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LOCAL AND GLOBAL SURVIVAL FOR NONHOMOGENEOUS RANDOM WALK SYSTEMS ON $\mathbb{Z}$

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ABSTRACT. We study an interacting random walk system on $\mathbb{Z}$ where at time 0 there is an active particle at 0 and one inactive particle on each site $n \geq 1$. Particles become active when hit by another active particle. Once activated they perform an asymmetric nearest neighbour random walk which depends only on the starting location of the particle. We give conditions for global survival, local survival and infinite activation both in the case where all particles are immortal and in the case where particles have geometrically distributed lifespan (with parameter depending on the starting location of the particle). In particular, in the immortal case, we prove a 0-1 law for the probability of local survival when all particles drift to the right. Besides that, we give sufficient conditions for local survival or local extinction when all particles drift to the left. In the mortal case, we provide sufficient conditions for global survival, local survival and local extinction. Analysis of explicit examples is provided.

Keywords: inhomogeneous random walks, frog model, egg model, local survival, global survival.
AMS subject classification: 60K35, 60G50.

1. Introduction

We study an interacting random walk system on $\mathbb{Z}$ where at time 0 there is one active particle at 0 and one inactive particle at each vertex of $\mathbb{N} \setminus \{0\} = \{1, 2, \ldots\}$. Particles become active if an active particle jumps to their location. The behaviour of the system depends on two sequences $\{l_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ of numbers in $(0, 1)$ and $[0, 1]$ respectively. The particle which at time 0 was at n, once activated, has a geometrically distributed lifespan with parameter $1 - p_n$ and while alive performs a nearest neighbour random walk with probability l_n of jumping to the left and $1 - l_n$ of jumping to the right. If $p_n = 1$ we say that the particle is immortal, otherwise it is mortal. We are interested in establishing, depending on the parameters, whether the process *survives globally, locally* and if there is *infinite activation* or not. Local and global survival have been studied for several processes; among these it is worth mentioning the *Contact Process* and the *Branching Random Walks* in continuous and discrete time (see for instance [4, 5, 6, 12, 13, 14, 18]).

To be precise, if L_0 is the event that site 0 is visited infinitely many times, we say that there is *local survival* if L_0 has positive probability and *almost sure local survival* if L_0 has probability 1. When there is no local survival, that is, when L_0 has probability zero, we also say that there is *local extinction.* We say that there is *global survival* if, with positive probability, at any time there is at least one active particle, and we say that there is *infinite activation* if, with positive probability, at arbitrarily large times there are particles which turn from inactive to active.

This process can be seen as a model for information or disease spreading: every active particle has some information and it shares that information with all particles it encounters on its way. In the last decade, different versions of this model have been studied, often under the name *frog model* or *egg model*. In [17] the authors prove almost sure local survival for a system of simple random walkers on $\mathbb{Z}^d$. This result has been extended in [16] to the case of a random initial configuration ($d \geq 3$) and in [8] for random walks on $\mathbb{Z}$ with right drift. Shape theorems on $\mathbb{Z}^d$ can be found in [2, 3]. Phase transitions for the model where particles have a $\mathcal{G}(1-p)$-distributed lifespan, are investigated in [2, 7, 11, 15]. Recently, in [10], global survival of an asymmetric inhomogeneous random walk system on $\mathbb{Z}$ has been studied (note that in that model particles die after L steps without activation).

Here is the outline of the paper and of its main results. We first deal, in Section 2, with the case where all particles are immortal (that is, $p_n = 1$ for all $n \geq 0$). It is obvious that in this case there is always global survival, but infinite activation is trivial only in the case where at least one particle has $l_n \leq 1/2$. Local survival is nontrivial in any case. In order to understand what the difficulties one encounters are, think of the case where all particles drift to the right (we refer to this situation as the *right drift* case): infinite activation is guaranteed but local survival is not. Theorem 2.1(a) states that, in this case, the probability of local survival obeys a 0–1 law. Roughly speaking (see Corollary 2.2) in the *right drift* case, if $l_n \uparrow \frac{1}{2}$ sufficiently fast, then we have almost sure local survival, otherwise we have local extinction. On the other hand, if all particles drift to the left (*left drift* case), local survival and infinite activation have the same probability (see Theorem 2.1(b)). Proposition 2.5 and Remark 2.6 provide sufficient conditions for infinite activation (thus also for local survival) in the *left drift* case. Example 2.7 shows that the condition of Remark 2.6 is not necessary. Proposition 2.8 states that if $\inf_{n\in\mathbb{N}} l_n > 1/2$ then there is no infinite activation (thus no local survival), but Examples 2.11 and 2.12 show that if $\inf_{n\in\mathbb{N}} l_n = 1/2$ nothing can *a priori* be said about infinite activation. Theorems 2.9 and 2.10 give a sufficient condition for local survival.

Section 3 is devoted to the case where all particles are mortal and have geometrical lifespan with parameter $1 - p_n$, $p_n \in [0, 1)$. Since any particle disappears almost surely after a finite number of steps, global survival is no longer guaranteed and, even if all particles have right drift, so is infinite activation. Indeed in this case global survival and infinite activation have the same probability. In Subsection 3.1 we give sufficient conditions for global survival (Theorems 3.2 and 3.3). In particular we show that in order to survive it is necessary that $\limsup_n p_n = 1$ and if $p_n \to 1$ and $l_n \to 1/2$ with a certain speed then there is global survival. In Subsection 3.2 we deal with the problem of local survival of the process. Theorems 3.4, 3.5 and 3.6 give some sufficient conditions for local extinction and local survival respectively. Example 3.7 shows how our results apply to some explicit cases.

All the proofs are to be found in Section 5, while in Section 4 we comment on some further questions which could be investigated.

2. Immortal particles

In this section, all particles are immortal, that is, $p_n = 1$ for all $n \geq 0$. This assumption guarantees global survival, nevertheless local survival and infinite activation need additional conditions on the sequence $\{l_n\}_{n\geq 0}$. Clearly, if for some $n \in \mathbb{N}$, $l_n = 1/2$ then there is local survival and infinite activation (with positive probability the initial particle reaches n and the random walk associated to n is recurrent). Therefore we assume that $l_n \neq 1/2$ for all n.

Let A_n be the event that the particle at n ever visits 0 (provided that it is activated), and B_n the event that the particle at n is activated sooner or later (clearly $A_n \subseteq B_n$). Note that $\{A_n \text{ i.o.}\} \subseteq L_0$ and $\mathbb{P}(L_0 \setminus \{A_n \text{ i.o.}\}) = 0$. Moreover if there exists n such that $l_n < 1/2$ then $\mathbb{P}(B_n) > 0$ and $\mathbb{P}(B_m | B_n) = 1$ for all $m > n$, thus in this case there is infinite activation.

For any choice of $\{l_n\}_{n\geq 0}$, reasoning as in [8, Section 2] we obtain that

$$\mathbb{P}(A_n|B_n) = \begin{cases} 1 & \text{if } l_n > 1/2; \\ \left(\frac{l_n}{1-l_n}\right)^n & \text{if } l_n < 1/2. \end{cases}$$

Let $B_\infty = \bigcap_{n=1}^\infty B_n$ be the event that all the particles are activated sooner or later; B_∞ represents infinite activation. The following theorem includes the particular case of [8, Theorem 2.2] when $\eta_1 = 1$ a.s. (there $l_n = 1 - p$ for all n). Theorem 2.1 characterizes the *right drift* case in terms of the sequence $\{l_n\}_{n\geq 0}$ and shows that in the *left drift* case the probability of local survival is equal to the probability of infinite activation.

Theorem 2.1. *(a) Suppose that $l_n < 1/2$ for all n (right drift case). The probability of local survival obeys a 0-1 law:*

$$\mathbb{P}(A_n \text{ i.o.}) = \begin{cases} 0 & \text{if } \sum_n \left(\frac{l_n}{1-l_n}\right)^n < +\infty; \\ 1 & \text{otherwise.} \end{cases}$$

(b) Suppose that $l_n > 1/2$ for all n (left drift case). Then $\mathbb{P}(B_\infty \triangle (A_n \text{ i.o.})) = 0$.

The following corollary gives some conditions which are easy to check that imply convergence or divergence of the characterizing series of Theorem 2.1(a).

Corollary 2.2. *In the right drift case ($l_n < 1/2$ for all n):*

(1) if $\limsup_n l_n < 1/2$ then $\mathbb{P}(A_n \text{ i.o.}) = 0$;
(2) if $n(1/2 - l_n) \not\to +\infty$ then $\mathbb{P}(A_n \text{ i.o.}) = 1$;
(3) if there exists $\lambda < 4$ such that $\sum_n \exp(-\lambda n(1/2 - l_n)) < +\infty$ then $\mathbb{P}(A_n \text{ i.o.}) = 0$.

