the quadratic form of the commutator $[D_\alpha, L_f]$, defined on the domain $\mathrm{dom}(D_\alpha) := \mathrm{dom}(\partial_\alpha^*) \oplus \mathcal{F}_\alpha$:

$$\begin{aligned}
\big(\xi'\oplus g'|[D_\alpha, L_f]\xi\oplus g\big) &= \big(D_\alpha(\xi'\oplus g')|L_f(\xi\oplus g)\big) - \big(L_{f^*}(\xi'\oplus g')|D_\alpha(\xi\oplus g)\big) \\
&= \big(\partial_\alpha g'\oplus\partial_\alpha^*\xi'|f\xi\oplus fg\big) - \big(f^*\xi'\oplus f^*g'|\partial_\alpha g\oplus\partial_\alpha^*\xi\big) \\
&= (\partial_\alpha g'|f\xi) + (\partial_\alpha^*\xi'|fg) - (\xi'|f\partial_\alpha g) - (f^*g'|\partial_\alpha^*\xi) \\
&= (f^*\partial_\alpha g'|\xi) + (\xi'|\partial_\alpha(fg)) - (\xi'|f\partial_\alpha g) - (\partial_\alpha(f^*g')|\xi) \\
&= (\xi'|(\partial_\alpha f)g) - ((\partial_\alpha f^*)g'|\xi)\,. \\
&= (\xi'|S_f g) - (S_{f^*}g'|\xi)\,.
\end{aligned}$$

Hence

$$[D_\alpha, L_f] = \begin{pmatrix} 0 & S_f \\ -S_{f^*}^* & 0 \end{pmatrix}, \qquad a \in \mathcal{A}_\alpha, \tag{3.9}$$

therefore $[D_\alpha, L_f]$ extends to a bounded operator on $\mathcal{K}_\alpha$ if and only if $f \in \mathcal{A}_\alpha$, and $\|[D_\alpha, L_f]\| = \|S_f\| = p_\alpha(f)$.
(iii) It is easily seen that $\mathcal{A}_\alpha$ is a subalgebra of $C(\mathbb{T})$, and, since, by Proposition A.3, it contains $C^{0,\gamma}(\mathbb{T})$, for $\gamma > \alpha$, it is uniformly dense in $C(\mathbb{T})$. □

Corollary 3.8. *Let $\alpha \in (0,1]$. Then, the distance d_D induced on $\mathbb{T}$ by the spectral triple $(\mathcal{A}_\alpha, \mathcal{K}_\alpha, D_\alpha)$, satifies, for any $\varepsilon > 0$, $d_D(x,y) \ge \frac{1}{c_\varepsilon}|x-y|^{\alpha+\varepsilon}$, $x, y \in \mathbb{T}$. Moreover, if $\alpha \ge \frac{1}{2}$, $d_D(x,y) \le \frac{1}{\tilde{c}_\alpha}|x-y|^\alpha$, $x, y \in \mathbb{T}$. Here, c_ε and $\tilde{c}_\alpha$ are as in Proposition A.3.*

Proof. Observe that, using the notation of Proposition A.3,

$$\begin{aligned} d_D(x,y) &= \sup\{|f(x)-f(y)| : \|[D_\alpha, f]\| \le 1\} = \sup\{|f(x)-f(y)| : p_\alpha(f) \le 1\} \\ &\ge \frac{1}{c_\varepsilon} \sup\left\{|f(x)-f(y)| : \|f\|_{C^{0,\alpha+\varepsilon}(\mathbb{T})}\right\} = \frac{1}{c_\varepsilon}|x-y|^{\alpha+\varepsilon}, \end{aligned}$$

and, if $\alpha \ge \frac{1}{2}$,

$$\begin{aligned} d_D(x,y) &= \sup\{|f(x)-f(y)| : p_\alpha(f) \le 1\} \\ &\le \frac{1}{\tilde{c}_\alpha} \sup\left\{|f(x)-f(y)| : \|f\|_{C^{0,\alpha}(\mathbb{T})}\right\} = \frac{1}{\tilde{c}_\alpha}|x-y|^{\alpha}. \end{aligned}$$

□

3.4. **The pairing with K-theory.** In this section we show that the phase of the Dirac operator gives rises to a Fredholm module that pairs nontrivially with respect to K-theory on the circle.

Proposition 3.9. *Let $\alpha \in (0,1]$, $\mathcal{K}_\alpha := L^2(\Omega^*_\alpha(\mathbb{T}))$ as constructed above, π the action of $C(\mathbb{T})$ on $\mathcal{K}_\alpha$ described in Theorem 3.7, F the phase of the Dirac operator D_α, γ the usual grading $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,*

$$\varepsilon = -i\left(\begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} + P_0\right),$$

where P_0 is the projection on the kernel of F, and V is given by $Ve_j = \mathrm{sgn}(j)e'_j = \mathrm{sgn}(j)|j|^{-\alpha}\partial_\alpha e_j$, $j \neq 0$, $Ve_0 = 0$. Then

(1) $\mathcal{F} = (\pi, \mathcal{K}_\alpha, F, \gamma, \varepsilon)$ is a tamely degenerate 1-graded Fredholm module on $C(\mathbb{T})$, namely $\tilde{\mathcal{F}} = (\pi, \mathcal{K}_\alpha, \widetilde{F})$ is a tamely degenerate odd Fredholm module on $C(\mathbb{T})$, where we set $\widetilde{F} = i\varepsilon F$;

(2) the pairing gives $\langle \mathcal{F}, e_k\rangle = \frac{1}{2}\langle \tilde{\mathcal{F}}, e_k\rangle = k$.

Proof. Let us observe that $F = \begin{pmatrix} 0 & W \\ W^* & 0 \end{pmatrix}$, where W is the partial isometry given by $We_i = e'_i$, $i \neq 0$, $We_0 = 0$. As a consequence, setting $Se_j = \mathrm{sgn}(j)e_j$, we get $V = WS$, hence $i\varepsilon = (I \oplus S)F(I \oplus S) + P_0$. A direct computation shows $P_0 = [\ker(F)] = [\ker(F^2)] = 1 - F^2 = Q_0 \oplus (1 - S^2)$, where Q_0 is the projection on $\ker(\partial^*_\alpha)$. Then the support of $(I \oplus S)F(I \oplus S)$ coincides with the support of F, which is $1 - P_0$. Therefore $\varepsilon^*\varepsilon = \big((I \oplus S)F(I \oplus S)\big)^2 + P_0 = F^2 + P_0 = 1$. We then compute

$$\tilde{F} = i\varepsilon F = \begin{pmatrix} WSW^* & 0 \\ 0 & S \end{pmatrix} = iF\varepsilon\,, \tag{3.10}$$

hence $[\varepsilon, F] = 0$. Since ε is clearly skew-adjoint, we get a (possibly degenerate) 1-graded Fredholm module if we prove the compactness of $[\varepsilon, \pi(f)]$. Indeed,

$$\begin{aligned} [i\varepsilon, \pi(f)] =& [(I \oplus S)F(I \oplus S) + P_0, \pi(f)] \\ =& [(I \oplus S), \pi(f)]F(I \oplus S) + (I \oplus S)[F, \pi(f)](I \oplus S) + (I \oplus S)F[(I \oplus S), \pi(f)] \\ &+ [P_0, \pi(f)]\,. \end{aligned}$$

The compactness of $[F, \pi(f)]$ follows by the spectral triple properties (cf. Proposition 2.5), and the compactness of $[P_0, \pi(f)]$ follows by Proposition 2.8 (ii). The compactness of $[(I \oplus S), \pi(f)] = 0 \oplus [S, f]$ follows by the Toeplitz theory.

We now prove the tame degeneracy. By equation (3.10), the spectral projection $\tilde{P}_+$ of $\tilde{F}$ for the eigenvalue 1 is $WQ_+W^* \oplus Q_+$, where Q_+ is the projection on span $\{e_p : p > 0\}$. Therefore

$$\mathrm{Ind}(\tilde{P}_+\pi(u)\tilde{P}_+) = \mathrm{Ind}(Q_+W^*\pi(u)WQ_+) + \mathrm{Ind}(Q_+\pi(u)Q_+).$$

We shall show that the two summands coincide. We start by proving that for any $p \in \mathbb{Z}$, $W^*\pi(e_p)W - \pi(e_p)$, or, equivalently, $W^*\pi(e_p)W\pi(e_{-p}) - 1$, is compact. Let us observe that, making use of (3.4), $e_n, W^*\pi(e_p)W\pi(e_{-p})e_m) = (e'_n, e_p e'_{m-p}) = \lambda_m \delta_{n,m}$, where

$$\lambda_m = \begin{cases} 0, & m = 0 \vee m = p, \\ \frac{|m|^{2\alpha}+|m-p|^{2\alpha}-|p|^{2\alpha}}{2|m|^\alpha|m-p|^\alpha}, & \text{otherwise}. \end{cases}$$

This shows that $\{e_m : m \in \mathbb{Z}\}$ is a basis of eigenvectors for $W^*\pi(e_p)W\pi(e_{-p}) - 1$, with eigenvalues $\{\lambda_m - 1 : m \in \mathbb{Z}\}$. Since

$$\lim_{|m|\to\infty} \frac{|m|^{2\alpha}+|m-p|^{2\alpha}-|p|^{2\alpha}}{2|m|^\alpha|m-p|^\alpha} = 1,$$

the compactness follows. By linearity and continuity, this implies $W^*\pi(f)W-\pi(f)$ is compact for any $f \in C(\mathbb{T})$. Then,

$$\mathrm{Ind}(Q_+W^*\pi(u)WQ_+) = \mathrm{Ind}(Q_+\pi(u)Q_+).$$

The equality for Q_- follows analogously, giving

$$\mathrm{Ind}(\tilde{P}_+\pi(u)\tilde{P}_+) = 2\,\mathrm{Ind}(Q_+\pi(u)Q_+), \quad \mathrm{Ind}(\tilde{P}_-\pi(u)\tilde{P}_-) = 2\,\mathrm{Ind}(Q_-\pi(u)Q_-).$$

The other results now follow by the Toeplitz theory. □

4. Spectral triples on the Sierpinski gasket

4.1. **Sierpinski Gasket and its Dirichlet form.** We denote by K the Sierpinski gasket, a prototype of self-similar fractal sets. It was introduced in [39] as a curve with a dense set of ramified points and has been the object of various investigations in Analysis [32], Probability [34], [4] and Theoretical Physics [37].
Let $p_0, p_1, p_2 \in \mathbb{R}^2$ be the vertices of an equilateral triangle of unit length and consider the contractions $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane: $w_i(x) := p_i + \frac{1}{2}(x - p_i) \in \mathbb{R}^2$. Then K is the unique fixed-point w.r.t. the contraction map $E \mapsto w_0(E) \cup w_1(E) \cup w_2(E)$ in the set of all compact subsets of $\mathbb{R}^2$, endowed with the Hausdorff metric. Two ways of approximating K are shown in Figures 1 and 2.

Let us denote by $\Sigma_m := \{0,1,2\}^m$ the set of words of length $m \geq 0$ composed by m letters chosen in the alphabet of three letters $\{0,1,2\}$ and by $\Sigma := \bigcup_{m\geq 0} \Sigma_m$ the whole vocabulary (by definition $\Sigma_0 := \{\emptyset\}$). A word $\sigma \in \Sigma_m$ has, by definition, length m and this is denoted by $|\sigma| := m$. For $\sigma = \sigma_1\sigma_2\ldots\sigma_m \in \Sigma_m$ let us denote by w_σ the contraction $w_\sigma := w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_m}$.

Let $V_0 := \{p_0, p_1, p_2\}$ be the set of vertices of the equilateral triangle and $E_0 := \{e_0, e_1, e_2\}$ the set of its edges, with e_i opposite to p_i. Then, for any $m \geq 1$, $V_m := \bigcup_{|\sigma|=m} w_\sigma(V_0)$ is the set of vertices of a finite graph (*i.e.* a one-dimensional simplex) denoted by (V_m, E_m) whose edges are given by $E_m := \bigcup_{|\sigma|=m} w_\sigma(E_0)$ (see Figure 2). The self-similar set K can be reconstructed also as an Hausdorff limit either of the increasing sequence V_m of vertices or of the increasing sequence E_m of edges, of the above finite graphs. Set $V_* := \cup_{m=0}^\infty V_m$, and $E_* := \cup_{m=0}^\infty E_m$.

