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DIFFERENTIAL 1-FORMS, THEIR INTEGRALS AND POTENTIAL THEORY ON THE SIERPINSKI GASKET

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Abstract. We provide a definition of differential 1-forms on the Sierpinski gasket K and their integrals on paths. We show how these tools can be used to build up a Potential Theory on K. In particular, we prove: i) a de Rham re-construction of a 1-form from its periods around lacunas in K; ii) a Hodge decomposition of 1-forms with respect to the Hilbertian energy norm; iii) the existence of potentials of elementary 1-forms on suitable covering spaces of K. We then apply this framework to the topology of the fractal K, showing that each element of the dual of the first Čech homology group $\check{H}_1(K)$ is represented by a suitable harmonic 1-form.

1. Introduction

1.1. **Purpose of the work.** The aim of this work is to develop, on the fractal set K known as *Sierpinski gasket*, a notion of *differential 1-form* ω and *line integral*

$$\int_\gamma \omega$$

along oriented paths γ in K. The purpose for doing this is twofold: on the one hand we wish to set up tools for a potential theory on K; on the other hand, we would like to use them to have local representations, i.e. by integrals, of topological invariants. Our main results are: the construction of a space $\Omega^1(K)$ of differential forms for which the integral along oriented paths makes sense; a de Rham (first and second) Theorem, proving that the sequence of periods around lacunas gives rise to a unique form (up to an exact one); then a Hodge Theorem, namely that any form $\omega \in \Omega^1(K)$ has a unique harmonic representative in cohomology; and finally, the construction of an (abelian) projective covering, where potentials of 1-forms will be defined; and the establishment of a pairing between the cohomology of forms and the Čech homology group of the gasket (de Rham duality theorem).

The classical framework we refer to is that of harmonic integrals on differentiable manifolds, developed by de Rham [5] and Hodge [9]. There, the notions of differential 1-form and line integral are direct outcome of the notion of tangent bundle. The analytic tool of exterior differentiation of forms then naturally provides homotopy invariants by means of the differential complex and its associated cohomology groups. The notion of line integral on the manifold M allows to establish a local pairing first between closed 1-forms and 1-cycles, and then between the first cohomology group $H^1(M)$ and the first singular homology group $H_1(M)$. Furthermore, the choice of a Riemannian metric on M allows to introduce the notions

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of co-closed and harmonic forms in such a way that each cohomology class in $H^1(M)$ has a unique harmonic representative.

Trying to develop the above framework on the Sierpinski gasket K, two main problems have to be tackled.

The first is that K is not a manifold: it was originally introduced in [19] as an example of space with a dense set of ramification points so that it has no open sets homeomorphic to Euclidean domains. This is the reason why a notion of differentiable structure on K has to be introduced in an unconventional way. We choose to do this by using the notion of energy or Dirichlet form, a sort of generalized Dirichlet integral, developed by Beurling and Deny [2], that can be considered on arbitrary topological spaces. In particular, we consider the so called standard Dirichlet form $\mathcal{E}$ considered by Kusuoka [14] in his construction of a diffusion process on K, and studied by Kigami [13] in the framework of his harmonic theory on self-similar fractal sets like K. The primary role of $\mathcal{E}$ is to provide the class of finite energy functions $\mathcal{F}$, which is a dense subalgebra of the algebra of continuous functions $C(K)$, and plays the role of a Sobolev space on the gasket. More importantly, there exists a canonical first order differential calculus associated to Dirichlet forms, as developed in [3]. It is represented by a closed derivation ∂, defined on $\mathcal{F}$ with values in a Hilbert $C(K)$-module $\mathcal{H}$, which is a differential square root of the Dirichlet form in the sense that $\mathcal{E}[a] = \|\partial a\|^2_{\mathcal{H}}$. One of the main technical issues will be the proof that the integral along oriented paths makes sense on (suitably regular) elements of $\mathcal{H}$. As we shall see below, this will force us to a long detour: the introduction of the bimodule of universal 1-forms $\Omega^1(\mathcal{F})$ on the Dirichlet algebra $\mathcal{F}$, the definition of line-integrals on it, and then the proof that an element of $\Omega^1(\mathcal{F})$ with zero Hilbert norm has zero integral along all edges, namely the integral makes sense on the quotient. What we get then is an $\mathcal{F}$-module $\Omega^1(K)$, which densely embeds in $\mathcal{H}$, thus furnishing the smooth subspace on which line integrals make sense.

The second problem is that K is a topological space which is not semilocally simply connected, so that it has no universal covering, i.e. a simply connected covering space [15]. This fact affects the development of a potential theory on K. In an ordinary manifold M, any closed form ω has a pull back $\widetilde{\omega}$ on the universal covering space $\widetilde{M}$, which is obviously still closed but also exact, since $\widetilde{M}$ is simply connected. Hence, any closed form on a manifold admits a primitive function U on $\widetilde{M}$, in the sense that $dU = \widetilde{\omega}$. Moreover, the primitive U is a potential of ω in the sense that its line integral along a path γ in M can be computed by the formula

$$\int_\gamma \omega = U(p) - U(q)$$

where $q, p \in \widetilde{M}$ are the initial and final points, respectively, of any lifting $\widetilde{\gamma}$ in $\widetilde{M}$ of γ.

For the needs of a potential theory on the gasket K, the role played by the universal covering of a manifold, acted upon by its fundamental group, will be played by a specific natural projective covering $\widetilde{L}$, acted upon by the first Čech homology group $\check{H}_1(K)$, which is a projective limit of finitely generated abelian discrete groups. In particular, the potentials U of 1-forms on K will be affine functions on $\widetilde{L}$.

1.2. **Main results.** We now come to a closer look at our results. Our first step is the construction of the bimodule of universal 1-forms $\Omega^1(\mathcal{F})$ on the Dirichlet algebra $\mathcal{F}$ and the definition of line integrals of its elements along elementary paths in K, namely finite unions of consecutive oriented edges in K. Also, we define a quadratic form Q on $\Omega^1(\mathcal{F})$ such that

$Q[df] = \mathcal{E}[f]$, as in the tangent bimodule construction. Now we have two natural quotients of $\Omega^1(\mathcal{F})$ to take, either w.r.t. the intersection of the kernels of the functionals $\omega \mapsto \int_e \omega$, where e is any edge, or w.r.t. the kernel of the quadratic form Q. A main task will be to show that the kernels coincide, hence both the integrals and Q make sense on the quotient. While the proof that Q makes sense on the space $\Omega^1(K)$ of forms modulo forms with zero integral on edges is quite direct, the converse is not at all trivial. What we do is to analyze periods of forms in $\Omega^1(K)$ around the lacunas of the gasket, and show that, given such periods, we may construct another form with the same periods in a canonical way as a series of a suitable sequence of forms dz_σ, parametrized by lacunas of K. We then prove that the difference between the original form and the series is an exact form dU, thus showing at once the first and second De Rham theorem for the gasket, namely the fact that one may build a form given its periods, and the fact that such a form is indeed unique, up to exact forms. In the same time, since the forms dz_σ are harmonic, we obtain a Hodge theorem, i.e. we show that any form has a harmonic representative in the space of (closed) forms modulo exact ones. Finally, since the decomposition of a form $\omega \in \Omega^1(K)$

$$\omega = dU + \sum_\sigma k_\sigma dz_\sigma$$

consists of pairwise orthogonal summands w.r.t. Q, we have that $Q[\omega] = 0$ implies $k_\sigma = 0$ for all σ, and $\mathcal{E}[U] = 0$, namely $\omega = 0$, thus proving that $\Omega^1(K)$ densely embeds in the tangent module $\mathcal{H}$. As a further outcome of our analysis, it turns out that the only natural definition of an external differential on 1-forms giving a differential complex is the trivial one, namely all 1-forms are closed, in accordance with the fact that the gasket is topologically one-dimensional.

A second major issue of our paper is the attempt of extending the integral of a form from elementary paths to general ones, and set up a potential theory for 1-forms. In order to work out a space on which potentials of 1-forms may be defined, we consider the projective limit $\widetilde{L}$ of a sequence of regular abelian covering spaces $\widetilde{L}_n$, where all loops around lacunas of order up to n are unfolded. The group Γ of deck transformations of such pro-covering happens to coincide with the Čech homology group $\check{H}_1(K, \mathbb{Z})$ of the gasket.

It turns out that, in contrast with the classical situation, the space of locally exact forms is a proper subspace of the space of (closed) forms. This subspace is the natural one from the point of view of algebraic topology, first because the integral of such forms extends naturally to all curves in the gasket; second, because any locally exact form ω has a potential U_ω on $\widetilde{L}$, such that the integral of ω along a path γ coincides with the variation of U_ω at the end-points of a lifting of γ to $\widetilde{L}$. Moreover, the potential U_ω is Γ-affine, namely it is associated with a homomorphism $\varphi_\omega : \Gamma \to \mathbb{R}$ such that $U_\omega(gx) = U_\omega(x) + \varphi_\omega(g)$. The pairing $\langle \omega, g\rangle = \varphi_\omega(g)$ extends to a de Rham duality between $\check{H}_1(K, \mathbb{R})$ and $H^1_{dR}(K, \mathbb{R})$, which we define as the quotient $\Omega^1_{\text{loc}}(K)/B^1(K, \mathbb{R})$, where $B^1(K, \mathbb{R})$ denotes the space of exact forms.

Finally, we try to extend the previous results to the space of all elementary 1-forms. The main question here is how to select the paths on which all elementary 1-forms can be integrated. The tool we use for that is a pseudo-metric d on $\widetilde{L}$, inducing a topology stronger than the projective limit one, and giving rise to a partition of $\widetilde{L}$ in connected d-components made of points with finite mutual distance. Also, this pseudo-metric produces a length function on Γ, and the subgroup Γ_d of elements with finite length. Then, forms in $\Omega^1(K)$ can be integrated on all paths which are contained in the same d-components, and potentials are Γ_d-affine functions on d-components of $\widetilde{L}$.

1.3. **Organization of the paper.** Concerning the structure of the work, the second section is dedicated to recall the definition and main properties of the gasket, and in particular to illustrate the construction and some properties of the standard Dirichlet form on it, used in the remaining part of the work.

In the third section we consider the module of universal 1-forms $\Omega^1(\mathcal{F})$ on the Dirichlet algebra $\mathcal{F}$, which is algebraically generated by elements $f dg$, with $f, g \in \mathcal{F}$. Then we carefully define line integrals of its elements along elementary paths in K. Later in that section, identifying forms in $\Omega^1(\mathcal{F})$ having the same integral on elementary paths, we construct the space $\Omega^1(K)$ of *elementary forms*, which consists of forms integrable along elementary paths, and on which the quadratic form Q is well defined. Elements ω in $\Omega^1(K)$ which are differentials of elements U in $\mathcal{F}$ by the derivation d are called *exact forms*, their space is denoted by $B^1(K, \mathbb{R})$ and U is called a *potential* of ω. Such notion may be localized on any open set in K and the notion of *locally exact forms*, denoted by $\Omega^1_{\text{loc}}(K)$, and their corresponding *local potentials* are well defined.

The fourth section, after the introduction of *co-closed* and *harmonic forms*, is devoted to the construction of an orthogonal system of locally exact *harmonic forms* $\{dz_\sigma : \sigma \in \Sigma\} \subset \Omega^1_{\text{loc}}(K)$ associated to the family of lacunas $\{\ell_\sigma : \sigma \in \Sigma\}$ of K. A suitable finite linear combination ω^σ of the dz_σ's may then be used to describe the winding number of a path γ around a lacuna ℓ_σ as the integral $\int_\gamma \omega^\sigma$.

In section five, by studying the properties of *periods* of forms in $\Omega^1(K)$ around elementary cycles in K and using the above system of harmonic forms, we show that locally exact forms, i.e. forms in $\Omega^1_{\text{loc}}(K)$, admit a unique decomposition as a sum of an exact form and a harmonic one which is a finite superposition of the dz_σ's. The above decomposition extends to forms in $\Omega^1(K)$ with the sum replaced by an infinite series, converging with respect to a norm which makes line integrals continuous. This is a de Rham characterization of forms in $\Omega^1(K)$ by their periods around lacunas. In the same section, we prove also a Hodge orthogonal decomposition for all elementary 1-forms, proving that $\Omega^1(K)$ densely embeds in the tangent module $\mathcal{H}$, with respect to its Hilbertian topology. Then we observe that the space $B^1(K, \mathbb{R})$ is closed in the Hilbert norm, hence the Hodge orthogonal decomposition extends to all elements in $\mathcal{H}$.

In the sixth section we introduce the pro-covering $\widetilde{L}$, show that all locally exact forms have a potential there, and prove an analogue of a second fundamental result of de Rham on the Sierpinski gasket: the line integral provides a duality between the first Čech homology group $\check{H}_1(K, \mathbb{R})$ and the first cohomology group $H^1_{dR}(K, \mathbb{R})$. The system of harmonic forms $\{dz_\sigma : \sigma \in \Sigma\}$ provides representatives for the cohomology classes.

In the seventh section we develop a Potential Theory for elementary 1-forms on K. We start by introducing a pseudo-metric d on the homological pro-covering $\widetilde{L}$ which gives rise to a subgroup Γ_d of the group of deck transformations and to a notion of effective length of paths. Then we prove that elementary 1-forms on K admit potentials which are Γ_d-affine functions on d-components of $\widetilde{L}$, and define the integral along a path of finite effective length as the variation of its potential along the lifting of the path to $\widetilde{L}$.

In an appendix to the work we confine the technical result on the coincidence of the projective limit topology of the homological pro-covering $\widetilde{L}$, with the topology generated by the potentials z_σ of the harmonic forms dz_σ associated to the family of all lacunas ℓ_σ in K, which implies that the topology induced by d is stronger than the projective limit topology.

2. Preliminary notions

We denote by K the Sierpinski gasket, a prototype of self-similar fractal sets. It was introduced in [19] as a curve with a dense set of ramified points and has been the object of various investigations in Probability [14] and Theoretical Physics [17].
Let $p_0 := (0,0)$, $p_1 := (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $p_2 := (1,0)$ be the vertices of an equilateral triangle and consider the contractions w_i of the plane: $x \in \mathbb{R}^2 \to p_i + \frac{1}{2}(x - p_i) \in \mathbb{R}^2$. Then K is the unique fixed-point w.r.t. the contraction map $E \mapsto \cup_{i=0}^2 w_i(E)$ in the set of all compact subsets of $\mathbb{R}^2$, endowed with the Hausdorff metric. Two ways of approximating K are shown in Figures 1 and 2.

Let us denote by $\Sigma_m := \{0,1,2\}^m$ the set of words of length $m \geq 0$ composed by m letters chosen in the alphabet of three letters $\{0,1,2\}$ and by $\Sigma := \bigcup_{m\geq 0} \Sigma_m$ the whole vocabulary (by definition $\Sigma_0 := \{\emptyset\}$). A word $\sigma \in \Sigma_m$ has, by definition, length m and this is denoted by $|\sigma| := m$. For $\sigma = \sigma_1\sigma_2 \dots \sigma_m \in \Sigma_m$ let us denote by w_σ the contraction $w_\sigma := w_{\sigma_1} \circ w_{\sigma_2} \circ \dots \circ w_{\sigma_m}$.

Let $V_0 := \{p_0, p_1, p_2\}$ be the set of vertices of the equilateral triangle and $E_0 := \{e_0, e_1, e_2\}$ the set of its edges, with e_i opposite to p_i. Then, for any $m \geq 1$, $V_m := \bigcup_{|\sigma|=m} w_\sigma(V_0)$ is the set of vertices of a finite graph (*i.e.* a one-dimensional simplex) (V_m, E_m) whose edges are given by $E_m := \bigcup_{|\sigma|=m} w_\sigma(E_0)$ (see Figure 2). The self-similar set K can be reconstructed also as an Hausdorff limit either of the increasing sequence V_m of vertices or of the increasing sequence E_m of edges, of the above finite graphs. Set $V_* := \cup_{m=0}^\infty V_m$, and $E_* := \cup_{m=0}^\infty E_m$.

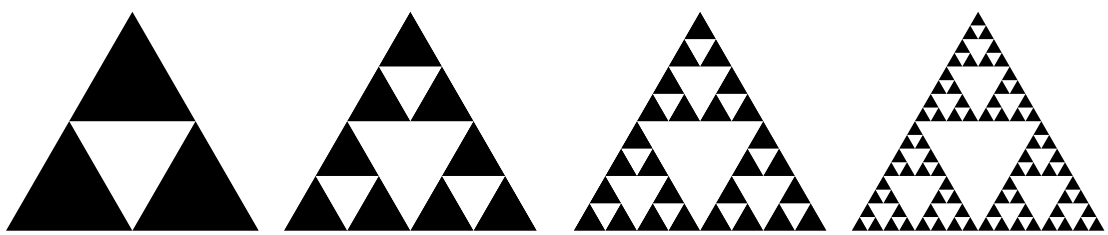


Figure 1. Approximations from above of the Sierpinski gasket.

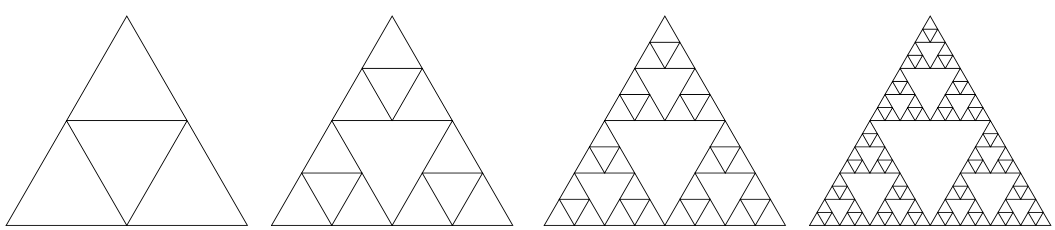


Figure 2. Approximations from below of the Sierpinski gasket.

In the present work a central role is played by the quadratic form $\mathcal{E} : C(K) \to [0, +\infty]$ given by

$$\mathcal{E}[f] = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{e\in E_m} |f(e_+) - f(e_-)|^2,$$

where each edge e has been arbitrarily oriented, and e_-, e_+ denote its source and target. It is a regular Dirichlet form since it is lower semicontinuous, densely defined on the subspace

$\mathcal{F} := \{f \in C(K) : \mathcal{E}[f] < \infty\}$ and satisfies the *Markovianity property*

$$\mathcal{E}[f \wedge 1] \leq \mathcal{E}[f] \qquad f \in C(K)^1. \tag{2.1}$$

The existence of the limit above and the mentioned properties are consequences of the theory of *harmonic structures* on self-similar sets developed by Kigami [13]. As a result of the theory of Dirichlet forms [2, 7], the domain $\mathcal{F}$ is an involutive subalgebra of $C(K)$ and, for any fixed $f, h \in \mathcal{F}$, the functional

$$\mathcal{F} \ni g \mapsto \frac{1}{2}\big(\mathcal{E}(f, gh) - \mathcal{E}(fh, g) + \mathcal{E}(h, fg)\big) \in \mathbb{R} \tag{2.2}$$

extends to a finite Radon measure called the *energy measure* (or *carré du champ*) *of f and h* and denoted by $\Gamma(f, h)$. In particular, for $f \in \mathcal{F}$, $\Gamma(f, f)$ is a nonnegative measure and one has the representation

$$\mathcal{E}[f] = \int_K 1\, d\Gamma(f, f) = \Gamma(f, f)(K) \qquad f \in \mathcal{F}.$$

In applications, f may represent a configuration of a system, $\mathcal{E}[f]$ its corresponding total energy and $\Gamma(f, f)$ represents its distribution. In homological terms, Γ is (up to the constant $1/2$) the Hochschild co-boundary of the 1-cocycle $\phi(f_0, f_1) := \mathcal{E}(f_0, f_1)$ on the algebra $\mathcal{F}$.