For instance if there exist $\lambda < 4$ and $\delta > 0$ such that $l_n \leq 1/2 - \frac{(1+\delta)}{\lambda n} \log(n)$ eventually as $n \to \infty$, then $\mathbb{P}(A_n \text{ i.o.}) = 0$ (that is, there is not local survival if l_n does not converge to 1/2 sufficiently fast). This corollary is useful for the analysis of the following explicit example.

Example 2.3. *Let $l_n := 1/2 - 1/n^\alpha$ (where $\alpha > 0$). It is clear that if $\alpha \geq 1$ then $n(1/2 - l_n) = n^{1-\alpha} \not\to +\infty$ thus there is almost sure local survival; conversely, if $\alpha \in (0,1)$ then $\sum_n \exp(-n(1/2 - l_n)) = \sum_n \exp(-n^{1-\alpha}) < +\infty$, hence there is local extinction.*

It is easy to extend Theorem 2.1 to the cases where there are both particles with right drift and ones with left drift, as we note in the following remark, which allows us to focus only on the two "pure" cases where all particles drift towards the same direction.

Remark 2.4. *If all but a finite number of particles have right drift then by Theorem 2.1(a) $\sum_n \left(\frac{l_n}{1-l_n}\right)^n < +\infty$ implies local extinction. If the series diverges, then we have local survival, since $\mathbb{P}(A_i \text{ i.o.}) = \mathbb{P}(B_\infty) = \mathbb{P}(B_j)$ where $j = \min\{n : l_n < 1/2\}$.*
On the other hand, if there is an infinite number of particles with left drift and at least one with right drift, then again we have local survival, since $\mathbb{P}(A_n \text{ i.o.}) = \mathbb{P}(B_j)$, where $j = \min\{n : l_n < 1/2\}$.

In the *left drift* case, Theorem 2.1(b) tells us that local survival and infinite activation have the same probability. Thus it is interesting to find conditions for $\mathbb{P}(B_\infty) > 0$. One idea, implemented in the following proposition and used throughout the whole paper, is to partition $\mathbb{N}$ into blocks. In each block we consider a sub-block and consider the event that, for each $j \geq 1$, at least one particle of the j-th sub-block visits all the sites of the $(j+1)$-th sub-block (see Figure 1 for an idea of how the blocks and the sub-blocks are constructed). This event is clearly a subset of B_∞, thus if this event has positive probability, then there is infinite activation, which, in the *left drift* case, also means local survival. Using the fact that the probability that the n-th particle, if activated, reaches $m > n$, is $\left(\frac{1-l_n}{l_n}\right)^{m-n}$, one gets the following proposition, which gives sufficient conditions for infinite activation in the *left drift* case. We note that a similar technique will be used also in the case of mortal particles replacing $(1-l_n)/l_n$ by the r.h.s. of equation (6.6) (see Section 3 for further details).

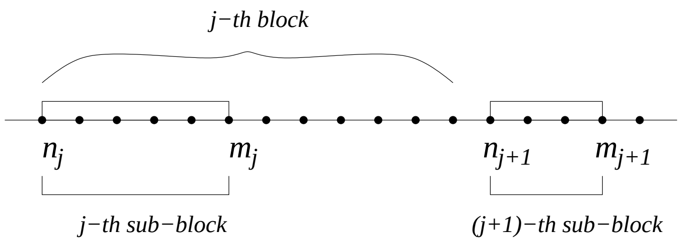

Figure 1.

Proposition 2.5. *Suppose that $l_n > 1/2$ for all $n \geq 0$. A sufficient condition for $\mathbb{P}(B_\infty) > 0$ is the existence of two strictly increasing sequences $\{n_j\}_{j\geq 0}$ and $\{m_j\}_{j \geq 0}$ in $\mathbb{N}$ such that $n_0 := 0$, $m_0 := 0$ and $n_j \leq m_j \leq n_{j+1} - 1$ for all $j \geq 0$ which satisfy any of the following conditions:*

(a) $\prod_{j=0}^\infty \left(1 - \prod_{i=n_j}^{m_j} \left(1 - \left(\frac{1-l_i}{l_i}\right)^{m_{j+1}-i}\right)\right) > 0;$

(b) $\sum_{j=0}^{\infty}\prod_{i=n_j}^{m_j}\left(1-\left(\frac{1-l_i}{l_i}\right)^{m_{j+1}-i}\right)<+\infty$*;*
(c) $\sum_{j=0}^{\infty}\prod_{i=n_j}^{m_j}(m_{j+1}-i)\frac{2l_i-1}{l_i}<+\infty$*;*
(d) $\sum_{j=0}^{\infty}\sum_{i=n_j}^{m_j}\frac{(m_{j+1}-i)^{m_j-n_j+1}}{m_j-n_j+1}\left(\frac{2l_i-1}{l_i}\right)^{m_j-n_j+1}<+\infty.$
Moreover, (a) is equivalent to (b) and (d) implies (c) which in turn implies (b).

The following remark gives a sufficient condition for $\mathbb{P}(B_\infty)>0$, in the *left drift* case, using a particular choice of sub-blocks in Proposition 2.5, namely sub-blocks of cardinality one. To be precise, we exploit the fact that if there exists a subsequence $\{n_j\}_{j\geq 0}$ in $\mathbb{N}$, such that the event "the n_j-th particle visits the n_{j+1}-th vertex, for all $j\geq 0$" (which is a sub-event of B_∞) has positive probability, then also B_∞ has positive probability.

Remark 2.6. *Suppose that all particles have left drift ($l_n>1/2$ for all n). Take $m_j=n_j$ for all $j\geq 0$ in Proposition 2.5. In this case the probability $\mathbb{P}(B_\infty)$ is bounded from below by*

$$\prod_{j=0}^{\infty}\left(\frac{1-l_{n_j}}{l_{n_j}}\right)^{n_{j+1}-n_j}>0. \tag{2.1}$$

By Lemma 6.1(2), an equivalent condition is

$$\sum_{j=0}^{\infty}(n_{j+1}-n_j)\frac{2l_{n_j}-1}{l_{n_j}}<+\infty. \tag{2.2}$$

It is clear that it is not possible to satisfy (2.2) *if $l_{n_j}\not\to 1/2$, but it is possible for sequences l_n going to 1/2 with sufficiently high speed. Note that, by Lemma 6.2, a sufficient condition for* (2.2) *is $\sum_{j=0}^{\infty}\min\{(2l_n-1)/l_n : n\leq j\}<+\infty$. In particular if $\{l_n\}_{n\geq 0}$ is nonincreasing then there exists $\{n_j\}_{j\geq 0}$ such that equation* (2.2) *holds if and only if $\sum_{j=0}^{\infty}\frac{2l_j-1}{l_j}<+\infty$.*

It is worth noting that even if, for all possible subsequences $\{n_j\}_{j\geq 0}$ in $\mathbb{N}$, the event "the n_j-th particle visits the n_{j+1}-th vertex, for all $j\geq 0$" has probability zero, nevertheless B_∞ may have positive probability, as the following example shows.

Example 2.7. *We can easily construct a class of random walk systems with left drift where there is no sequence $\{\bar n_j\}_{j\in\mathbb{N}}$ such that $\prod_{j=0}^{\infty}\left(\frac{1-l_{\bar n_j}}{l_{\bar n_j}}\right)^{\bar n_{j+1}-\bar n_j}>0$ nevertheless $\mathbb{P}(B_\infty)>0$. We use again a block argument. We consider a partition of $\mathbb{N}$ into blocks and we follow the notation of Proposition 2.5. Given $\{n_i\}_i$ and $\{m_i\}_i$ such that $n_0=m_0:=0$ and $n_i\leq m_i\leq n_{i+1}-1$ for all i, let $\mathcal{A}_i:=\{n_i,\dots,n_{i+1}-1\}$, $\mathcal{B}_i:=\{n_i,\dots,m_i\}$ for all $i\in\mathbb{N}$. Define $q_i:=\max\{l_n : n\in\mathcal{B}_i\}$. Now we estimate the conditional probability that, once all the particles in $\mathcal{B}_i$ are activated, at least one of them travels at least to the rightmost point of the set $\mathcal{B}_{i+1}$. This implies that all the particles in $\mathcal{B}_{i+1}$ will be activated.*

This conditional probability is larger than the probability that, if we have m_i-n_i+1 independent random walkers (with probability of moving to the left equal to q_i) starting at the leftmost vertex