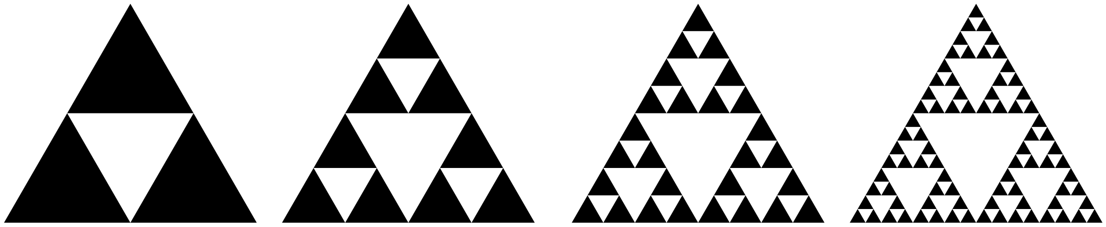

FIGURE 1. Approximations from above of the Sierpinski gasket.

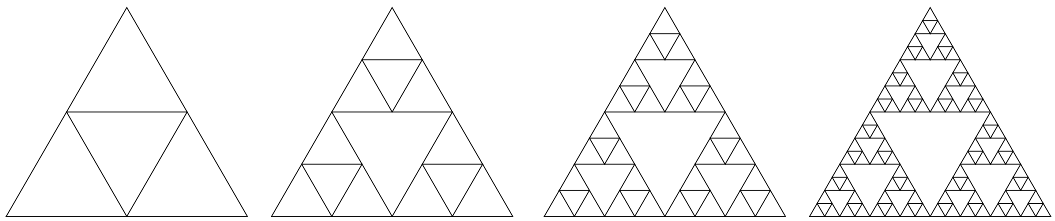

FIGURE 2. Approximations from below of the Sierpinski gasket.

In the present work a central role is played by the quadratic form $\mathcal{E} : C(K) \to [0, +\infty]$ given by

$$\mathcal{E}[f] = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{e\in E_m} |f(e_+) - f(e_-)|^2,$$

where each edge e has been arbitrarily oriented, and e_-, e_+ denote its source and target. It is a regular Dirichlet form since it is lower semicontinuous, densely defined on the subspace $\mathcal{F} := \{f \in C(K) : \mathcal{E}[f] < \infty\}$ and satisfies the *Markovianity property*

$$\mathcal{E}[f \wedge 1] \le \mathcal{E}[f] \qquad f \in C(K)^1. \tag{4.1}$$

The existence of the limit above and the mentioned properties are consequences of the theory of *harmonic structures* on self-similar sets developed by Kigami [32].

As a result of the theory of Dirichlet forms [8, 20], the domain $\mathcal{F}$ is an involutive subalgebra of $C(K)$ and, for any fixed $f, g \in \mathcal{F}$, the functional

$$\mathcal{F} \ni h \mapsto \Gamma(f,g)(h) := \frac{1}{2}\big(\mathcal{E}(f, hg) - \mathcal{E}(fg, h) + \mathcal{E}(g, fh)\big) \in \mathbb{R} \tag{4.2}$$

defines a finite Radon measure called the *energy measure* (or *carré du champ*) *of f and g*. In particular, for $f \in \mathcal{F}$, the measure $\Gamma(f,f)$ is nonnegative and one has the representation

$$\mathcal{E}[f] = \int_K 1\, d\Gamma(f,f) = \Gamma(f,f)(K) \qquad f \in \mathcal{F}\,.$$

In applications, f may represent a configuration of a system, $\mathcal{E}[f]$ its corresponding total energy and $\Gamma(f,f)$ represents its distribution. In homological terms, Γ is (up to the constant $1/2$) the Hochschild co-boundary of the 1-cocycle $\phi(f_0, f_1) := \mathcal{E}(f_0, f_1)$ on the algebra $\mathcal{F}$.

The Dirichlet or energy form $\mathcal{E}$ should be considered as a Dirichlet integral on the gasket. It is lower semicontinuous on the space $L^2(K, m)$, finite on the subspace $\mathcal{F}$, with respect to a wide range of positive Borel measures on K and, once the measure m has been chosen, it

[1] Here and in the following, we will denote by $C(K)$ the space of real valued continuous functions. As a consequence, the quadratic Dirichlet form $\mathcal{E}$ will be considered as a symmetric bilinear form over $\mathcal{F}$.

is the quadratic form of a positive, self-adjoint operator on $L^2(K,m)$, which may be thought of as a Laplace-Beltrami operator on K. However, since in the present work the Dirichlet form solely will play a role, the Laplace-Beltrami operator we need will be understood as the operator $\Delta : \mathcal{F} \to \mathcal{F}^*$ such that

$$\langle \Delta f, g\rangle := \mathcal{E}(f,g) \quad f,g \in \mathcal{F}\,.$$

A function $f \in \mathcal{F}$ is said to be *harmonic* in a open set $A \subset K$ if, for any $g \in \mathcal{F}$ vanishing on the complementary set A^c, one has

$$\mathcal{E}(f,g) = 0\,.$$

As a consequence of the Markovianity property (4.1), a Maximum Principle holds true for harmonic functions on the gasket [32]. In particular, one calls 0-*harmonic* a function u on K which is harmonic in V_0^c. Equivalently, for given boundary values on V_0, u is the unique function in $\mathcal{F}$ such that $\mathcal{E}[u] = \min\{\mathcal{E}[v] : v \in \mathcal{F}, v|_{V_0} = u\}$. More generally, one may call m-*harmonic* a function that, given its values on V_m, minimizes the energy among all functions in $\mathcal{F}$. For such functions we have

$$\mathcal{E}[u] = \left(\frac{5}{3}\right)^m \sum_{e\in E_m} |u(e_+) - u(e_-)|^2\,.$$

It is not difficult to check that $f \in \mathcal{F}$ is m-harmonic if and only if Δf is a linear combination of Dirac measures supported on the vertices V_m.

Definition 4.1. (Cells, lacunas) For any word $\sigma \in \Sigma_m$, define a corresponding *cell* in K as follows

$$C_\sigma := w_\sigma(K)\,,$$

its *perimeter* by $\pi C_\sigma = w_\sigma(E_0)$, its (combinatorial) *boundary* by $\partial C_\sigma = w_\sigma(V_0)$ and its (combinatorial) *interior* by $C_\sigma^o = C_\sigma \setminus \partial C_\sigma$. We will also define the *lacuna* $\ell_\emptyset$, see Fig. 3, as the boundary of the first removed triangle according to the approximation in Fig. 1. For any $\sigma \in \Sigma$, the lacuna ℓ_σ is defined as $\ell_\sigma := w_\sigma(\ell_\emptyset)$. We shall use the notation $\mathcal{E}_{C_\sigma}[u] = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{e\in E_m, e\subset C_\sigma} |u(e_+) - u(e_-)|^2$.

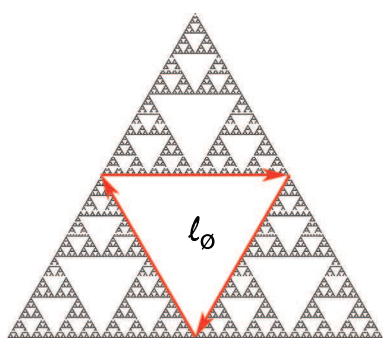

FIGURE 3. The lacuna $\ell_\emptyset$

4.2. **The dimensional spectrum and volume.** Choose $\alpha \in (0,1]$, and let $\ell_\emptyset$ be the main lacuna of the gasket, identified isometrically with $\mathbb{T}$, and consider the triple $(\mathcal{A}, \mathcal{H}_\emptyset, D_\emptyset)$, where $\mathcal{H}_\emptyset$ is the space $L^2(\Omega^*_\alpha(\ell_\emptyset))$, $D_\emptyset$ is the Dirac operator D_α, through the isometric identification of $\ell_\emptyset$ with $\mathbb{T}$, and $\mathcal{A}$ is the algebra of continuous functions on the gasket acting via the representation $\pi_\emptyset$ given by the restriction to $\ell_\emptyset$ and then by the action on $\mathcal{H}_\emptyset$ described before.

Then, choose $\beta > 0$, and, for any $\sigma \in \cup_n \{0,1,2\}^n$, consider the triple $(\mathcal{A}, \mathcal{H}_\sigma, D_\sigma)$, where $\mathcal{H}_\sigma = \mathcal{H}_\emptyset$, $D_\sigma = 2^{\beta|\sigma|} D_\emptyset$, and $\mathcal{A}$ acts via the representation π_σ, with $\pi_\sigma(f) = \pi_\emptyset(f \circ w_\sigma)$.

Definition 4.2. Let us consider the following triple: $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H} = \oplus_{\sigma\in\Sigma}\mathcal{H}_\sigma$, $D = \oplus_{\sigma\in\Sigma} D_\sigma$, and $\mathcal{A}$ acts on $\mathcal{H}$ via the representation $\pi = \oplus_{\sigma\in\Sigma}\pi_\sigma$. According to the prescriptions of noncommutative geometry, we set $\oint f := tr_\omega(f|D|^{-d})$, where tr_ω is the (logarithmic) Dixmier trace, and d is the abscissa of convergence of the zeta function $s \to tr(|D|^{-s})$.

Theorem 4.3. *The zeta function $\mathcal{Z}_D$ of $(\mathcal{A}, \mathcal{H}, D)$, i.e. the meromorphic extension of the function $s \in \mathbb{C} \mapsto tr(|D|^{-s})$, is given by*

$$\mathcal{Z}_D(s) = \frac{4\zeta(\alpha s)}{1 - 3 \cdot 2^{-\beta s}}\,,$$

where ζ denotes the Riemann zeta function. Therefore, the dimensional spectrum of the spectral triple is

$$\mathcal{S}_{dim} = \{\alpha^{-1}\} \cup \{\frac{\log 3}{\beta \log 2}\left(1 + \frac{2\pi i}{\log 3}k\right) : k \in \mathbb{Z}\} \subset \mathbb{C}\,.$$

As a consequence, the metric dimension d_D of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$, namely the abscissa of convergence of its zeta function, is $\max(\alpha^{-1}, d)$, *with $d = \frac{\log 3}{\beta \log 2}$.*

When $0 < \beta < \alpha\frac{\log 3}{\log 2}$, i.e. $d_D = \frac{\log 3}{\beta\log 2}$, $\mathcal{Z}_D$ has a simple pole in d_D, and the volume measure coincides with a multiple of the Hausdorff measure H_γ (normalized on the gasket), with $\gamma = \frac{\log 3}{\log 2}$:

$$\mathrm{vol}(f) \equiv \int_K f\, d\,\mathrm{vol} := tr_\omega(f|D|^{-d}) = \frac{4d}{\log 3}\frac{\zeta(d)}{(2\pi)^d}\int_K f\, dH_\gamma \qquad f \in C(K).$$

Proof. The non vanishing eigenvalues of $|D_\sigma|$ are exactly $\{2^{\beta|\sigma|}(2\pi k^\alpha)\}$, each one with multiplicity 4.

Hence $tr(|D_\sigma|^{-s}) = 4 \cdot 2^{-s\beta|\sigma|}(2\pi)^{-s}\sum_{k>0}(k^\alpha)^{-s} = 4(2\pi)^{-s}2^{-s\beta|\sigma|}\zeta(\alpha s)$ and for $\mathrm{Re}\, s > d$ we have

$$\begin{aligned} tr(|D|^{-s}) &= \sum_\sigma tr(|D_\sigma|^{-s}) = 4(2\pi)^{-s}\zeta(\alpha s)\sum_\sigma 2^{-s\beta|\sigma|} \\ &= 4(2\pi)^{-s}\zeta(\alpha s)\sum_{n\geq 0} 2^{-s\beta n}\sum_{|\sigma|=n} 1 \\ &= 4(2\pi)^{-s}\zeta(\alpha s)\sum_{n\geq 0}\left(3\cdot 2^{-s\beta}\right)^n = 4(2\pi)^{-s}\zeta(\alpha s)(1 - 3\cdot 2^{-s\beta})^{-1}\,. \end{aligned}$$

As the Riemann zeta function has just one pole at $s = 1$ we have $\mathcal{S}_{dim} = \{\alpha^{-1}\} \cup \{d\left(1 + \frac{2\pi i}{\log 3}k\right) : k \in \mathbb{Z}\} \subset \mathbb{C}$. Now we assume that $0 < \beta < \alpha\frac{\log 3}{\log 2}$, i.e. $d_D = \frac{\log 3}{\beta \log 2}$, and prove that the volume measure is a multiple of the Hausdorff measure H_γ.

Clearly, the functional $\mathrm{vol}(f) = tr_\omega(f|D|^{-d})$ makes sense also for bounded Borel functions on K, and we recall that the logarithmic Dixmier trace may be calculated as a residue (cf.