The Dirichlet or energy form $\mathcal{E}$ should be considered as a Dirichlet integral on the gasket. It is closable with respect to a wide range of Borel measures on K and, once the measure m has been chosen, it gives rise to a positive, self-adjoint operator on $L^2(K, m)$, which may be thought of as a Laplace-Beltrami operator on K. However, since in the present work the Dirichlet form solely will play a role, the Laplace-Beltrami operator we need will be understood as the operator $\Delta : \mathcal{F} \to \mathcal{F}^*$ such that

$$\langle \Delta f, g\rangle := \mathcal{E}(f, g) \quad f, g \in \mathcal{F}.$$

A function $f \in \mathcal{F}$ is said to be *harmonic* in a open set $A \subset K$ if, for any $g \in \mathcal{F}$ vanishing on A^c, one has

$$\mathcal{E}(f, g) = 0.$$

As a consequence of the Markovianity property 2.1, a Maximum Principle holds true for harmonic functions on the gasket [13]. In particular, one calls 0-*harmonic* a function u on K which is harmonic in V_0^c. Equivalently, for given boundary values on V_0, u is the unique function in $\mathcal{F}$ such that $\mathcal{E}[u] = \min\{\mathcal{E}[v] : v \in \mathcal{F}, v|_{V_0} = u\}$. More generally, one may call m-*harmonic* a function that, given its values on V_m, minimizes the energy among all functions in $\mathcal{F}$. For such functions we have

$$\mathcal{E}[u] = \left(\frac{5}{3}\right)^m \sum_{e \in E_m} |u(e_+) - u(e_-)|^2.$$

It is not difficult to check that $f \in \mathcal{F}$ is m-harmonic if and only if Δf is a linear combination of Dirac measures supported on the vertices V_m.

Definition 2.1. (Cells, lacunas) For any word $\sigma \in \Sigma_m$, define a corresponding *cell* in K as follows

$$C_\sigma = w_\sigma(K),$$

its *perimeter* by $\pi C_\sigma = w_\sigma(E_0)$, its (combinatorial) *boundary* by $\partial C_\sigma = w_\sigma(V_0)$ and its (combinatorial) *interior* by $C_\sigma^o = C_\sigma \setminus \partial C_\sigma$. We will also define the *lacuna* $\ell_\emptyset$, see Fig. 3, as

[1] Here and in the following, we will denote by $C(K)$ the space of real valued continuous functions. As a consequence, the quadratic Dirichlet form $\mathcal{E}$ will give rise to a symmetric bilinear form over $\mathcal{F}$.

the boundary of the first removed triangle according to the approximation in Fig. 1. For any $\sigma \in \Sigma$, the lacuna ℓ_σ is defined as $\ell_\sigma = w_\sigma(\ell_\emptyset)$.

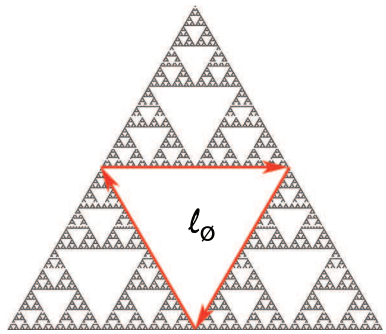


FIGURE 3. The lacuna $\ell_\emptyset$

For a function f on K, let us define its *oscillation on a closed subset* $T \subseteq K$ as

$$\mathrm{Osc}(f)(T) := \max_{x,y \in T} |f(x) - f(y)| = \max_T f - \min_T f \, .$$

Lemma 2.2. *Let f be harmonic in the interior of a cell C and let C_1 be one of its three sub-cells. Then* $\mathrm{Osc}(f)(C_1) \leq \frac{3}{5} \mathrm{Osc}(f)(C)$.

Proof. Since f is harmonic, its maximum and minimum are attained on the boundary. If f_0, f_1, f_2 are the values of f on the vertices of C, and x is one of the new vertices of the subdivision of C, then $f(x)$ is a convex combination of f_0, f_1, f_2 with coefficients $\frac{1}{5}, \frac{2}{5}, \frac{2}{5}$. The thesis then follows from a direct computation. $\square$

3. THE SPACE OF 1-FORMS

Let us denote with $\Omega^1(\mathcal{F})$ the $\mathcal{F}$-bimodule of *universal* 1*-forms* [10], that is $\Omega^1(\mathcal{F})$ is the sub-$\mathcal{F}$-bimodule of $\mathcal{F} \otimes \mathcal{F}$ generated by elements of the form $f dg$, where the differential operator d is defined by $df := f \otimes 1 - 1 \otimes f$, $f \in \mathcal{F}$, and the bimodule operations are $f\, dg = f(g \otimes 1 - 1 \otimes g) := fg \otimes 1 - f \otimes g$, and $dg\, f := d(gf) - g\, df = g \otimes f - 1 \otimes gf$, $f, g \in \mathcal{F}$. There is a natural pairing between elements of $\mathcal{F} \otimes \mathcal{F}$ and oriented edges which is given by $(f \otimes g)(e) := f(e_+)g(e_-)$ on elementary tensors. As a consequence,

$$dg(e) = g(e_+) - g(e_-) \tag{3.1}$$

$$(f\, dg)(e) = f(e_+) dg(e) \tag{3.2}$$

$$(dg\, f)(e) = f(e_-) dg(e). \tag{3.3}$$

3.1. Integrating 1-forms along elementary paths.

Definition 3.1. A path in K given by a finite union of consecutive oriented edges in E_* is called *elementary*.

Let γ be an oriented elementary path in K and $\omega = \sum_{i \in I} f_i dg_i \in \Omega^1(\mathcal{F})$. For $n \in \mathbb{N}$, define

$$I_n(\gamma)(\omega) = \sum_{e \in E_n(\gamma)} \omega(e),$$

where $E_n(\gamma)$ denotes the set of oriented edges of level n contained in γ.

Definition 3.2. We define the integral of a 1-form ω along an elementary path γ as the limit $\int_\gamma \omega = \lim_{n \to \infty} I_n(\gamma)(\omega)$. The existence of such limit is proved below.

Theorem 3.3. *Let $\omega \in \Omega^1(\mathcal{F})$ be a 1-form and γ an elementary path in K. Then*

(*i*) *the integral $\int_\gamma \omega$ is well defined,*
(*ii*) *the integral is a bimodule trace, namely*

$$\int_\gamma h\,\omega = \int_\gamma \omega\, h \qquad h \in \mathcal{F},$$

(*iii*) *for all $h \in \mathcal{F}$, the following approximation holds true:*

$$\int_\gamma h\,\omega = \lim_n \sum_{e\in E_n(\gamma)} h(e_+) \int_e \omega\,.$$

Proof. It is not restrictive to assume $\omega = fdg$.
(i) For given n and $e \in E_n(\gamma)$, let $e^0 \in V_{n+1}$ be the middle point of the edge e. One computes

$$\begin{aligned} I_{n+1}(fdg) &= \sum_{e\in E_n(\gamma)} f(e_+)\big(g(e_+) - g(e^0)\big) + \sum_{e\in E_n(\gamma)} f(e^0)\big(g(e^0) - g(e_-)\big) \\ &= I_n(fdg) + \sum_{e\in E_n(\gamma)} \big(f(e^0) - f(e_+)\big)\big(g(e^0) - g(e_-)\big)\,, \end{aligned} \tag{3.4}$$

so that

$$|I_{n+1}(fdg) - I_n(fdg)| \le \Bigg(\sum_{e\in E_n(\gamma)} \big|f(e^0) - f(e_+)\big|^2\Bigg)^{1/2} \Bigg(\sum_{e\in E_n(\gamma)} \big|g(e^0) - g(e_-)\big|^2\Bigg)^{1/2}$$

$$\le \Bigg(\sum_{e\in E_{n+1}(\gamma)} |df(e)|^2\Bigg)^{1/2} \Bigg(\sum_{e\in E_{n+1}(\gamma)} |dg(e)|^2\Bigg)^{1/2} \tag{3.5}$$

$$\le \frac{1}{2} \sum_{e\in E_{n+1}(\gamma)} (|df(e)|^2 + |dg(e)|^2) \tag{3.6}$$

$$\le \frac{1}{2}\left(\frac{3}{5}\right)^{n+1} (\mathcal{E}[f] + \mathcal{E}[g])\,. \tag{3.7}$$

Hence,

$$|I_n(\gamma)(fdg) - I_{n+p}(\gamma)(fdg)| \le \sum_{k=n}^{n+p-1} I_{k+1}(fdg) - I_k(fdg)| \le \frac{3}{4}(\mathcal{E}[f] + \mathcal{E}[g])\left(\frac{3}{5}\right)^n,$$

namely the sequence $I_n(\gamma)(fdg)$ converges.
(ii) The result follows form

$$I_n(\gamma)(h\,fdg) - I_n(\gamma)(fdg\,h)| \le \|f\|_\infty \sum_{e\in E_n(\gamma)} |dh(e)|\,|dg(e)| \le \frac{1}{2}\|f\|_\infty(\mathcal{E}[h] + \mathcal{E}[g])\Big(\frac{3}{5}\Big)^n.$$

(iii) The thesis follows from

$$\begin{aligned}\Big|I_n(\gamma)(h\omega) - \sum_{e\in E_n(\gamma)} h(e_+)\int_e \omega\Big| &\le \sum_{e\in E_n(\gamma)} |h(e_+)|\ \Big|\omega(e) - \int_e \omega\Big| \\ &\le \|h\|_\infty \sum_{e\in E_n(\gamma)} \sum_{p=0}^{\infty} |I_{p+n+1}(e)(fdg) - I_{p+n}(e)(fdg)| \\ &\le \frac{1}{2}\|h\|_\infty \sum_{p=0}^{\infty} \sum_{e\in E_n(\gamma)} \sum_{e'\in E_{p+n+1}(e)} (|df(e')|^2 + |dg(e')|^2) \\ &\le \frac{1}{2}\|h\|_\infty \sum_{p=0}^{\infty} \sum_{e'\in E_{p+n+1}(\gamma)} (|df(e')|^2 + |dg(e')|^2) \\ &\le \frac{1}{2}\|h\|_\infty (\mathcal{E}[f] + \mathcal{E}[g]) \sum_{p=0}^{\infty} \left(\frac{3}{5}\right)^{p+n+1} \le \frac{3}{4}\|h\|_\infty (\mathcal{E}[f] + \mathcal{E}[g]) \left(\frac{3}{5}\right)^{n}.\end{aligned}$$

□

In the definition of integral of 1-forms, we used a kind of Riemann-Stieltjes integral conditioned to diadic partitions of edges. Unfortunately, while the classical result of Young [24] for $\int f dg$ requires Hölder continuity of f and g with sum of the exponents > 1, restrictions to edges of finite energy functions on the gasket are known to be only β-Hölder, with $\beta < 1/2$ (cf. e.g. [11]), therefore we cannot use Young result. Also, restrictions to edges of finite energy functions are not of bounded variation in general, therefore we cannot use Lebesgue-Stieltjes integral either.

Nevertheless, on identifying an edge $e \in E_*$ with $[0,1]$, the bilinear form $(f, Dg)_e$ on $L^2(e)$, which coincides with $\int_0^1 f(x)g'(x)\,dx$ for smooth functions, naturally extends to a bounded form on $H^{1/2}(e)$, hence makes sense also for $f, g \in \mathcal{F}$ since, by results of Jonsson [12], traces of finite energy functions on edges $e \in E_*$ belong to the fractional Sobolev space $H^\alpha(e)$ for any $\alpha \le \alpha_0$, $\alpha_0 = \frac{\log(10/3)}{\log 4} \sim 0.87$.

The two notions indeed coincide, as shown below.

Proposition 3.4. *Let e be an edge in K, f, g finite energy functions on K. Then*

$$\int_e f dg = (f, Dg)_e. \tag{3.8}$$

Proof. Let us consider the continuous piecewise-linear approximation f_n of a function f which coincides with f on diadic points of the edge e identified with the interval $[0,1]$:

$$f_n(x) = \sum_{j=1}^{2^n} \chi_{[(j-1)2^{-n}, j2^{-n})}(x)\Big(f((j-1)2^{-n}) + \frac{f(j2^{-n}) - f((j-1)2^{-n})}{2^{-n}}(x - (j-1)2^{-n})\Big).$$

Since eq. (3.8) clearly holds for continuous piecewise-linear functions, it is sufficient to show that both terms in (3.8) are continuous w.r.t. the approximation above. By definition, $I_k(fdg) = I_k(f_n dg_n)$, $n \ge k$, therefore

$$\Big|\int_e f dg - \int_e f_n dg_n\Big| \le \Big|\int_e f dg - I_n(fdg)\Big| + \Big|I_n(f_n dg_n) - \int_e f_n dg_n\Big| \to 0,$$

since the first summand goes to 0 by the preceding Theorem 3.3, and, setting $|e| = p$,

$$\left| I_n(f_n dg_n) - \int_e f_n dg_n \right| = \sum_{e' \in E_{p+n}(e)} df(e')\, dg(e') \leq \frac{1}{2} \left(\frac{3}{5}\right)^{n+p} (\mathcal{E}[f] + \mathcal{E}[g]).$$

As for the bilinear form, it is sufficient to show that $f_n \to f$ in $H^{1/2}(e)$. According to [12], a norm for the Sobolev spaces $H^\alpha[0,1]$, $1/2 < \alpha < 1$, is

$$\|f\|_{H^\alpha} = (f(0)^2 + f(1)^2)^{1/2} + \Big(\sum_{n=0}^{\infty} 2^{n(2\alpha-1)} E_n(f) \Big)^{1/2},$$

where

$$E_n(f) = \sum_{j=1}^{2^n} \big(f(j2^{-n}) - f((j-1)2^{-n}) \big)^2.$$

Therefore,

$$\|f - f_k\|^2_{H^\alpha} = \sum_{n=k+1}^{\infty} 2^{n(2\alpha-1)} E_n(f - f_k) \leq 2 \sum_{n=k+1}^{\infty} 2^{n(2\alpha-1)} E_n(f) + 2 \sum_{n=k+1}^{\infty} 2^{n(2\alpha-1)} E_n(f_k).$$

If $\alpha \leq \alpha_0$, the first summand is a remainder of a convergent series, hence goes to 0, as $k \to \infty$. As for the second, since f_k has constant slope on diadic intervals of length 2^{-k}, a direct computation shows that, for $n > k$, $E_n(f_k) = 2^{k-n} E_k(f)$, therefore

$$\sum_{n=k+1}^{\infty} 2^{n(2\alpha-1)} E_n(f_k) = (2^{2-2\alpha} - 1)^{-1} 2^{k(2\alpha-1)} E_k(f) \to 0$$

since $2^{k(2\alpha-1)} E_k(f)$ is the generic term of a convergent series. This shows that, for $\alpha \in (1/2, \alpha_0]$, $f_k \to f$ in $H^\alpha[0,1]$. The convergence in $H^{1/2}[0,1]$ then follows. □

3.2. **An inner product for 1-forms.** The aim of this section is to generalize the quadratic form $Q[df] := \mathcal{E}[f]$ from *exact* forms df to general 1-forms in $\Omega^1(\mathcal{F})$. Since $\mathcal{E}[f] = \lim_n \mathcal{E}_n[f]$, with

$$\mathcal{E}_n[f] = (5/3)^n \sum_{e \in E_n} |df(e)|^2 = (5/3)^n \sum_{e \in E_n} \Big| \int_e df \Big|^2 ,$$

we shall consider the quadratic forms

$$Q_n[\omega] = (5/3)^n \sum_{e \in E_n} \left| \int_e \omega \right|^2, \quad \widetilde{Q}_n[\omega] = (5/3)^n \sum_{e \in E_n} |\omega(e)|^2 , \qquad \omega \in \Omega^1(\mathcal{F}) . \tag{3.9}$$

We shall show that the limits $\lim_n Q_n[\omega]$ and $\lim_n \widetilde{Q}_n[\omega]$ exist and are equal for any 1-form.

Theorem 3.5. *For any $\omega \in \Omega^1(\mathcal{F})$, $\widetilde{Q}[\omega] := \lim_n \widetilde{Q}_n[\omega]$ exists and is finite. Moreover,*
(i) $\widetilde{Q}[f\, dg - dg\, f] = 0$.
(ii) $\widetilde{Q}(dg, f dh) = \frac{1}{2}\Big(\mathcal{E}(g, fh) - \mathcal{E}(gh, f) + \mathcal{E}(h, fg) \Big)$. *In particular we have the identities*

$$\widetilde{Q}(f dg, dh) = \widetilde{Q}(dg, f dh) = \int_K f\, d\Gamma(g,h) \qquad f, g, h \in \mathcal{F} , \tag{3.10}$$

where $\Gamma(g,h)$ is the energy measure of the Dirichlet form $\mathcal{E}$, associated to $g, h \in \mathcal{F}$, cf. (2.2).

Proof. We have

$$\widetilde{Q}_n[f\,dg - dg\,f] = \left(\frac{5}{3}\right)^n \sum_{e\in E_n} df(e)^2 dg(e)^2 \le \mathcal{E}_n[f] \max_{e\in E_n} dg(e)^2.$$

Since the last term tends to 0 by uniform continuity of g, we get (i).
A straightforward computation gives

$$\widetilde{Q}_n(dg, f\,dh) + \widetilde{Q}_n(dg, dh\,f) = \mathcal{E}_n(g, fh) - \mathcal{E}_n(gh, f) + \mathcal{E}_n(h, fg),$$

therefore (ii) follows from (i). The first equality in (3.10) follows by (ii) and the symmetry of $\widetilde{Q}$ and $\mathcal{E}$. The main statement now follows by linearity. □

Remark 3.6. In the following, we use the shorthand notation $\mathcal{E}_C[f] := \mathcal{E}[f|_C]$, for any cell C in K.

Theorem 3.7. *There is a well defined quadratic form Q on $\Omega^1(\mathcal{F})$, given by*

$$Q[\omega] = \lim_{n\to\infty} Q_n[\omega] = \lim_{n\to\infty} \widetilde{Q}_n[\omega] = \widetilde{Q}[\omega].$$

In particular, we have

$$Q(f dg, dh) = Q(dg, f dh) = \int_K f\, d\Gamma(g,h) \qquad f, g, h \in \mathcal{F}\,, \tag{3.11}$$

where $\Gamma(g,h)$ is the energy measure of the Dirichlet form $\mathcal{E}$, associated to $g,h\in\mathcal{F}$. We shall define a Hilbertian seminorm on $\Omega^1(\mathcal{F})$ by $\|\omega\|_2 = Q[\omega]^{1/2}$.

Proof. For sequences $x = \{x_e : e \in E_*\}$, we introduce the seminorms

$$\Phi_n(x) := \left(\frac{5}{3}\right)^{n/2} \left(\sum_{e\in E_n} |x_e|^2\right)^{1/2}. \tag{3.12}$$

In particular, $\widetilde{Q}_n[\omega] = \Phi_n(\omega(e))^2$ and $Q_n[\omega] = \Phi_n(\int_e \omega)^2$. Let us denote with $C(e)$ the cell having e as one of its boundary segments. We get, by inequality (3.5),

$$\begin{aligned}
\Phi_n\left((f_i dg_i)(e) - \int_e f_i dg_i\right)^2 &= \left(\frac{5}{3}\right)^n \sum_{e\in E_n} \left|I_n(e)(f_i dg_i) - \lim_{k\to\infty} I_k(e)(f_i dg_i)\right|^2 \\
&\le \left(\frac{5}{3}\right)^n \sum_{e\in E_n} \left(\sum_{j=n}^{\infty} \left|I_{j+1}(e)(f_i dg_i) - I_j(e)(f_i dg_i)\right|\right)^2 \\
&\le \left(\frac{5}{3}\right)^n \sum_{e\in E_n} \left(\sum_{j=n}^{\infty} \left(\frac{3}{5}\right)^{j+1} \mathcal{E}_{C(e)}[f_i]^{1/2} \mathcal{E}_{C(e)}[g_i]^{1/2}\right)^2 \\
&= \frac{9}{4}\left(\frac{3}{5}\right)^n \sum_{e\in E_n} \mathcal{E}_{C(e)}[f_i]\ \mathcal{E}_{C(e)}[g_i] \le \frac{27}{4}\left(\frac{3}{5}\right)^n \mathcal{E}[f_i]\mathcal{E}[g_i].
\end{aligned}$$

As a consequence, for $\omega = \sum_{i\in I} f_i dg_i$,

$$\begin{aligned}
\left|\widetilde{Q}_n[\omega]^{1/2} - Q_n[\omega]^{1/2}\right| &= \left|\Phi_n(\omega(e)) - \Phi_n(\int_e \omega)\right| \le \left|\Phi_n(\omega(e) - \int_e \omega)\right| \\
&\le \sum_{i\in I} \left|\Phi_n((f_i dg_i)(e) - \int_e f_i dg_i)\right| \le \frac{3\sqrt{3}}{2}\left(\frac{3}{5}\right)^{n/2} \sum_{i\in I} \mathcal{E}[f_i]^{1/2}\mathcal{E}[g_i]^{1/2}.
\end{aligned}$$

The thesis follows. □

Definition 3.8. Let us now introduce the equivalence relation on $\Omega^1(\mathcal{F})$ given by $\omega \sim \omega' \iff \int_e (\omega - \omega') = 0$, for all $e \in E_*$, and consider the quotient space $\Omega^1(K) := \Omega^1(\mathcal{F})/\sim$. We call *elementary 1-forms* the elements of $\Omega^1(K)$. We endow $\Omega^1(K)$ with the norm

$$\|\omega\|_{2,\infty} = \sup_n Q_n[\omega]^{1/2}. \tag{3.13}$$

Since $\|\omega\|_{2,\infty} = 0 \Rightarrow Q_n[\omega] = 0,\ \forall n \Rightarrow \int_e \omega = 0,\ \forall e \in E_*$, the norm property follows.