$\mathcal{B}_i$, at least one of them reaches the rightmost vertex of $\mathcal{B}_{i+1}$, that is,

$$\zeta_i := 1 - \left(1 - \left(\frac{1-q_i}{q_i}\right)^{m_{i+1}-n_i}\right)^{m_i-n_i+1}.$$

If $\prod_{i=1}^\infty \zeta_i > 0$ then there is global (and local) survival. Clearly

$$\prod_{i=1}^\infty \zeta_i > 0 \iff \sum_{i=1}^\infty \left(1 - \left(\frac{1-q_i}{q_i}\right)^{m_{i+1}-n_i}\right)^{m_i-n_i+1} < +\infty.$$

Suppose that $m_{i+1} - n_i = O(q_i/(2q_i-1))$. For every $\varepsilon \in (0,1)$ we have, for all i sufficiently large and $C \geq \sup_i (m_{i+1}-n_i)\frac{2q_i-1}{q_i}$,

$$\begin{aligned}\left(\frac{1-q_i}{q_i}\right)^{m_{i+1}-n_i} &= \left(\left(1 - \frac{2q_i-1}{q_i}\right)^{\frac{q_i}{2q_i-1}}\right)^{(m_{i+1}-n_i)\frac{2q_i-1}{q_i}} \\ &\geq ((1-\varepsilon)/e)^C.\end{aligned}$$

Suppose that, for some $\varepsilon > 0$, $\sum_{i\in\mathbb{N}} \left(1 - ((1-\varepsilon)/e)^C\right)^{m_i-n_i} < +\infty$ then

$$\sum_{i\in\mathbb{N}} \left(1 - \left(\frac{1-q_i}{q_i}\right)^{m_{i+1}-n_i}\right)^{m_i-n_i+1} \leq \sum_{i\in\mathbb{N}} \left(1 - ((1-\varepsilon)/e)^C\right)^{m_i-n_i+1} < +\infty$$

which implies $\mathbb{P}(B_\infty) > 0$ (that is, infinite activation and local survival).

Given a sequence $\{\bar n_j\}_{j\in\mathbb{N}}$ (which has nothing to do with $\{n_i\}_{i\in\mathbb{N}}$), let $V_{\{\bar n_j\}_{j\in\mathbb{N}}}$ be the event "the $\bar n_j$-th particle visits the $\bar n_{j+1}$-th vertex, for all $j \geq 0$". Assume that $\{l_i\}_{i\in\mathbb{N}}$ is constant within each block (that is, $l_i := q_j$ for all $i \in \mathcal{A}_j$ and for all $j \in \mathbb{N}$). If $q_i \downarrow 1/2$ and $\sum_{i\in\mathbb{N}}(n_{i+1}-n_i)(2q_i-1)/q_i = +\infty$ then $\mathbb{P}(V_{\{\bar n_j\}_{j\in\mathbb{N}}}) = 0$ for any possible sequence $\{\bar n_j\}_{j\in\mathbb{N}}$. Indeed by Remark 2.6, in this case $\mathbb{P}(V_{\{\bar n_j\}_{j\in\mathbb{N}}}) > 0$ if and only if $\sum_{i\in\mathbb{N}}(\bar n_{j+1} - \bar n_j)(2l_{\bar n_j} - 1)/l_{\bar n_j} = +\infty$. According to Lemma 6.2(4) (using $\alpha_i := (2l_i-1)/l_i$), since $\{(2l_i-1)/l_i\}_{i\in\mathbb{N}}$ is nonincreasing, the existence of sequence $\{\bar n_j\}_{j\in\mathbb{N}}$ such that $\sum_{j=0}^\infty (\bar n_{j+1} - \bar n_j)\frac{2l_{\bar n_j}-1}{l_{\bar n_j}} < +\infty$ is equivalent to $\sum_{i=0}^\infty \frac{2l_i-1}{l_i} < +\infty$. But in this case, clearly, $\sum_{i=0}^\infty \frac{2l_i-1}{l_i} = \sum_{i\in\mathbb{N}}(n_{i+1}-n_i)(2q_i-1)/q_i = +\infty$.

An explicit example is given by $q_i := 1/2 + 1/i^2$, $n_i := i^3$ and $m_i := i^3 + i$.

So far we have seen, in the *left drift* case, sufficient conditions for $\mathbb{P}(B_\infty) > 0$. We now give a sufficient condition for $\mathbb{P}(B_\infty) = 0$, whose proof makes use of a random walk approach.

Proposition 2.8. *In the left drift case($l_n > 1/2$ for all n), if $\liminf_{n\to\infty} l_n > 1/2$ then $\mathbb{P}(B_\infty) = 0$ and there is local extinction.*

We observe that, in the *left drift* case, if $\inf_{n\in\mathbb{N}} l_n = 1/2$ then both $\mathbb{P}(B_\infty) > 0$ or $\mathbb{P}(B_\infty) = 0$ are possible, see Example 2.11 and Example 2.12 respectively. Example 2.11 makes use of Theorem 2.9, therefore we place these examples after that statement. Theorem 2.9 gives sufficient conditions for the local survival of the process, which apply also to the general case where there are both particles with left drift and particles with right drift. We note that, in order to have local survival, with positive probability we need to activate all the particles and at the same time an infinite number of them must visit the origin. To this aim the idea is to divide $\mathbb{N}$ into consecutive connected blocks of fixed length L. If with positive probability the particles of the odd labelled blocks take care of activation and the remaining particles (say, at least one per block) visit the origin then we have local survival and infinite activation. Note that the same idea is used in Theorem 3.5 in the case of mortal particles.

Theorem 2.9. *If there exists $L \in \mathbb{N}$ such that $\sum_n \prod_{k=2nL}^{(2n+1)L-1} (l_k - 1/2)^+ < +\infty$ and $\sum_n \prod_{k=(2n+1)L}^{2(n+1)L-1} k(1/2 - l_k)^+ < +\infty$ then there is local survival.*

In particular if there exists $L \in \mathbb{N}$ such that $\sum_{n: l_n > 1/2} (l_n - 1/2)^L < +\infty$ and $\sum_{n: l_n < 1/2} n^L (1/2 - l_n)^L < +\infty$ then there is local survival.

We note that to apply the previous theorem it suffices that $l_n \to 1/2$ sufficiently fast. In Example 2.11 we show that $l_n \to 1/2$ is not necessary. Indeed, if instead of blocks of length L we use blocks of length L_1 and, in each block, we require that a particular subset of particles of cardinality L is responsible for the action (the visit to the rightmost vertex of the next block or the visit to the origin) then we obtain the following result.

Theorem 2.10. *If there exists $L, L_1 \in \mathbb{N}$ such that, $L \le L_1$ and*

$$\sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq \mathcal{A}_{2n}: \#\mathcal{B}=L} \prod_{k\in\mathcal{B}} (l_k - 1/2)^+ < +\infty, \qquad \sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq \mathcal{A}_{2n+1}: \#\mathcal{B}=L} \prod_{k\in\mathcal{B}} k(1/2 - l_k)^+$$

(where $\mathcal{A}_i$ is the i-th block of length L_1) then there is local survival.
In particular if

$$\sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq \mathcal{A}_{2n}: \#\mathcal{B}=L} \sum_{k\in\mathcal{B}} ((l_k - 1/2)^+)^L < +\infty, \qquad \sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq \mathcal{A}_{2n+1}: \#\mathcal{B}=L} \sum_{k\in\mathcal{B}} k^L ((1/2 - l_k)^+)^L < +\infty$$

then there is local survival.

Example 2.11. *Take $l_i = 1/2 + 1/i^\alpha$ where $\alpha > 0$; hence $l_i \downarrow 1/2$ and, if $L > 1/\alpha$, $\sum_i (l_i - 1/2)^L < \infty$ thus, by Theorem 2.9, we have local survival and $\mathbb{P}(B_\infty) > 0$ for all $\alpha > 0$.*
Note that when $\{l_i\}_{i\in\mathbb{N}}$ satisfies Theorem 2.9 then a "mild" modification of that sequence, such as $\{\bar{l}_i\}_{i\in\mathbb{N}}$ where $\bar{l}_{2i} := l_i$, satisfies Theorem 2.10; thus there is local survival even if $\lim_i \bar{l}_i \neq 1/2$ (take for instance $\bar{l}_{2i} = 1/2 + 1/i$ and $\bar{l}_{2i+1} = 3/4$ for all $i \in \mathbb{N}$).