[16]): $tr_\omega(f|D|^{-d}) = \text{Res}_{s=d}\, tr(f|D|^{-s})$, when the latter exists. Then, for any multi-index τ,

$$\begin{aligned} tr_\omega(\chi_{C_\tau}|D|^{-d}) &= \text{Res}_{s=d}\, tr(\chi_{C_\tau}|D|^{-s}) \\ &= \lim_{s\to d^+}(s-d)\, tr(\chi_{C_\tau}|D|^{-s}) \\ &= \lim_{s\to d^+}(s-d)\sum_\sigma tr(\chi_{C_\tau}\circ w_\sigma|D_\sigma|^{-s}), \end{aligned}$$

and we note that $\chi_{C_\tau}\circ w_\sigma$ is not zero either when $\sigma<\tau$ or when $\sigma\geq\tau$. In the latter case, $\chi_{C_\tau}\circ w_\sigma = 1$. Since $d>1$, $tr(\chi_{C_\tau}|D_\sigma|^{-s}) \leq tr(|D_\sigma|^{-s}) = 4(2\pi)^{-s}2^{-s\beta|\sigma|}\zeta(\alpha s) \to 4(2\pi)^{-d}3^{-|\sigma|}\zeta(\alpha d)$ when $s\to d^+$, hence $\lim_{s\to d^+}(s-d)tr(\chi_{C_\tau}|D_\sigma|^{-s}) = 0$. Therefore we may forget about the finitely many $\sigma<\tau$, and get

$$\begin{aligned} tr_\omega(\chi_{C_\tau}|D|^{-d}) &= \lim_{s\to d^+}(s-d)\sum_{\sigma\geq\tau} tr(|D_\sigma|^{-s}) \\ &= \lim_{s\to d^+}(s-d)4(2\pi)^{-s}\zeta(\alpha s)\sum_\sigma 2^{-s\beta(|\sigma|+|\tau|)} \\ &= 4\frac{\zeta(\alpha d)}{(2\pi)^d}2^{-d\beta|\tau|}\lim_{s\to d^+}\frac{(s-d)}{1-3\cdot 2^{-s\beta}} \\ &= \frac{4d}{\log 3}\frac{\zeta(\alpha d)}{(2\pi)^d}\left(\frac{1}{3}\right)^{|\tau|} = \frac{4d}{\log 3}\frac{\zeta(\alpha d)}{(2\pi)^d}H_\gamma(C_\tau)\,. \end{aligned}$$

This implies that for any $f\in\mathcal{C}(K)$ for which $f\leq\chi_{C_\tau}$, $\text{vol}(f)\leq\frac{4d}{\log 3}\frac{\zeta(\alpha d)}{(2\pi)^d}\left(\frac{1}{3}\right)^{|\tau|}$, therefore points have zero volume, and $\text{vol}(\chi_{\dot{C}_\tau}) = \text{vol}(\chi_{C_\tau})$, where $\dot{C}_\tau$ denotes the interior of C_τ. As a consequence, for the simple functions given by finite linear combinations of characteristic functions of cells or vertices, $\text{vol}(\varphi) = \frac{4d}{\log 3}\frac{\zeta(\alpha d)}{(2\pi)^d}\int\varphi\, dH_\gamma$. Since continuous function are Riemann integrable w.r.t. such simple functions, the thesis follows. $\square$

4.3. **The commutator condition and Connes metric.** In this section we will show that for $\beta\in(0,1]$ the triple $(\mathcal{A},\mathcal{H},D)$ considered above is a spectral triple in the sense of Connes [16], up to the infinite dimensionality of $\ker(D)$. Moreover, the commutator $\|[D,f]\|$ gives a Lip-norm in the sense of Rieffel [38]. Such condition for spectral triples has been recently considered in [5], where these triples are called spectral metric spaces.

Definition 4.4. We shall consider the following seminorms on functions defined on lacunas ℓ_σ:

$$L_{\sigma\eta}(f) = \|f\|_{C^{0,\eta}(\ell_\sigma)}2^{|\sigma|(\beta-\eta)}$$

Proposition 4.5. *If $\eta\in(\alpha,1]$, $\beta\leq\eta$, $f\in C^{0,\eta}$, and c_ε is given in Proposition A.3, then*

$$\|[D,f]\| = \sup_{\sigma\in\Sigma}2^{\beta|\sigma|}p_\alpha(f\circ w_\sigma) \leq c_{\eta-\alpha}\sup_{\sigma\in\Sigma}L_{\sigma\eta}(f)\leq c_{\eta-\alpha}\|f\|_{C^{0,\eta}(K)}.$$

Proof. By equation (3.8) and Proposition A.3

$$\begin{aligned} \|[D,f]\| &= \|\bigoplus_{\sigma\in\Sigma}[D_\sigma,\pi_\sigma(f)]\| = \sup_{\sigma\in\Sigma}\|[D_\sigma,\pi_\sigma(f)]\| \\ &= \sup_{\sigma\in\Sigma}2^{\beta|\sigma|}\|[D_\alpha,\pi_\emptyset(f\circ w_\sigma)]\| = \sup_{\sigma\in\Sigma}2^{\beta|\sigma|}\|S_{f\circ w_\sigma}\| = \sup_{\sigma\in\Sigma}2^{\beta|\sigma|}p_\alpha(f\circ w_\sigma) \\ &\leq c_{\eta-\alpha}\sup_{\sigma\in\Sigma}2^{\beta|\sigma|}\|f\circ w_\sigma\|_{C^{0,\eta}(\ell_\emptyset)} = c_{\eta-\alpha}\sup_{\sigma\in\Sigma}\|f\|_{C^{0,\eta}(\ell_\sigma)}2^{|\sigma|(\beta-\eta)}. \end{aligned}$$

$\square$

The previous Proposition gives an estimate from above of the norm of the commutator. However, by making use of Lemma A.1, we may get an estimate from below.

Lemma 4.6. *Let $\tilde{c}_\alpha$ be as in Proposition A.3. Then*

$$\|[D,f]\| \geq \tilde{c}_\alpha \sup_{\sigma\in\Sigma} L_{\sigma\alpha}(f).$$

Proof.

$$\|[D,f]\| = \sup_{\sigma\in\Sigma} 2^{\beta|\sigma|} p_\alpha(f\circ w_\sigma) \geq \tilde{c}_\alpha \sup_{\sigma\in\Sigma} 2^{\beta|\sigma|} \|f\circ w_\sigma\|_{C^{0,\alpha}(\ell_\emptyset)} = \tilde{c}_\alpha \sup_{\sigma\in\Sigma} \|f\|_{C^{0,\alpha}(\ell_\sigma)} 2^{|\sigma|(\beta-\alpha)}.$$

□

Proposition 4.7. *There exists a constant $k(\alpha,\beta)$ such that*

$$\|f\|_{C^{0,\beta}(K)} \leq k(\alpha,\beta)\|[D,f]\|. \tag{4.3}$$

Proof. Our aim is to estimate $|f(x)-f(y)|$ for a continuous function f for which $\|[D,f]\| < \infty$.
1^{st} step. Let C_σ be a cell of level m, x a point in C_σ. We now construct inductively a sequence of cells $C_{\sigma(j,x)}$, $j\geq 1$, such that $x\in C_{\sigma(j,x)}$, $C_{\sigma(1,x)} := C_\sigma$, $C_{\sigma(j+1,x)} \subset C_{\sigma(j,x)}$, $|\sigma(j,x)| = m+j-1$ (if x is not a vertex such sequence is uniquely determined). We then construct a sequence $\{x_j\}_{j\geq 1}$ of points as follows: x_1 is a vertex of ℓ_σ contained in $C_{\sigma(2,x)}$, x_j is the unique point in $\ell_{\sigma(j-1,x)} \cap \ell_{\sigma(j,x)}$, $j>1$. By construction, $x_j \to x$ and the points x_j, x_{j+1} belong to the lacuna $\ell_{\sigma(j,x)}$.

We now observe that, by Lemma 4.6,

$$\begin{aligned}|f(x_{j+1}) - f(x_j)| &\leq \|f\|_{C^{0,\alpha}(\ell_{\sigma(j,x)})} d(x_{j+1},x_j)^\alpha \leq L_{\sigma(j,x),\alpha}(f) 2^{-(m+j-1)(\beta-\alpha)} (\mathrm{diam}(\ell_{\sigma(j,x)}))^\alpha \\ &\leq \tilde{c}_\alpha^{-1} 2^{-\alpha} \|[D,f]\| 2^{-\beta(m+j-1)}.\end{aligned}$$

As a consequence,

$$|f(x_1) - f(x)| \leq \sum_{j\geq 1} |f(x_{j+1}) - f(x_j)| \leq \tilde{c}_\alpha^{-1} 2^{-\alpha} (1-2^{-\beta})^{-1} \|[D,f]\| 2^{-m\beta}.$$

2^{nd} step. If x_0 is a vertex of level $n\neq 0$, and $m\geq n$, the butterfly shaped neighborhood $W(x_0,m)$ is the union of the two cells of level m containing x. For $x,y\in K$, let $W(x_0,m)$ be a minimal butterfly shaped neighborhood containing them. Observe that, by minimality, at least one of the points, say x, does not belong to $W(x_0,m+1)$, hence $\rho_{geo}(x,y) \geq \rho_{geo}(x,x_0) \geq 2^{-(m+1)}$.

Let us now choose $W(x_1,m+1)$ contained in one of the wings of $W(x_0,m)$ and containing both x and x_0. Reasoning as in the first step,

$$|f(x_0) - f(x)| \leq |f(x_0) - f(x_1)| + |f(x_1) - f(x)| \leq 2\tilde{c}_\alpha^{-1} 2^{-\alpha} (1-2^{-\beta})^{-1} \|[D,f]\| 2^{-m\beta},$$

hence,

$$\begin{aligned}|f(y) - f(x)| &\leq |f(y) - f(x_0)| + |f(x_0) - f(x)| \leq 4\tilde{c}_\alpha^{-1} 2^{-\alpha} (1-2^{-\beta})^{-1} \|[D,f]\| 2^{-m\beta} \\ &\leq 8\tilde{c}_\alpha^{-1} 2^{-\alpha} (1-2^{-\beta})^{-1} \|[D,f]\| (\rho_{geo}(x,y))^\beta.\end{aligned}$$

The thesis follows. □

Corollary 4.8. *For any $\alpha,\beta\in(0,1]$ the triple $(\mathcal{A},\mathcal{H},D)$ is a spectral triple, and the seminorm $f\to\|[D,f]\|$ is a Lip-norm according to Rieffel* [38]*. Therefore, the metric*

$$\rho_D(x,y) = \sup_{f\in\mathcal{A}} \frac{|f(x)-f(y)|}{\|[D,f]\|}$$

induces the original topology on K. Let ρ_{geo} denote the Euclidean geodesic metric on K. Then, if $\beta > \alpha$, $\|[D,f]\|$ and $\|f\|_{C^{0,\beta}(K)}$ are equivalent seminorms, and the metric ρ_D is bi-Lipschitz w.r.t. the metric $(\rho_{geo})^\beta$ on K.
In particular, if $\beta = 1$, the metric ρ_D is bi-Lipschitz w.r.t. ρ_{geo}.

Proof. Choosing $\eta = 1$ in Proposition 4.5 we prove the density of $\mathcal{A}$ in $C(K)$ for any $\beta \leq 1$, which, together with the results in the previous Sections, give the spectral triple property. The Lip-norm property follows by Proposition 4.7, cf. [38]. Indeed, it implies that functions for which $\|[D,f]\| \leq 1$ are equicontinuous, which gives the compactness property of the set of elements for which $\|[D,f]\| \leq 1$ and $\|f\| \leq 1$, and it also implies that $\|[D,f]\|$ vanishes only on constant functions.

When $\beta > \alpha$, we may choose $\eta = \beta$ in Proposition 4.5, thus getting the equivalence of the seminorms. The other results easily follow. $\square$

4.4. **The gasket in K-homology.** Let $\alpha, \beta \in (0,1]$, and denote by $(\mathcal{A}, \mathcal{H}, D)$ the spectral triple for the gasket considered above. Let F be the phase of D, and, for any $\sigma \in \Sigma$, denote by $\gamma_\sigma, \varepsilon_\sigma$, a copy of the gradings γ, ε associated, by Proposition 3.9, to the lacuna $\ell_\emptyset$, identified with $\mathbb{T}$. Finally, let $\gamma = \oplus_{\sigma\in\Sigma}\gamma_\sigma$, $\varepsilon = \oplus_{\sigma\in\Sigma}\varepsilon_\sigma$. As in Proposition 3.9, we set $\widetilde{F} = i\varepsilon F$.

