BACKWARD STOCHASTIC RICCATI EQUATIONS AND INFINITE HORIZON L-Q OPTIMAL CONTROL WITH INFINITE DIMENSIONAL STATE SPACE AND RANDOM COEFFICIENTS

GIUSEPPINA GUATTERI AND GIANMARIO TESSITORE

Abstract. We study the Riccati equation (BSRE) arising in a class of quadratic optimal control problems with infinite dimensional stochastic differential state equation and infinite horizon cost functional. We allow the coefficients, both in the state equation and in the cost, to be random.

In such a context BSREs are backward stochastic differential equations in the whole positive real axis that involve quadratic non-linearities and take values in a non-Hilbertian space. We prove existence of a minimal non-negative solution and, under additional assumptions, its uniqueness. We show that such a solution allows to perform the synthesis of the optimal control and investigate its attractivity properties. Finally the case where the coefficients are stationary is addressed and an example concerning a controlled wave equation in random media is proposed.

Key words. Infinite horizon, backward stochastic Riccati equation, linear quadratic optimal control, stochastic coefficients

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1. Introduction

In this paper we will be concerned with a linear quadratic control problem with stochastic coefficients and infinite horizon cost. We consider a system governed by the following infinite dimensional *state equation* :

$$\begin{cases} dy(t) = (Ay(t) + A_\sharp(t)y(t) + B(t)u(t))\,dt + C(t)y(t)\,dW(t), \quad t \geq 0, \\ y(0) = x. \end{cases} \tag{1.1}$$

where the state y takes values in an Hilbert space H, the control u takes values in another Hilbert space U, the operator A is unbounded and all the other coefficients are allowed to be random. More precisely $A_\sharp$, B, C are assumed to be bounded adapted stochastic processes taking values in suitable spaces of bounded linear operators. Our aim is to minimize an infinite horizon cost functional as:

$$J_\infty(0,x,u) = \mathbb{E}\int_0^{+\infty} [\langle S(s)y(s), y(s)\rangle_H + |u(s)|^2]ds \tag{1.2}$$

where S is again a bounded adapted processes taking values in the cone $\Sigma^+(H)$ of bounded symmetric non negative operators in H.

Linear quadratic optimal control problems with stochastic coefficients and the corresponding backward stochastic Riccati equations (BSREs) were recently intensively studied in the finite-horizon and finite-dimensional case, see [2], [3], [10], [11], [12], [13], [16], [17]. On the other side linear quadratic optimal control problems with deterministic coefficients have been treated in the infinite dimensional case both with finite and infinite horizon case, see [6], [15], [21] covering a large class of optimal control problems for concrete stochastic partial differential equations with deterministic coefficients. Finally the infinite dimensional case with stochastic coefficients but with finite horizon was studied in [8]. We recall that, in this last situation, the Riccati equation is a backward stochastic differential equation (with final condition at finite time $T > 0$) that takes values in the space $L(H)$ of linear and bounded operators. Clearly this is not an Hilbert space as soon as H is infinite dimensional thus new difficulties are introduced in the picture since some essential tools in stochastic calculus can not be used. To cope with this difficulty in [8] we have proposed the notion of *generalized* solution to the BSRE that will be widely used in this paper. We notice that this definition characterizes only the process P. This is nevertheless enough to perform the synthesis of the optimal control.

As in the deterministic coefficients case, see for instance [21], a "finite cost" condition (or "stabilizability" condition) has to be required in order to ensure that the infinite horizon cost is not trivially equal to $+\infty$. When coefficients are allowed to be random the choice of the notion of stabilizability is not completely obvious. Indeed it turns out here that the right formulation is to impose that for all $t \in [0, +\infty)$ and all $x \in H$ there exists an adapted control $u \in L^2([t, +\infty) \times \Omega; U)$ such that

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} (|\sqrt{S}(s) y^{t,x,u}(s)|^2 + |u(s)|^2)\, ds < M_{t,x} \qquad \mathbb{P} - a.s. \tag{1.3}$$

where $M_{t,x}$ is a suitable constant that depends on t ad x.