Example 2.12. *We now construct an example where $l_n \downarrow 1/2$ so slowly that $\mathbb{P}(B_\infty) = 0$. Indeed, consider a decreasing sequence $\{q_i\}_{i\in\mathbb{N}}$ such that $q_i \downarrow 1/2$ and fix $\delta > 0$. The idea is to construct an increasing sequence of integers $\{n_k\}_{k\in\mathbb{N}}$ and to define $l_i = q_k$ if $j \in \{n_k, \ldots, n_{k+1} - 1\}$ (we call $\{n_k, \ldots, n_{k+1} - 1\}$ the k-th block). We construct $\{n_k\}_{k\in\mathbb{N}}$ iteratively, in such a way that $\mathbb{P}(B_\infty) = 0$ (clearly $l_n \downarrow 1/2$). Suppose we defined n_i for all $i \le k$. Consider the models $\mathcal{M}_k$ with left jump* *probabilities* $\{\widehat{l}_i(k)\}_{i\in\mathbb{N}}$ *where*

$$\widehat{l}_i(k) = \begin{cases} q_j & \text{if } j \in \{n_j, \dots, n_{j+1}-1\},\ \forall j < k \\ q_k & \text{if } i \geq n_k. \end{cases}$$

We know that, since $\inf_i \widehat{l}_i(k) > 1/2$*, by Proposition 2.8, almost surely for the model* $\mathcal{M}_k$ *we will have only a finite number of activations (i.e. there is no local survival). Hence it is possible to find* n_{k+1} *large enough such that, with probability at least* δ*, no particles in* $\{i : i \geq n_{k+1}\}$ *will be activated in the model* $\mathcal{M}_k$*.*

Note that for a model $\mathcal{M}$ *satisfying* $l_i = q_k$ *if* $j \in \{n_k, \dots, n_{k+1}-1\}$ *(for all* k*), the conditional probability of activating particles in the* $(k+1)$*-th block given that all the particles in* $\{1, \dots, n_k - 1\}$ *have been activated is at most* $1-\delta$*. This is due to the fact that, since* $l_i = \widehat{l}_i(k)$ *for all* $i < n_{k+1}$*, before the activation of a particle in the* $(k+1)$*-th block,* $\mathcal{M}$ *and* $\mathcal{M}_k$ *have the same behaviour. Hence, with probability 1, sooner or later there will be no new activations and* $\mathbb{P}(B_\infty) = 0$*.*

3. Particles with geometrical lifespan

We now suppose that the particle at n survives, at each step, with probability p_n, thus it has a lifespan which is $\mathcal{G}(1-p_n)$-distributed. The main differences between the immortal particle case are that here global survival is not guaranteed (but it has the same probability as the event of infinite activations) and that the knowledge of the drift, *a priori*, plays a minor role. Indeed particles with right drift will activate a finite number of sites almost surely and particles with left drift have a positive (but strictly smaller than 1) probability of visiting the origin. We note that if $p_n = 1$ for some n, then there is global survival, since there is a positive probability of activating those particles. Thus we assume in this whole section that $p_n \in [0, 1)$ for all n, that is, that all particles are mortal (clearly $p_0 > 0$ otherwise the process would not start at all). Observe that if $p_n = 0$ for some $n \in \mathbb{N}$ then those particles do not participate in the evolution of the system therefore it is like having a system with empty vertices.

Remark 3.1. *If we focus only on the left and right drift cases, by coupling with the case of immortal particles, it is clear that the most interesting situations are* $\sup l_n = 1/2$ *(if right drift) and* $\inf l_n = 1/2$ *(if left drift). Indeed if* $\sup l_n < 1/2$ *then, according to Corollary 2.2(1), then there is local extinction even for an immortal particle system, thus there is no local survival in the mortal case. If* $\inf l_n > 1/2$*, by Proposition 2.8, even in the immortal case we activate only a finite number of particles almost surely, thus there is global extinction in the mortal case.*

3.1. Conditions for global survival.

In this case global survival is not guaranteed and has the same probability of B_∞, that is, the event of infinite activation. Note that to activate infinitely many sites, in any case we need the action of infinitely many particles. For instance it is no longer true, as it was in the case of immortal particles, that it suffices that there exists a particle with right drift to have infinite activation (that particle would still be activated with positive probability but in the mortal case it will almost surely activate a finite number of particles). The following theorem gives some sufficient conditions for global survival and states that $\limsup_n p_n = 1$ is a necessary condition.

Theorem 3.2. *(a) If there exists $L \in \mathbb{N}$ such that $\sum_n \prod_{k=nL}^{(n+1)L-1} S_k < +\infty$, where*

$$S_k := \begin{cases} \sqrt{1-p_k} & \text{if } p_k l_k \le 1/2 \\ \sqrt{1-p_k} + 2p_k(l_k - 1/2) & \text{if } p_k l_k > 1/2 \end{cases}$$

then there is global survival.

(b) If $\sum_n (1-p_n)^{L/2} < +\infty$, $\sum_{n: p_n l_n > 1/2} (l_n - 1/2)^L < +\infty$ then there is global survival.

(c) If $\sup_n p_n < 1$ then $\mathbb{P}(B_\infty) = 0$ and there is no global survival almost surely.

Note that condition (b) of the previous theorem implies for instance that if $p_n \to 1$ and $(l_n - 1/2)^+ \to 0$ sufficiently fast, then there is global survival. In particular, if all but a finite number of particles have right drift, then a sufficient condition for global survival is the existence of $L \in \mathbb{N}$ such that $\sum_n (1-p_n)^{L/2} < +\infty$. By using the same trick as in Theorem 2.10, it is clear that the conditions in the previous theorem do not need to be satisfied by the whole sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{l_n\}_{n\in\mathbb{N}}$ as the following theorem states.

Theorem 3.3. *(a) If there exists $L, L_1 \in \mathbb{N}$ such that, $L \le L_1$ and*

$$\sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq\mathcal{A}_n : \#\mathcal{B}=L} \prod_{k\in\mathcal{B}} S_k < +\infty,$$

where S_k is the same as in Theorem 3.2 and $\mathcal{A}_n$ is the n-th block of length L_1, then there is global survival.

(b) If

$$\sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq\mathcal{A}_n : \#\mathcal{B}=L} \Big(\sum_{k\in\mathcal{B}} (1-p_k)^{L/2} + \sum_{k\in\mathcal{B}: p_k l_k > 1/2} (l_k - 1/2)^L \Big) < +\infty,$$

then there is global survival.

3.2. **Conditions for local survival.** From now on we deal with local survival. Our first result gives sufficient conditions for local extinction. The first assertion is similar to Theorem 2.1(a): note that (as explained in the proof of the theorem) the probability that the n-th particle, if activated, ever visits the site 0, is

$$\left(\frac{1 - \sqrt{1 - 4p_n^2 l_n (1-l_n)}}{2p_n(1-l_n)} \right)^n. \tag{3.3}$$

Theorem 3.4. *If $\sum_n \left(\frac{1-\sqrt{1-4p_n^2 l_n(1-l_n)}}{2p_n(1-l_n)} \right)^n < +\infty$ then $\mathbb{P}(A_n \text{ i.o.}) = 0$. In particular*

(1) if $\sum_n p_n^n (1 - (2l_n - 1)^+)^n < +\infty$ then $\mathbb{P}(A_n \text{ i.o.}) = 0$;

(2) if $\sum_n p_n^n < +\infty$ (for instance, if $\sup_n p_n < 1$) then $\mathbb{P}(A_n \text{ i.o.}) = 0$ for any choice of $\{l_n\}_n$.

The tricky part is finding conditions for local survival: note that on one hand we need that all particles get activated sooner or later and on the other hand that infinitely many of them visit the origin. To avoid dealing with situations where a particle is required both to activate a certain number of sites and to visit the origin, we exploit once again the idea of partitioning $\mathbb{N}$ into blocks and sub-blocks. Some sub-blocks will take care of activation and the others of local survival. We partition $\mathbb{N}$ into subsets of length $2L$: the event where, for all j, at least one of the particles between position $2jL$ and $(2j+1)L-1$ visits site $(2j+3)L-1$ and at least one of the particles between position $(2j+1)L$ and $2(j+1)L-1$ visits 0 (see Figure 2), is a subset of the event of local survival. Thus a sufficient condition for local survival is that this event has positive probability.

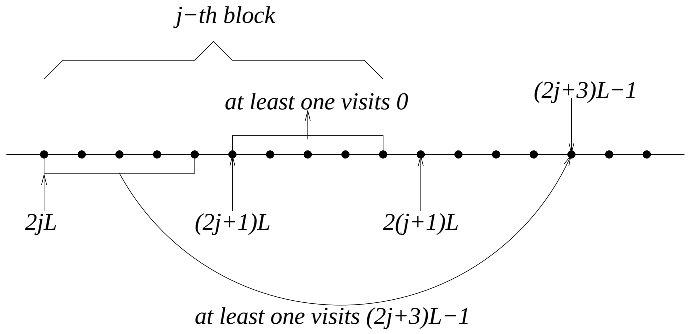

Figure 2.