Theorem 4.9. *The quintuple $(\pi, \mathcal{H}, F, \gamma, \varepsilon)$ is a tamely degenerate 1-graded Fredholm module on $\mathcal{A}$ which is non trivial in K-homology. In particular, it pairs non trivially with the generators of the (odd) K-theory of the gasket associated with the lacunas.*

Proof. First step. To check the compactness of $[\widetilde{F}, \pi(f)]$, it is enough to check that of $[\varepsilon, \pi(f)] = \oplus_\sigma[\varepsilon_\sigma, \pi_\emptyset(f \circ w_\sigma)]$. As in the proof of Proposition 3.9, this amounts to prove that $\oplus_\sigma[S_\sigma, \pi^0_\emptyset(f \circ w_\sigma)]$ is compact, where $\pi^i_\emptyset$ denotes the action of $C(\ell_\emptyset)$ on $L^2(\Omega^i(\ell_\emptyset))$, $i = 0, 1$, and, for each $\sigma \in \Sigma$, S_σ denotes a copy of the operator S in the proof of Proposition 3.9. Even though each summand is compact, the compactness of the direct sum is not obvious.

We first consider an affine function f in the plane, restricted to the gasket, and observe that consequently $f \circ w_\sigma|_{\ell_\emptyset} = const + 2^{-|\sigma|}f|_{\ell_\emptyset}$. Let us denote by $\{s_n\}$ the sequence of the singular values with multiplicity, arranged in a non increasing order, of $[S_\emptyset, \pi^0_\emptyset(f)]$. Then, for any given σ,

$$[S_\sigma, \pi^0_\emptyset(f \circ w_\sigma)] = 2^{-|\sigma|}[S_\emptyset, \pi^0_\emptyset(f)],$$

namely the sequence of singular values for $\oplus_\sigma[S_\sigma, \pi_\emptyset(f \circ w_\sigma)]$ is $\{2^{-|\sigma|}s_n : \sigma \in \Sigma, n \in \mathbb{N}\}$, showing that $[\varepsilon, \pi(f)]$ is compact.

Now, for any given $n \in \mathbb{N}$, consider the piece-wise affine functions $\mathrm{Aff}_n(K)$ on the gasket, which are affine when restricted to cells of level n. Reasoning as above, we obtain that, for $|\sigma| = n$, the operator $\oplus_{\tau\geq\sigma}[\varepsilon_\sigma, \pi_\emptyset(f \circ w_\tau)]$ is compact, from which the compactness of $[\varepsilon, \pi(f)]$ follows again. Since $\cup_n \mathrm{Aff}_n(K)$ is dense in $\mathcal{A}$, the thesis is proved.

Second step. We recall that the proof of the tame degeneracy on $\mathbb{T}$, given in Proposition 3.9, was a consequence of the compactness of $W^*\pi(f)W - \pi(f)$ on $L^2(\Omega^0(\mathbb{T}))$. In the present case, the compactness of $\oplus_\sigma W^*\pi^1_\emptyset(f \circ w_\sigma)W - \pi^0_\emptyset(f \circ w_\sigma)$ follows as in the first step, implying tame degeneracy as in the case of $\mathbb{T}$. $\square$

4.5. **The Dirichlet form.** Let us recall that the integral $\oint a$ of a element $a \in \mathcal{A}$ in noncommutative geometry is defined as the Dixmier trace $\mathrm{tr}_\omega(a|D|^d)$, where d is the metric dimension of the triple. Such integral may be computed, for a positive a, in three equivalent ways:

$$\lim_n \frac{\sigma_n(|D|^{-d/2}a|D|^{-d/2})}{\log n}; \tag{4.4}$$

$$\lim_{s\to 1}(s-1)\,\mathrm{tr}(|D|^{-d/2}a|D|^{-d/2})^s; \tag{4.5}$$

$$\lim_{s\to 1}(s-1)\,\mathrm{tr}(a|D|^{-sd}) = d^{-1}\lim_{t\to d}(t-d)\,\mathrm{tr}(a|D|^{-t}); \tag{4.6}$$

when such limits exist, cf. [16] Proposition 4 p.306, and [10] Corollary 3.7.

Our aim here is to use the NCG translation table to define a Dirichlet energy on spectral triples. Starting with the classical formula $f \mapsto \int |\nabla f|^2\, d\mathrm{vol}$, we replace ∇f with $[D, f]$, the integral $\oint$ as explained above, and get $a \mapsto \mathrm{tr}_\omega\left(|[D, a]|^2|D|^{-\delta}\right)$, for a suitable δ. As above, we may hope to compute the energy as

$$\lim_n \frac{\sigma_n(|D|^{\delta/2}|[D,a]|^2|D|^{\delta/2})}{\log n}; \tag{4.7}$$

$$\lim_{s\to 1}(s-1)\,\mathrm{tr}(|D|^{\delta/2}|[D,a]|^2|D|^{\delta/2})^s; \tag{4.8}$$

$$\lim_{s\to 1}(s-1)\,\mathrm{tr}(|[D,a]|^2|D|^{s\delta}) = \delta^{-1}\lim_{t\to \delta}(t-\delta)\,\mathrm{tr}(|[D,a]|^2|D|^{t}). \tag{4.9}$$

We first observe that: 1) we cannot expect the energy to be finite for all elements of $\mathcal{A}$, but only for elements in some suitable Sobolev space H^1; 2) elements for which such energy is finite do not necessarily have bounded commutator with D, the latter property requiring Lipschitz regularity, and even the converse is not necessarely true; 3) while formulas (4.7) and (4.8) necessarily coincide when $|D|^{\delta/2}|[D, a]|^2|D|^{\delta/2} \in \mathcal{L}^{1,\infty}$, the third may not; indeed, the proof in [10], Proposition 3.6 would require the boundedness of $[D, a]$; 4) the number δ does not necessarily coincide with the metric dimension d of the triple; formula (4.9) suggests to compute δ as the abscissa of convergence of $t \mapsto \mathrm{tr}(|[D, a]|^2|D|^t)$.

For sufficiently regular (noncommutative) spaces we may expect δ to coincide with d. In this case, for elements $a \in A$, [10] Corollary 3.7 applies, and formulae (4.7), (4.8), (4.9) coincide, where A consists of elements with bounded commutators with D. Under reasonable conditions, the quadratic form we get is closable and its closure is a Dirichlet form.

However, things change when spectral triples for the gasket are considered. In that case, as we shall see below, the abscissa of convergence δ for functions with finite Kigami energy is different from d, and formula (4.9) gives rise to a multiple of the Kigami energy on the gasket. Therefore we propose formula (4.9) as the right replacement for the Dirichlet energy in noncommutative geometry. We relate the inequality $\delta < d$ to the classical result of Kusuoka stating the singularity of the energy measures w.r.t. to the Hausdorff volume measure on the gasket. Only for a subalgebra of the algebra of finite energy functions we are able to show that formulas (4.7) and (4.8) reproduce Kigami energy.

In the following Theorem, a result of Jonsson [30], on the regularity of the trace of a finite energy function on an edge of the gasket, will imply that the Kigami energy on the gasket can be recovered via the spectral triple only if α is not too close to 1. In this section, when f is a continuous function on the gasket, $\mathcal{E}[f]$ denotes the Kigami energy. Let us first observe that

$$\mathrm{Tr}(|[D,f]|^2|D|^{-s}) = \sum_\sigma \mathrm{Tr}(|[D_\sigma,\pi_\sigma(f)]|^2|D_\sigma|^{-s}) = \sum_\sigma 2^{\beta(2-s)|\sigma|}\,\mathrm{Tr}(|[D_\emptyset,\pi_\emptyset(f\circ w_\sigma)]|^2|D_\emptyset|^{-s}).$$

However, the following holds.

Lemma 4.10. *Let $s > \frac{1}{\alpha}$. Then,* $\mathrm{Tr}(|[D_\emptyset, \pi_\emptyset(g)]|^2 |D_\emptyset|^{-s})$ *is finite if and only if* $g \in H^\alpha(\ell_\emptyset)$.

Proof. By (3.9), and the definition of $D_\emptyset$, we get

$$\begin{aligned}\mathrm{Tr}(|[D_\emptyset, \pi_\emptyset(g)]|^2 |D_\emptyset|^{-s}) &= \mathrm{Tr}\left(\begin{pmatrix} S_g S_g^* & 0 \\ 0 & S_{g^*}^* S_{g^*}\end{pmatrix} |D_\emptyset|^{-s}\right) \\ &= \mathrm{Tr}(S_{g^*}^* S_{g^*} (\partial_\alpha^* \partial_\alpha)^{-s/2}) + \mathrm{Tr}(S_g S_g^* (\partial_\alpha \partial_\alpha^*)^{-s/2}).\end{aligned}$$

As a consequence Lemma 3.6 implies

$$2\zeta(\alpha s)\|\partial_\alpha g\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} \leq \mathrm{Tr}(|[D_\emptyset, \pi_\emptyset(g)]|^2 |D_\emptyset|^{-s}) \leq 4\zeta(\alpha s)\|\partial_\alpha g\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)}\,. \tag{4.10}$$

□

We then recall that, according to [30] Thm. 3.1, the restrictions of finite energy functions to edges of the gasket belong to H^α, for $\alpha \leq \alpha_0 = \frac{\log(10/3)}{\log 4}$. We can then prove the following.

Theorem 4.11. *Assume as above that $\beta > 0$, $\frac{1}{2} < \alpha \leq \alpha_0$, with $\alpha_0 = \frac{\log(10/3)}{\log 4} \approx 0.87$, and assume f has finite Kigami energy* [2]. *Then the abscissa of convergence of the function* $\mathrm{tr}(|[D, f]|^2|D|^{-s})$ *is* $\max(\alpha^{-1}, \delta)$, *with $\delta := 2 - \frac{\log 5/3}{\beta \log 2}$. If $\delta > \alpha^{-1}$, i.e. $\beta(2 - \alpha^{-1}) > \frac{\log(5/3)}{\log 2}$,* $\mathrm{tr}(|[D, f]|^2|D|^{-s})$ *has a simple pole in δ, and*

$$\mathrm{Res}_{s=\delta}\, \mathrm{tr}(|[D, f]|^2|D|^{-s}) = const\ \mathcal{E}[f]. \tag{4.11}$$

Proof. According to the discussion above,

$$\mathrm{tr}(|[D, f]|^2|D|^{-s}) \leq 4\zeta(\alpha s)\sum_\sigma 2^{\beta(2-s)|\sigma|}\|\partial_\alpha(f\circ w_\sigma)\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)}\,. \tag{4.12}$$

By the trace theorem of Jonsson [30], the restriction map from $\mathcal{F}$ to $H^\alpha(\ell_\emptyset)$ is continuous (for $\alpha \leq \alpha_0$), implying in particular that, for a suitable constant K_1,

$$\|\partial_\alpha g\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} \leq K_1 \mathcal{E}[g]. \tag{4.13}$$

Hence,

$$\sum_{|\sigma|=n}\|\partial_\alpha(f\circ w_\sigma)\|^2_{L^2(\ell_\emptyset)} \leq K_1\sum_{|\sigma|=n}\mathcal{E}[f\circ w_\sigma] = K_1\left(\frac{3}{5}\right)^n\sum_{|\sigma|=n}\mathcal{E}_{C_\sigma}[f] = K_1\left(\frac{3}{5}\right)^n\mathcal{E}[f]. \tag{4.14}$$

As a consequence, if $s > \max(\alpha^{-1}, 2 - \frac{\log 5/3}{\beta\log 2})$,

$$tr(|[D, f]|^2|D|^{-s}) \leq 4K_1\zeta(\alpha s)\sum_n\left(\frac{3}{5}2^{\beta(2-s)}\right)^n\mathcal{E}[f] = \frac{4K_1\zeta(\alpha s)}{1 - \frac{3}{5}2^{\beta(2-s)}}\mathcal{E}[f]. \tag{4.15}$$

Now we compute the residue. Let us consider the linear map which associates to any vector $\vec{v} = (v_0, v_1, v_2) \in \mathbb{R}^3$, the 0-harmonic function $f = \varphi(\vec{v})$ on the gasket taking values v_i, $i = 0, 1, 2$, on the extreme points of the lacuna $\ell_\emptyset$. Clearly $\vec{v} \to \|\partial_\alpha\varphi(\vec{v})\|^2_{L^2(\ell_\emptyset)}$ is a quadratic form on $\mathbb{R}^3$, invariant under permutations of the components and vanishing only on constant vectors. As a consequence, $\|\partial_\alpha\varphi(\vec{v})\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)}$ is a multiple of $\sum_{i,j}|v_i - v_j|^2$, which in turn is a multiple of $\mathcal{E}[f]$, since f is 0-harmonic. We proved that, for 0-harmonic functions, there exists a non-zero constant K_2 for which

$$\|\partial_\alpha f\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} = K_2\mathcal{E}[f]. \tag{4.16}$$

[2] The conditions $\beta > 0$ and $2 - \frac{\log 5/3}{\beta\log 2} > \alpha^{-1}$ indeed imply $\alpha > \frac{1}{2}$.