Then one needs to give meaning to the following BSRE equation on the whole real axis and with values in $L(H)$

$$\begin{aligned} -dP(t) = {} & (A^* P(t) + P(t)A + A_\sharp^*(t)P(t) + P(t)A_\sharp(t) - P(t)B(t)B^*(t)P(t) + S(t))\, dt + \\ & \mathrm{Tr}[C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t)]\, dt + Q(t)\, dW(t), \qquad t \in [0, +\infty) \end{aligned} \tag{1.4}$$

For us a solution of equation (1.4) will be any adapted, strongly continuous, process P with values in $\Sigma^+(H)$ such that, for all $T > 0$, P restricted to $[0, T]$ is the unique generalized solution, in the sense of [8], to the BSRE with final condition, at time T, equal to $P(T)$ (that is: equal to the solution itself computed at time T). We notice that the above defined solution of BSRE is not necessarily bounded in time, differently from the usual requirement in the theory of infinite horizon backward stochastic equations, see [9] for the infinite dimensional case and [18], with the bibliography therein, for the finite dimensional case.

Once the above definitions are suitably given we can prove, by a monotonicity argument, that stabilizability is equivalent to the existence of a solution to the BSRE (1.4) in $[0, \infty[$. Moreover we show that, in the above case, a minimal non negative solution of (1.4) exists and that such minimal solution determines the optimal cost and allows to characterize the optimal feedback (now stochastic) of the problem. In other words we achieve the synthesis of the optimal control generalizing classical results holding in the deterministic coefficients case (see for instance [14] and [21] and references within) with the backward stochastic Riccati equation (1.4) that plays the role that was played in the quoted papers by the algebraic Riccati equation.

Then we address the problem of the uniqueness of the non negative solution of equation (1.4). We prove that if the minimal non-negative solution of equation (1.4) stabilizes the state equation with respect to the identity uniformly with respect to time (see definition 4.3) then it is also maximal and therefore it is the unique solution of (1.4) (at least among all bounded, adapted, $\Sigma^+(H)$ valued processes). The condition stated in definition 4.3 is a kind of observability requirement and always holds if $S(t) \geq \epsilon I$ for all $t > 0$ and P is bounded uniformly in time. Under the same assumption we show that the unique non negative solution of equation (1.4) enjoys attractivity properties among non-negative solutions of finite-horizon Riccati equations.

We also investigate special features of the stationary case. We choose the same framework as in [5] and [22] in which stationarity is expressed in terms of cocycle, measure preserving, transformations of the underlying probability space, see definition 5.1. We show that if all the coefficients are stationary then the minimal non-negative solution of the Riccati equation is stationary. This simplifies the formulation of many of the results holding in the general case.

Finally we show that our general results can be applied to the concrete case of the optimal, infinite horizon, linear-quadratic control of a wave equation in random media. We prove stabilizability using a simple, Lyapunov type, sufficient condition based on existence of a deterministic algebraic super-solution of equation (2.8) (see proposition 3.8).

The paper is organized as follows: in section 2 we state the problem and fix notations and general framework. We also recall previous results included the definition of generalized solution to the backward stochastic Riccati equation in finite horizon and infinite dimensions taken from [8]. In section 3 we state and prove the main results on existence of a solution to the BSRE (1.4) and on the synthesis of the optimal control for the infinite horizon problem. In section 4 we study uniqueness and attractivity properties of the solution to equation (1.4). Section 5 is devoted to the stationary case and section 6 to the example on wave equation in random media.

2. Notation, Assumptions and Preliminary results

By H, U and Ξ we will always indicate real separable Hilbert spaces.

If K is an Hilbert space its inner scalar product and norm will be denoted by $\langle\cdot,\cdot\rangle_K$ and $|\cdot|_K$ omitting the K when no confusion is possible.

For any Banach space E by $\mathcal{B}(E)$ we denote its Borel σ-field.