Let us observe that the integrals $\omega \to \int_e \omega$ and the seminorm $\omega \to \|\omega\|_2$ are continuous w.r.t. the norm $\|\cdot\|_{2,\infty}$.

Theorem 3.9. *The space $\Omega^1(K)$ is an $\mathcal{F}$-bimodule, and d becomes a derivation on $\mathcal{F}$ with values in $\Omega^1(K)$. The integral along an elementary path and the seminorm $\|\cdot\|_2$ are well defined on it. Also, the left and right module multiplications coincide.*

Proof. The kernel of the quotient map is an $\mathcal{F}$-bimodule because of Theorem 3.3 (ii), hence the quotient is a bimodule, too. Denoting by π the quotient map, we have the Leibniz property $\pi(d(fg)) = \pi(f\,dg) + \pi(df\,g) = f\pi(dg) + \pi(df)g$, namely d is a derivation with values in the bimodule $\Omega^1(K)$. The seminorm $\|\cdot\|_2$ is well defined by Theorem 3.7. Left and right module multiplications coincide because of Theorem 3.3 (ii). □

Let us observe that, up to now, $\|\cdot\|_2$ is only a seminorm. The norm property will be proved later on, as a consequence of the Hodge decomposition in Theorem 5.8.

Remark 3.10. In a general Dirichlet space over a locally compact Hausdorff space X, the positivity properties of the Dirichlet form and, more specifically, those of the carré du champ, give rise to a Hilbertian seminorm on $\Omega^1(\mathcal{F})$. By separation and completion, this gives rise to a Hilbert space $\mathcal{H}$ which is in fact a Hilbert $C_0(X)$-bimodule called the *tangent bimodule associated to* $\mathcal{E}$ and whose elements are called *square integrable forms*, [3]. In the present case of the Sierpinski gasket, since the Dirichlet form is strongly local, the right and left actions coincide so that $\mathcal{H}$ is a Hilbert $C(K)$-module.

3.3. **Locally exact 1-forms.**

Definition 3.11. Let ω be a form in $\Omega^1(K)$, $\mathcal{V} \subseteq K$. A continuous function f on $\mathcal{V}$ will be called a *local potential* of ω on $\mathcal{V}$ if $df = \omega|_{\mathcal{V}}$, i.e.

$$\int_e \omega = f(e_+) - f(e_-), \qquad \forall e \in E_*(\mathcal{V}).$$

The form ω will be called *locally exact* if, $\forall x \in K$, there exists a pair $(\mathcal{V}_x, f_x)$, where $\mathcal{V}_x$ is a neighborhood of x and f_x is a local potential of ω on $\mathcal{V}_x$. We denote by $\Omega^1_{loc}(K)$ the set of such forms.
The form ω will be called *n-exact* if it has a local potential f_σ on any cell C_σ, $|\sigma| = n$.

Theorem 3.12.
(i) *A form is locally exact iff it is n-exact for some $n \in \mathbb{N}$.*
(ii) *If f is a local potential on the open set $\mathcal{V}$ of $\omega \in \Omega^1(K)$, then f has finite local energy, namely*

$$\exists\ \mathcal{E}_{\mathcal{V}}[f] = \lim_n \left(\frac{5}{3}\right)^n \sum_{C \in \mathrm{Cell_n}(\mathcal{V})} \sum_{e \in E_n(C)} |df(e)|^2 < \infty,$$

where $\mathrm{Cell_n}(\mathcal{V})$ *is the set of cells of level n contained in $\mathcal{V}$.*

(iii) A collection $\{f_\sigma\}_{|\sigma|=n}$ of functions with finite local energy on the cells C_σ, $|\sigma| = n$, uniquely determines an n-exact form in $\Omega^1(K)$.

Proof. (i) We denote by $\{\mathcal{U}_n(x) : n \in \mathbb{N}\}$ a neighbourhood basis for $x \in K$, where, if $x \notin V_*$, $\mathcal{U}_n(x)$ denotes the unique open n-cell containing x, whereas, if $x \in V_*$, $\mathcal{U}_n(x)$ denotes the neighbourhood of x consisting of x and of the (at most two) open n-cells bounding x. Let us say that $\mathcal{U}_n(x)$ has level n.
($\Longrightarrow$) For any $x \in K$, let (U_x, f_x) be a basic neighbourhood of x, and a local potential for ω in $\overline{U_x}$. Because $\{U_x : x \in K\}$ is an open cover of K, we can extract a finite cover $\{U_1, \ldots, U_k\}$. Let n denote the maximum level of the neighbourhoods $U_1, \ldots, U_k$. Then ω is n-exact.
($\Longleftarrow$) Let $\{f_\sigma\}_{|\sigma|=n}$ be the collection of local potentials on the n-cells $\{C_\sigma : |\sigma| = n\}$. If $x \in K \setminus (V_n \setminus V_0)$, there is a unique σ such that $x \in C_\sigma \supset \mathcal{U}_n(x)$, so that f_σ is a local potential on $\mathcal{U}_n(x)$. If $x \in V_n \setminus V_0$, there are σ_1, σ_2 such that $|\sigma_1| = |\sigma_2| = n$ and $x \in C_{\sigma_1} \cap C_{\sigma_2}$. Set $c := f_{\sigma_1}(x) - f_{\sigma_2}(x)$ and $f_x(y) := \begin{cases} f_{\sigma_1}(y), & y \in C_{\sigma_1}, \\ f_{\sigma_2}(y) + c, & y \in C_{\sigma_2}, \end{cases}$ so that f_x is a continuous function, and a local potential for ω on $\mathcal{U}_n(x)$.
(ii) If C is a cell of level n, $\sum_{e\in E_n(C)} |df(e)|^2 \leq 5/3 \sum_{e\in E_{n+1}(C)} |df(e)|^2$, hence

$$\begin{aligned} \left(\frac{5}{3}\right)^n \sum_{C\in\mathrm{Cell_n}(\mathcal{V})} \sum_{e\in E_n(C)} |df(e)|^2 &\leq \left(\frac{5}{3}\right)^{n+1} \sum_{C\in\mathrm{Cell_n}(\mathcal{V})} \sum_{e\in E_{n+1}(C)} |df(e)|^2 \\ &\leq \left(\frac{5}{3}\right)^{n+1} \sum_{C\in\mathrm{Cell_{n+1}}(\mathcal{V})} \sum_{e\in E_{n+1}(C)} |df(e)|^2, \end{aligned}$$

showing that the sequence is increasing. Finally,

$$\begin{aligned} \lim_n \left(\frac{5}{3}\right)^n \sum_{C\in\mathrm{Cell_n}(\mathcal{V})} \sum_{e\in E_n(C)} |df(e)|^2 &= \lim_n \left(\frac{5}{3}\right)^n \sum_{C\in\mathrm{Cell_n}(\mathcal{V})} \sum_{e\in E_n(C)} |\int_e \omega|^2 \\ &\leq \lim_n \left(\frac{5}{3}\right)^n \sum_{e\in E_n} |\int_e \omega|^2 = Q[\omega] < \infty. \end{aligned}$$

(iii) Let f_σ be a function with finite energy defined on the cell C_σ. We may associate with it an element in $\Omega^1(K)$ as follows: let $A \supset C_\sigma$ be an open set in K such that $(K \setminus C_\sigma^o) \cap \overline{A}$ consists of (at most) three cells, each containing exactly one boundary vertex of C_σ; let $\tilde{f}_\sigma$ be a function in $\mathcal{F}$ which coincides with f_σ in C_σ and is constant on each connected component of $(K \setminus C_\sigma^o) \cap \overline{A}$; and let χ_σ be a function in $\mathcal{F}$ which is 1 on C_σ and has support contained in A. If we set $\omega_\sigma = \chi_\sigma d\tilde{f}_\sigma$, then

$$\int_e \omega_\sigma = \lim_{n\to\infty} \sum_{\substack{e'\in E_n \\ e'\subset e}} \chi_\sigma(e'_+)(\tilde{f}_\sigma(e'_+) - \tilde{f}_\sigma(e'_-)).$$

Now, if e intersects C_σ at most in one vertex, we get $\int_e \omega_\sigma = 0$, because $\tilde{f}_\sigma$ is constant on any $e' \in E_n$, $e' \subset e$. If, on the contrary, $e \subset C_\sigma$, then $\chi_\sigma(e'_+) = 1$, for any such e', while $\tilde{f}_\sigma = f_\sigma$, so that $\int_e \omega_\sigma = \lim_{n\to\infty} \sum_{\substack{e'\in E_n \\ e'\subset e}} (f_\sigma(e'_+) - f_\sigma(e'_-)) = f_\sigma(e_+) - f_\sigma(e_-)$. Clearly $\sum_{|\sigma|=n} \omega_\sigma$ is the required n-exact form. $\square$

Theorem 3.13.

(i) Let ω be a k-exact 1-form generated by the local potentials f_σ, $|\sigma| = k$. Then $Q[\omega] = \sum_{|\sigma|=k} \mathcal{E}_{C_\sigma}[f_\sigma]$.
(ii) $\|\cdot\|_2$ is a norm on $\Omega^1_{loc}(K)$.

Proof. (i) For any $n > k$, we get

$$Q_n(\omega) = \left(\frac{5}{3}\right)^n \sum_{e\in E_n} \left|\int_e \omega\right|^2 = \left(\frac{5}{3}\right)^n \sum_{|\tau|=k} \sum_{e\in E_n(C_\tau)} \left|\int_e \omega\right|^2 = \sum_{|\tau|=k} \mathcal{E}_n[f_\tau].$$

Therefore, $Q(\omega) = \lim_{n\to\infty} Q_n(\omega) = \sum_{|\sigma|=k} \lim_{n\to\infty} \mathcal{E}_n[f_\sigma] = \sum_{|\sigma|=k} \mathcal{E}[f_\sigma]$.

(ii) Indeed, $0 = Q(\omega) = \sum_{|\sigma|=k} \mathcal{E}[f_\sigma] \implies f_\sigma$ is constant on C_σ, for any $\sigma \implies \omega = 0$. □

3.4. **The completion of the space of elementary 1-forms.** We denote by $\overline{\Omega^1(K)}$ the completion of $(\Omega^1(K), \|\cdot\|_{2,\infty})$.

Theorem 3.14. *The quadratic forms Q, Q_n, $n \in \mathbb{N}$, extend to $\overline{\Omega^1(K)}$ by continuity. Moreover, if $\{\omega_k\}_{k\in\mathbb{N}} \subset \Omega^1(K)$ and $\omega_k \to \omega$ in $\overline{\Omega^1(K)}$, then*

$$\lim_{k\to\infty} Q[\omega_k] = \lim_{n\to\infty} Q_n[\omega]. \tag{3.14}$$

Proof. The first statement is obvious since $\omega \to \|\omega\|_2$ and $\omega \to \int_e \omega$ are continuous w.r.t. $\|\cdot\|_{2,\infty}$. As a consequence, the first limit in (3.14) exists and is finite. Then,

$$\begin{aligned} |Q[\omega_k]^{1/2} - Q_n[\omega]^{1/2}| &\le |Q[\omega_k]^{1/2} - Q_n[\omega_k]^{1/2}| + |Q_n[\omega_k]^{1/2} - Q_n[\omega]^{1/2}| \\ &\le |Q[\omega_k]^{1/2} - Q_n[\omega_k]^{1/2}| + \|\omega_k - \omega\|_{2,\infty} \end{aligned}$$

The thesis follows. □

Proposition 3.15. *$\|\cdot\|_2$ is not a norm on $\overline{\Omega^1(K)}$.*

Proof. Let p_i, $i = 0, 1, 2$, be the external vertices of the gasket, e_i be the edge in E_0 opposite to p_i, $i = 0, 1, 2$, and let g be the 0-harmonic function taking value $-1/2$ on x_0, 0 on x_1 and $1/2$ on x_2. Then, for any given n, let us consider the n-exact form ω_n determined by the functions f_σ, $|\sigma| = n$, where

$$f_\sigma = \begin{cases} 2^{-n} g \circ w_\sigma & \text{if } \sigma \in \{0, 2\}^n \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Observe that, for any edge $e \in E_k$,

$$\lim_n \int_e \omega_n = \begin{cases} 2^{-k} & \text{if } e = w_\sigma e_1, \sigma \in \{0, 2\}^k \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

On the one hand

$$\|\omega_n\|_2^2 \equiv Q[\omega_n] = \sum_{\sigma\in\{0,2\}^n} \mathcal{E}[2^{-n} g \circ w_\sigma], = 2^{-n}(5/3)^n \mathcal{E}[g],$$

namely ω_n converges to 0 w.r.t. $\|\cdot\|_2$. We now prove that $\omega_n \to \omega$ in $\|\cdot\|_{2,\infty}$, where $\omega \in \overline{\Omega^1(K)}$ is non-trivial.

Define ω by its values $\int_e \omega := \begin{cases} 2^{-k} & \text{if } e = w_\sigma e_1, \sigma \in \{0, 2\}^k \\ 0 & \text{otherwise.} \end{cases}$

Fix $n \in \mathbb{N}$, and compute $Q_k[\omega - \omega_n]$. If $k < n$,

$$Q_k[\omega - \omega_n] = \left(\frac{5}{3}\right)^k \sum_{\substack{e \in E_k \\ e \not\subset e_1}} \Big| \int_e \omega_n \Big|^2 = \left(\frac{5}{3}\right)^k \cdot 2^{k+1}(2^{-n-1})^2 = \frac{1}{2}\left(\frac{10}{3}\right)^k 4^{-n}.$$

If $k \geq n$, $Q_k[\omega - \omega_n]^{1/2} \leq Q_k[\omega]^{1/2} + Q_k[\omega_n]^{1/2}$. Now, $Q_k[\omega] = \left(\frac{5}{3}\right)^k 2^{-2k} = \left(\frac{5}{12}\right)^k$, and $Q_k[\omega_n] = Q[\omega_n]$, because each edge $e \in E_k$ is contained in only one cell C_σ, where ω_n has a potential f_σ, so that

$$Q_k[\omega_n] = \left(\frac{5}{3}\right)^k \sum_{|\sigma|=n} \sum_{e \in E_k(C_\sigma)} \Big| \int \omega_n \Big|^2 = \sum_{|\sigma|=n} \mathcal{E}_k[f_\sigma|_{C_\sigma}] = \sum_{|\sigma|=n} \mathcal{E}_{C_\sigma}[f_\sigma] = Q[\omega_n].$$

Therefore, $Q_k[\omega - \omega_n] \leq 2Q_k[\omega] + 2Q_k[\omega_n] \leq 2\left(\frac{5}{12}\right)^k + 2\mathcal{E}[g]\left(\frac{5}{6}\right)^n$. Hence,

$$Q[\omega - \omega_n] = \sup_k Q_k[\omega - \omega_n] \leq \frac{1}{2}\left(\frac{5}{6}\right)^n + 2\left(\frac{5}{12}\right)^n + 2\mathcal{E}[g]\left(\frac{5}{6}\right)^n \to 0, \quad n \to \infty,$$

namely ω_n converges in $\|\cdot\|_{2,\infty}$ to the non-trivial 1-form $\omega \in \overline{\Omega^1(K)}$. □

4. Harmonic 1-forms

4.1. **1-forms associated with lacunas.** In this section we introduce a distinguished system of locally exact 1-forms associated with lacunas. In the forthcoming sections, their properties will play a fundamental role in the proof of a de Rham characterization and a Hodge decomposition for 1-forms.

Definition 4.1. For any $n \geq 0$ and $|\sigma| = n$, define dz_σ as the $(n+1)$-exact form which minimizes the norm $\|\cdot\|_2$ among those $(n+1)$-exact 1-forms ω satisfying $\int_{\ell_\sigma} \omega = 1$.

By definition dz_σ is exact in any of the cells $C_{\sigma i}$, hence $\int_{\pi C_\sigma} dz_\sigma = -1$ (lacunas are traversed clockwise and boundaries of cells are traversed counter-clockwise, according to the standard convention, as they constitute the boundary of the union of the convex hulls of the cells $C_{\sigma i}$, $i = 1, 2, 3$). The minimization request implies that dz_σ vanishes in any cell C_τ with $\tau \neq \sigma$, $|\tau| = n$, and that dz_σ is symmetric for rotations of $\frac{2}{3}\pi$ around ℓ_σ.

Proposition 4.2. *The forms dz_σ are weakly co-closed, i.e. orthogonal to all exact 1-forms, and pairwise orthogonal, with*

$$\|dz_\sigma\|_2^2 = \frac{5}{6}\left(\frac{5}{3}\right)^{|\sigma|}.$$

Proof. A simple calculation shows that for any cell $C_{\sigma i}$, the local potential z_σ^i on such cell is the harmonic function determined (up to an additive constant) by the values $\frac{1}{6}, 0, -\frac{1}{6}$ on the vertices x_1, x_2, x_3, where x_3, x_1 is the edge bounding the lacuna. Therefore, Δz_σ^i may be canonically identified with the measure given by the linear combination $\frac{1}{2}\delta_{x_1} - \frac{1}{2}\delta_{x_3}$. As a consequence, for any $f \in \mathcal{F}$,

$$Q(df, dz_\sigma) = \sum_{i=1,2,3} Q(df, dz_\sigma^i) = \sum_{i=1,2,3} \mathcal{E}(f, z_\sigma^i) = \sum_{i=1,2,3} \int_K f d(\Delta z_\sigma^i) = 0.$$

If $\tau < \sigma$ the orthogonality follows as above; if τ and σ are not ordered, dz_σ and dz_τ have disjoint support. The value of the norm follows from a direct computation. □

Similarly to the case of an ordinary compact smooth manifold, we introduce the following definition.

Definition 4.3. 1-forms which are locally exact and co-closed will be termed *harmonic*. Hence $\{dz_\sigma : \sigma \in \Sigma\}$ is a orthogonal system of harmonic 1-forms.

Remark 4.4. In the next section, as a consequence of a Hodge decomposition, we will show that locally exact forms are dense in the tangent $\mathcal{F}$-module $\mathcal{H}$ of square integrable 1-forms. Hence, the only way to define on $\mathcal{H}$ a closable operator $(d_1, \Omega^1(K))$, with values in another non degenerate, Hilbertian $\mathcal{F}$-module $\Omega^2(K)$, to get a complex $0 \to \mathcal{F} \to \Omega^1(K) \to \Omega^2(K)$, is by the zero map $d_1 = 0$. This forces $\Omega^2(K) = \{0\}$, so that non vanishing 2-forms cannot exist on K. This supports, homologically, the fact that K is, topologically, one dimensional.