By using (3.3) and the fact that the probability that the n-th particle, if activated, ever visits the site $m > n$ is

$$\left(\frac{1-\sqrt{1-4p_n^2 l_n(1-l_n)}}{2p_n l_n}\right)^{m-n}, \tag{3.4}$$

one gets a lower bound for the probability of local survival: this is the main idea in Theorem 3.5.

Theorem 3.5. *If there exists* $L \in \mathbb{N}$ *such that* $\sum_n \prod_{k=2nL}^{(2n+1)L-1} S_k < +\infty$ *and* $\sum_n \prod_{k=(2n+1)L}^{2(n+1)L-1} k\widetilde{S}_k < +\infty$*, where*

$$S_k := \begin{cases} \sqrt{1-p_k} & \text{if } p_k l_k \le 1/2 \\ \sqrt{1-p_k} + 2p_k(l_k - 1/2) & \text{if } p_k l_k > 1/2 \end{cases}$$

$$\widetilde{S}_k := \begin{cases} \sqrt{1-p_k} & \text{if } p_k(1-l_k) \le 1/2 \\ \sqrt{1-p_k} + 2p_k(1/2 - l_k) & \text{if } p_k(1-l_k) > 1/2, \end{cases}$$

then there is local survival.

In particular if there exists $L \in \mathbb{N}$ *such that* $\sum_n n^L(1-p_n)^{L/2} < +\infty$, $\sum_{n: p_n l_n > 1/2}(l_n - 1/2)^L < +\infty$ *and* $\sum_{n: p_n(1-l_n)>1/2} n^L (1/2 - l_n)^L < +\infty$ *then there is local survival.*

Again, the same trick used in Theorems 2.10 and 3.3 yields the following generalization.

Theorem 3.6. *If there exists* $L, L_1 \in \mathbb{N}$ *such that,* $L \le L_1$ *and*

$$\sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq\mathcal{A}_{2n}:\#\mathcal{B}=L} \prod_{k\in\mathcal{B}} S_k < +\infty, \qquad \sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq\mathcal{A}_{2n+1}:\#\mathcal{B}=L} \prod_{k\in\mathcal{B}} k\widetilde{S}_k < +\infty,$$

where S_k and $\widetilde{S}_k$ are the same as in Theorem 3.5 and $\mathcal{A}_i$ is the i-th block of length L_1, then there is local survival.

In particular if

$$\begin{cases} \sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq\mathcal{A}_{2n}:\#\mathcal{B}=L} \Big(\sum_{\substack{k\in\mathcal{B}\\ p_k l_k>1/2}} (l_k-1/2)^L + \sum_{k\in\mathcal{B}} (1-p_k)^{L/2}\Big) < +\infty \\ \sum_{n\in\mathbb{N}} \min_{\mathcal{B}\subseteq\mathcal{A}_{2n+1}:\#\mathcal{B}=L} \Big(\sum_{\substack{k\in\mathcal{B}\\ p_k(1-l_k)>1/2}} k^L(1/2-l_k)^L + \sum_{k\in\mathcal{B}} k^L(1-p_k)^{L/2}\Big) < +\infty \end{cases}$$

then there is local survival.

Example 3.7. *Suppose that $p_n := 1-1/n^\beta$ and $l_n := 1/2+1/n^\alpha$ (where α, $\beta>0$). According to Theorem 3.2 for all $\alpha,\beta>0$ there is global survival. By Theorem 3.5 there is local survival if $\beta>2$ and $\alpha>0$ while, by Theorem 3.4 there is local extinction if $\beta<1$ and $\alpha>0$. Note that, if $\beta<1$ then there is local extinction for any choice of $\{l_n\}_{n\in\mathbb{N}}$.*

On the other hand, if $p_n := 1-1/n^\beta$ and $l_n := 1/2-1/n^\alpha$, we still have global survival for all $\alpha,\beta>0$ and local extinction if $\beta<1$ and $\alpha>0$. If $\beta>2$ and $\alpha>1$ then Theorem 3.5 guarantees local survival.

We note that local survival does not imply $\liminf_n l_n \geq 1/2$ or $\lim_n p_n = 1$; analogously, global survival does not imply $\limsup_n l_n \leq 1/2$ or $\lim_n p_n = 1$. Indeed one can proceed as in Example 2.11 by modifying the sequences of Example 3.7 and by using Theorem 3.6.

4. Final Remarks

First of all, let us discuss briefly the case where there are particles on the whole line $\mathbb{Z}$. When we say that the left (respectively right) process survives globally (respectively locally) we are talking about the process which involves just the particle in the left (respectively right) side of the line (the origin is included).

Clearly if either the right or the left process survives (globally or locally) then the whole process survives (globally or locally).

We discuss mainly the immortal particle case for simplicity and we sketch the differences with the mortal case. Suppose that all the particles in the left (respectively right) process are activated then the conditional probability of local survival is 1 or 0 depending on the divergence or convergence of the series $\sum_n \min\left(1, \frac{l_n}{1-l_n}\right)$ (respectively $\sum_n \min\left(1, \frac{1-l_n}{l_n}\right)$).

Clearly if the conditional probabilities of local survival of both the left and right processes are 0, then there is local extinction for the whole process as well. Indeed there might be cooperation between the particles in two half lines in order to improve the activation process but nothing can be done for local survival.

Hence if either the left process or the right one can survive globally, then there is a positive probability of local survival if and only if at least one of the two process survives locally (once all the particle are activated). Here we are not saying that one of the process survives locally by itself but that it might survive once all its particles are activated (maybe by one particle from the other side). We observe that in the mortal particle case, the local survival of the whole process is equivalent to the local survival of one of the two half processes by itself.

If both processes cannot survive globally then there might still be global survival; in order to survive globally it is sufficient (and necessary as well) that an infinite number of particle from each side crosses the origin and goes to the other side. Thus, in this case global survival is equivalent to local survival.

Another question is what can be said in random environment, that is the case where $\{l_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables taking values in $(0,1)$ (also the sequence $\{p_n\}_{n\in\mathbb{N}}$ may be randomly chosen). The analysis of the random environment case exceeds the purpose of this paper, nevertheless some results may be deduced, for instance in the immortal case it is not difficult to see that if $\sum_n \mathbb{P}\left(l_1 > 1/2 - \frac{(1+\delta)}{\lambda n}\log(n)\right) < +\infty$ for some $\lambda < 4$ and $\delta > 0$ then there is local extinction (see Corollary 2.2(3)).

5. Proofs

Proof of Theorem 2.1. (a) Let C_0 be the event that the particle which starts at 0 visits all vertices $n \geq 1$: since $l_0 < 1/2$, then $\mathbb{P}(B_\infty) = \mathbb{P}(C_0) = 1$. Moreover, with respect to $\mathbb{P}(\cdot|C_0)$, $\{A_n\}_{n\geq 1}$ is an independent family of events; $\mathbb{P}(A_n) = \mathbb{P}(A_n|B_n) = \mathbb{P}(A_n|C_0)$ for $n \geq 1$. Clearly in this case, $\{A_n\}_{n\geq 1}$ is independent with respect to $\mathbb{P}$. Thus

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \left(\frac{l_n}{1-l_n}\right)^n.$$

The claim follows by Borel-Cantelli lemma.

(b) If all particles have a drift to the left, each particle visits 0 a.s. only a finite number of times. Hence in order to have local survival, we need to activate all particles. But infinite activation is also a sufficient condition since starting at $n > 0$ each particle visits 0 a.s. at least once.

□

Proof of Corollary 2.2. (1) Since $l_n < 1/2$ for all $n \in \mathbb{N}$ then $\limsup_n l_n < 1/2$ is equivalent to $\sup_n l_n < 1/2$. Thus $\sup_n l_n/(1-l_n) < 1$ and the series $\sum_{n=1}^{\infty} (l_n/(1-l_n))^n$ converges.

(2) Let $\{n_j\}$ be such that $n_j(1/2 - l_{n_j}) \leq \delta$ for all $j \in \mathbb{N}$. In this case $l_{n_j} \to 1/2$ as $j \to \infty$ and, for every $\varepsilon > 0$, eventually we have that

$$\begin{aligned}\left(\frac{l_{n_j}}{1-l_{n_j}}\right)^{n_j} &= \left[\left(1 - \frac{1-2l_{n_j}}{1-l_{n_j}}\right)^{(1-l_{n_j})/(1-2l_{n_j})}\right]^{2n_j(1/2-l_{n_j})/(1-l_{n_j})} \\ &\geq \left(\frac{1-\varepsilon}{e}\right)^{4n_j(1/2-l_{n_j})} \geq \left(\frac{1-\varepsilon}{e}\right)^{4\delta}\end{aligned}$$

which implies the divergence of $\sum_{n=1}^{\infty} (l_n/(1-l_n))^n$.