Assume now f to be q-harmonic, and $s > \delta = \max(\alpha^{-1}, 2 - \frac{\log 5/3}{\beta \log 2})$. Then, making use of the equalites in (3.5) and (4.16),

$$\begin{aligned}\sum_{|\sigma|\geq q} 2^{\beta(2-s)|\sigma|} tr(S^*_{f\circ w_\sigma} S_{f\circ w_\sigma}(\partial^*_\alpha \partial_\alpha)^{-s/2}) &= 2\zeta(\alpha s)\sum_{|\sigma|\geq q} 2^{\beta(2-s)|\sigma|}\|\partial_\alpha(f\circ w_\sigma)\|_{L^2(\ell_\emptyset\times\ell_\emptyset)}\\ &= 2K_2\zeta(\alpha s)\sum_{n\geq q}\left(\frac{3}{5}2^{\beta(2-s)}\right)^n \mathcal{E}[f]\\ &= 2K_2\zeta(\alpha s)\mathcal{E}[f]\left(\frac{3}{5}2^{\beta(2-s)}\right)^q\left(1-\frac{3}{5}2^{\beta(2-s)}\right)^{-1}.\end{aligned}$$

Now we pass to the second part of equation (4.12). Reasoning as above and taking into account formula (3.6), for a 0-harmonic function g, $\|S^*_g\partial_\alpha e_k\|^2$ is again a multiple of the Kigami energy of g, namely for any k there exists a constant C_k such that

$$\|S^*_g\partial_\alpha e_k\|^2 = C_k\mathcal{E}[g]. \tag{4.17}$$

Formula (3.6) may be used to show that $0 < C_k \leq |k|^{2\alpha}$, if $k \neq 0$. Then, for an n-harmonic function f, and using (3.7), we get

$$\begin{aligned}\sum_{|\sigma|=n} 2^{\beta(2-s)n} tr(S_{f\circ w_\sigma} S^*_{f\circ w_\sigma}(\partial_\alpha \partial^*_\alpha)^{-s/2}) &= 2^{\beta(2-s)n}\sum_{|\sigma|=n}\sum_{k\neq 0}|k|^{-(s+2)\alpha}\|S^*_{f\circ w_\sigma}\partial_\alpha e_k\|^2\\ &= 2^{\beta(2-s)n}\sum_{|\sigma|=n}\sum_{k\neq 0} C_k|k|^{-(s+2)\alpha}\mathcal{E}[f\circ w_\sigma]\\ &= \left(\frac{3}{5}2^{\beta(2-s)}\right)^n\sum_{|\sigma|=n}\mathcal{E}_{C_\sigma}[f]\sum_{k\neq 0}C_k|k|^{-(s+2)\alpha} = \left(\frac{3}{5}2^{\beta(2-s)}\right)^n\mathcal{E}[f]C(s),\end{aligned}$$

where we set $C(s) = \sum_k C_k|k|^{-(s+2)\alpha}$. Assuming now f to be q-harmonic, we get

$$\begin{aligned}\sum_{|\sigma|\geq q} 2^{\beta(2-s)|\sigma|} tr(S_{f\circ w_\sigma} S^*_{f\circ w_\sigma}(\partial_\alpha \partial^*_\alpha)^{-s/2}) &= \sum_{n\geq q}\left(\frac{3}{5}2^{\beta(2-s)}\right)^n\mathcal{E}[f]C(s)\\ &= C(s)\mathcal{E}[f]\left(\frac{3}{5}2^{\beta(2-s)}\right)^q\left(1-\frac{3}{5}2^{\beta(2-s)}\right)^{-1}.\end{aligned}$$

We note that, by relations (3.6), (4.16), (4.17),

$$\mathcal{E}[g]\sum_k C_k|k|^{-(s+2)\alpha} = \sum_k |k|^{-(s+2)\alpha}\|S^*_g\partial_\alpha e_k\|^2 \leq 2\zeta(s\alpha)\|\partial_\alpha g\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)} = 2K_2\zeta(s\alpha)\mathcal{E}[g],$$

for any 0-harmonic function g, namely $C(s) \leq K_2\zeta(\alpha s)$. This shows that, when $2 - \frac{\log 5/3}{\beta\log 2} > \alpha^{-1}$, the function $tr(|[D,f]|^2|D|^{-s})$ has a simple pole in $s = \delta$, with residue $(\beta\log 2)^{-1}(2K_2\zeta(\alpha\delta) + C(\delta))\mathcal{E}[f]$. Equation (4.11) now follows for q-harmonic functions, the result for general functions depending on continuity. □

4.6. **Kigami Energy and Dixmier traces.** In this section we re-construct the Kigami energy form on the Sierpinski gasket using the Dixmier trace. In particular, the self-similar energy of a function in a suitable form core, coincides with the evaluation, by the Dixmier trace, of the square of the modulus of its commutator with the Dirac operator D times a weight proportional to a negative power of $|D|$.

Definition 4.12. For any $\varepsilon > 0$, we shall consider the set $\mathcal{B}_\varepsilon$ defined as follows:

$$\mathcal{B}_\varepsilon := \{f \in \mathcal{F} : \exists c > 0 \text{ such that } \mathcal{E}_{C_\sigma}[f] \leq c\, e^{-\varepsilon|\sigma|}\mathcal{E}[f], \sigma \in \Sigma\},$$

and set $\mathcal{B} := \cup_{\varepsilon>0}\mathcal{B}_\varepsilon$.

Lemma 4.13. *Let f be a k-harmonic function. Then $f \in \mathcal{B}_{\log(5/3)}$. More precisely, for any $\sigma \in \Sigma$, $\mathcal{E}_{C_\sigma}[f] \leq (3/5)^{(|\sigma|-k)}\mathcal{E}[f]$.*

Proposition 4.14. *Each $\mathcal{B}_\varepsilon$, $\varepsilon > 0$, is a vector space, $\mathcal{B}$ is an algebra.*

Proof. We first prove additivity. The case $f_1 + f_2 = 0$ being trivial, we assume $\mathcal{E}[f_1 + f_2] \neq 0$. Then

$$\begin{aligned}\mathcal{E}_{C_\sigma}[f_1 + f_2] &\leq 2\mathcal{E}_{C_\sigma}[f_1] + 2\mathcal{E}_{C_\sigma}[f_2]\\ &\leq 2(c_1\mathcal{E}[f_1] + c_2\mathcal{E}[f_2])e^{-\varepsilon|\sigma|} = c\, e^{-\varepsilon|\sigma|}\mathcal{E}[f_1 + f_2],\end{aligned}$$

where $c = \mathcal{E}[f_1+f_2]^{-1}2(c_1\mathcal{E}[f_1]+c_2\mathcal{E}[f_2])$. As for multiplicativity, assuming as before $\mathcal{E}[f_1f_2] \neq 0$, we get

$$\begin{aligned}\mathcal{E}_{C_\sigma}[f_1f_2] &\leq \|f_2|_{C_\sigma}\|_\infty\mathcal{E}_{C_\sigma}[f_1] + \|f_1|_{C_\sigma}\|_\infty\mathcal{E}_{C_\sigma}[f_2]\\ &\leq \|f_2\|_\infty c_1\mathcal{E}[f_1]e^{-\varepsilon_1|\sigma|} + \|f_1\|_\infty c_2\mathcal{E}[f_2]e^{-\varepsilon_2|\sigma|} = c\, e^{-\varepsilon|\sigma|}\mathcal{E}[f_1f_2],\end{aligned}$$

where $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ and $c = \mathcal{E}[f_1f_2]^{-1}(\|f_2\|_\infty c_1\mathcal{E}[f_1] + \|f_1\|_\infty c_2\mathcal{E}[f_2])$. □

Lemma 4.15. *Assume $\alpha^{-1} < \delta$, $f \in \mathcal{B}_\varepsilon$. There exists $s \in (\alpha^{-1}, \delta)$ such that $[D, f]|D|^{-\frac{1}{2}s}$ is bounded.*

Proof. Making use of (4.10), (4.13), we get:

$$\begin{aligned}\|[D, f]\ |D|^{-\frac{1}{2}s}\|^2 &= \sup_\sigma \|[D_\sigma, f]|D_\sigma|^{-\frac{1}{2}s}\|^2\\ &= \sup_\sigma 2^{\beta|\sigma|(2-s)}\|[D_\emptyset, f\circ w_\sigma]|D_\emptyset|^{-\frac{1}{2}s}\|^2\\ &\leq \sup_\sigma 2^{\beta|\sigma|(2-s)}\operatorname{tr}(|[D_\emptyset, f\circ w_\sigma]|^2 D_\emptyset|^{-s})\\ &\leq 4\sup_\sigma 2^{\beta|\sigma|(2-s)}\zeta(\alpha s)\|\partial_\alpha(f\circ w_\sigma)\|^2_{L^2(\ell_\emptyset\times\ell_\emptyset)}\\ &\leq 4K_1\zeta(\alpha s)\sup_\sigma 2^{\beta|\sigma|(2-s)}\mathcal{E}[f\circ w_\sigma]\\ &\leq 4K_1\zeta(\alpha s)\sup_\sigma 2^{\beta|\sigma|(2-s)}\left(\frac{3}{5}\right)^{|\sigma|}\mathcal{E}_{C_\sigma}[f]\\ &\leq 4K_1\zeta(\alpha s)\sup_n \exp((\beta(2-s)\log 2 + \log(3/5))n)\max_{|\sigma|=n}\mathcal{E}_{C_\sigma}[f]\\ &\leq 4cK_1\zeta(\alpha s)\sup_n \exp((\beta(2-s)\log 2 + \log(3/5) - \varepsilon)n)\mathcal{E}[f].\end{aligned}$$

We get a non trivial bound when $\beta(2-s)\log 2 + \log(3/5) - \varepsilon \leq 0$, namely

$$s \geq 2 - \frac{\log 5/3 + \varepsilon}{\beta\log 2} = \delta - \frac{\varepsilon}{\beta\log 2}$$

We have proved that, for $\max(\alpha^{-1}, \delta - \dfrac{\varepsilon}{\beta\log 2}) < s < \delta$, the thesis is satisfied, more precisely,

$$\||D|^{-\frac{1}{2}s}|[D, f]|^2|D|^{-\frac{1}{2}s}\| = \|[D, f]|D|^{-\frac{1}{2}s}\|^2 \leq 4cK_1\zeta(\alpha s)\mathcal{E}[f].$$

□

Theorem 4.16. *The set* $\{|D|^{-\frac{1}{2}\delta}|[D,f]|^2|D|^{-\frac{1}{2}\delta} \in \mathcal{B}(\mathcal{H}) : f \in \mathcal{B}\}$ *is contained in* $\mathcal{L}^{1,\infty}(\mathcal{H})$*, namely* $\exists M > 0$ *such that* $f \in \mathcal{B}$ *implies* $\mathrm{tr}_\omega\left(|[D,f]|^2|D|^{-\delta}\right) \le M\mathcal{E}[f]$*.*

Proof. We shall use Lemma 4.5 in [19] with the contraction U given by the operator $[D,f]|D|^{-\frac{1}{2}s}$ suitably normalized, the positive operator T given by $|D|^{-(\delta-s)}$, and the convex function $f(x) = x^{1+t}$, with $t > 0$. Then, with s as in the previous Lemma,

$$\begin{aligned}
&\mathrm{tr}\left((|D|^{-\frac{1}{2}\delta}|[D,f]|^2|D|^{-\frac{1}{2}\delta})^{1+t}\right)\\
&\qquad = \mathrm{tr}\left((|D|^{-\frac{1}{2}(\delta-s)}|D|^{-\frac{1}{2}s}|[D,f]|^2|D|^{-\frac{1}{2}s}|D|^{-\frac{1}{2}(\delta-s)})^{1+t}\right)\\
&\qquad = \||D|^{-\frac{1}{2}s}|[D,f]|^2|D|^{-\frac{1}{2}s}\|^{1+t}\,\mathrm{tr}\left((T^{\frac{1}{2}}U^*UT^{\frac{1}{2}})^{1+t}\right)\\
&\qquad = \|[D,f]|D|^{-\frac{1}{2}s}\|^{2(1+t)}\,\mathrm{tr}\left((UTU^*)^{1+t}\right)\\
&\qquad \le \|[D,f]|D|^{-\frac{1}{2}s}\|^{2(1+t)}\,\mathrm{tr}(UT^{1+t}U^*)\\
&\qquad = \|[D,f]|D|^{-\frac{1}{2}s}\|^{2(1+t)}\,\mathrm{tr}(T^{\frac{1}{2}(1+t)}U^*UT^{\frac{1}{2}(1+t)})\\
&\qquad = \|[D,f]|D|^{-\frac{1}{2}s}\|^{2t}\,\mathrm{tr}(|D|^{-\frac{1}{2}(\delta-s)(1+t)}|D|^{-\frac{1}{2}s}|[D,f]|^2|D|^{-\frac{1}{2}s}|D|^{-\frac{1}{2}(\delta-s)(1+t)})\\
&\qquad = \|[D,f]|D|^{-\frac{1}{2}s}\|^{2t}\,\mathrm{tr}(|[D,f]|^2|D|^{-(\delta+t(\delta-s))}).
\end{aligned}$$