Proposition 4.5. *Any n-exact form has a unique orthogonal decomposition as the sum of an exact form plus a finite linear combination of dz_τ, $|\tau| < n$.*

Proof. Let us observe that the orthogonal complement of n-exact forms into $(n+1)$-exact forms is generated by the dz_σ, $|\sigma| = n$. Indeed, by Theorem 3.13 (ii), the seminorm $\|\cdot\|_2$ is a norm on locally exact forms, hence the orthogonal complement is well defined. We now note that, for any cell C_σ, an $(n+1)$-exact form ω on C_σ is indeed n-exact if and only if $\int_{\ell_\sigma} \omega = 0$, since in this case the three local potentials on the three sub-cells may glue to a continuous function on C_σ. Therefore, any $(n+1)$-exact form ω supported in C_σ may be written as

$$\omega = (\omega - c_\sigma dz_\sigma) + c_\sigma dz_\sigma, \qquad c_\sigma := \int_{\ell_\sigma} \omega,$$

namely, for any cell C_σ, the codimension of n-exact forms into $(n+1)$-exact forms supported in C_σ is 1. This shows that exact forms and the dz_τ, $|\tau| < n$, generate the n-exact forms, hence the thesis. □

Remark 4.6. Proposition 4.5 is a characterization of n-exact forms. As we shall see in Corollary 5.4, Proposition 4.5 generalizes to elementary 1-forms, finite linear combinations being replaced by infinite series.

4.2. **Winding numbers and a combinatoric way to describe lacunas bounding cells.** Since dz_σ is invariant under rotations of $\frac{2}{3}\pi$ around the lacuna ℓ_σ, the integral along any edge e bounding C_σ is equal to $-1/3$. We now consider the integral $\int_{\ell_\tau} dz_\rho$. It is not difficult to see that such integral does not vanish only if $\tau \le \rho$ (τ is a truncation of ρ), more precisely,

$$\int_{\ell_\tau} dz_\rho = \begin{cases} 1 & \text{if } \tau = \rho, \\ -1/3 & \text{if } \ell_\tau \cap \pi C_\rho \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, we can find numbers a^σ_ρ, $\rho \le \sigma$, such that the 1-form

$$\omega^\sigma := \sum_{\rho \le \sigma} a^\sigma_\rho dz_\rho$$

has the property

$$\int_{\ell_\tau} \omega^\sigma = \delta_{\sigma\tau} \qquad \sigma\,, \tau \in \Sigma\,. \tag{4.1}$$

In other words, the 1-form ω^σ detects only the lacuna ℓ_σ.

Remark 4.7 (Winding number). It follows directly by the observations above that, for any closed elementary path γ in K,

$$\int_\gamma \omega^\sigma$$

is the *winding number of the path γ around the lacuna ℓ_σ*. This interpretation extends to general closed paths, according to Definition 6.3 below.

Observe that, for any multi-index σ, we get $a^\sigma_\sigma = 1$ and, for $\tau < \sigma$, $0 = \int_{\ell_\tau}(\sum a^\sigma_\rho dz_\rho) = \sum_{\rho\leq\sigma} a^\sigma_\rho \int_{\ell_\tau} dz_\rho$, namely

$$a^\sigma_\tau = \frac{1}{3}\sum_{\tau<\rho\leq\sigma} A(\tau,\rho)a^\sigma_\rho, \quad \text{where} \quad A(\tau,\rho) = \begin{cases} 1 & \text{if } \ell_\tau \cap \pi C_\rho \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Even if the system of equations can be solved iteratively, by successive truncations of σ, we will need in the sequel just the following bound.

Lemma 4.8. *With the notation above, $0 \leq a^\sigma_\tau \leq 1$, $\tau \leq \sigma$.*

Proof. For a given σ, let us rename indices and variables as follows: replace the n-th truncation $\sigma^{(n)}$ of σ with n, so that the order is reversed, and rename $a^\sigma_{\sigma^{(n)}}$ as v_n. Then the equation above becomes

$$v_p = \frac{1}{3}\sum_{j=0}^{p-1} A_{pj}v_j, \text{ when } p \neq 0, \qquad v_0 = 1.$$

Denoting by P the projection on the 0-th component, we get $v = (\frac{1}{3}A + P)v$. Recall that A_{ij} may be non zero for at most three indices i following j, and observe that A is a lower triangular matrix, hence $(A^p)_{jk}$ does not vanish only if $k \leq j - p$, and $PA = 0$. Therefore we get

$$v = \left(\frac{1}{3}A + P\right)^p v = 3^{-p}A^p v + \sum_{j=0}^{p-1}\left(\frac{1}{3}\right)^j A^j P v,$$

and, since $v_0 = 1$,

$$v_p = 3^{-p}(A^p)_{p0}v_0 + \sum_{j=0}^{p-1}\left(\frac{1}{3}\right)^j (A^j)_{p0}v_0 = \sum_{j=0}^{p}\left(\frac{1}{3}\right)^j (A^j)_{p0}.$$

We now interpret A as the adjacency matrix of an oriented simple graph, where the vertices are the indices $0, 1, \ldots$ and an oriented edge goes from j to i if $A_{ij} = 1$. Then, $(A^j)_{p0}$ is equal to the number of oriented paths of length j joining 0 with p. Since from any vertex may depart at most three edges, if there is an edge joining 0 with p, then there are at most 2 oriented paths of length 2 joining 0 with p, at most 6 oriented paths of length 3 joining 0 with p, and so on. So, denoting with n_i the number of oriented paths of length i joining 0 with p, we have

$$\begin{cases} n_1 \leq 1 \\ n_1 + n_2 \leq 3 \\ 3n_1 + n_2 + n_3 \leq 9 \\ \ldots \\ \sum_{i=1}^{q-1} 3^{q-1-i}n_i + n_q \leq 3^{q-1}. \end{cases} \tag{4.3}$$

As a consequence, for $q \geq 1$, we have

$$\begin{aligned} v_q = \sum_{i=1}^{q} 3^{-i} n_i &= 3^{-q} n_q + 3^{1-q} \left(\sum_{i=1}^{q-1} 3^{q-1-i} n_i \right) \\ &\leq 3^{-q} n_q + 3^{1-q} \left(3^{q-1} - n_q \right) \leq 1 - \frac{2}{3} 3^{1-q} n_q \leq 1. \end{aligned}$$

□

5. Hodge orthogonal decomposition of elementary 1-forms

The first result of this section is a version of de Rham first theorem, namely the construction of a form on the gasket given its periods around all lacunas. More precisely, since the sequence of periods is infinite, we first establish a bound for the periods of an elementary form, and then, given a sequence of periods satisfying this bound, construct a harmonic form with those periods. This will provide a Hodge type orthogonal decomposition for elementary 1-forms, namely a decomposition of an elementary form ω as a sum of an exact one $\omega_E = dU_E$ and a harmonic one ω_H. We will also provide a criterion for local exactness and an explicit construction of a non locally exact form.

5.1. **Reconstructing 1-forms via their periods.** Our aim now is to associate to any elementary 1-form ω a series of the dz_σ's which sums to a harmonic 1-form ω_H having exactly the periods $c_\sigma = \int_{\ell_\sigma} \omega$. Using the coefficients (4.2) and setting

$$k_\tau = \sum_{\sigma \geq \tau} a_\tau^\sigma c_\sigma, \qquad \omega_H = \sum_\tau k_\tau dz_\tau, \tag{5.1}$$

we will prove in Theorem 5.3 that the series defining ω_H converges in $\overline{\Omega^1(K)}$ and, $\forall \rho \in \Sigma$,

$$\int_{\ell_\rho} \omega_H = \int_{\ell_\rho} \omega. \tag{5.2}$$

Lemma 5.1. *For any elementary 1-form ω there exist finitely many functions f_i in $\mathcal{F}$ such that*

$$|c_\sigma| \leq \frac{1}{2} \sum_{k=|\sigma|+1}^{\infty} \sum_{e \in E_k(\ell_\sigma)} \sum_i df_i(e)^2. \tag{5.3}$$

Proof. It is enough to prove the result for $\omega = f dg$. Observe that

$$|c_\sigma| = |\lim_n I_n(\ell_\sigma)(fdg)| \leq |I_{|\sigma|+1}(\ell_\sigma)(fdg)| + \sum_{k=|\sigma|+1}^{\infty} |I_{k+1}(\ell_\sigma)(fdg) - I_k(\ell_\sigma)(fdg)|$$

Since ℓ_σ is a closed curve, $|I_{|\sigma|+1}(\ell_\sigma)(fdg)| = |I_{|\sigma|+1}(\ell_\sigma)((f - const)dg)|$. Denoting by x_1, x_2, x_3 the vertices of ℓ_σ, and choosing $const = f(x_1)$, we get

$$|I_{|\sigma|+1}(\ell_\sigma)(fdg)| = |df(x_1,x_2)dg(x_1,x_2) + df(x_1,x_3)dg(x_2,x_3)| \leq \frac{1}{2} \sum_{e \in E_{|\sigma|+1}(\ell_\sigma)} df(e)^2 + dg(e)^2.$$

The thesis follows by eq. (3.6). □

Lemma 5.2. *Let $\{c_\sigma\}$ be a sequence satisfying estimate* (5.3) *for suitable functions $f_i \in \mathcal{F}$, and let k_σ be as in* (5.1). *Then there exists a positive finite measure μ on the gasket such that*

$$|k_\tau| \leq \left(\frac{3}{5}\right)^{|\tau|} \mu(C_\tau). \tag{5.4}$$

Proof. Indeed,

$$\begin{aligned}\sum_{\sigma\geq\tau} |c_\sigma| &\leq \frac{1}{2} \sum_{\sigma\geq\tau} \sum_{k=|\sigma|+1}^{\infty} \sum_{e\in E_k(\ell_\sigma)} \sum_i df_i(e)^2 \leq \frac{1}{2} \sum_{k=|\tau|+1}^{\infty} \sum_{e\in E_k(C_\tau)} \sum_i df_i(e)^2 \\ &\leq \frac{1}{2} \sum_{k=|\tau|+1}^{\infty} \left(\frac{3}{5}\right)^k \sum_i \mathcal{E}_{C_\tau}[f_i] \leq \frac{3}{4} \left(\frac{3}{5}\right)^{|\tau|} \sum_i \mathcal{E}_{C_\tau}[f_i].\end{aligned} \tag{5.5}$$

As a consequence, by Lemma 4.8,

$$|k_\tau| \leq \sum_{\sigma\geq\tau} |c_\sigma| \leq \frac{3}{4} \left(\frac{3}{5}\right)^{|\tau|} \sum_i \mu_{f_i}(C_\tau), \tag{5.6}$$

where μ_{f_i} denotes the energy measure associated with f_i. □

Theorem 5.3 (De Rham first theorem)**.** (*i*) *Let $\{c_\sigma\}$ be a sequence satisfying estimate* (5.3) *for suitable functions $f_i \in \mathcal{F}$, and let k_σ be as in* (5.1). *Then, the series $\sum_\sigma k_\sigma dz_\sigma$ converges to a form in $\overline{\Omega^1(K)}$ having the c_σ's as its periods.*
(*ii*) *For any $\omega \in \Omega^1(K)$, the form $\omega_H := \sum_\sigma k_\sigma dz_\sigma \in \overline{\Omega^1(K)}$ satisfies eq.* (5.2).

Proof. (*i*) A simple calculation shows that $Q_n(dz_\sigma) \leq (5/3)^{|\sigma|}$, therefore, by Lemma 5.2,

$$Q_n(k_\sigma dz_\sigma) \leq |k_\sigma|^2 (5/3)^{|\sigma|} \leq (3/5)^{|\sigma|} \mu(C_\sigma)^2.$$

Then the series converges absolutely in $\overline{\Omega^1(K)}$, since:

$$\sum_\sigma (3/5)^{|\sigma|/2} \mu(C_\sigma) = \sum_k (3/5)^{k/2} \sum_{|\sigma|=k} \mu(C_\sigma) = \left(1 - \sqrt{3/5}\right)^{-1} \mu(K).$$

(*ii*) Now we prove eq. (5.2). Indeed

$$\int_{\ell_\rho} \omega_H = \sum_\tau \sum_{\sigma\geq\tau} a_\tau^\sigma c_\sigma \int_{\ell_\rho} dz_\tau = \sum_\sigma c_\sigma \int_{\ell_\rho} \left(\sum_{\tau\leq\sigma} a_\tau^\sigma dz_\tau \right) = \int_{\ell_\rho} \omega,$$

where the first equality follows from (5.5), the convergence of $\sum_\sigma k_\sigma dz_\sigma$ with respect to the $\|\cdot\|_{2,\infty}$ norm and the continuity of the integral with respect to this topology. □

Theorem 5.4. *Any elementary 1-form $\omega \in \Omega^1(K)$ may be uniquely decomposed as*

$$\omega = dU_E + \sum_\sigma k_\sigma dz_\sigma,$$

where $U_E \in \mathcal{F}$, the k_σ's are defined above, and the convergence takes place with respect to the $\|\cdot\|_{2,\infty}$-norm. As a consequence, $\omega_H = \sum_\sigma k_\sigma dz_\sigma$ is a harmonic form in $\Omega^1(K)$.

The proof of the Theorem relies on some preliminary Propositions and Lemmas.

Proposition 5.5. *Let ω be an elementary 1-form. Then, for any σ,*

$$\int_{\pi C_\sigma} \omega = -\sum_{\tau\geq\sigma} \int_{\ell_\tau} \omega = -\sum_{\tau\geq\sigma} \int_{\ell_\tau} \omega_H = \int_{\pi C_\sigma} \omega_H.$$

Proof. As above, we may assume $\omega = fdg$. As for the first equation, we have, for any $n \geq |\sigma|$,

$$\int_{\pi C_\sigma} fdg = -\sum_{\tau\geq\sigma, |\tau|\leq n} \int_{\ell_\tau} fdg + \sum_{\tau\geq\sigma, |\tau|=n+1} \int_{\pi C_\tau} fdg.$$

Therefore we have to prove that the second summand goes to 0 when $n \to \infty$. It is not restrictive to assume $\sigma = \emptyset$. With estimates similar to those in Lemma 5.1, we get

$$\begin{aligned}\sum_{|\tau|=n+1} \int_{\pi C_\tau} fdg &\leq \frac{1}{2} \sum_{|\tau|=n+1} \sum_{k=n+1}^{\infty} \sum_{e\in E_k(\pi C_\tau)} df(e)^2 + dg(e)^2 \\ &\leq \frac{1}{2} \sum_{k=n+1}^{\infty} \left(\frac{3}{5}\right)^k (\mathcal{E}[f] + \mathcal{E}[g]) \leq \frac{3}{4} \left(\frac{3}{5}\right)^n (\mathcal{E}[f] + \mathcal{E}[g]).\end{aligned}$$

The second equation follows by eq. (5.2), and the third by absolute convergence. □

Lemma 5.6. *Let $\omega, k_\sigma, \omega_H$ be as above, μ_ω the finite positive measure associated with ω as in Lemma 5.2, and let γ be an elementary simple path contained in the cell C_σ, $|\sigma| = n$. Then,*

$$\left| \int_\gamma \omega_H \right| \leq \mu_\omega(K)(n+3) \left(\frac{3}{5}\right)^n.$$

Proof. It is easy to see that $\int_\gamma dz_\tau$ can be non-zero only if either $\tau < \sigma$ or $\tau \geq \sigma$. Moreover, since γ has no loops, $|\int_\gamma dz_\tau| \leq 1$.

When $\tau < \sigma$, choosing i such that $\tau i \leq \sigma$, dz_τ is exact in $C_{\tau i}$, hence in C_σ, with $\mathrm{Osc}_{C_{\tau i}}(z_\tau) = 1/3$. According to Lemma 2.2, we get

$$\left| \int_\gamma dz_\tau \right| \leq \mathrm{Osc}_{C_\sigma}(z_\tau) \leq \frac{1}{3} \left(\frac{3}{5}\right)^{|\sigma|-|\tau|-1}. \tag{5.7}$$

Hence, making use of eq. (5.4), we have

$$\begin{aligned}\left| \int_\gamma \omega_H \right| &\leq \sum_\tau k_\tau \left| \int_\gamma dz_\tau \right| \leq \sum_{\tau\geq\sigma} k_\tau + \frac{1}{3} \sum_{\tau<\sigma} k_\tau \left(\frac{3}{5}\right)^{n-|\tau|-1} \\ &\leq \sum_{\tau\geq\sigma} \left(\frac{3}{5}\right)^{|\tau|} \mu_\omega(C_\tau) + \frac{1}{3} \sum_{\tau<\sigma} \left(\frac{3}{5}\right)^{n-1} \mu_\omega(C_\tau) \\ &\leq \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^{n+k} \sum_{|\tau|=k} \mu_\omega(C_{\sigma\circ\tau}) + \frac{1}{3}\mu_\omega(K) \sum_{\tau<\sigma} \left(\frac{3}{5}\right)^{n-1} \leq \mu_\omega(K)(n+3) \left(\frac{3}{5}\right)^n\end{aligned}$$

□

Let us now consider the form $\omega_1 = \omega - \omega_H$, which has trivial integral along the perimeter of any cell C_σ. For any n, denoting by S_n the 1-skeleton of the n-th approximation of K, given two points $x, y \in S_n$, and a path γ in S_n joining them, the integral $\int_\gamma (\omega - \omega_H)$ depends only on the end points x, y, namely we get a primitive function U_E^n on S_n, i.e,

$$\forall e \in E_n, \qquad \int_e (\omega - \omega_H) = dU_E^n(e). \tag{5.8}$$

Lemma 5.7. *Let $\omega = fdg$, ω_H and U_E^n be as above. Set $|\sigma| = n$, and choose $x_0 \in V_n \cap C_\sigma$, $x \in V_{n+p} \cap C_\sigma$. Then there exists a constant c such that*

$$|U_E^{n+p}(x) - U_E^{n+p}(x_0)| \leq \|f\|_\infty \mathrm{Osc}_{C_\sigma}(g) + c(\mathcal{E}[f] + \mathcal{E}[g])(n+3)(3/5)^n.$$

Proof. **First step**. Let $\sigma^0, \sigma^1, \dots \sigma^p$ be the subsequent multi-indices of length $n+j$, $\sigma^0 = \sigma$, such that $x \in C_{\sigma^j}$, $j = 0, \cdots, p$. We shall construct inductively a path γ, joining x_0 with x, given by vertices $x_0, \dots x_{p+1} = x$, such that

- $x_j \in V_{n+j}$ for $j \le p$, $x_{p+1} \in V_{n+p}$;
- $x_j \in C_{\sigma^j}$, $j \le p$;
- either $x_{j-1} = x_j$, or x_{j-1}, x_j are joined by an edge e_j, with $e_j \in E_{n+j}$ if $0 < j \le p$, and $e_{p+1} \in E_{n+p}$. In the first case we set e_j to be the trivial edge.