(3) Note that $(1-\frac{1-2l_n}{1-l_n})^{\frac{1-l_n}{1-2l_n}} \le 1/e$. Then

$$\left(\frac{l_n}{1-l_n}\right)^n \le \exp\left(-\frac{2n(1/2-l_n)}{1-l_n}\right).$$

We divide the sum into two disjoint convergent series

$$\sum_n \left(\frac{l_n}{1-l_n}\right)^n \le \sum_{n:2/(1-l_n)\le\lambda} \left(\frac{l_n}{1-l_n}\right)^n + \sum_{n:2/(1-l_n)>\lambda} \exp\left(-\frac{2n(1/2-l_n)}{1-l_n}\right) < \infty$$

and this yields the conclusion.

□

Proof of Proposition 2.5. Let $\{n_j\}_{j\ge0}$ and $\{m_j\}_{j\ge0}$ be two strictly increasing sequences in $\mathbb{N}$ such that $n_0 := 0$ and $n_j \le m_j \le n_{j+1}-1$ for all $j \ge 0$. We define j-th block of vertices the set which spans from position n_j to position $n_{j+1}-1$ (included) while we refer to the set of vertices $\{n_j,\dots,m_j\}$ as the j-th sub-block. A sufficient condition for $\mathbb{P}(B_\infty) > 0$ is that, with positive probability, for every sub-block there exists at least one particle visiting all the vertices in the following sub-block. Since the probability that the n-th particle, if activated, reaches $m > n$, is $\left(\frac{1-l_n}{l_n}\right)^{m-n}$, the probability that, for every sub-block there exists at least one particle visiting all the vertices in the following sub-block, is condition (a), that is

$$\prod_j \left(1-\prod_{i=n_j}^{m_j}\left(1-\left(\frac{1-l_i}{l_i}\right)^{m_{j+1}-i}\right)\right) > 0$$

which, by Lemma 6.1, is equivalent to (b)

$$\sum_j \prod_{i=n_j}^{m_j}\left(1-\left(\frac{1-l_i}{l_i}\right)^{m_{j+1}-i}\right) < +\infty.$$

Since $1-x^n \le n(1-x)$ for all $x \ge 0$, the previous inequality is implied by (c)

$$\sum_j \prod_{i=n_j}^{m_j} (m_{j+1}-i)\frac{2l_i-1}{l_i} < +\infty$$

which, using the inequality between arithmetic and geometric means, is in turn implied by (d)

$$\sum_j \sum_{i=n_j}^{m_j} \frac{(m_{j+1}-i)^{m_j-n_j+1}}{m_j-n_j+1}\left(\frac{2l_i-1}{l_i}\right)^{m_j-n_j+1} < +\infty.$$

□

Proof of Proposition 2.8. Note that, since $l_n > 1/2$ for all $n \in \mathbb{N}$, then $\liminf_n l_n > 1/2$ is equivalent to $\inf_n l_n > 1/2$. We associate to the process a random walk on a subset of $\mathbb{N}\times\mathbb{N}$. To this aim we define the generation 0 as the set containing only the initial active particle and, recursively, the generation $n+1$ as the set of vertices visited by at least one particle of generation n. We denote by j_{n+1} the rightmost position reached by a particle of a generation $i \le n$. Hence the generation n is nonempty if and only if $j_n > j_{n-1}$, in this case it contains all the particles starting in the set of positions $\{j_{n-1}+1,\dots,j_n\}$. It is clear that if the n-th generation is empty then all generations $m \ge n$ are empty as well. The system survives locally if and only if all the particles are activated, that is, if and only if every generation contains at least one particle.

As a warm-up we start with the simpler case of an homogeneous system: $l_n = l > 1/2$ for every n. We associate to this process the random walk $\{\Delta_n\}_n$ which counts the particles of the generation n, which is $\Delta_n = j_n - j_{n-1}$. The origin is the only absorbing state of this Markov chain. It is easy to compute the probability of absorption (or local extinction)

$$\begin{aligned}\mathbb{P}(\Delta_n = 0 | \Delta_{n-1} = h) &= \left(1 - \frac{1-l}{l}\right)\left(1 - \left(\frac{1-l}{l}\right)^2\right) \cdots \left(1 - \left(\frac{1-l}{l}\right)^h\right) \\ &\ge \prod_{i=1}^{\infty} \left(1 - \left(\frac{1-l}{l}\right)^i\right)\end{aligned}$$

which is strictly positive according to Lemma 6.1. This implies, in particular, that the Markov chain $\{\Delta_n\}_n$ is absorbed in 0 a.s., whence $\mathbb{P}(B_\infty) = 0$.

In the general case of an inhomogeneous system, $\{\Delta_n\}_n$ is no longer a Markov process. In order to be able to mimic the steps above, we must consider the Markov chain $\{(\Delta_n, j_n)\}_n$. In this case

$$\begin{aligned}\mathbb{P}(\Delta_n = 0 | (\Delta_{n-1}, j_{n-1}) = (h,k)) &= \prod_{i=k-h+1}^{k} \left(1 - \left(\frac{1-l_i}{l_i}\right)^{k-i+1}\right) \\ &\ge \inf_{h,k \in \mathbb{N}: h \le k} \prod_{i=k-h+1}^{k} \left(1 - \left(\frac{1-l_i}{l_i}\right)^{k-i+1}\right) \\ &= \inf_{k \in \mathbb{N}} \prod_{i=1}^{k} \left(1 - \left(\frac{1-l_i}{l_i}\right)^{k-i+1}\right).\end{aligned}$$

Note that $\inf_{k\in\mathbb{N}} \prod_{i=1}^{k} \left(1 - \left(\frac{1-l_i}{l_i}\right)^{k-i+1}\right) > 0$ is equivalent to $\inf_{i\in\mathbb{N}} l_i > 1/2$ and implies $\mathbb{P}(B_\infty) = 0$. □

Proof of Theorem 2.9. The proof can be easily adapted from the proof of Theorem 3.5. □

Proof of Theorem 2.10. The proof is an easy adaptation of the proof of Theorem 3.6. □

Proof of Theorem 3.2. We note that under the conditions of the theorem we have that, if we partition $\mathbb{N}$ into blocks of length L, in all but a finite number (say N_0) of blocks at least one particle has a strictly positive lifetime parameter p_n. Otherwise the series $\sum_n \prod_{k=nL}^{(n+1)L-1} S_k$ (or $\sum_{n\in\mathbb{N}} \sqrt{1-p_n}$) would be divergent. Since there is always a positive probabilty that the particle at 0 reaches $(N_0+1)L$, then we can assume without loss of generality that in every block there is at least one particle with strictly positive lifetime parameter.

(a) Consider the (mortal) random walk with $p(j,j-1) = p_n l_n$, $p(j,j+1) = p_n(1-l_n)$, $p(j,D) = 1-p_n$ for all $j \in \mathbb{Z}$, $p(D,D) = 1$ (D represents the absorbing state where the particle is considered dead). Define

$$f_n^{(k)}(x,y) = \mathbb{P}(\text{the } n\text{-th RW visits } y \text{ for the first time at time } k+h|\text{the RW is at } x \text{ at time } h).$$

Let $F_n(x,y|z) = \sum_k f_n^{(k)}(x,y)z^k$. Then

$$F_n(x-1,x|z) = p_n(1-l_n)z + p_n l_n z F_n(x-1,x+1|z). \tag{6.5}$$

Noting that $F_n(x-1,x+1|z) = (F_n(x-1,x|z))^2$ we get an equation for $F_n(x-1,x|z)$ whose acceptable solution is

$$F_n(x-1,x|z) = \frac{1-\sqrt{1-4z^2p_n^2 l_n(1-l_n)}}{2zp_n l_n} = \frac{2zp_n(1-l_n)}{1+\sqrt{1-4z^2p_n^2 l_n(1-l_n)}}. \tag{6.6}$$

Hence the probability for a mortal particle starting from j_n to ever reach j_{n+1} is $F_{j_n}(x-1,x|1)^{j_{n+1}-j_n}$.