The previous inequality, together with equation (4.15), gives

$$\mathrm{tr}\left((|D|^{-\frac{1}{2}\delta}|[D,f]|^2|D|^{-\frac{1}{2}\delta})^{1+t}\right) \le \|[D,f]|D|^{-\frac{1}{2}s}\|^{2t}4K_1\zeta(\alpha s)\mathcal{E}[f]\left(1-\frac{3}{5}2^{\beta(2-\delta-t(\delta-s))}\right)^{-1}$$

hence

$$\limsup_{t\to 0} t\,\mathrm{tr}\left((|D|^{-\frac{1}{2}\delta}|[D,f]|^2|D|^{-\frac{1}{2}\delta})^{1+t}\right) \le \frac{4K_1\zeta(\alpha s)}{\beta(\delta-s)\log 2}\mathcal{E}[f]. \tag{4.18}$$

The thesis follows by Theorem 4.5 (i) in [9]. □

Lemma 4.17. *Let* B *be a positive, possibly unbounded, operator on* $\mathcal{H}$*,* $T \in \mathcal{B}(\mathcal{H})_+$ *such that* $\mathrm{Tr}(T^s)$ *is finite, for* $s > d$*,* $\mathrm{Tr}(BT^sB)$ *is finite, for* $s > \delta$*, and* $\lim_{s\to\delta^+}(s-\delta)\,\mathrm{Tr}(BT^sB)$ *exists and is finite. Then* $BT^\delta B \in \mathcal{L}^{1,\infty}(\mathcal{H})$ *and*

$$\lim_{s\to\delta^+}(s-\delta)\,\mathrm{Tr}(BT^sB) \le d\cdot\mathrm{Tr}_\omega(BT^\delta B). \tag{4.19}$$

Proof. Let us set $A = BT^{\delta/2}$. For $r > 0$, Hölder inequality for the conjugate exponents $r+1$, $\frac{r+1}{r}$ gives

$$\mathrm{Tr}(BT^sB) = \mathrm{Tr}\left(A^*AT^{s-\delta}\right) \le \left(\mathrm{Tr}(A^*A)^{1+\frac{1}{r}}\right)^{\frac{r}{r+1}}\left(\mathrm{Tr}(T^{(s-\delta)(r+1)})\right)^{\frac{1}{r+1}}.$$

Setting $r = \dfrac{d+\varepsilon+\delta-s}{s-\delta}$ for $\varepsilon > 0$, i.e. $(s-\delta)(r+1) = d+\varepsilon$, we get

$$\begin{aligned}
\lim_{s\to\delta^+}(s-\delta)\,\mathrm{Tr}(BT^sB) &\le \lim_{r\to\tilde\omega}\frac{d+\varepsilon}{r+1}\left(\mathrm{Tr}(A^*A)^{1+\frac{1}{r}}\right)^{\frac{r}{r+1}}\left(\mathrm{Tr}(T^{(d+\varepsilon)})\right)^{\frac{1}{r+1}}\\
&\le (d+\varepsilon)\lim_{r\to\tilde\omega}\frac{r}{r+1}\left(\frac{1}{r}\ \mathrm{Tr}(A^*A)^{1+\frac{1}{r}}\right)^{\frac{r}{r+1}}\\
&= (d+\varepsilon)\,\mathrm{Tr}_\omega(A^*A) = (d+\varepsilon)\,\mathrm{Tr}_\omega(BT^\delta B).
\end{aligned}$$

The thesis follows by the arbitrariness of ε. □

Proposition 4.18. *The quadratic form* $f \to \mathrm{Tr}_\omega(|[D,f]|^2|D|^{-\delta})$ *is self-similar and invariant under rotations of* $\frac{2}{3}\pi$*.*

Proof. Let us prove self-similarity.

$$\begin{aligned}
\sum_{i=1,2,3} \mathrm{Tr}_\omega(|[D, f\circ w_i]|^2|D|^{-\delta}) &= \sum_{i=1,2,3} \mathrm{Tr}_\omega\left(\bigoplus_\sigma |[D_\sigma, f\circ w_i\circ w_\sigma]|^2|D_\sigma|^{-\delta}\right)\\
&= \mathrm{Tr}_\omega\left(\bigoplus_{i=1,2,3}\bigoplus_\sigma 2^{\beta(2-\delta)|\sigma|}|[D_\emptyset, f\circ w_i\circ w_\sigma]|^2|D_\emptyset|^{-\delta}\right)\\
&= \mathrm{Tr}_\omega\left(\bigoplus_{\tau\neq\emptyset} 2^{\beta(2-\delta)(|\tau|-1)}|[D_\emptyset, f\circ w_\tau]|^2|D_\emptyset|^{-\delta}\right)\\
&= 2^{-\beta(2-\delta)}\big(\mathrm{Tr}_\omega(|[D,f]|^2|D|^{-\delta}) - \mathrm{Tr}_\omega(|[D_\emptyset,f]|^2|D_\emptyset|^{-\delta})\big)\\
&= \frac{3}{5}\,\mathrm{Tr}_\omega(|[D,f]|^2|D|^{-\delta}),
\end{aligned}$$

where $\mathrm{Tr}_\omega(|[D_\emptyset, f]|^2|D_\emptyset|^{-\delta})$ vanishes since, as shown in Lemma 4.10, $|[D_\emptyset, f]|^2|D_\emptyset|^{-\delta}$ is trace class. Rotation invariance can be proved along the same lines. □

Corollary 4.19. *On the algebra $\mathcal{B}$, the quadratic form $f \to \mathrm{Tr}_\omega(|[D,f]|^2|D|^{-\delta})$ coincides with Kigami energy up to a multiplicative constant.*

Proof. Let $f \in \mathcal{B}_\varepsilon$. Then, Theorem 4.11, inequalities (4.18) and (4.19) show that $\mathrm{Tr}_\omega(|[D,f]|^2|D|^{-\delta})$ is finite non-vanishing *iff* $\mathcal{E}[f]$ is finite non-vanishing. Therefore, by the Proposition above, they coincide (up to a constant) on finitely harmonic functions. Approximating $f \in \mathcal{B}_\varepsilon$ with finitely harmonic functions, we get the thesis, again using inequality (4.18). □

4.7. **The (β, α) plane.**

- The whole construction makes sense only if $0 < \alpha \leq 1$, $\beta \in \mathbb{R}$.
- If $\beta > 0$, the inverse of D on the orthogonal complement of the kernel is compact.
- if $\beta > 0$ and $\alpha > \beta/d_S$, the noncommutative volume measure coincides (up to a constant factor) with the Hausdorff measure H_{d_S}, $d_S = \frac{\log 3}{\log 2}$ = the similarity dimension. The metric dimension is $d_D = \frac{d_S}{\beta}$.
- If $0 < \beta \leq 1$, $\|[D, f]\|$ is a densely defined Lip-norm, $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple and $(\pi, \mathcal{H}, F, \varepsilon_0, \varepsilon)$ is a 1-graded Fredholm module. The latter has non-trivial pairing with $K^1(K)$.
- If $\beta \leq 1$ and $\alpha < \beta$, ρ_D is bi-Lipschitz w.r.t. $(\rho_{geo})^\beta$. Moreover, if $\beta = 1$ and $0.79 \approx \frac{\log 2}{\log(12/5)} < \alpha \leq \frac{\log(10/3)}{\log 4} \approx 0.87$, d_D coincides with the similarity dimension of the gasket and ρ_D is bi-Lipschitz with ρ_{geo}.
- If $\alpha_0 < \beta \leq 1$ and $\delta^{-1} < \alpha \leq \alpha_0$, with $\delta = 2 - \frac{\log(5/3)}{\log 2}\beta^{-1}$, then the residue in $s = \delta$ of $\mathrm{tr}([D, f]^2 |D|^{-s})$ (noncommutative energy) gives the Kigami energy, up to a constant factor. Observe that in this case the inequality $\beta > \alpha$ is automatically satisfied.

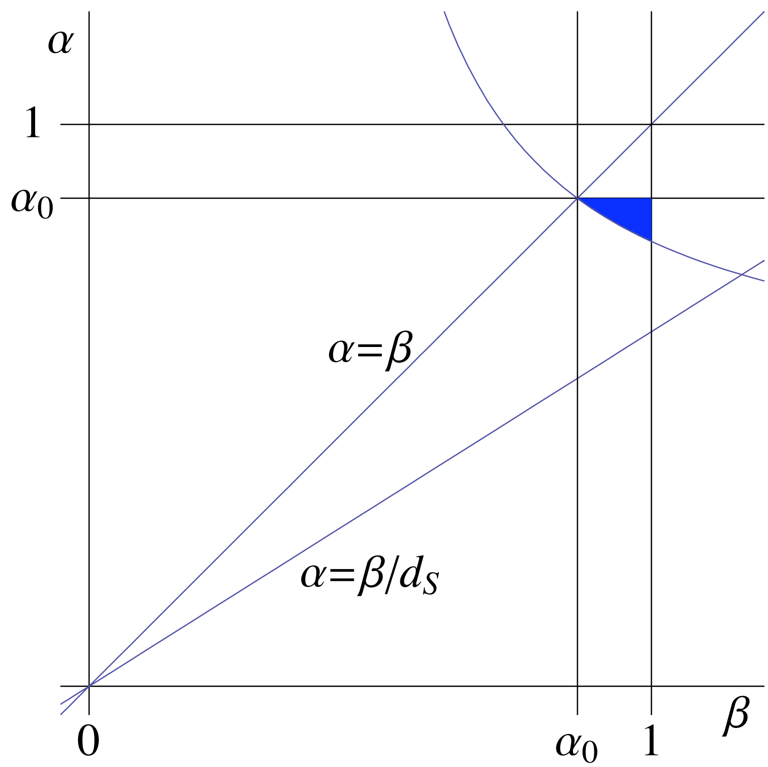

FIGURE 4. The (β, α) plane

Appendix A. Estimates on the Clausen function

According to ([36], p. 236, [18] section 1.11) the analytic extension of the polylogarithm function of order s on the whole complex plane with the line $[1,+\infty)$ removed is given by

$$\mathrm{Li}_s(z) = -\frac{z\Gamma(1-s)}{2\pi i}\int_\gamma \frac{(-t)^{s-1}}{e^t - z}\,dt,$$

where γ is a path as in figure 5.

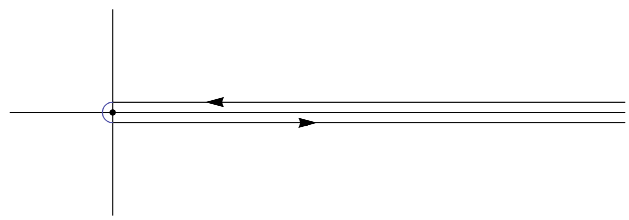

Figure 5. Path used for the analytic extension of polylogarithm.