Since x_0 is given, we only need to describe the inductive step. Suppose we have x_{j-1}, $j \le p$. If $x_{j-1} \in C_{\sigma^j}$, we set $x_j := x_{j-1}$. If not, it is connected by an edge $e_j \in E_{n+j}$ to a vertex $x_j \in V_{n+j} \cap C_{\sigma^j}$. Finally, x_p and x_{p+1} are both vertices in $V_{n+p} \cap C_{\sigma^p}$, hence either coincide or are joined by an edge e_{p+1}.
Second step. There exists a constant c_1 such that

$$|\int_\gamma f dg| \le \|f\|_\infty \operatorname{Osc}_{C_\sigma}(g) + c_1 \left(\frac{3}{5}\right)^n (\mathcal{E}[f] + \mathcal{E}[g]).$$

We decompose the restriction of f to γ as $f = \sum_{k=0}^{p+1} f_k$, with $f_0 = f(x_0)$ constantly, and, for $0 < k \le p+1$,

$$f_k(t) = \begin{cases} 0 & t \in e_j, j < k, \\ f(t) - f(x_{k-1}) & t \in e_k, \\ f(x_k) - f(x_{k-1}) & t \in e_j, j > k. \end{cases}$$

We then get

$$\begin{aligned} \int_\gamma f dg &= \int_\gamma f_0 dg + \sum_{k=1}^{p+1} \sum_{j=k}^{p+1} \int_{e_j} f_k dg \\ &= f(x_0)(g(x) - g(x_0)) + \sum_{k=1}^{p+1} \int_{e_k} f_k dg + \sum_{k=1}^{p} \sum_{j=k+1}^{p+1} df(e_k) dg(e_j) \end{aligned}$$

As for the first summand, we clearly have $|f(x_0)(g(x) - g(x_0))| \le \|f\|_\infty \operatorname{Osc}_{C_\sigma}(g)$. We now estimate the second summand. First observe that

$$\begin{aligned} \left| \int_{e_k} f_k dg \right| &\le \left(I_{n+k}(e_k)(f_k dg) + \sum_{r=n+k+1}^{\infty} |I_r(e_k)(f dg) - I_{r-1}(e_k)(f dg)| \right) \\ &\le \sum_{r=n+k}^{\infty} \left(\sum_{e \in E_r} df(e)^2 \right)^{1/2} \left(\sum_{e \in E_r} dg(e)^2 \right)^{1/2} \le \frac{5}{4} \left(\frac{3}{5}\right)^{n+k} (\mathcal{E}[f] + \mathcal{E}[g]). \end{aligned}$$

Therefore,

$$\left| \sum_{k=1}^{p+1} \int_{e_k} f_k dg \right| \le \sum_{k=1}^{p+1} \frac{5}{4} \left(\frac{3}{5}\right)^{n+k} (\mathcal{E}[f] + \mathcal{E}[g]) \le \frac{15}{8} \left(\frac{3}{5}\right)^n (\mathcal{E}[f] + \mathcal{E}[g]).$$

We now consider the third summand. Since, $\forall e \in E_m$, $|df(e)| \leq (3/5)^{m/2}\mathcal{E}[f]^{1/2}$, we get

$$
\begin{aligned}
|\sum_{k=1}^{p}\sum_{j=k+1}^{p+1} df(e_k)dg(e_j)| \leq & \mathcal{E}[f]^{1/2}\mathcal{E}[g]^{1/2}\sum_{k=1}^{\infty}\left(\frac{3}{5}\right)^{(n+k)/2}\sum_{j=k+1}^{\infty}\left(\frac{3}{5}\right)^{(n+j)/2} \\
= & \frac{3}{4}\frac{\sqrt{3}}{\sqrt{5}-\sqrt{3}}\left(\frac{3}{5}\right)^{n}(\mathcal{E}[f]+\mathcal{E}[g]).
\end{aligned}
$$

The thesis follows.

Conclusion. Since $|U_E^{n+p}(x) - U_E^{n+p}(x_0)| = |\int_\gamma(fdg - \omega_H)| \leq |\int_\gamma fdg| + |\int_\gamma \omega_H|$, the result follows by Step 2 and Lemma 5.6. □

Proof of Theorem 5.4. As usual, it is not restrictive to assume $\omega = fdg$. Clearly, the functions U_E^n constructed above are defined up to an additive constant, therefore we choose a vertex x in V_0 and set $U_E^n(x) = 0$ for any n. Let us now observe that the functions U_E^n satisfy, for $m \geq n$, $U_E^m|_{S_n} = U_E^n$, therefore they define a function U_E on $S := \cup_n S_n$. By Lemma 5.7, U_E is uniformly continuous on a dense subset of K, hence it extends to a continuous function on K, and, by definition, $\int_e(\omega - \omega_H) = dU_E(e)$. This shows at once that $\mathcal{E}[U_E] = \|\omega - \omega_H\|_2^2 < \infty$, and $\omega - \omega_H = dU_E$ as elements of $\overline{\Omega^1(K)}$. □

5.2. **Hodge orthogonal decomposition.** We have proved that any elementary 1-form ω may be uniquely decomposed as $\omega = dU_E + \sum_\sigma k_\sigma dz_\sigma$ and all summands in the second term are pairwise orthogonal. In particular, the harmonic part $\omega_H = \sum_\sigma k_\sigma dz_\sigma = \omega - dU_E$, being a linear combination of elementary forms, is itself elementary.

Theorem 5.8.

(i) *An elementary 1-form ω vanishes if and only if $\|\omega\|_2 = 0$, namely elementary 1-forms give a pre-Hilbert space.*

(ii) *(Hodge decomposition). Any elementary 1-form can be uniquely decomposed as an orthogonal sum of an exact form and a harmonic form, the exact part coinciding with dU_E, the harmonic part with $\omega_H = \sum_\sigma k_\sigma dz_\sigma$.*

(iii) *An elementary 1-form is locally exact iff the k_σ's are eventually zero.*

(iv) *(De Rham second theorem) An elementary 1-form is exact iff all periods c_σ vanish.*

Proof. (i) Let $\omega = dU_E + \sum_\sigma k_\sigma dz_\sigma$. Since, by Proposition 4.2, this decomposition is an orthogonal decomposition w.r.t. the form Q, then $\|\omega\|_2 = 0$ implies $\|dU_E\|_2^2 = \mathcal{E}[U_E] = 0$ and $k_\sigma = 0$ for any $\sigma \in \Sigma$. As a consequence ω vanishes.
(ii) Since $\|\cdot\|_2$ is a norm on elementary 1-forms, and the space of exact 1-forms is closed w.r.t. this norm, we may uniquely decompose an elementary 1-form ω as $\omega = \omega_E \oplus \omega_H$, where ω_E is the projection on the subspace of exact forms.
(iii) Immediately follows by Proposition 4.5.
(iv) The implication $(\Rightarrow)$ is obvious. As for the other one, $c_\sigma = 0$, $\forall\sigma$, implies, by equation (5.1), $k_\sigma = 0$, $\forall\sigma$, hence the result follows from (ii). □

Remark 5.9. (1) An equivalent way to formulate Hodge decomposition theorem is that each cohomology class has a (unique) harmonic representative.
(2) Hodge decomposition allows us to define a gradient d^* on forms:

$$d^*\omega = d^*(dU_E + \omega_H) = \Delta U_E.$$

Observe that the domain and the range of d^* depend on the corresponding data for Δ.
(3) Even though the dz_σ's are parametrized by lacunas, they are not the dual basis of the lacunas, considered as a basis for the homology vector space, as follows by eq. (4.1).

Corollary 5.10. *The space $\Omega^1(K)$ embeds isometrically (w.r.t. the norm $\|\cdot\|_2$) in the tangent bimodule $\mathcal{H}$ of vector fields for Dirichlet forms as constructed in* [3]*. The operator $\mathcal{F} \ni f \to df \in \Omega^1(K)$ from $\mathcal{F}$ to $\mathcal{H}$ is the derivation associated to the Dirichlet form*

$$\mathcal{E}[f] = \|df\|^2_{\mathcal{H}} \qquad f \in \mathcal{F}\,. \tag{5.9}$$

The space $B^1(K,\mathbb{R})$ of exact forms is norm closed, hence the decomposition of Theorem 5.4 extends to the whole space $\mathcal{H}$.

Proof. The first statement follows by eq. (2.2), Theorem 3.5, and statement (i) in the Theorem above. The map $d : \mathcal{F} \to \mathcal{H}$ is a derivation on the algebra $\mathcal{F}$ with values in the $C(K)$-module $\mathcal{H}$, i.e. satisfies the Leibniz rule, since it is induced by the universal derivation defined, in section 3, on $\mathcal{F}$ with values in the $\mathcal{F}$-module $\Omega^1(\mathcal{F})$. Moreover, the representation (5.9) follows from (3.10). The fact that the $C(K)$-module $\mathcal{H}$, being the completion of the module $\Omega^1(K)$, is non degenerate and the uniqueness (up to unitary equivalence) of the minimal derivation representing the Dirichlet form (see [3] Theorem 8.3) imply the second statement.
The closedness of $B^1(K,\mathbb{R})$ has been argued in [4] and we show it here for the sake of completeness.
The space of exact forms $B^1(K,\mathbb{R})$ is the range $d(\mathcal{F})$ of the derivation $d : \mathcal{F} \to \mathcal{H}$. Since the space of 0-harmonic functions on K is three dimensional, it is enough to prove that the image $d(\mathcal{F}_0)$ of the subspace $\mathcal{F}_0 := \{f \in \mathcal{F} : f \text{ vanishes on } V_0\}$ of finite energy functions vanishing at the boundary V_0 of K, is closed in $\mathcal{H}$. By the inequality

$$\|u\|_\infty \le c\sqrt{\mathcal{E}[u]} \qquad u \in \mathcal{F}_0$$

(holding for a finite constant $c > 0$, see [13] Chapter 2), if $\{u_n \in \mathcal{F}_0 : n \ge 1\}$ is a sequence such that $\{du_n \in \mathcal{H} : n \ge 1\}$ has the Cauchy property, then $\{u_n \in \mathcal{F}_0 : n \ge 1\}$ is itself a Cauchy sequence in $\mathcal{F}_0$ with respect to the uniform norm and we may consider its limit $u \in \mathcal{F}_0$. As the quadratic form $\mathcal{E}$ comes from an harmonic structure on K (see [13] Example 3.1.5), it is the pointwise monotone limit of bounded quadratic forms on $C(K)$ and, in particular, it is lower semicontinuous. Then, if for a fixed $\varepsilon > 0$, $N \ge 1$ is such that $\mathcal{E}[u_n - u_m] < \varepsilon$, for all $n, m \ge N$, then

$$\|du - du_m\|^2_{\mathcal{H}} = \mathcal{E}[u - u_m] \le \liminf_n \mathcal{E}[u_n - u_m] < \varepsilon \qquad m \ge N$$

so that the sequence $\{du_n \in \mathcal{H} : n \ge 1\}$ converges to $du \in \mathcal{H}$. □

5.3. **On the existence of non-locally exact forms.** On a manifold, all closed forms are locally exact, namely the difference between closed and exact forms cannot be detected locally. Due to its exotic topology, this is no longer true on the gasket, as we show below.

Lemma 5.11. *Let f_i be the 0-harmonic function on the gasket taking value 1 on the vertex p_i and 0 on the others, and consider the scalar products $a_{ijk} := Q(df_i, f_j df_k)$, $i, j, k = 0, 1, 2$. Then*

$$a_{ijk} = \begin{cases} 1 & \text{if } i = j = k; \\ -\frac{1}{2} & \text{if } i = j \ne k \text{ or } i \ne j = k; \\ \frac{1}{2} & \text{if } i = k \ne j; \\ 0 & \text{if the indices are pairwise different.} \end{cases}$$

Proof. The result directly follows from Theorem 3.5 (*ii*), together with the relation

$$2Q(df_i, f_j df_j) = Q(df_i, d(f_j^2)) = \langle \Delta f_i, f_j^2 \rangle = \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } i \neq j; \end{cases}$$

where we recall that Δf_i is the sum of twice the Dirac measure concentrated on the vertex p_i minus the Dirac measures concentrated on the other vertices. □

Lemma 5.12. *With the notation of the previous Lemma,*

$$Q(dz_\emptyset, f_0 df_1) = \frac{1}{15}.$$

Proof. Since $dz_\emptyset$ is invariant under $2\pi/3$ rotations, we have $Q(dz_\emptyset, f_0 df_1) = Q(dz_\emptyset, f_i df_{i+1})$ for any $i = 0, 1, 2$, hence

$$\begin{aligned} Q(dz_\emptyset, f_0 df_1) &= \frac{1}{3} \sum_{i=0,1,2} Q(dz_\emptyset, f_i df_{i+1}) = \frac{5}{9} \sum_{i,j=0,1,2} Q(dz_\emptyset \circ w_j, (f_i df_{i+1}) \circ w_j) \\ &= \frac{5}{3} \sum_{i=0,1,2} Q(dz_\emptyset \circ w_1, (f_i df_{i+1}) \circ w_1), \end{aligned}$$

where, in the last equality, we used the fact that $\sum_{i=0,1,2} Q(dz_\emptyset \circ w_j, (f_i df_{i+1}) \circ w_j)$ does not depend on j. A simple computation shows that $dz_\emptyset \circ w_1 = dg$, with $g = \frac{1}{6}(-f_0 + f_2)$, $f_0 \circ w_1 = \frac{1}{5}(2f_0 + f_2)$, $f_1 \circ w_1 = \frac{1}{5}(2 + 3f_1)$, $f_2 \circ w_1 = \frac{1}{5}(f_0 + 2f_2)$. As a consequence,

$$\begin{aligned} Q(dz_\emptyset, f_0 df_1) &= \frac{1}{15} Q(dg, 2df_0 + 4df_2 + 3f_1 df_0 + 6f_1 df_2 + 6f_0 df_1 + 3f_2 df_1 + \\ &\qquad + 2f_0 df_0 + 2f_2 df_2 + f_0 df_2 + 4f_2 df_0) \\ &= \frac{1}{15} Q(dg, 2df_2 + 3f_1 df_2 - 3f_2 df_1 + 3f_2 df_0) \\ &= \frac{2}{15} \langle \Delta f_2, g \rangle + \frac{1}{30} Q(d(-f_0 + f_2), f_1 df_2 - f_2 df_1 + f_2 df_0) \end{aligned}$$

where in the second equality we used the invariance of the scalar product under the reflection of the gasket which fixes p_1. By Lemma 5.11 the second summand vanishes, while $\langle \Delta f_2, g \rangle = 1/2$, proving the thesis. □

Corollary 5.13. *The form $f_0 df_1$ is not locally exact, indeed all the coefficients k_σ of the decomposition of Corollary 5.4 are non-zero.*

Proof. Set $\alpha(g, h) = Q(dz_\emptyset, gdh)$. Since $dz_\emptyset$ is harmonic,

$$\alpha(g, h) = Q(dz_\emptyset, gdh) = Q(dz_\emptyset, d(gh)) - Q(dz_\emptyset, hdg) = -Q(dz_\emptyset, hdg) = -\alpha(h, g).$$

Restricting this bilinear form to 0-harmonic functions, we get a bilinear antisymmetric form on $\mathbb{R}^3$ such that $\alpha(g, const) = 0$ for any g. Moreover it is non-trivial since, by Lemma 5.12, $\alpha(f_0, f_1) = 1/15$. As a consequence, $\alpha(g, h) = 0$ *iff* $ag + bh = 1$, for some constants a, b. For any index σ we get

$$Q(dz_\sigma, f_0 df_1) = \left(\frac{5}{3}\right)^{|\sigma|} Q(dz_\emptyset, f_0 \circ w_\sigma d(f_1 \circ w_\sigma)) = \left(\frac{5}{3}\right)^{|\sigma|} \alpha(f_0 \circ w_\sigma, f_1 \circ w_\sigma).$$

By harmonicity of f_i, the map $f_i \to f_i \circ w_\sigma$ is injective and linear, therefore $\alpha(f_0, f_1) \neq 0$ $\Leftrightarrow$ f_0 and f_1 do not generate constants $\Leftrightarrow$ $f_0 \circ w_\sigma$ and $f_1 \circ w_\sigma$ do not generate constants $\Leftrightarrow$

$\alpha(f_0 \circ w_\sigma, f_1 \circ w_\sigma) \neq 0$. Finally, by Theorem 5.4, we have

$$Q(dz_\sigma, f_0 df_1) = k_\sigma Q[dz_\sigma],$$

namely $k_\sigma \neq 0$ for any σ. □

6. de Rham duality for cohomology of locally exact forms

The main goal of this section is to prove a de Rham duality between the first homology group of K and the first cohomology group, which we will define in terms of locally exact forms on K.

6.1. **Extension of the notion of integral of locally exact forms.** Let us set $T_n = \bigcup_{|\sigma|=n} w_\sigma(T)$ (see Figure 1 for T_1 - T_4), where T is the convex hull of K in $\mathbb{R}^2$, namely the full triangle.

Lemma 6.1. *Let γ_1, γ_2 be elementary paths in K which are equivalent in the singular homology group $H_1(T_n)$. Then, for any n-exact form ω, $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$.*

Proof. By assumption, the class $[\gamma_1 - \gamma_2]$ in $H_1(T_n)$ is a boundary of a 2-chain, namely a linear combination of 2-cells c_i in T_n. Any such cell is contained in only one of the triangles constituting T_n, where ω is exact. The thesis follows. □

Lemma 6.2. *For any $n \in \mathbb{N}$, and any path γ in K, there exists an elementary path γ_n, consisting of edges of level n, such that γ and γ_n are equivalent in $H_1(T_n)$.*

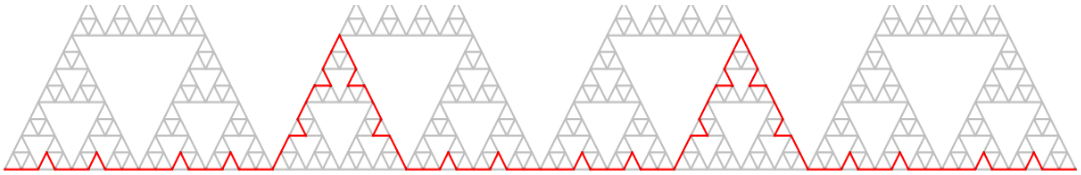
Proof. We construct γ_n as follows. Set t_0 as the first time for which $\gamma(t_0) \in V_n$, t_2 as the first time for which $\gamma(t_2) \in V_n$ with $\gamma(t_2) \neq \gamma(t_0)$, t_4 as the first time for which $\gamma(t_4) \in V_n$ with $\gamma(t_4) \neq \gamma(t_2)$, and so on. We then set t_1 as the last time in $[t_0, t_2)$ for which $\gamma(t_1) = \gamma(t_0)$, and so on. Let us observe that:

- The sequence $\{t_k\}$ is finite. If not, on the one hand $\gamma(t_k)$ would converge, on the other hand V_n is discrete, giving a contradiction. We call $2p$ the last even index.
- For any $x \in V_n$, let us denote by $\mathcal{U}(x)$ the neighborhood of x consisting of x and of the (at most two) open cells of size n bounding x. Then $\gamma(t) \in \mathcal{U}(\gamma(t_{2k}))$ when $t \in (t_{2k-1}, t_{2k+2})$, where we set $t_{-1} = 0$ and $t_{2p+2} = t_{2p+1} = 1$.

The continuous path γ_n is uniquely defined by the following properties: it traverses the edge $[\gamma(t_{2k-1}), \gamma(t_{2k})]$ (with linear parametrization) when $t \in [t_{2k-1}, t_{2k}]$, for $0 < k \leq 2p$, and is constant otherwise. By definition, γ_n is an elementary path. Finally, when $t \in (t_{2k-1}, t_{2k+2})$, both $\gamma(t)$ and $\gamma_n(t)$ belong to $\mathcal{U}(\gamma(t_{2k}))$. The thesis follows. □

Definition 6.3. Let $\gamma \subset K$ be a path in and ω a locally exact 1-form. Then the integral $\int_\gamma \omega$ is defined as $\int_{\gamma_n} \omega$ (n large enough), where γ_n is an elementary path γ_n such that γ and γ_n are equivalent in $H_1(T_n)$. The definition is well posed because of Lemmas 6.1, 6.2.

Remark 6.4 (A path in K containing no edges). Let p_i, $i = 0, 1, 2$, be the external vertices of the gasket, e_i be the edge in E_0 opposite to p_i, $i = 0, 1, 2$, and let r be the rotation of $\frac{2}{3}\pi$ on the gasket. Then, consider the following set of similitudes: $\{w_{000}, w_{002}, w_{020} \circ r, w_{020} \circ r^2, w_{022}, w_{200}, w_{202} \circ r, w_{202} \circ r^2, w_{220}, w_{222}\}$. The selfsimilar fractal in K determined by such similitudes is a von Koch-like curve, and does not contain any edge, see the picture below.



6.2. **Coverings with finitely generated group of deck transformations.** Let us recall that for any n, the Sierpinski gasket K can be written as

$$K = \bigcup_{|\sigma|=n} w_\sigma(K).$$

Denote by T the triangle which is the convex envelope of K, let $T_n = \bigcup_{|\sigma|=n} w_\sigma(T)$, and $\iota_n : K \hookrightarrow T_n$ be the embeddings.