Consider now the partition in blocks of length L. The probability that, in each block, there exists at least one particle which visits all the site of the following block is

$$\prod_{j=1}^{\infty}\left(1-\prod_{k=jL}^{(j+1)L-1}\left(1-\left(\frac{1-\sqrt{1-4p_k^2 l_k(1-l_k)}}{2p_k l_k}\right)^{(j+2)L-1-k}\right)\right). \tag{6.7}$$

By Lemma 6.1 a sufficient condition for the positivity of the product in equation (6.7) is

$$\sum_j \prod_{k=jL}^{(j+1)L-1}\left(1-\left(\frac{1-\sqrt{1-4p_k^2 l_k(1-l_k)}}{2p_k l_k}\right)^{(j+2)L-1-k}\right) < +\infty; \tag{6.8}$$

the fact that in each block there is at least one particle, say at n, with $p_n > 0$ implies that each term in the product (6.7) is strictly positive and Lemma 6.1 applies. Since $1-x^n \le n(1-x)$ (for all $n \in \mathbb{N}$) and by using the following estimates

$$\begin{aligned} 0 \le 1 - \frac{2p_n(1-l_n)}{1+\sqrt{1-4p_n^2 l_n(1-l_n)}} &= \frac{1+\sqrt{(2p_n l_n-1)^2+4p_n(1-p_n)l_n} - 2p_n(1-l_n)}{1+\sqrt{1-4p_n^2 l_n(1-l_n)}} \\ \le 1-2p_n(1-l_n)+2\sqrt{1-p_n}+|2p_n l_n - 1| \le W_n &:= \begin{cases} 2(1-p_n)+2\sqrt{1-p_n} & \text{if } p_n l_n \le 1/2 \\ 4p_n(l_n-1/2)+2\sqrt{1-p_n} & \text{if } p_n l_n > 1/2. \end{cases} \end{aligned}$$

we have that equation (6.8) is implied by $\sum_n \prod_{k=nL}^{(n+1)L-1} W_k < +\infty$ (note that the exponent $(j+2)L-1-k$ in equation (6.8) is bounded above by $2L-1$, uniformly in k and j) which, in turn, is implied by $\sum_n \prod_{k=nL}^{(n+1)L-1} S_k < +\infty$ since

$$W_k \le \begin{cases} 4S_k & \text{if } p_k l_k \le 1/2 \\ 2S_k & \text{if } p_k l_k > 1/2 \end{cases}$$

whence $\prod_{k=nL}^{(n+1)L-1} W_k \le 4^L \prod_{k=nL}^{(n+1)L-1} S_k$ and we are done.

(b) By using the inequality between arithmetic and geometric means we have that $\sum_{n=0}^{\infty}\prod_{k=nL}^{(n+1)L-1} S_k \leq \frac{1}{L}\sum_{n=0}^{\infty}\sum_{k=nL}^{(n+1)L-1}(S_{nL+k})^L \equiv \frac{1}{L}\sum_n S_n^L$. Hence $\sum_n S_n^L < +\infty$ implies global survival. Using, on one hand, the Minkowski inequality and, on the other, the fact that S_n is the sum of the two nonnegative functions $\sqrt{1-p_n}$ and $2p_n(l_n-1/2)\mathbb{1}_{(0,+\infty)}(l_n-1/2)$, we have that $\sum_n S_n^L < +\infty$ is equivalent to $\sum_n (1-p_n)^{L/2} < +\infty$, $\sum_{n: p_n l_n > 1/2}(l_n - 1/2)^L < +\infty$ (since $p_n \to 1$ in both cases).

(c) Suppose that $\sup_n p_n = p < 1$ and that n dormient particles are activated in n consecutive vertices, say $i, i+1, \ldots, i+n-1$. The probability that the lifespan of all these particles is so short that neither of them can possibly reach the vertex $i+n$ (and activate more particles) is

$$\prod_{j=0}^{n-1}(1-p_{i+j}^{n-j}) \geq \prod_{j=1}^{\infty}(1-p^j) > 0, \qquad \forall n \in \mathbb{N}.$$

which implies the result.

□

Proof of Theorem 3.3. We fix a subset $\mathcal{B}$ of cardinality L in each block of length L_1. We require that, in each block, at least one particle of the fixed subset visits all the vertices of the fixed subset of the following block. It is easy to see that, mimicking the proof of Theorem 3.2, the best choice for the subsets $\mathcal{B}$ in each block is the one which minimizes the summands. □

Proof of Theorem 3.4. We note that in this case, switching l_n and $1-l_n$ in equation (6.6)

$$\mathbb{P}(A_n|B_n) = \left(\frac{1-\sqrt{1-4p_n^2 l_n(1-l_n)}}{2p_n(1-l_n)}\right)^n = \left(\frac{2p_n(1-l_n)}{1+\sqrt{1-4p_n^2 l_n(1-l_n)}}\right)^n. \tag{6.9}$$

Now, since $A_n \subset B_n$, $\mathbb{P}(A_n) \leq \mathbb{P}(A_n|B_n)$ and and by Borel-Cantelli we have that $\sum_n \mathbb{P}(A_n|B_n) < +\infty$ implies $\mathbb{P}(A_i \text{ i.o.}) = 0$.

In particular,

$$\frac{2p_n(1-l_n)}{1+\sqrt{1-4p_n^2 l_n(1-l_n)}} \leq \frac{2p_n(1-l_n)}{1+|2p_n l_n - 1|} = \begin{cases} \dfrac{1-l_n}{l_n}, & \text{if } p_n l_n \geq 1/2 \\ 1-\dfrac{1-p_n}{1-p_n l_n}, & \text{if } p_n l_n < 1/2. \end{cases}$$
$$\leq p_n(1-(2l_n-1)^+)$$

then $\sum_n p_n^n(1-(2l_n-1)^+)^n < +\infty$ implies $\sum_n \mathbb{P}(A_n|B_n) < +\infty$. The last part is straightforward.

□

Proof of Theorem 3.5. We follow closely the proof of Theorem 3.2. We consider $\mathbb{N}$ partitioned into subsets of length $2L$; as in Theorem 3.2 we can assume, without loss of generality, that in each block of length L there exists at least one particle with strictly positive lifetime parameter p_n.

The probability of local survival is larger or equal to the probability that, for all j, at least one of the particles between position $2jL$ and $(2j+1)L-1$ visits site $(2j+3)L-1$ and at least one of the particles between position $(2j+1)L$ and $2(j+1)L-1$ visits 0. By (3.3) and (3.4), this event has probability

$$\begin{aligned}&\prod_{j=1}^{\infty}\left(1-\prod_{k=2jL}^{(2j+1)L-1}\left(1-\left(\frac{1-\sqrt{1-4p_k^2l_k(1-l_k)}}{2p_kl_k}\right)^{(2j+3)L-1-k}\right)\right)\\&\quad\cdot\prod_{j=1}^{\infty}\left(1-\prod_{k=(2j+1)L}^{2(j+1)L-1}\left(1-\left(\frac{1-\sqrt{1-4p_k^2l_k(1-l_k)}}{2p_k(1-l_k)}\right)^{k}\right)\right).\end{aligned} \tag{6.10}$$

By Lemma 6.1 an equivalent condition for the positivity of the product in equation (6.10) is

$$\begin{cases}\sum_j\prod_{k=2jL}^{(2j+1)L-1}\left(1-\left(\frac{1-\sqrt{1-4p_k^2l_k(1-l_k)}}{2p_kl_k}\right)^{(2j+3)L-1-k}\right)<+\infty,\\ \sum_j\prod_{k=(2j+1)L}^{2(j+1)L-1}\left(1-\left(\frac{1-\sqrt{1-4p_k^2l_k(1-l_k)}}{2p_k(1-l_k)}\right)^{k}\right)<+\infty.\end{cases} \tag{6.11}$$

Since $1-x^n\le n(1-x)$ (for all $n\in\mathbb{N}$) and by using the following estimates

$$0\le 1-\frac{2p_n(1-l_n)}{1+\sqrt{1-4p_n^2l_n(1-l_n)}}\le W_n:=\begin{cases}2(1-p_n)+2\sqrt{1-p_n} & \text{if } p_nl_n\le 1/2\\ 4p_n(l_n-1/2)+2\sqrt{1-p_n} & \text{if } p_nl_n>1/2.\end{cases}$$

$$0\le 1-\frac{2p_nl_n}{1+\sqrt{1-4p_n^2l_n(1-l_n)}}\le \widetilde{W}_n:=\begin{cases}2(1-p_n)+2\sqrt{1-p_n} & \text{if } p_n(1-l_n)\le 1/2\\ 4p_n(1/2-l_n)+2\sqrt{1-p_n} & \text{if } p_n(1-l_n)>1/2.\end{cases}$$

we have that equation (6.11) is implied by $\sum_n\prod_{k=2nL}^{(2n+1)L-1}W_k<+\infty$ and $\sum_n\prod_{k=(2n+1)L}^{2(n+1)L-1}k\widetilde{W}_k<+\infty$ which, in turn, is implied by $\sum_n\prod_{k=2nL}^{(2n+1)L-1}S_k<+\infty$ and $\sum_n\prod_{k=(2n+1)L}^{2(n+1)L-1}k\widetilde{S}_k<+\infty$ since

$$W_k\le\begin{cases}4S_k & \text{if } p_kl_k\le 1/2\\ 2S_k & \text{if } p_kl_k>1/2\end{cases}\qquad \widetilde{W}_k\le\begin{cases}4\widetilde{S}_k & \text{if } p_k(1-l_k)\le 1/2\\ 2S_k & \text{if } p_k(1-l_k)>1/2\end{cases}$$

whence $\prod_{k=2nL}^{(2n+1)L-1}W_k\le 4^L\prod_{k=2nL}^{(2n+1)L-1}S_k$ and $\prod_{k=2nL}^{(2n+1)L-1}k\widetilde{W}_k\le 4^L\prod_{k=(2n+1)L}^{2(n+1)L-1}k\widetilde{S}_k$ and the first part of the theorem is proved.