Therefore the Clausen cosine function $\mathrm{Ci}_s(t)$ can be defined as

$$\mathrm{Ci}_s(\vartheta) = -\operatorname{Re}\frac{\Gamma(1-s)}{2\pi i}\int_\gamma \frac{(-t)^{s-1}}{e^{t-i\vartheta}-1}\,dt,$$

Lemma A.1. *When* $\operatorname{Re} s < 1$, $\lim_{t\to 0^\pm} |t|^{1-s}\,\mathrm{Li}_s(e^{it}) = \Gamma(1-s)e^{\pm i\pi(1-s)/2}$, *as a consequence, for* $\alpha \in (0,1]$,

$$\lim_{t\to 0}|t|^{1+2\alpha}\,\mathrm{Ci}_{-2\alpha}(t) = -\Gamma(1+2\alpha)\sin\pi\alpha.$$

Moreover, when $\alpha\in[\frac12,1)$ *and* $|t|\le\frac{\pi}{4}$, $\mathrm{Ci}_{-2\alpha}$ *is strictly negative, and*

$$|\,\mathrm{Ci}_{-2\alpha}(t) + \Gamma(1+2\alpha)\sin\pi\alpha\,|t|^{-(2\alpha+1)}| \le \frac{31}{2\pi^2}\Gamma(1+2\alpha)\sin\pi\alpha, \tag{A.1}$$

$$\frac{1}{32}\sin(\pi\alpha)\Gamma(1+2\alpha) \le -\,\mathrm{Ci}_{-2\alpha}(t)|t|^{2\alpha+1} \le \frac{63}{32}\sin(\pi\alpha)\Gamma(1+2\alpha). \tag{A.2}$$

Finally, when $|t|\ge\pi/4$,

$$|\,\mathrm{Ci}_{-2\alpha}(t)|\,|t|^{2\alpha+1} \le 23.$$

Proof. Let $0<|\vartheta|\le\pi/4$. Then we may choose γ in figure 5 as $\gamma_0-\sigma$ where γ_0 is made of the half lines $\sqrt{\pi^2-\varepsilon^2}+t\pm i\varepsilon$, $t>0$, and (most of) the circle of radius π centered at the origin, and σ is a suitably small positively oriented cycle surrounding the point $i\vartheta$. Then

$$\begin{aligned}
\mathrm{Li}_s(e^{i\vartheta}) &= -\frac{e^{i\vartheta}\Gamma(1-s)}{2\pi i}\int_{\gamma_0}\frac{(-t)^{s-1}}{e^t-e^{i\vartheta}}\,dt + \frac{e^{i\vartheta}\Gamma(1-s)}{2\pi i}\int_\sigma \frac{(-t)^{s-1}}{e^t-e^{i\vartheta}}\,dt\\
&= -\frac{\Gamma(1-s)}{2\pi i}\int_{\gamma_0}\frac{(-t)^{s-1}}{e^{(t-i\vartheta)}-1}\,dt + \Gamma(1-s)\,\mathrm{Res}_{t=i\vartheta}\frac{(-t)^{s-1}}{e^{(t-i\vartheta)}-1}\\
&= -\frac{\Gamma(1-s)}{2\pi i}\int_{\gamma_0}\frac{(-t)^{s-1}}{e^{(t-i\vartheta)}-1}\,dt + \Gamma(1-s)e^{i\,\mathrm{sgn}(\vartheta)\pi(1-s)/2}|\vartheta|^{s-1}.
\end{aligned}$$

In particular,

$$\mathrm{Ci}_{-2\alpha}(\vartheta) = \operatorname{Re}\mathrm{Li}_{-2\alpha}(e^{i\vartheta}) = -\frac{\Gamma(1+2\alpha)}{2\pi}\operatorname{Im}\left(\int_{\gamma_0}\frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right) - \Gamma(1+2\alpha)\sin\pi\alpha\;|\vartheta|^{-(2\alpha+1)},$$

from which the first relations hold. Moreover,

$$-\operatorname{Ci}_{-2\alpha}(\vartheta) = \Gamma(1+2\alpha)\sin\alpha\pi\ |\vartheta|^{-2\alpha-1} + \frac{\Gamma(1+2\alpha)}{2\pi}\ \operatorname{Im}\left(\int_{\gamma_0} \frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right),$$

hence

$$|\operatorname{Ci}_{-2\alpha}(\vartheta) + \Gamma(1+2\alpha)\sin\alpha\pi\ |\vartheta|^{-2\alpha-1}| = \frac{\Gamma(1+2\alpha)}{2\pi}\left|\operatorname{Im}\left(\int_{\gamma_0} \frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right)\right|.$$

We now assume $\alpha \geq \frac{1}{2}$, and observe that the part of the path constituted by the half lines $\sqrt{\pi^2-\varepsilon^2}+t\pm i\varepsilon$, $t>0$ is invariant under reflection w.r.t. to the real axis, which sends the variable of integration to its conjugate. Therefore,

$$\int_{\gamma_0} \frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt = \int_{|z|=\pi} \frac{(-z)^{-(1+2\alpha)}}{e^{(z-i\vartheta)}-1}\,dz + 2i\sin(2\pi\alpha)\int_\pi^\infty \frac{t^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\ .$$

As for the second integral, we have

$$\begin{aligned}
&\left|2\sin(2\pi\alpha)\ \operatorname{Im}\left(i\int_\pi^\infty \frac{t^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right)\right| \\
&\qquad = 2|\sin(2\pi\alpha)|\ \left|\int_\pi^\infty \frac{\big(e^t\cos\vartheta-1\big)t^{-(1+2\alpha)}}{|e^{(t-i\vartheta)}-1|^2}\,dt\right| \\
&\qquad \leq 2|\sin(2\pi\alpha)|\,\pi^{-(1+2\alpha)}\int_\pi^\infty \frac{e^t}{e^{2t}+1-2e^t\cos\vartheta}\,dt \\
&\qquad \leq 2|\sin(2\pi\alpha)|\,\pi^{-(1+2\alpha)}\int_\pi^\infty \frac{e^t}{(e^t-1)^2}\,dt \\
&\qquad \leq 4|\sin(\pi\alpha)|\,\pi^{-(1+2\alpha)}(e^\pi-1)^{-1}, \qquad \text{(A.3)}
\end{aligned}$$

where in the first inequality we used $|\vartheta|\leq\frac{\pi}{4}$, which implies $|e^t\cos\vartheta-1|\leq e^t$ for $t\geq\pi$. We now come to the first integral. Let us observe that, when $\alpha\in\mathbb{Z}$, it is a contour integral of a meromorphic function, therefore it may be computed via residues. In particular, when $\alpha\neq 0$, the only residue comes from $z=i\vartheta$, whose real part vanishes, as shown above. To get an estimate which is small for α close to 1, we set

$$\psi(\alpha,\vartheta) = \operatorname{Im}\left(\int_{|z|=\pi} \frac{(-z)^{-(1+2\alpha)}}{e^{(z-i\vartheta)}-1}\,dz\right),$$

so that we have

$$|\psi(\alpha,\vartheta)| = \left|\int_\alpha^1 \frac{\partial\psi}{\partial\alpha}(s,\vartheta)ds\right| \leq \int_\alpha^1 \left|\frac{\partial\psi}{\partial\alpha}(s,\vartheta)\right| ds \leq (1-\alpha)\sup_{\alpha\leq s\leq 1}\left|\frac{\partial\psi}{\partial\alpha}(s,\vartheta)\right|. \tag{A.4}$$

Moreover,

$$\begin{aligned}
|\partial_\alpha\psi(s,\vartheta)| &= \left|\operatorname{Im}\left(\int_{|z|=\pi} -2\log(-z)\frac{(-z)^{-(1+2s)}}{e^{(z-i\vartheta)}-1}\,dz\right)\right| \\
&= \left|\operatorname{Im}\left(\int_0^{2\pi} -2(\log\pi+i(t-\pi))\pi^{-2s}\frac{e^{-i(t-\pi)(1+2s)}}{e^{(\pi e^{it}-i\vartheta)}-1}i\,e^{it}\,dt\right)\right| \\
&\leq 4\pi^{1-2\alpha}(\log\pi+\pi)\left(\min_{t\in[0,2\pi]}|e^{(\pi e^{it}-i\vartheta)}-1|^2\right)^{-1/2}, \quad \alpha\leq s\leq 1.
\end{aligned}$$

We now consider the two cases $\pi|\sin t| \leq |\vartheta| + \pi/2$, and $\pi|\sin t| \geq |\vartheta| + \pi/2$.
If $\pi|\sin t| \leq |\vartheta| + \pi/2$, $|\cos t| \geq (1 - (|\vartheta|/\pi + 1/2)^2)^{1/2}$, and

$$\begin{aligned}|e^{(\pi e^{it} - i\vartheta)} - 1|^2 &= e^{2\pi\cos t} + 1 - 2e^{\pi\cos t}\cos(\pi\sin t - \vartheta) \\ &\geq (e^{\pi\cos t} - 1)^2 \geq (1 - e^{-(\pi^2 - (|\vartheta| + \pi/2)^2)^{1/2}})^2.\end{aligned}$$

If $\pi|\sin t| \geq |\vartheta| + \pi/2$, $\frac{3}{2}\pi \geq |\pi\sin t - \vartheta| \geq |\pi\sin t| - |\vartheta| \geq \pi/2$, therefore $\cos(\pi\sin t - \vartheta) \leq 0$, and

$$|e^{(e^{it} - i\vartheta)} - 1|^2 = e^{2\pi\cos t} + 1 - 2e^{\pi\cos t}\cos(\pi\sin t - \vartheta) \geq 1.$$

We have proved that

$$|\psi(\alpha,\vartheta)| \leq 4(1-\alpha)\pi^{1-2\alpha}(\log\pi + \pi)(1 - e^{-(\pi^2 - (|\vartheta| + \pi/2)^2)^{1/2}})^{-1}, \tag{A.5}$$

hence, by inequalities (A.3), (A.4), (A.5), and since $\alpha \geq 1/2$ implies $2(1-\alpha) \leq \sin\pi\alpha$,

$$\left|\operatorname{Im}\left(\int_{\gamma_0} \frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)} - 1}\,dt\right)\right| \leq \frac{2\sin(\pi\alpha)}{\pi^{(1+2\alpha)}}\Big(\frac{2}{e^\pi - 1} + \frac{\pi^2(\log\pi + \pi)}{1 - e^{-(\pi^2 - (|\vartheta| + \pi/2)^2)^{1/2}}}\Big). \tag{A.6}$$

Then,

$$\left|\frac{\operatorname{Ci}_{-2\alpha}(\vartheta)|\vartheta|^{2\alpha+1}}{\sin(\pi\alpha)\Gamma(1+2\alpha)} + 1\right| \leq \frac{|\vartheta|^{2\alpha+1}}{2\pi\sin(\pi\alpha)}\left|\operatorname{Im}\left(\int_{\gamma_0} \frac{(-t)^{-(1+2\alpha)}}{e^{(t-i\vartheta)} - 1}\,dt\right)\right| \leq \left(\frac{|\vartheta|}{\pi}\right)^{2\alpha+1} h\left(\frac{|\vartheta|}{\pi}\right),$$

where the function $h(r)$, $r \in (0, 1/4]$ is given by

$$h(r) = \frac{\pi(\log\pi + \pi)}{1 - \exp\left(-\pi\sqrt{1 - (r + 1/2)^2}\right)} + \frac{2}{\pi(e^\pi - 1)}.$$

Since h is increasing, it attains its maximum for $r = \frac{1}{4}$, where $h(\frac{1}{4}) < \frac{31}{2}$. Hence,

$$\left(1 - \frac{31}{2}\left(\frac{|\vartheta|}{\pi}\right)^{2\alpha+1}\right) \leq \frac{-\operatorname{Ci}_{-2\alpha}(\vartheta)|\vartheta|^{2\alpha+1}}{\sin(\pi\alpha)\Gamma(1+2\alpha)} \leq \left(1 + \frac{31}{2}\left(\frac{|\vartheta|}{\pi}\right)^{2\alpha+1}\right), \tag{A.7}$$

which implies (A.1). Now, since $|\vartheta| \leq \pi/4$ and $\alpha \geq 1/2$, we get $\frac{31}{2}(|\vartheta|/\pi)^{2\alpha+1} \leq \frac{31}{32}$, hence

$$\frac{1}{32}\sin(\pi\alpha)\Gamma(1+2\alpha) \leq -\operatorname{Ci}_{-2\alpha}(\vartheta)|\vartheta|^{2\alpha+1} \leq \frac{63}{32}\sin(\pi\alpha)\Gamma(1+2\alpha),$$

showing in particular that $-\operatorname{Ci}_{-2\alpha}(\vartheta)$ is strictly positive for $|\vartheta| \leq \pi/4$.