Proposition 6.5.
(i) The universal cover $(\widetilde{T}_n, \widetilde{p}_n, T_n)$ *induces via* ι_n *a regular (also called normal or Galois) covering* $(\widetilde{K}_n, p'_n, K)$*,*
(ii) the map ι_n *induces a map between fundamental groups* $\iota_{n*} : \pi_1(K) \to \pi_1(T_n)$ *which is an epimorphism such that* $\ker(\iota_{n*}) = p'_{n*}(\pi_1(\widetilde{K}_n))$*,*
(iii) the group $\mathrm{deck}(\widetilde{K}_n) \equiv \mathrm{deck}(\widetilde{K}_n, p'_n, K)$ *of deck transformations is isomorphic to* $\pi_1(T_n)$*, hence is a free group, with as many generators as the number of lacunas* ℓ_σ *with* $|\sigma| \le n-1$*.*
(iv) The family $\{(\widetilde{K}_n, p'_n, K) : n \in \mathbb{N}\}$ *is projective, that is, for any* $n \in \mathbb{N}$*, there is a map* $\pi'_n : \widetilde{K}_n \to \widetilde{K}_{n-1}$ *such that* $p'_n = p'_{n-1} \circ \pi'_n$*. Moreover, the map* π'_n *is a covering map, hence surjective, for any* $n \in \mathbb{N}$*.*

Proof. (i) It follows from [15], pp. 150, 178, 179. Observe that $\widetilde{K}_n = (\widetilde{p}_n)^{-1}(K)$ and $p'_n = \widetilde{p}_n|_{\widetilde{K}_n}$.
(ii) It follows from [15], p. 179.
(iii) It follows from [15], p. 163, that $\mathrm{deck}(\widetilde{K}_n) \equiv \mathrm{deck}(\widetilde{K}_n, p'_n, K)$ is isomorphic to

$$\frac{\pi_1(K)}{p'_{n*}(\pi_1(\widetilde{K}_n))} \cong \frac{\pi_1(K)}{\ker i_{n*}} \cong \mathrm{Im}\, i_{n*} = \pi_1(T_n). \tag{6.1}$$

Finally, since T_n is homotopic to a graph, the thesis follows.
(iv) It follows from [15], pp. 159, 160. ☐

Remark 6.6 (Fractafolds and coverings)*.* Let us notice that since the projection $p'_n : \widetilde{K}_n \to K$ is a local homeomorphism, the covering $\widetilde{K}_n$ is a non compact *fractafold* with boundary based on the gasket in the sense of Strichartz [20] (see also [21, 22]). Indeed, in [21], Strichartz studies many covering fractafolds, called blow-ups of the gasket, even though they are coverings of the Octahedron Fractafold, OSG, which is a compact fractafold (without boundary) based on the Sierpinski gasket. Such blow-ups may be described in analogy with what we did above. First embed the OSG into a space X in which any copy of the gasket is replaced by a full triangle. Then the universal covering space of X induces a regular covering $\widetilde{\mathrm{OSG}}$ of OSG. The blow-ups described in [21] are intermediate coverings determined by subgroups of $\pi_1(X)$. However, such coverings are non-normal, namely the group of deck transformations does not act transitively on the fibers. In fact, Strichartz shows that the group of deck transformations is trivial.

The main differences w.r.t. our coverings are the following: (1) our fractafolds, being coverings of the gasket, have boundary; (2) our coverings are normal, hence they are acted upon by a (large) group of transformations, making the notion of periodic function an elementary one, in contrast with Strichartz situation; (3) the blow-ups of Strichartz are equipped with a tower of compact coverings of OSG, while the tower of the $\widetilde{K}_n$'s consists of non-compact spaces; (4) while in Strichartz's case the groups associated with the coverings are subgroups of $\pi_1(X)$, the groups associated with our coverings are *a fortiori* subgroups of $\pi_1(K)$, hence the cells of the covering are in one-to-one correspondence with sub-cells of the gasket (cf. the notion of cellular reconstruction in [20], Theorem 2.1). In other terms, while a fundamental domain in Strichartz's blow-ups is given by just four cells, for our coverings the number of cells of a fundamental domain in $\widetilde{K}_n$ increases with n. (5) As a consequence, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on K extends in a unique way to a Dirichlet form $(\widetilde{\mathcal{E}}_n, \widetilde{\mathcal{F}}_n)$ on the covering space $\widetilde{K}_n$ (see [8]) in such a way that the covering map $p'_n : \widetilde{K}_n \to K$ transforms the restriction of $\widetilde{\mathcal{E}}_n$ on any cell of $\widetilde{K}_n$ to the multiple $\left(\frac{3}{5}\right)^n \mathcal{E}$ of the Dirichlet form of the base K.

By the general theory of Dirichlet forms (see for example [7]), the space of *locally finite energy functions* $\widetilde{\mathcal{F}}_{n,\mathrm{loc}}$ is then defined as those functions on $\widetilde{K}_n$ which coincide, on any open set of a suitable open cover of $\widetilde{K}_n$, with a finite energy function in $\widetilde{\mathcal{F}}_n$. Locally finite energy functions on $\widetilde{K}_n$ are, in particular, continuous. Potentials of locally exact forms on K, which we shall introduce below, will be locally finite energy functions on the above considered covers.

6.3. **Affine functions.** We now construct a potential function on $\widetilde{K}_n$ for any locally-exact 1-form on K. Let us observe that any form ω lifts, in an obvious way, to a $\mathrm{deck}(\widetilde{K}_n)$-periodic form $\widetilde{\omega}$ on $\widetilde{K}_n$.

Lemma 6.7. *Let ω be an n-exact form, γ be a path in $\widetilde{K}_n$ joining two points x_1, x_2 in $\widetilde{K}_n$. Then, $\int_\gamma \widetilde{\omega}$ only depends on the end-points x_1, x_2.*

Proof. Since the universal covering $\widetilde{T}_n$ is simply connected, two paths joining x_1 with x_2 are homotopic in $\widetilde{T}_n$, hence equivalent in $H_1(\widetilde{T}_n)$. The thesis follows as in Lemmas 6.1, 6.2. □

Definition 6.8. (Potentials of locally exact forms) Let ω be an n-exact form and choose $x_0 \in \widetilde{K}_n$. Then, we call the function $f_\omega : \widetilde{K}_n \to \mathbb{R}$ defined as

$$f_\omega(x) = \int_{x_0}^{x} \widetilde{\omega}, \qquad x \in \widetilde{K}_n, \tag{6.2}$$

the *potential of* ω (vanishing at $x_0 \in \widetilde{K}_n$). Since $\widetilde{K}_n$ is arcwise connected, $f_\omega(x)$ is defined for any $x \in \widetilde{K}_n$. Clearly, changing the reference point $x_0 \in \widetilde{K}_n$, amounts to changing the potential f_ω by an additive constant only. Notice also that $f_\omega \in C(\widetilde{K}_n)$, i.e. potentials of locally exact forms are continuous functions.

Remark 6.9. Observe that if ω is n-exact, γ is a path in K, and $\tilde{\gamma}$ is a lifting of γ to $\widetilde{K}_n$, then $\int_{\tilde{\gamma}} \widetilde{\omega} = \int_\gamma \omega$. Also, if f^k_ω is the potential on $\widetilde{K}_k$, $k \geq n$, f_k is the periodic lifting of f_n to $\widetilde{K}_k$.

Definition 6.10. (Affine functions) Let G be a topological group acting on a space X. A continuous function f on X is *G-affine* if there exists a continuous group homomorphism $\varphi : G \to (\mathbb{R}, +)$ such that $f(gx) = f(x) + \varphi(g)$ for all $(g, x) \in G \times X$.

Let us observe that, since the group homomorphisms φ associated to affine functions are valued in the *abelian* group $(\mathbb{R}, +)$, they vanish on commutators. In particular, let $[\mathrm{deck}(\widetilde{K}_n), \mathrm{deck}(\widetilde{K}_n)]$ and $\Gamma_n := \mathrm{Ab}(\mathrm{deck}(\widetilde{K}_n)) = \mathrm{deck}(\widetilde{K}_n)/[\mathrm{deck}(\widetilde{K}_n), \mathrm{deck}(\widetilde{K}_n)]$ be the commutator subgroup and the abelianization, respectively, of the group $\mathrm{deck}(\widetilde{K}_n)$. Then, a $\mathrm{deck}(\widetilde{K}_n)$-affine function on $\widetilde{K}_n$ can be considered as a Γ_n-affine function on the quotient space

$$\widetilde{L}_n := \widetilde{K}_n/[\mathrm{deck}(\widetilde{K}_n), \mathrm{deck}(\widetilde{K}_n)]\,,$$

which is an abelian covering $(\widetilde{L}_n, p_n, K)$ of K (cf. e.g. [18], p. 423, or [23], Theorem 2.2.10) such that, making use also of eq. (6.1),

$$\mathrm{deck}(\widetilde{L}_n) = \Gamma_n = H_1(T_n). \tag{6.3}$$

Let us observe that Γ_n is a free abelian group with as many generators as the number of lacunas ℓ_σ, $|\sigma| \leq n-1$. We also mention that the abelian coverings $\widetilde{L}_n$ are fractafolds as are their non-abelian counterparts. See figure 4 for a portion of $\widetilde{L}_2$, which is an example of a fundamental domain in the sense of Proposition A.4. Notice that x_i and x_i' project to the same point on K.

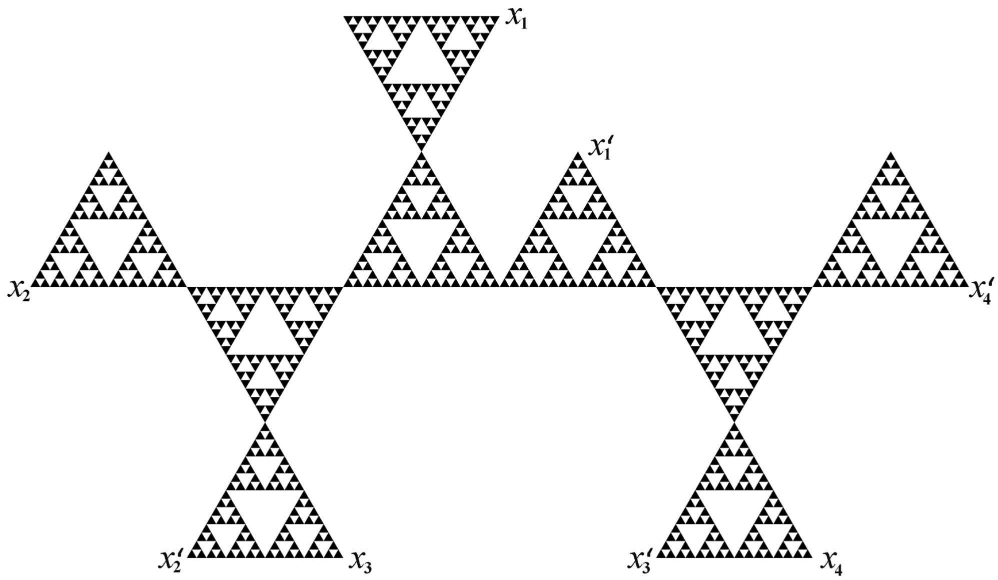


FIGURE 4. A fundamental domain for $\widetilde{L}_2$.

Theorem 6.11. *The potential f_ω of an n-exact 1-form ω, constructed above, is a* $\mathrm{deck}(\widetilde{K}_n)$*-affine function on the covering space $\widetilde{K}_n$, hence it can be considered as a Γ_n-affine function on the abelian covering space $\widetilde{L}_n$. Moreover, for any path γ in K we have*

$$\int_\gamma \omega = f_\omega(x_1) - f_\omega(x_0)\,, \tag{6.4}$$

where x_0, x_1 are the end-points of a lifting of γ to $\widetilde{L}_n$.

Proof. Since the r.h.s. in (6.2) is clearly $\mathrm{deck}(\widetilde{K}_n)$-invariant, we obtain

$$f(x) - f(x_0) = f(gx) - f(gx_0), \quad \forall g \in \mathrm{deck}(\widetilde{K}_n),$$

or, equivalently, $f(gx_0) - f(x_0) = f(gx) - f(x)$, namely the quantity $\varphi(g) = f(gx) - f(x)$ only depends on the group element g, and gives rise to a function on the group $\mathrm{deck}(\widetilde{K}_n)$, which is automatically continuous as this group is discrete. Moreover, for $g, h \in \mathrm{deck}(\widetilde{K}_n)$,

$$\varphi(gh) = f(ghx) - f(x) = (f(ghx) - f(hx)) + (f(hx) - f(x)) = \varphi(g) + \varphi(h),$$

that is φ is a homomorphism from $\operatorname{deck}(\widetilde{K}_n)$ to $(\mathbb{R},+)$. The continuity of f_ω and (6.2) follow directly by the definition of f_ω. $\square$

6.4. **Projective coverings.** It follows from Proposition 6.5 that the family $\big\{(\widetilde{K}_n, p'_n, K) : n \in \mathbb{N}\big\}$ is projective, and has a non-empty projective limit, which we denote by $\widetilde{K} = \varprojlim \widetilde{K}_n$, and we denote by $p' : \widetilde{K} \to K$ and $q'_n : \widetilde{K} \to \widetilde{K}_n$ the continuous maps such that $p'_n \circ q'_n = p'$, and $\pi'_n \circ q'_n = \pi'_{n-1}$.

The space $\widetilde{K}$ is not a covering space of K, but just a pro-covering, namely a projective limit of covering spaces. However,

Lemma 6.12. *Pro-coverings of a topological space X have the unique path-lifting property.*

Proof. We have to prove that if $\{(X_n, p_n)\}_{n\in\mathbb{N}}$ is a tower of coverings of X and $(\widetilde{X}, p)$ is the associated pro-covering, then for any path γ in X starting in x, and any $\tilde{x} \in \widetilde{X}$ with $p(\tilde{x}) = x$, there is a unique path $\tilde{\gamma}$ in $\widetilde{X}$, starting in $\tilde{x}$, such that $p \circ \tilde{\gamma} = \gamma$. Let q_n be the projection map from $\widetilde{X}$ to X_n, $\tilde{x}_n = q_n(\tilde{x})$, $\tilde{\gamma}_n$ the unique path starting from $\tilde{x}_n$ and lifting the path γ to X_n. We may set $\tilde{\gamma}$ as the inverse limit of $\tilde{\gamma}_n$, and any lifting $\tilde{\gamma}$ of γ to $\widetilde{X}$ starting from $\tilde{x}$ is of this form. $\square$

Remark 6.13. Even though K is not semilocally simply connected, hence it has no universal covering space [15], $\widetilde{K}$ has some properties of a universal covering space. First of all, being a pro-covering, it has a covering projection with the unique path-lifting property. Second, the homotopy group $\pi_1(K)$ embeds injectively in the group of deck tranformations for $\widetilde{K}$. Indeed, the group $\operatorname{deck}(\widetilde{K})$ of deck transformations for $\widetilde{K}$ coincides with the projective limit of the groups $\operatorname{deck}(\widetilde{K}_n)$, hence is the so called *Čech homotopy group* $\check{\pi}_1(K)$ of K, cf. [1], Proposition 2.8.

Analogously, the family $\big\{(\widetilde{L}_n, p_n, K) : n \in \mathbb{N}\big\}$ is projective, because, for any $n \in \mathbb{N}$, there is a covering map $\pi_n : \widetilde{L}_n \to \widetilde{L}_{n-1}$ such that $p_n = p_{n-1} \circ \pi_n$. The group Γ of deck transformations for $\widetilde{L}$ coincides with the projective limit of the groups $\Gamma_n = H_1(T_n)$, and is therefore the first *Čech homology group* (cf. e.g. [6], Theorem X.3.1, p. 261), which we shall denote by $\check{H}_1(K)$.

The group Γ of deck transformations of $\widetilde{L}$ is the abelianized of $\operatorname{deck}(\widetilde{K})$ and is the direct product of countably many copies of $\mathbb{Z}$, where generators can be identified with lacunas.

Definition 6.14. We call *homological pro-covering* of K the projective limit $\widetilde{L} = \varprojlim \widetilde{L}_n$, topologized by the projective limit topology.

Lemma 6.15. *For any Γ-affine function f on $\widetilde{L}$ there exists $n \in \mathbb{N}$ and a Γ_n-affine function f_n on $\widetilde{L}_n$ such that f_n lifts to f.*

Proof. This is the same as saying that the homomorphism φ associated with f satisfies $\varphi(g_\sigma) = 0$, for $|\sigma|$ large enough, where g_σ denotes the homotopy class of the lacuna ℓ_σ. Assume, by contradiction, that φ is continuous, and non-trivial on infinitely many elements $g_n = g_{\sigma_n}$. Recall that a sequence h_n in Γ converges to h in the projective limit topology iff, for any $k \in \mathbb{N}$, $q_k(h_n) = q_k(h)$ for sufficiently large n, where $q_k : \Gamma \to \Gamma_k$ is the projection; therefore, for any sequence $\{k_n\} \subset \mathbb{Z}$, $\lim_N \prod_{n=1}^N g_n^{k_n} = \prod_{n=1}^\infty g_n^{k_n}$ in the projective limit topology. As a consequence,

$$\varphi\left(\prod_{n=1}^{\infty} g_n^{k_n}\right) = \sum_{n=1}^{\infty} k_n \varphi(g_n).$$

However, one may always find a sequence of integers $\{k_n\}_{n\in\mathbb{N}}$ such that the series above diverges. □

Theorem 6.16.
(i) The homological pro-covering $(\widetilde{L}, p, K)$ has the unique path-lifting property.
(ii) A quadratic (energy) form $\mathcal{E}_\Gamma : \mathcal{A}(\Gamma, \widetilde{L}) \to [0, +\infty]$ is well defined on the space $\mathcal{A}(\Gamma, \widetilde{L})$ of Γ-affine functions on the covering space $\widetilde{L}$ by

$$\mathcal{E}_\Gamma[f] = \lim_n \left(\frac{5}{3}\right)^n \sum_{e\in E_n} |\partial f(e)|^2 , \tag{6.5}$$

where the quantity $\partial f(e) := f(\tilde{e}_+) - f(\tilde{e}_-)$ does not depend on the choice of the lifting $\tilde{e} \subset \widetilde{L}$ of $e \in E_*(K)$.
(iii) The energy of a Γ-affine function f is finite if and only if f is the potential of a locally exact form ω on K, and, in that case, $\mathcal{E}_\Gamma[f] = \|\omega\|_2^2$. We shall write $df = \omega$.

Proof. (i) It follows directly from Lemma 6.12.
(ii) Let $\tilde{e}^1, \tilde{e}^2$ be two liftings, $\tilde{e}^1_n, \tilde{e}^2_n$ the corresponding projections on $\widetilde{L}_n$, $g_n \in \Gamma_n$ be such that $g_n(\tilde{e}^1_n) = \tilde{e}^2_n$. The family $\{g_n\}$ is a projective sequence of deck transformations, which defines a deck transformation g on $\widetilde{L}$ satisfying $g(\tilde{e}^1) = \tilde{e}^2$. Since f is Γ-affine its variation is the same for all liftings. Since f is the lifting of a continuous function on $\widetilde{L}_m$ for some m, the sequence above is increasing for $n > m$, and this shows the second statement.
(iii) If f is a Γ-affine function of finite energy then, by Lemma 6.15, f is the lifting of a Γ_n-affine function f_n on $\widetilde{L}_n$. Set $f_\sigma = f_n|_{C_\sigma}$ for $|\sigma| = n$. Since the covering projection from $\widetilde{L}_n$ to K is 1:1 on cells of level n, we get the desired form by glueing the df_σ's. Conversely, the existence of a potential of a locally exact form has been already shown above, and the equality $\mathcal{E}_\Gamma[f] = \|\omega\|^2$ follows by Theorem 3.7. □

Notice that the quadratic form just defined on Γ-affine functions on the covering space $\widetilde{L}$ reduces to the standard Dirichlet form on the gasket K when evaluated on Γ_0-affine functions, i.e. on (liftings of) functions on K. This is also the reason why the notation $df = \omega$ is consistent with the usual notation for the derivation of a finite energy function on K.

Summarizing the results above, we have

Theorem 6.17. *There is a 1:1 correspondence between locally exact forms $\omega \in \Omega^1_{loc}(K)$ and and their potentials, i.e. Γ-affine, finite energy functions $f \in \mathcal{A}(\Gamma, \widetilde{L})$ on $\widetilde{L}$ such that*

(i) $df = \omega$;
(ii) $\int_\gamma \omega = f(x_1) - f(x_0)$ for any path $\gamma \subset K$, where x_0, x_1 are the end-points of a lifting of γ to $\widetilde{L}$.