For any pair K_1 and K_2 of separable real Hilbert spaces we denote by $L(K_1,K_2)$ the Banach space of linear and bounded operators from K_1 to K_2 endowed by the norm $|T|_{L(K_1,K_2)} = \sup_{\{x\in K_1, |x|_{K_1}=1\}} |Tx|_{K_2}$ (as usual $L(H) = L(H,H)$).

By $\Sigma(H)$ we denote the subspace of all symmetric and bounded operators and by $\Sigma^+(H)$ the cone of $\Sigma(H)$ that contains all positive semidefinite operators.

$L_2(K,H)$ denotes the Hilbert space of Hilbert-Schmidt operators from K to H, endowed with the Hilbert-Schmidt norm $|T|^2_{L_2(K,H)} = \sum_{i=1}^{\infty} |Te_i|^2_H$ ($\{e_i : i \in \mathbb{N}\}$ being an orthonormal basis in K) and we set $L_2(H,H) = L_2(H)$. $\Sigma_2(H)$ is the subset of $L_2(H)$ that consists in all linear and symmetric operators and $\Sigma_2^+(H)$ is the cone of $\Sigma_2(H)$ that consists in all non-negative operators.

The cylindrical Wiener Process

We fix a probability basis $(\Omega,\mathcal{F},\mathbb{P})$. A cylindrical Wiener process with value in Ξ is a family $W(t)$, $t \geq 0$, of linear mappings $\Xi \to L^2(\Omega)$ such that:

i) for every $h \in \Xi$, $\{W(t)h,\ t\geq 0\}$ is a real (continuous) Wiener process;

ii) for every $h, k \in \Xi$ and $t, s \geq 0$, $\mathbb{E}(W(t)h \cdot W(s)k) = (t\wedge s)\langle h,k\rangle_\Xi$.

We denote by $\mathcal{F}_t$ its natural filtration augmented with the set $\mathcal{N}$ of $\mathbb{P}$-null sets of $\mathcal{F}$. As it is well known the filtration $\mathcal{F}_t$ satisfies the usual conditions. By $\mathbb{E}^{\mathcal{F}_t}$ we denote the conditional expectation with respect to $\mathcal{F}_t$.

Finally by $\mathcal{P}$ we denote the predictable σ-field on $\Omega \times [0,T]$.

Some classes of stochastic process

Let K be any separable Hilbert space and let $\mathcal{B}(K)$ be its Borel σ-field on K. The following classes of processes will be used in this work:

- $L^p_{\mathcal{P}}(\Omega\times[0,T];K)$, $p \in [1,+\infty]$ denotes the subset of $L^p(\Omega\times[0,T];K)$, given by all equivalence classes admitting a predictable version. This space is endowed with the natural norm
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega\times[0,T];K)} = \mathbb{E}\int_0^T |Y_s|^p_K\, ds$$
Elements of this space are defined up to modification.
- $L^p_{\mathcal{P}}(\Omega;L^2([0,T];K))$ denotes the space of equivalence classes of processes Y, admitting a predictable version such that the norm:
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega;L^2([0,T];K))} = \mathbb{E}\Big(\int_0^T |Y_s|^2_K\, ds\Big)^{p/2}$$
is finite. Elements of this space are defined up to modification.
- $C_{\mathcal{P}}([0,T];L^p(\Omega;K))$ denotes the space of K-valued processes Y such that $Y : [0,T] \to L^p(\Omega,K)$ is continuous and Y has a predictable modification, endowed with the norm:
$$|Y|^p_{C_{\mathcal{P}}([0,T];L^p(\Omega;K))} = \sup_{t\in[0,T]} \mathbb{E}|Y_t|^p_K$$
Elements of $C_{\mathcal{P}}([0,T];L^p(\Omega;K))$ are identified up to modification.
- $L^p_{\mathcal{P}}(\Omega;C([0,T];K))$ denotes the space of predictable processes Y with continuous paths in K, such that the norm
$$|Y|^p_{L^p_{\mathcal{P}}(\Omega;C([0,T];K))} = \mathbb{E}\sup_{t\in[0,T]} |Y_t|^p_K$$
is finite. Elements of this space are defined up to indistinguishability.