As before, the inequality between arithmetic and geometric means implies $\sum_n\prod_{k=2nL}^{(2n+1)L-1}S_k\le\sum_{n=0}^{\infty}\frac{1}{L}\sum_{k=0}^{L-1}(S_{2nL+k})^L$ and a similar one for $\widetilde{W}_k$. Hence

$$\sum_{n\in\mathbb{N},k=0,\dots,L-1}(S_{2nL+k})^L<+\infty \text{ and } \sum_{n\in\mathbb{N},k=0,\dots,L-1}(((2n+1)L+k)\widetilde{S}_{(2n+1)L+k})^L<+\infty$$

imply local survival. Again, the Minkowski inequality yields the result. □

Proof of Theorem 3.6. As in the proof of Theorem 3.3, we fix a subset $\mathcal{B}$ of cardinality L in each block of length L_1. We require that, in odd-labelled blocks, at least one particle of the fixed subset visits all the vertices of the fixed subset of the following block; on the other hand we require that, in even-labelled blocks, at least one particle of the fixed subset visits the origin. It is easy to see that, mimicking the proof of Theorem 3.5, the best choice for the subsets $\mathcal{B}$ in each block is the one which minimizes the summands. □

Lemma 6.1. *Let $\{\alpha_i\}_{i\in\mathbb{N}}$ and $\{k_i\}_{i\in\mathbb{N}}$ be such that $\alpha_i \in (-\infty, 1)$ and $k_i \geq 0$ for all $i \in \mathbb{N}$.*

(1)

$$\sum_{i\in\mathbb{N}} k_i\alpha_i < +\infty \Longleftarrow \prod_{i\in\mathbb{N}} (1-\alpha_i)^{k_i} > 0;$$

(2) moreover if $\alpha_i \in [0,1)$ and $k_i \geq 1$ eventually as $i \to \infty$ then

$$\sum_{i\in\mathbb{N}} k_i\alpha_i < +\infty \Longleftrightarrow \prod_{i\in\mathbb{N}} (1-\alpha_i)^{k_i} > 0;$$

(3) If $\alpha_i(j) \in [0, 1-\epsilon]$ (for some $\epsilon > 0$) and $k_i(j) \geq 1$ for all $i, j \in \mathbb{N}$ then

$$\sup_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} k_i(j)\alpha_i(j) < +\infty \Longleftrightarrow \inf_{j\in\mathbb{N}} \prod_{i\in\mathbb{N}} (1-\alpha_i(j))^{k_i(j)} > 0.$$

Proof. Clearly $\prod_{i\in\mathbb{N}}(1-\alpha_i)^{k_i} > 0$ if and only if $\sum_{i\in\mathbb{N}} k_i \log(1-\alpha_i) > -\infty$.

(1) Observe that $\log(1-x) \leq -x$ for all $x < 1$ hence

$$\sum_{i\in\mathbb{N}} k_i\alpha_i \leq -\sum_{i\in\mathbb{N}} k_i \log(1-\alpha_i) < \infty. \tag{6.12}$$

(2) In this case, since $k_i \geq 1$ both sides imply $\alpha_i \to 0$. Thus $\log(1-\alpha_i) \sim -\alpha_i$ and

$$\sum_{i\in\mathbb{N}} k_i \log(1-\alpha_i) > -\infty \Longleftrightarrow \sum_{i\in\mathbb{N}} k_i\alpha_i < \infty.$$

(3) If $\inf_{j\in\mathbb{N}} \prod_{i\in\mathbb{N}}(1-\alpha_i(j))^{k_i(j)} > 0$ then using the first inequality in equation (6.12) we obtain $\sup_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} k_i(j)\alpha_i(j) < +\infty$. Conversely, it suffices to note that there exists $\delta \in (0,1)$ such that $-\delta\alpha_i(j) \leq \log(1-\alpha_i)$.

□

Lemma 6.2. *Let $\{\alpha_i\}_i$ be a sequence of nonnegative numbers. Define $\bar{\alpha}_n := \min\{\alpha_i : i \leq n\}$; the following are equivalent:*

(1) there exists an increasing sequence $\{n_i\}$ such that $\sum_i (n_{i+1}-n_i)\alpha_{n_i} < +\infty$;

(2) it is possible to define recursively an infinite, increasing sequence $\{r_j\}_j$ by

$$\begin{cases} r_0 = 0 \\ r_{n+1} = \min\{i > r_n : \alpha_i \leq \alpha_{r_n}\} \end{cases} \tag{6.13}$$

and $\sum_i (r_{i+1}-r_i)\alpha_{r_i} < +\infty$;

(3) there exists an increasing sequence $\{\bar{n}_i\}$ such that $\{\alpha_{\bar{n}_i}\}$ is nonincreasing and $\sum_i \bar{\alpha}_i < +\infty$.

Moreover if $\alpha_i > 0$ for all $i \in \mathbb{N}$ then the previous assertions are equivalent to

(4) $\sum_i \bar{\alpha}_i < +\infty$.

Proof. (1) $\implies$ (2). If (1) the previous inequality hold then $\lim_i \varepsilon_i = 0^-$ hence it is possible to define recursively the sequence $\{r_n\}$ and clearly we have

$$\alpha_i \geq \alpha_{r_n}, \qquad \forall i < r_{n+1}. \tag{6.14}$$

We show now that for all increasing sequences $\{n_i\}_i$ we have

$$\sum_i (n_{i+1} - n_i)\alpha_{n_i} \geq \sum_i (r_{i+1} - r_i)\alpha_{r_i}$$

which implies easily (2). Indeed, note that if we define $\gamma_j = \alpha_{r_i}$ for all $j \in [r_i, r_{i+1})$ then

$$\sum_i (r_{i+1} - r_i)\alpha_{r_i} = \sum_j \gamma_j; \tag{6.15}$$

similarly if $\gamma'_j = \alpha_{n_i}$ for all $j \in [n_i, n_{i+1})$ then

$$\sum_i (n_{i+1} - n_i)\alpha_{n_i} = \sum_j \gamma'_j.$$

Let us fix $j \in \mathbb{N}$ and suppose that $j \in [r_i, r_{i+1}) \cap [n_l, n_{l+1})$, then $n_l < r_{i+1}$ whence equation (6.14) implies that

$$\gamma'_j = \alpha_{n_l} \geq \alpha_{r_i} = \gamma_j.$$

Thus $\gamma'_j \geq \gamma_j$ for all $j \in \mathbb{N}$.

(2) $\implies$ (1). It is straightforward.

(2) $\implies$ (3). Let us define $\bar{n}_i = r_i$ and let $\{\gamma_i\}$ as before. The sequence $\{\alpha_{\bar{n}_i}\}$ is clearly nonincreasing. Using equation (6.15), we just need to prove that $\gamma_n = \bar{\alpha}_n$ for all n. Indeed, if $n \in [\bar{n}_i, \bar{n}_{i+1})$ then

$$\gamma_n = \alpha_{r_i} = \alpha_{\bar{n}_i} \leq \alpha_j$$

for all $j < r_{i+1} = \bar{n}_{i+1}$. Hence, $\gamma_n = \alpha_{\bar{n}_i} = \min\{\alpha_j : j \leq n\} = \bar{\alpha}_n$.

(3) $\implies$ (1). It is straightforward.

(3) $\implies$ (4). Clearly if $\alpha_i > 0$ for all $i \in \mathbb{N}$ and $\sum_i \bar{\alpha}_i < +\infty$ then there exists an increasing sequence $\{\bar{n}_i\}$ such that $\{\alpha_{\bar{n}_i}\}$ is nonincreasing. □

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