Definition 6.18. We define $B^1(K, \mathbb{R})$ as the space of exact forms on K, and

$$H^1_{dR}(K, \mathbb{R}) = \frac{\Omega^1_{loc}(K)}{B^1(K, \mathbb{R})}$$

as the *de Rham cohomology group for the Sierpinski gasket.*

Remark 6.19. Since the group $\Gamma = \check{H}_1(K)$ has no torsion, its homological information is fully recovered by the group $\check{H}_1(K, \mathbb{R}) = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 6.20 (de Rham cohomology theorem)**.** *The pairing $\langle\gamma,\omega\rangle = \int_\gamma \omega$ between continuous paths and locally exact forms gives rise to a nondegenerate pairing between elements of the group $\check{H}_1(K,\mathbb{R})$ and elements of the de Rham cohomology $H^1_{dR}(K,\mathbb{R})$. Such a pairing is indeed a duality.*

Proof. Let γ be a continuous closed path in K. For any n, we may associate with γ its singular homology class $[\gamma]_n \in H_1(T_n)$, and then its projective limit $[\gamma] = \varprojlim [\gamma]_n \in \Gamma = \varprojlim H_1(T_n)$. If ω is k-exact, and φ_ω the associated homomorphism, then

$$\varphi_\omega([\gamma]) = \langle \varprojlim [\gamma]_n, \omega\rangle = \langle [\gamma]_k, \omega\rangle = \int_\gamma \omega.$$

Since the pairing above is trivial when the form is exact, we get a pairing $\Gamma \times H^1_{dR}(K,\mathbb{R}) \to \mathbb{R}$. Such pairing clearly extends to a pairing $\check{H}_1(K,\mathbb{R}) \times H^1_{dR}(K,\mathbb{R}) \to \mathbb{R}$.
Now we prove the duality relation. On the one hand, $H^1_{dR}(K,\mathbb{R})$ is isomorphic to $\varinjlim H^1_{dR}(T_n,\mathbb{R})$, topologized with the direct limit topology. On the other hand $\check{H}_1(K,\mathbb{R}) = \varprojlim H_1(T_n,\mathbb{R})$, topologized with the projective limit topology. The thesis follows. □

Remark 6.21. Summarizing the main results we get so far, we have that
1. any locally exact form has a potential defined on the whole covering $\widetilde{L}$ which is Γ-affine;
2. the integral of a locally exact form along a path can be read as the variation of the potential at the end points of the lifting of the path;
3. the pairing above gives rise to a nondegenerate pairing between $\check{H}_1(K,\mathbb{R})$ and $H^1_{dR}(K,\mathbb{R})$;
4. such pairing is indeed a duality.

We shall investigate below how such results generalize to the class of elementary 1-forms.

Before closing this section, we show that $\widetilde{L}$ is arcwise connected.

Proposition 6.22. *Let $(\widetilde{L},p,K)$ be the homological pro-covering of K. Then $\widetilde{L}$ is arcwise connected.*

Proof. Let us fix $x_0 \in V_1 \subset K$. Let $\widehat{x},\widehat{y} \in \widetilde{L}$, and $x := p(\widehat{x})$, $y := p(\widehat{y}) \in K$. Since K is arcwise connected, there are continuous paths γ_x,γ_y in K from x,y to x_0. Let $\widetilde{\gamma}_x,\widetilde{\gamma}_y$ be the unique liftings of γ_x,γ_y starting at $\widehat{x},\widehat{y}$, respectively. Then we only need to construct a continuous path $\widetilde{\gamma}$ from $\widetilde{x} := \widetilde{\gamma}_x(1)$ to $\widetilde{y} := \widetilde{\gamma}_y(1)$. Since $p(\widetilde{x}) = p(\widetilde{y}) = x_0$, there is $g \in \Gamma$ such that $g(\widetilde{x}) = \widetilde{y}$. Moreover, $g = \prod_{\sigma\in\Sigma} e_\sigma^{n_\sigma}$, where, for any σ, $n_\sigma \in \mathbb{Z}$, and e_σ is a homology class representing the lacuna ℓ_σ traversed clockwise. Since $\widetilde{L}$ is a projective limit, $\widetilde{x} = \{x_n\}$, $\widetilde{y} = \{y_n\}$, where $x_n, y_n \in \widetilde{L}_n$, $p_n(x_n) = p_n(y_n) = x_0$. We will construct the path $\widetilde{\gamma}$ as the unique lifting, starting at $\widetilde{x}$, of a closed path γ in K, which in turn is obtained as uniform limit of a sequence of closed paths γ_n based at x_0.

To ease the description of the construction of the γ_n's, let us first construct, for any $\sigma \in \Sigma$, $T > 0$, and a given $x_\sigma \in \ell_\sigma \cap V_{|\sigma|}$ (*i.e.* one of the three vertices of the triangle ℓ_σ), a closed path $\gamma_\sigma \equiv \gamma_\sigma^{x_\sigma,n_\sigma,T}$, based at x_σ, which is a representative of the homology class $e_\sigma^{n_\sigma}$. Define γ_σ as the continuous closed path which: (0) is defined on $[0,T]$; (1) starts at x_σ; (2) traverses the lacuna ℓ_σ a number n_σ of times in the positive (*i.e.* clockwise) direction in the time interval $[0,t_\sigma]$, where $t_\sigma := \frac{T}{8}$; (3) traverses clockwise the first two edges and the first half of the third, until it reaches the middle point, in the time interval $[t_\sigma, 2t_\sigma]$; (4) stays still in this middle point for a time interval t_σ; (5) comes back (*i.e.* in the counterclockwise direction) until it reaches the middle point of the second edge, in the time interval $[3t_\sigma, 4t_\sigma]$; (6) stays still in this middle point for a time interval t_σ; (7) continues to come back, until it reaches the middle point of the first edge, in the time interval $[5t_\sigma, 6t_\sigma]$; (8) stays still in this middle point for a time interval t_σ; (9) comes back to x_0, in the time interval $[7t_\sigma, 8t_\sigma]$. For ease of reference, we call the path constructed in $(2)-(3)$ the head of γ_σ, and that constructed in $(4)-(9)$ the tail.

Define $\gamma_0 := \gamma_\emptyset^{x_0,n_\emptyset,1}$. To define γ_1, modify the tail of γ_0, by attaching, to each of the vertices $x_\sigma \in V_2 \setminus V_1$ traversed by γ_0, the closed path $\gamma_\sigma^{x_\sigma,n_\sigma,8^{-1}}$, and call tails of γ_1 the tails of the three γ_σ's. Therefore, γ_1 is the path obtained by traversing first the head of γ_0, and then the tail of γ_0 modified in such a way that, instead of waiting for 8^{-1} time units at each vertex in $V_2 \setminus V_1$, we go around each path γ_σ, $|\sigma| = 1$.

To define γ_2, modify the tails of γ_1, by attaching, to each of the vertices $x_\sigma \in V_3 \setminus V_2$, the closed path $\gamma_\sigma^{x_\sigma,n_\sigma,8^{-2}}$, and call tails of γ_2 the tails of the nine γ_σ's.

In general, assume that γ_n has already been defined. Then, γ_{n+1} is obtained by modifying the 3^n tails of γ_n, by attaching, to each of the vertices $x_\sigma \in V_{n+2} \setminus V_{n+1}$, the closed path $\gamma_\sigma^{x_\sigma,n_\sigma,8^{-n-1}}$. Call tails of γ_{n+1} the tails of the 3^{n+1} γ_σ's.

It is now easy to prove that the sequence $\{\gamma_n\}$ converges uniformly to a continuous closed path γ in K, based at x_0.

It follows from Lemma 6.12 that there is a unique path $\widetilde{\gamma}$ in $\widetilde{L}$ starting at $\widetilde{x}$, and covering γ. We only need to prove that $\widetilde{\gamma}(1) = \widetilde{y}$. Indeed, because of Lemma 6.12 there are unique paths $\hat{\gamma}_n$, $\widetilde{\gamma}_n$ in $\widetilde{L}_n$, starting at x_n, and covering γ_n and γ, respectively. It is easy to see that $\widetilde{\gamma}_n(1) = \hat{\gamma}_n(1) = y_n$. But this implies that $\widetilde{\gamma}(1) = \widetilde{y}$. ☐

7. Potential Theory of elementary 1-forms

In this section we introduce compatible metrics d_N (associated to seminorms N on numerical sequences) on the homological pro-covering $\widetilde{L}$ of the Sierpinski gasket K, and a corresponding subgroups Γ_N of the group of its deck transformations. This will allow to select a class of paths in K, having a lifting of finite length in each metric component of $\widetilde{L}$. By suitably choosing the seminorm N, we prove all elementary 1-forms admit potentials (or primitives) as Γ_N-affine functions on metric components of $\widetilde{L}$.

7.1. Compatible pseudo-metric on the homological pro-covering $\widetilde{L}$.

Definition 7.1. A pseudo-metric d on a space X is a metric which is allowed to be infinite. A *d-component* of X is a subset of points in X with mutually finite distance. A pseudo-metric is finer than a topology $\mathcal{T}$ on X if the topology induced by d is finer than $\mathcal{T}$.

A pseudo-metric d on $\widetilde{L}$ is Γ-invariant if $d(\gamma x, \gamma y) = d(x, y)$ for any γ in the group Γ of deck transformations on $\widetilde{L}$.
We now denote by z_σ the Γ_n-affine potential on $\widetilde{L}_n$ of the n-exact form dz_σ, $n = |\sigma| + 1$, and by $\varphi_\sigma : \Gamma \to \mathbb{R}$ the corresponding homomorphism. Let us observe that $\varphi_\sigma(\ell_\tau) \neq 0$ implies $\tau \leq \sigma$.
If $a = \{a_\sigma\}_{\sigma\in\Sigma} \in c_c(\Sigma)$ is a finitely supported, real valued sequence on Σ, denote by $a^k := \{a^k_\sigma\}_{\sigma\in\Sigma}$ its *k-th truncation*, namely $a^k_\sigma = a_\sigma$ if $|\sigma| \leq k$ and $a^k_\sigma = 0$ otherwise.

Definition 7.2. Let N be a norm on the space $c_c(\Sigma)$ such that $N(a^k) \leq N(a)$ for all $a \in c_c(\Sigma)$, and extend N to the space $\mathbb{R}^\Sigma$ of all sequences via $N(a) = \lim_k N(a^k)$, thus getting a norm on the subspace $\{a \in \mathbb{R}^\Sigma : N(a) < \infty\}$.

For $x, y \in \widetilde{L}$, consider the sequence $z_\bullet(x) - z_\bullet(y) \in c_c(\Sigma)$ defined by $(z_\bullet(x) - z_\bullet(y))_\sigma := z_\sigma(x) - z_\sigma(y)$ for all $\sigma \in \Sigma$ and set

$$d_N(x, y) = N\big(z_\bullet(x) - z_\bullet(y)\big)\,.$$

Theorem 7.3. *The function d_N is a Γ-invariant pseudo-metric which is finer than the projective limit topology.*

Proof. The value $d_N(x, y)$ is obtained by composing the norm N on sequences indexed by Σ with the (semi-definite) distances $d_\sigma(x, y) = |z_\sigma(y) - z_\sigma(x)|$. Therefore, on the one hand d is a (possibly semi-definite) pseudo-metric on $\widetilde{L}$, on the other hand the topology induced by d_N is stronger than the weak topology induced by the z_σ's, which is the projective limit topology, by Lemma A.4 in the Appendix. Since the projective limit topology is Hausdorff, this shows at once that d_N is positive definite and that is a finer pseudo-metric. Finally, for all $g \in \Gamma$ we have

$$d_N(gx, gy) = N\big(z_\bullet(gy) - z_\bullet(gx)\big) = N\big(z_\bullet(y) - z_\bullet(x)\big) = d_N(x, y)\,.$$

□

Remark 7.4. We observe that, since $\widetilde{L}$ is arcwise connected, it is also connected in the projective limit topology. On the other hand, it is not connected in general in the topology induced by d_N.

Proposition 7.5. *Let x be a point in $\widetilde{L}$, $g \in \Gamma$. Then, the quantity $\ell_N(g) := d_N(x, gx)$ does not depend on x, and $\ell_N(g) = 0$ iff g is the identity. The set $\Gamma_N = \{g \in \Gamma : d_N(x, gx) < \infty\}$ does not depend on x, and is a subgroup of Γ.*

Proof. For any $\sigma \in \Sigma$, let $\varphi_\sigma \in \hom(\Gamma, \mathbb{R})$ be the homomorphism associated to the Γ-affine function z_σ on $\widetilde{L}$ in such a way that $z_\sigma(gx) - z_\sigma(x) = \varphi_\sigma(g)$ for all $g \in \Gamma$. Let us denote by $\varphi_\bullet(g) \in \mathbb{R}^\Sigma$ the sequence $\sigma \mapsto \varphi_\sigma(g)$. Definition 7.2 then shows that

$$d_N(x, gx) = N\big(z_\bullet(gx) - z_\bullet(x)\big) = N\big(\varphi_\bullet(g)\big),$$

and the first statement follows. Since d_N is a pseudo-metric, $\ell_N(g) = 0$ means $gx = x$ for any x, namely $g = e$. The last property is obvious. □

Remark 7.6. The function $g \in \Gamma \to \ell_N(g) \in [0, +\infty]$ is a generalized length function on Γ since it clearly satisfies

$$\ell_N(g_1 g_2) \le \ell_N(g_1) + \ell_N(g_2) \qquad g_1\, g_2 \in \Gamma$$

and is a length function on the subgroup Γ_N.

We shall say that a path $\gamma \subset K$ has *finite effective length* if

$$\lambda(\gamma) := d_N(\tilde\gamma(1), \tilde\gamma(0)) < \infty. \tag{7.1}$$

where $\tilde\gamma$ is the lifting of γ to $\widetilde{L}$.

Remark 7.7. Let us observe that, for any loop γ whose liftings are loops in $\widetilde{L}$, $\lambda(\gamma) = 0$. On the other hand, any such path gives rise to a zero chain in homology, namely $\lambda(\gamma)$ may be considered as a length for homology chains.

Lemma 7.8. *The projection map p restricted to a d_N-component is surjective $\iff$ for all $x, y \in K$ there is a continuous path γ in K between them which has finite effective length.*

Proof. ($\Longleftarrow$) Let us fix $\widetilde{x}_0 \in \widetilde{L}$, and let $x_0 := p(\widetilde{x}_0)$. Then, for any $x \in K$ there is a continuous path γ in K, starting in x_0 and ending in x, which has finite effective length. Denote by $\widetilde{\gamma}$ its unique lifting to a path in $\widetilde{L}$ starting at $\widetilde{x}_0 \in \widetilde{L}$. Then $\pi(\widetilde{\gamma}(1)) = x$, and $\widetilde{\gamma}(1)$ belongs to the same d_N-component of $\widetilde{x}_0$.
($\Longrightarrow$) Let $x, y \in K$. By assumption, there are $\widetilde{x}, \widetilde{y} \in \widetilde{L}$ such that $d_N(\widetilde{x}, \widetilde{y}) < \infty$ and $p(\widetilde{x}) = x$, $p(\widetilde{y}) = y$. Because of Proposition 6.22, there is a continuous path $\widetilde{\gamma}$ in $\widetilde{L}$ between $\widetilde{x}$ and $\widetilde{y}$. Set $\gamma := p \circ \widetilde{\gamma}$, which automatically has finite effective length. □

Lemma 7.9. *Let N be a norm on the space $c_c(\Sigma)$ of finitely supported sequences as in Definition 7.2, such that*

(1) *N is invariant under permutations $\pi : \Sigma \to \Sigma$ of indices preserving the length, namely*
$$N(a) = N(a_\pi), \quad \text{if} \quad (a_\pi)_\sigma := a_{\pi(\sigma)}, \quad |\pi(\sigma)| = |\sigma|, \quad \sigma \in \Sigma;$$
(2) *there exists $c > 0$ such that $N(S_i a) \le c\,N(a)$ for any finitely supported sequence $a = \{a_\sigma\}$, where the i-shift operator S_i, $i = 0, 1, 2$, acts as*
$$(S_i a)_\sigma = \begin{cases} a_\tau & \text{if } \sigma = i \cdot \tau \\ 0 & \text{otherwise.} \end{cases}$$

Then we have:

(a) *an edge has finite effective length iff any elementary path has finite effective length;*
(b) *if all edges have finite effective length and $c < 1$, then any d_N-component of $\widetilde{L}$ projects surjectively on K.*

Proof. First we observe that the effective length is sub-additive. Indeed, if γ_1, γ_2 are consecutive paths, $\widetilde{\gamma}_1$ is a lifting of γ_1 starting from some point $\widetilde{x}_0 \in \widetilde{L}$, and $\widetilde{\gamma}_2$ is a lifting of γ_2 starting from $x := \widetilde{\gamma}_1(1)$, then
$$\lambda(\gamma_1 \cdot \gamma_2) = d_N(\widetilde{\gamma}_1(0), \widetilde{\gamma}_2(1)) \le d_N(\widetilde{\gamma}_1(0), x) + d_N(x, \widetilde{\gamma}_2(1)) = \lambda(\gamma_1) + \lambda(\gamma_2).$$
Let now compute the sequence $a^{\tau,i}_\sigma = \int_{w_\tau e_i} dz_\sigma$, where e_i is the edge opposite to the vertex p_i. If σ and τ are not ordered, C_σ and $w_\tau e_i$ do not intersect, hence $\int_{w_\tau e_i} dz_\sigma = 0$. If $\sigma \ge \tau$, namely $\sigma = \tau \cdot \rho$, C_σ intersects $w_\tau e_i$ *iff* ρ does not contain the letter i, and in this case $\int_{w_\tau e_i} dz_\sigma = -1/3 = \int_{e_i} dz_\rho$. For $\sigma < \tau$, one can estimate the integral with the oscillation of z_σ on C_τ, which in turn is estimated by $(1/3)(3/5)^{|\tau|-|\sigma|-1}$. As a consequence,
$$a^{\tau,i}_\sigma = (S_\tau a^{\emptyset,i})_\sigma + b^{\tau,i}_\sigma,$$
where S_τ is the composition of $|\tau|$ shift operators and $b^{\tau,i}_\sigma$ satisfies
$$|b^{\tau,i}_\sigma| \begin{cases} \le \frac{1}{3}\left(\frac{3}{5}\right)^{|\tau|-|\sigma|-1} & \text{when } \sigma < \tau \\ = 0 & \text{otherwise.} \end{cases}$$
Since $b^{\tau,i}$ has finite support, the finiteness of $N(a^{\tau,i})$ is equivalent to the finiteness of $N(S_\tau a^{\emptyset,i})$, which only depends on $|\tau|$ by assumption (1), showing that the finiteness of $\lambda(w_\tau e_i) = N(a^{\tau,i})$ is equivalent to the finiteness of $\lambda(w_\rho e_j)$ for any $|\rho| = |\tau|$, and any $j = 0, 1, 2$. Then, if all edges of level k have finite effective length, all edges of level $\le k$ have finite effective length by sub-additivity. Finally, by assumption (2),
$$N(a^{\tau,i}) \le N(b^{\tau,i}) + c^{|\tau|}\lambda(e_i), \tag{7.2}$$
showing that finiteness propagates to edges of all levels. The finiteness of the effective length of elementary paths follows by sub-additivity, thus proving (a).