We finally estimate $|\operatorname{Ci}_{-2\alpha}(\vartheta)|$ for $|\vartheta| \geq \frac{\pi}{4}$. We simply choose the contour γ as the circle of radius $\lambda|\vartheta|$ around the origin and the half lines $\sqrt{\lambda^2\vartheta^2 - \varepsilon^2} + t \pm i\varepsilon$, $t > 0$, for $\frac{1}{2} < \lambda < 1$. As for the first integral, we get

$$\left|\int_{|z| = \lambda|\vartheta|} \frac{(-z)^{-(1+2\alpha)}}{e^{(z-i\vartheta)} - 1}\,dz\right| \leq 2\pi\lambda|\vartheta|(\lambda|\vartheta|)^{-(2\alpha+1)}\left(\min_{t\in[0,2\pi]} |e^{\lambda|\vartheta|e^{it} - i\vartheta} - 1|^2\right)^{-1/2}.$$

Since $|\lambda\sin t - 1|\,|\vartheta| \geq (1-\lambda)\,|\vartheta|$, we get $\cos((\lambda\sin t - 1)\vartheta) \leq \cos((1-\lambda)\,\vartheta)$, therefore

$$\begin{aligned}|e^{\lambda|\vartheta|e^{it} - i\vartheta} - 1|^2 &= e^{2\lambda|\vartheta|\cos t} + 1 - 2e^{\lambda|\vartheta|\cos t}\cos((\lambda\sin t - 1)\vartheta) \\ &\geq e^{2\lambda|\vartheta|\cos t} + 1 - 2e^{\lambda|\vartheta|\cos t}\cos((1-\lambda)\vartheta) \\ &\geq \sin^2[(1-\lambda)\,|\vartheta|].\end{aligned}$$

As a consequence,

$$\left|\int_{|z|=\lambda|\vartheta|} \frac{(-z)^{-(1+2\alpha)}}{e^{(z-i\vartheta)}-1}\,dz\right| \le (\lambda|\vartheta|)^{-(2\alpha+1)}\frac{2\pi\lambda|\vartheta|}{\sin[(1-\lambda)\,|\vartheta|]} \le (\lambda|\vartheta|)^{-(2\alpha+1)}\frac{2\lambda\pi^2}{\sin[(1-\lambda)\pi]}$$

The second integral is estimated, as above, by

$$\begin{aligned}\left|2i\sin(2\pi\alpha)\int_{\lambda|\vartheta|}^{\infty}\frac{t^{-(1+2\alpha)}}{e^{(t-i\vartheta)}-1}\,dt\right| &\le 4\sin(\pi\alpha)(\lambda|\vartheta|)^{-(1+2\alpha)}\int_{\lambda|\vartheta|}^{\infty}\frac{e^t+1}{(e^t-1)^2}\,dt\\ &\le 8\sin(\pi\alpha)(\lambda|\vartheta|)^{-(1+2\alpha)}(e^{\lambda|\vartheta|}-1)^{-1} \le 8(\lambda|\vartheta|)^{-(1+2\alpha)}(e^{\frac{\lambda\pi}{4}}-1)^{-1}\end{aligned}$$

whence

$$|\operatorname{Ci}_{-2\alpha}(\vartheta)| \le \frac{\Gamma(1+2\alpha)}{2\pi}(\lambda|\vartheta|)^{-(2\alpha+1)}\left(\frac{2\lambda\pi^2}{\sin[(1-\lambda)\pi]}+\frac{8}{e^{\frac{\lambda\pi}{4}}-1}\right).$$

Finally, for any $\lambda\in(0,1)$,

$$\begin{aligned}|\operatorname{Ci}_{-2\alpha}(\vartheta)|\,|\vartheta|^{2\alpha+1} &\le \frac{\Gamma(1+2\alpha)}{2\pi}\lambda^{-(2\alpha+1)}\left(\frac{2\lambda\pi^2}{\sin[(1-\lambda)\pi]}+\frac{8}{e^{\frac{\lambda\pi}{4}}-1}\right)\\ &\le \lambda^{-3}\left(\frac{2\lambda\pi}{\sin[(1-\lambda)\pi]}+\frac{8}{\pi(e^{\frac{\lambda\pi}{4}}-1)}\right).\end{aligned}$$

Suitably choosing λ, one gets $|\operatorname{Ci}_{-2\alpha}(\vartheta)|\,|\vartheta|^{2\alpha+1} < 23$. □

Proposition A.2. *Let f be an even $\mathcal{C}^\infty$ function on $\mathbb{T}$ vanishing in 0. Then, for $\alpha\in(0,1)$,*

$$\int_{-\pi}^{\pi}\operatorname{Ci}_{-2\alpha}(\vartheta)f(\vartheta)\,d\vartheta = \pi(\{k^{2\alpha}\},\{f_k\})_{\ell^2(\mathbb{N})},$$

where $f_k=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(k\vartheta)f(\vartheta)\,d\vartheta$.

Proof. Let us set $\operatorname{Ci}(s,\rho,\vartheta) := \operatorname{Re}(\operatorname{Li}_s(\rho e^{i\vartheta}))$. Then, reasoning as in the proof of Lemma A.1, it is not difficult to show that

$$\sup_{\rho\in[0,1],|\vartheta|\le\pi}|\operatorname{Ci}(-2\alpha,\rho,\vartheta)|\vartheta|^{2\alpha+1}| < \infty. \tag{A.8}$$

We may assume that f is real valued, namely $f(\vartheta)=\sum_{k\ge0}f_k\cos(k\vartheta)$, with $f_k\in\mathbb{R}$. The other properties of f amount to f_k rapidly decreasing and $\sum_k f_k=0$. Since f is even, it has indeed a zero of order 2 in $\vartheta=0$, hence, by (A.8), $\operatorname{Ci}(-2\alpha,\rho,\vartheta)f(\vartheta)$ is uniformly $L^1(\vartheta)$, for $\rho\in[0,1]$. By dominated convergence,

$$\begin{aligned}\int_{-\pi}^{\pi}\operatorname{Ci}_{-2\alpha}(\vartheta)f(\vartheta)\,d\vartheta &= \lim_{\rho\to1}\int_{-\pi}^{\pi}\operatorname{Ci}(-2\alpha,\rho,\vartheta)f(\vartheta)\,d\vartheta\\ &= \lim_{\rho\to1}\operatorname{Re}\left(-i\int_{|z|=\rho}\operatorname{Li}_{-2\alpha}(z)\left(\frac12\sum_{k=0}^{\infty}f_k(\rho^{-k}z^k+\rho^kz^{-k})\right)\frac{dz}{z}\right)\\ &= \lim_{\rho\to1}\operatorname{Re}\left(\frac{-i}{2}\sum_{k=0}^{\infty}f_k\rho^k\sum_{n=1}^{\infty}n^{2\alpha}\int_{|z|=\rho}z^{n-k}\,\frac{dz}{z}\right)\\ &= \pi\lim_{\rho\to1}\sum_{k=0}^{\infty}\rho^kf_k\;k^{2\alpha} = \pi\sum_{k=0}^{\infty}f_k\;k^{2\alpha}.\end{aligned}$$

□

Proposition A.3. *Let $\alpha \in (0,1)$, and consider the seminorm $p_\alpha(f)$, $f \in C(\mathbb{T})$, given by*

$$p_\alpha(f)^2 = \frac{1}{2\pi} \sup_{x\in\mathbb{T}} \int_{\mathbb{T}} \varphi_\alpha(x-y)|f(x)-f(y)|^2 dy < +\infty,$$

where $\varphi_\alpha(t) = -2\pi \operatorname{Ci}_{-2\alpha}(t)$, and denote by $\|f\|_{0,\alpha} = \sup_{x,y} \dfrac{|f(x)-f(y)|}{d(x,y)^\alpha}$ the Hölder seminorm. Then,

(i) $\forall \varepsilon > 0$, $p_\alpha(f) \leq c_\varepsilon \|f\|_{0,\alpha+\varepsilon}$, where $c_\varepsilon = \frac{1}{\sqrt{\varepsilon}}\left(\frac{\pi}{4}\right)^\varepsilon \left(4 + 23(4^{2\varepsilon}-1)\right)^{1/2}$,

(ii) for $\alpha \geq \frac12$, $\tilde c_\alpha \|f\|_{0,\alpha} \leq p_\alpha(f)$, where $\tilde c_\alpha = \frac{\sqrt{3\sin(\pi\alpha)}}{16\sqrt{2}}$.

Proof.

(i) If f is $(\alpha+\varepsilon)$-Hölder then

$$\begin{aligned}\sup_{x\in\mathbb{T}} \int_{\mathbb{T}} \varphi_\alpha(x-y)|f(x)-f(y)|^2 dy &\leq \|f\|^2_{0,\alpha+\varepsilon} \sup_{x\in\mathbb{T}} \int_{\mathbb{T}} \varphi_\alpha(x-y) d(x,y)^{2(\alpha+\varepsilon)}\, dy \\ &= 2\|f\|^2_{0,\alpha+\varepsilon} \int_0^\pi \varphi_\alpha(t) t^{2(\alpha+\varepsilon)}\, dt\,.\end{aligned}$$

Making use of the estimates in Lemma A.1, one gets

$$\begin{aligned}\int_0^\pi \varphi_\alpha(t)t^{2(\alpha+\varepsilon)}\,dt &= \int_0^{\pi/4} \varphi_\alpha(t)t^{2(\alpha+\varepsilon)}\,dt + \int_{\pi/4}^\pi \varphi_\alpha(t)t^{2(\alpha+\varepsilon)}\,dt \\ &\leq 2\pi\frac{63}{32}\sin(\pi\alpha)\Gamma(1+2\alpha)\int_0^{\pi/4} t^{2\varepsilon-1}\,dt + 2\pi\cdot 23\int_{\pi/4}^\pi t^{2\varepsilon-1}\,dt \\ &\leq 4\pi\sin(\pi\alpha)\Gamma(1+2\alpha)\frac{(\pi/4)^{2\varepsilon}}{2\varepsilon} + 46\pi\frac{\pi^{2\varepsilon}-(\pi/4)^{2\varepsilon}}{2\varepsilon} \\ &\leq \frac{\pi}{\varepsilon}\left(\frac{\pi}{4}\right)^{2\varepsilon}\left(4+23(4^{2\varepsilon}-1)\right).\end{aligned}$$

(ii) Assume $p_\alpha(f) < \infty$, let $x, y \in \mathbb{T}$, and denote by σ the distance between x and y, and by I_σ the arc of length σ with end-points x and y. By Lemma A.1, $\varphi_\alpha(t) > 0$ for $|t| \leq \frac{\pi}{4}$.

Then, for $\sigma \leq \frac{\pi}{4}$,

$$\begin{aligned}\left| f(x) - \frac{1}{\sigma}\int_{I_\sigma} f(z)\,dz\right| &\leq \frac{1}{\sigma}\int_{I_\sigma} |f(x)-f(z)|\,dz \\ &= \sigma^{-1}\int_{I_\sigma} |f(x)-f(z)|\varphi_\alpha(x-z)^{1/2}\varphi_\alpha(x-z)^{-1/2}\,dz \\ &\leq \sigma^{-1}\, p_\alpha(f)\sqrt{2\pi}\cdot\left(\int_0^\sigma \frac{1}{\varphi_\alpha(t)t^{1+2\alpha}} t^{1+2\alpha}\,dt\right)^{\frac12} \\ &\leq (2+2\alpha)^{-1/2}\left(\sup_{0<t\leq\frac{\pi}{4}} \frac{2\pi}{\varphi_\alpha(t)t^{1+2\alpha}}\right)^{1/2} \sigma^\alpha\, p_\alpha(f) \\ &\leq (2+2\alpha)^{-1/2}\left(\frac{32}{\sin(\pi\alpha)\Gamma(1+2\alpha)}\right)^{1/2}\sigma^\alpha\, p_\alpha(f) \\ &\leq 4\big((1+\alpha)\sin(\pi\alpha)\Gamma(1+2\alpha)\big)^{-1/2}\sigma^\alpha p_\alpha(f).\end{aligned}$$

Therefore, using the triangle inequality we obtain, for all x, y, such that $d(x,y) \leq \frac{\pi}{4}$,

$$\frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq \frac{8}{\sqrt{(1+\alpha)\sin(\pi\alpha)\Gamma(1+2\alpha)}}\, p_\alpha(f).$$

A direct computation then shows

$$\|f\|_{0,\alpha} \leq \frac{32 \cdot 4^{-\alpha}}{\sqrt{(1+\alpha)\sin(\pi\alpha)\Gamma(1+2\alpha)}}\, p_\alpha(f) \leq \frac{16\sqrt{2}}{\sqrt{3\sin(\pi\alpha)}}\, p_\alpha(f).$$

□

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(F.C.) Politecnico di Milano, Dipartimento di Matematica, piazza Leonardo da Vinci 32, 20133 Milano, Italy.
E-mail address: fabio.cipriani@polimi.it

(D.G.) Dipartimento di Matematica, Università di Roma "Tor Vergata", I–00133 Roma, Italy
E-mail address: guido@mat.uniroma2.it

(T.I.) Dipartimento di Matematica, Università di Roma "Tor Vergata", I–00133 Roma, Italy
E-mail address: isola@mat.uniroma2.it

(J.-L.S.) Institut de Mathématiques, CNRS-Université Pierre et Marie Curie, boite 191, 4 place Jussieu, F-75252 Paris Cedex 05, France
E-mail address: jlsauva@math.jussieu.fr