We now prove (b). As shown in Lemma 7.8, the thesis is equivalent to the connectedness of K by means of paths of finite effective length. We have shown in Lemma 5.7 that a vertex $v_0 \in V_0$ can be connected to any vertex of level p by an elementary path consisting of at most 1 edge of level j, $j \leq p$, proving that v_0 can indeed be connected to any point x in K by a path consisting of (possibly infinitely many) edges, at most 1 of them for any level. Denoting by τ^j the j-th truncation of τ, and by $\delta^\rho = \{\delta^\rho_\sigma\}$ the sequence equal to 1 for $\sigma = \rho$ and to 0 otherwise, we have, if $c \neq \frac{3}{5}$,

$$N(b^{\tau,i}) \leq \frac{1}{3}\sum_{j=0}^{|\tau|-1}\left(\frac{3}{5}\right)^{|\tau|-j-1} N(\delta^{\tau^j}) \leq \frac{N(\delta^\emptyset)}{3}\left(\frac{3}{5}\right)^{|\tau|-1}\sum_{j=0}^{|\tau|-1}\left(\frac{5c}{3}\right)^j \leq \frac{N(\delta^\emptyset)}{3}\frac{(3/5)^{|\tau|}-c^{|\tau|}}{3/5-c},$$

and $N(b^{\tau,i}) \leq \frac{1}{3}N(\delta^\emptyset)|\tau|(\frac{3}{5})^{|\tau|-1}$, if $c = 3/5$. By the inequality 7.2 we get, for an edge $e_n \in E_n$,

$$\lambda(e_n) \leq \begin{cases} \frac{1}{3}N(\delta^\emptyset)\frac{(3/5)^n-c^n}{3/5-c} + c^n\lambda(e_1), & c \neq \frac{3}{5}, \\ \frac{5}{9}N(\delta^\emptyset)n(\frac{3}{5})^n + c^n\lambda(e_1), & c = \frac{3}{5}. \end{cases} \tag{7.3}$$

The thesis follows since the series $\sum_{n=0}^\infty \lambda(e_n)$ converges for $c < 1$. □

7.2. **Potentials of elementary 1-forms.** We now choose a particular norm on sequences, and consider the distance d on $\widetilde{L}$ and the length ℓ on Γ associated to this norm. This choice is motivated by Theorem 7.11 below. Let us consider

$$N(a) = \sum_{n\geq 0}(3/5)^n \sup_{|\sigma|=n}|a_\sigma|.$$

Proposition 7.10. *Any d-component of $\widetilde{L}$ projects surjectively on K, and elementary paths have finite effective length. In particular, for an edge $e_n \in E_n$, the following estimate holds:*

$$\lambda(e_n) \leq \left(\frac{1}{3}n + \frac{3}{2}\right)\left(\frac{3}{5}\right)^{n-1}$$

Proof. Properties (1) and (2) of Lemma 7.9 are satisfied with $c = 3/5$, hence the statement follows. As for the estimate on the effective length, we observe that

$$N(\delta^\emptyset) = 1, \qquad \lambda(e_1) = \frac{5}{2},$$

hence eq. (7.3) becomes

$$\lambda(e_n) \leq \frac{1}{3}n(3/5)^{n-1} + \frac{5}{2}(3/5)^n = \left(\frac{1}{3}n + \frac{3}{2}\right)\left(\frac{3}{5}\right)^{n-1}.$$

□

For any elementary form $\omega = dU_E + \sum_\sigma k_\sigma dz_\sigma$, one has, by (5.4),

$$(5/3)^n \sum_{|\tau|=n}|k_\tau| \leq \sum_{|\tau|=n}\mu_\omega(C_\tau) = \mu_\omega(K). \tag{7.4}$$

Proposition 7.11. *To any elementary 1-form ω is associated a function $U = U_E + U_H$ in any d-component of $\widetilde{L}$, where U_E was described in Theorem 5.4 and U_H may be written as*

$$U_H(x) = \sum_\sigma k_\sigma(z_\sigma(x) - z_\sigma(x_0)),$$

for any fixed x_0 in the given d-component. The series defining U converges uniformly on compact sets. U is a d-continuous Γ_N-affine function on any d-component of $\widetilde{L}$.

Proof. Let us recall that U_H is defined up to an additive constant. Let us choose $x_0 \in \widetilde{L}$ and, for any x in the same d-component, we set

$$U_H(x) = \sum_\sigma k_\sigma (z_\sigma(x) - z_\sigma(x_0)).$$

Since ω satisfies (7.4), given two points x_1, x_2 in the same d-component, we have

$$\begin{aligned} |U_H(x_2) - U_H(x_1)| &= \left| \sum_\sigma k_\sigma (z_\sigma(x_2) - z_\sigma(x_1)) \right| = \left| \sum_n \sum_{|\sigma|=n} k_\sigma (z_\sigma(x_2) - z_\sigma(x_1)) \right| \\ &\leq \left(\sum_n (3/5)^n \sup_{|\sigma|=n} |z_\sigma(x_2) - z_\sigma(x_1)| \right) \left(\sup_n (5/3)^n \sum_{|\sigma|=n} |k_\sigma| \right) \quad (7.5) \\ &\leq \mu_\omega(K) d(x_1, x_2), \end{aligned}$$

where the last inequality follows by Lemma 5.2. As a consequence, U_H is Lipschitz d-continuous. In particular, if $\ell(g) < \infty$, then x and gx belong to the same d-component, and

$$U_H(gx) - U_H(x) = \sum_\sigma k_\sigma \varphi_\sigma(g),$$

namely U_H is Γ_N-affine. Since U_E is continuous on K, it lifts to a Γ-invariant function on $\widetilde{L}$, continuous in the projective limit topology, hence also in the (stronger) d-topology. □

Lemma 7.12. *Let $\omega = dU_E + \sum_\sigma k_\sigma dz_\sigma \in \Omega^1(K)$ be the decomposition of an elementary 1-form and γ a path in K with $\ell(\gamma) < \infty$. Then the following limit exists and is finite:*

$$\lim_n \int_\gamma dU_E + \sum_{|\sigma| \leq n} k_\sigma \int_\gamma dz_\sigma. \quad (7.6)$$

Proof. The thesis follows as in (7.5). □

Definition 7.13. Let ω be an elementary 1-form, γ a path in K with $\ell(\gamma) < \infty$. We define $\int_\gamma \omega$ by the limit in eq. (7.6).

Theorem 7.14. *The function U associated to the form ω in Proposition 7.11 is a potential for ω, namely, for any path γ in K with $\ell(\gamma) < \infty$, we have*

$$\int_\gamma \omega = U(\tilde{\gamma}(1)) - U(\tilde{\gamma}(0)) < \infty, \quad (7.7)$$

where $\tilde{\gamma}$ is a lifting of γ to $\widetilde{L}$.

Corollary 7.15. *We have proved that*

(1) *any form in $\Omega^1(K)$ has a potential defined on any d-component of $\widetilde{L}$ which is Γ_N-affine;*
(2) *the integral of a form in $\Omega^1(K)$ along a d_N-finite path can be read as the variation of the potential at the end points of a lifting of the path;*
(3) *the pairing above gives rise to a nondegenerate pairing between Γ_N and equivalence classes of forms in $\Omega^1(K)$ modulo $B^1(K, \mathbb{R})$.*

Remark 7.16.

(1) Corollary above may be considered as a weak form of a de Rham theorem for elementary 1-forms.

(2) The considerations above work for any choice of the norm N, up to the surjectivity of the d-components. Other norms will select larger or smaller d-components, and consequently smaller or larger classes of 1-forms admitting a finite primitive on d-components.

Appendix A. The projective limit topology on $\widetilde{L}$ is generated by potentials

Lemma A.1. *Let $C_{\sigma i}$ be one of the three subcells of the cell C_σ, denote by z^i_σ the potential of dz_σ on $C_{\sigma i}$, and by $x^i_\sigma = w_\sigma(p_i)$ the common vertex of $C_{\sigma i}$ and C_σ. Then*

- *(a) The set $\{x \in C_{\sigma i} : z^i_\sigma(x) = z^i_\sigma(x^i_\sigma)\}$ coincides with the intersection A^i_σ of $C_{\sigma i}$ with the axis of the edge $e^i_{\sigma i} = w_{\sigma i}(e_i)$ opposite to x^i_σ in $C_{\sigma i}$.*
- *(b) All points in A^i_σ are vertices.*

Proof. It is not restrictive to assume that $\sigma = \emptyset$, $i = 1$, $z(p_1) := z^1_\emptyset(p_1) = 0$. We first prove the following statement.

Claim A.2. *For any $n \in \mathbb{N}$, denote by $\mathbf{1}_n$ the multi-index of length n and taking only the value 1, and let $\Theta_n := \{\mathbf{1}_k : k = 1, \dots, n\}$. Then,*

$$C_1 = C_{\mathbf{1}_n} \cup \bigcup_{\rho \in \Theta_{n-1}} C_{\rho 0} \cup C_{\rho 2}. \tag{A.1}$$

(i) If $x \in C_{\rho 0}$, $\rho \in \Theta_{n-1}$, and $z(x) = 0$ then $x = w_{\rho 0}(p_2)$, hence is on the axis $A := A^1_\emptyset$. Analogously, if $x \in C_{\rho 2}$, $\rho \in \Theta_{n-1}$, and $z(x) = 0$ then $x = w_{\rho 2}(p_0) \in A$.
(ii) The values of z at the points $w_{\mathbf{1}_n}(p_0)$, $w_{\mathbf{1}_n}(p_2)$ are, respectively, $-1/6 \cdot 5^{-n+1}$, $1/6 \cdot 5^{-n+1}$.

Proof of the Claim. The statement clearly holds for $n = 1$. Suppose now it is true for some n. Since $C_{\mathbf{1}_n} = C_{\mathbf{1}_n 0} \cup C_{\mathbf{1}_n 1} \cup C_{\mathbf{1}_n 2}$, equality (A.1) still holds. By harmonic extension, the boundary values of z on $C_{\mathbf{1}_n 0}$ are $-1/6 \cdot 5^{-n+1}$, $-1/6 \cdot 5^{-n}$ and 0, hence, by the maximum principle, the value 0 is assumed only on the vertex, proving (i). The proof of (ii) also follows by harmonic extension. □

Now we turn to the proof of the Lemma. If $z(x) = 0$, either $x \in A$ or $x \in \cap_n C_{\mathbf{1}_n}$, which means $x = p_1 \in A$. Conversely, if $x \in A$, either x is a vertex and $z(x) = 0$ or $x \in \cap_n C_{\mathbf{1}_n}$, which means $x = p_1$ hence $z(x) = 0$. Both (a) and (b) then follow. □

Lemma A.3. *For any $g \in \Gamma_n$, there exists $|\sigma| < n$ such that $\varphi_\sigma(g)$ is a non-vanishing integer, where φ_σ is the homomorphism associated with the Γ_n-affine potential z_σ.*

Proof. The element g may be uniquely decomposed as $g = \prod_{|\tau|<n} g_\tau^{k_\tau}$, where g_τ denotes the homology class of the lacuna ℓ_τ according to the identification $\Gamma_n = H_1(T_n)$. If we choose σ of minimal length such that $k_\sigma \neq 0$, we have

$$\varphi_\sigma(g) = \sum_{|\sigma| \le |\tau| < n} k_\tau \varphi_\sigma(g_\tau) = \sum_{|\sigma| \le |\tau| < n} k_\tau \int_{\ell_\tau} dz_\sigma = k_\sigma,$$

where we used the fact that, as observed at the beginning of Subsection 4.2, $\int_{\ell_\tau} dz_\sigma$ is non-zero only if $\tau \le \sigma$. □

Proposition A.4. *The weak topology $\Upsilon(z_\sigma)$ induced by $\{z_\sigma : \sigma \in \Sigma\}$ on $\widetilde{L}$ coincides with the projective limit topology.*

Proof. We shall prove that, given a point $\tilde{x} \in \widetilde{L}$ and one of its neighborhoods $\widetilde{U}$ in the projective limit topology, there exists a set Ω, open in the weak topology induced by $\{z_\sigma : \sigma \in \Sigma\}$, such that $x \in \Omega \subseteq \widetilde{U}$. This proof will in some points split in three cases:

(c1) $p(\tilde{x}) \notin V_*$,
(c2) $p(\tilde{x}) \in V_0$,
(c3) $p(\tilde{x}) \in V_* \setminus V_0$,

where $p : \widetilde{L} \to K$ is the covering projection. In the course of the proof, we will use the stardard notation X°, resp. ∂X for the (topological) interior, resp. boundary, of $X \subset K$. To avoid misunderstanding, we will denote by C_σ^ι, resp. bC_σ, the combinatorial interior, resp. boundary, of a cell C_σ. Observe that $C_\sigma^\circ = C_\sigma^\iota$ and $\partial C_\sigma = bC_\sigma \iff C_\sigma$ doesn't contain one of the vertices p_0, p_1, p_2.

About the neighborhood $\widetilde{U}$. By definition, there exists $n \in \mathbb{N}$ such that $\widetilde{U}$ is the preimage in $\widetilde{L}$ of a neighborhood U of $x_0 \in \widetilde{L}_n$, where $\tilde{x}$ projects onto x_0. It is not restrictive to assume, possibly passing to a higher covering, that

(c1-c2) the open set U is the interior of a cell of level n in $\widetilde{L}_n$.
(c3) the open set U is a butterfly shaped neighborhood made of two cells of level n in $\widetilde{L}_n$ in such a way that $p_n(U)$ is not contained in a cell of level $n-1$, where $p_n : \widetilde{L}_n \to K$ is the covering projection.

The choice of a fundamental domain. As a closed fundamental domain $\mathcal{F}$ in $\widetilde{L}_n$, we pick a finite union of closed cells of level n in $\widetilde{L}_n$ such that $\mathcal{F}$ is connected, $p_n(\mathcal{F}) = K$ and $p_n|_{\mathcal{F}^\circ}$ is injective, and with the further property that, for any $|\tau| = n-1$, $p_n^{-1}(C_\tau) \cap \mathcal{F}$ is connected. We also require that

(c1-c2) the neighboring cells of U in $\widetilde{L}_n$, whose projection to K lie in the same cell of level $n-1$ containing $p_n(U)$, still belong to $\mathcal{F}$. If $p_n(U) = C_{\sigma i}^\circ$ [*i.e.* $p_n(U) = C_{\sigma i}^\iota$ or $p_n(U) = C_{\sigma i}^\iota \cup \{p_n(x_0)\}$], we get in particular that U is in the middle of the preimage $p_n^{-1}(C_\sigma) \cap \mathcal{F}$.
(c3) same as above for the two subcells of the butterfly neighborhood U. If $p_n(U) = (C_{\sigma i} \cup C_{\rho j})^\circ$ [where, by the above assumption, $\sigma \neq \rho$ and $i \neq j$], we get in particular that $U \cap p_n^{-1}(C_\sigma))$ is in the middle of the preimage $p_n^{-1}(C_\sigma) \cap \mathcal{F}$, and $U \cap p_n^{-1}(C_\rho)$ is in the middle of the preimage $p_n^{-1}(C_\rho) \cap \mathcal{F}$.

The normalization of the z_τ's. We have asked the preimage in $\mathcal{F}$ of any cell C_τ, $|\tau| = n-1$ to be connected. Since such preimage consists of three cells of level n, only one of them is intermediate, namely has a vertex in common with the others. For $|\tau| = n-1$, we set z_τ to be zero on the third vertex of such intermediate cell, so that the range of z_τ on $p_n^{-1}(C_\tau) \cap \mathcal{F}$ is $[-1/2, 1/2]$. We normalize the z_τ for $|\tau| \leq n-2$ such that, again, the range of z_τ on $p_n^{-1}(C_\tau) \cap \mathcal{F}$ is $[-1/2, 1/2]$. In particular,

(c1) the range of z_σ on $U = p_n^{-1}(C_{\sigma i}^\iota) \cap \mathcal{F}$ is $(-1/6, 1/6)$, because U is the intermediate cell, so that, by Lemma A.1, $z_\sigma(x_0) \neq 0$,
(c2) the range of z_σ on $U = p_n^{-1}(C_{\sigma i}^\iota \cup \{p_n(x_0)\}) \cap \mathcal{F}$ is $(-1/6, 1/6)$ and, by Lemma A.1, $z_\sigma(x_0) = 0$,
(c3) the ranges of z_σ and z_ρ on $U = p_n^{-1}((C_{\sigma i} \cup C_{\rho j})^\circ) \cap \mathcal{F}$ are equal to $(-1/6, 1/6)$ and, by Lemma A.1, $z_\sigma(x_0) = z_\rho(x_0) = 0$.

$\mathcal{F}^\circ$ is open in the topology $\mathcal{T}(z_\sigma)$. By definition of $\mathcal{F}$, for any $x \in \mathcal{F}^\circ$, and for any $|\tau| < n$, the position $z_\tau^{\mathcal{F}}(p_n(x)) := z_\tau(x)$ gives a well defined function on $p_n(\mathcal{F}^\circ)$. As a consequence, with the normalization above, $z_\tau^{\mathcal{F}}$ takes values in $(-1/2, 1/2)$ on the open cell C_τ°, and is constant on the other cells, with values $-1/3, 0, 1/3$. Therefore, for any $|\tau| < n$, $\{z_\tau(x) : x \in \mathcal{F}^\circ\} = (-1/2, 1/2)$. If $x \notin \mathcal{F}$, there exists $x' \in \mathcal{F}$ and a non trivial $g \in \Gamma_n$ such that $x = gx'$. By Lemma A.3 there exists $|\tau| < n$ such that $\varphi_\tau(g)$ is a non zero integer, hence $z_\tau(x) = \varphi_\tau(g) + z_\tau(x') \in (-\infty, -1/2] \cup [1/2, +\infty)$. Also, if $x \in \partial\mathcal{F}$, $p_n(x) \in V_n \setminus V_0$, hence

$\exists!\tau$, $|\tau| < n$ such that $p_n(x)$ is a vertex of ℓ_τ and $z_\tau(x) = \pm 1/2$. Then,

$$\{x \in \widetilde{L}_n : z_\tau(x) \in (-1/2, 1/2), |\tau| < n\} = \mathcal{F}^\circ \tag{A.2}$$

The construction of Ω.

(c1) Set $\Omega = \bigcap_{\tau\neq\sigma,|\tau|=n-1} z_\tau^{-1}(-1/2,1/2) \cap z_\sigma^{-1}\{(-1/6,0)\cup(0,1/6)\}$. The result above implies $\Omega \subset \mathcal{F}^\circ$. By the chosen normalization, the values of $z_\sigma^{\mathcal{F}}$ on the cells different from C_σ can only be $-1/3$, 0 or $1/3$, hence the values in $(-1/6,0)\cup(0,1/6)$ are only assumed in $C_{\sigma i}^\circ \equiv C_{\sigma i}^\iota$. Therefore $\Omega \subset U$.

(c2) Set $\Omega = \bigcap_{|\tau|=n-1} z_\tau^{-1}(-1/6,1/6)$. Again $\Omega \subset \mathcal{F}^\circ$. Since $p_n(x_0)$ is in V_0, the values of $z_\sigma^{\mathcal{F}}$ on the cells different from C_σ can only be $-1/3$ or $1/3$, namely the values $(-1/6,1/6)$ are only assumed in $C_{\sigma i}^\circ \equiv C_{\sigma i}^\iota \cup \{p_n(x_0)\}$. Therefore $\Omega \subset U$.

(c3) Set $\Omega = \bigcap_{|\tau|=n-1} z_\tau^{-1}(-1/6,1/6)$. Again $\Omega \subset \mathcal{F}^\circ$. By construction, the removal of the cell C_σ° disconnects $p_n(\mathcal{F}^\circ)$, and we call $D_\sigma(-1/3), D_\sigma(0), D_\sigma(1/3)$ the (connected) components according to the value of $z_\sigma^{\mathcal{F}}$ on them. In the same way, the removal of the cell C_ρ° disconnects $p_n(\mathcal{F}^\circ)$, and we call $D_\rho(-1/3), D_\rho(0), D_\rho(1/3)$ the components according to the value of $z_\rho^{\mathcal{F}}$ on them. Note that, by the simple connectedness of T_n, $D_\sigma(0) = \{p_n(x_0)\} \sqcup C_\rho^\circ \sqcup D_\rho(-1/3) \sqcup D_\rho(1/3)$ and $D_\rho(0) = \{p_n(x_0)\} \sqcup C_\sigma^\circ \sqcup D_\sigma(-1/3) \sqcup D_\sigma(1/3)$, where $\sqcup$ denotes disjoint union. Then, the prescription $z_\sigma^{\mathcal{F}}(y) \in (-1/6,1/6)$ selects $C_{\sigma i}^\circ \cup D_\sigma(0)$, the prescription $z_\rho^{\mathcal{F}}(y) \in (-1/6,1/6)$ selects $C_{\rho j}^\circ \cup D_\rho(0)$, both select $(C_{\sigma i} \cup C_{\rho j})^\circ$, implying $\Omega \subset U$.

Since in all three cases $\Omega \in \mathcal{T}(z_\sigma)$, we have proved the thesis.

□

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