Now let us consider the space $L(H)$ of linear and bounded operators from a separable Hilbert space H to H. We introduce the σ-field:

$$\mathcal{L}_S = \sigma\Big\{\{T \in L(H) : Tu \in A\}, \text{ where } u \in H \text{ and } A \in \mathcal{B}(H)\Big\}$$

Following again [7] the elements of $\mathcal{L}_S$ are called *strongly measurable*.

We notice that the maps $P \to |P|_{L(H)}$ and $(P, u) \to Pu$ are measurable from $(L(H), \mathcal{L}_S)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and from $(L(H) \times H, \mathcal{L}_S \otimes \mathcal{B}(H))$ to $(H, \mathcal{B}(H))$ respectively.

Moreover $\mathcal{L}_S$ is identical to the weak σ-field:

$$\mathcal{L}_S = \sigma\Big\{\{T \in L(H) : \langle Tu, x\rangle_H \in A\}, \text{ where } u, x \in H \text{ and } A \in \mathcal{B}(\mathbb{R})\Big\}$$

We define the following spaces:

- $L^\infty_{\mathcal{P},S}([0,T] \times \Omega; L(H))$ the space of essentially bounded, strongly measurable predictable processes $Y : \Omega \times [0,T] \to L(H)$. That is Y is measurable from $(\Omega \times [0,T], \mathcal{P})$ to $(L(H), \mathcal{L}_S)$ and the real valued random valued $|Y|_{L(H)}$ is in $L^\infty(\Omega \times [0,T]; \mathbb{R})$. By $|Y|_{L^\infty_{\mathcal{P},S}(\Omega\times[0,T];L(H))}$ we indicate the norm of $|Y|_{L(H)}$ in $L^\infty(\Omega \times [0,T]; \mathbb{R})$. Elements of this space are identified up to modification. Similar definition (with obvious modifications) is given for $L^\infty_{\mathcal{P},S}([0,\infty) \times \Omega; L(H))$
- $L^\infty_S(\Omega, \mathcal{F}_t; L(H))$ is the space of measurable maps $Y : (\Omega, \mathcal{F}_t) \to (L(H), \mathcal{L}_S)$ such that $|Y|_{L(H)}$ is in $L^\infty(\Omega; \mathbb{R})$. By $|Y|_{L^\infty_S(\Omega;L(H))}$ we indicate the norm of $|Y|_{L(H)}$ in $L^\infty(\Omega; \mathbb{R})$.
- $L^1_{\mathcal{P},S}([0,T]; L^\infty(\Omega, L(H)))$ is the space of predictable, strongly measurable processes such that $|Y|_{L(H)}$ is in $L^1([0,T]; L^\infty(\Omega; \mathbb{R}))$. By $|Y|_{L^1_{\mathcal{P},S}([0,T];L^\infty(\Omega,L(H)))}$ we indicate the norm of $|Y|_{L(H)}$ in $L^1([0,T]; L^\infty(\Omega; \mathbb{R}))$. Elements of this space are identified up to modification. Identical definition is given for $L^1_{\mathcal{P},S}([0,\infty); L^\infty(\Omega, L(H)))$

Similarly (with obvious changes) we define: $L^\infty_{\mathcal{P},S}(\Omega \times [0,T]; \Sigma^+(H))$, $L^\infty_{\mathcal{P},S}(\Omega \times [0,T]; L(U,H)))$, $L^1_{\mathcal{P},S}([0,T]; L^\infty(\Omega, \Sigma^+(H)))$ and $L^\infty_S(\Omega, \mathcal{F}_t; \Sigma^+(H))$. Elements of these spaces are identified up to modification.

Statement of the problem and general assumptions on the coefficients

We consider the following infinite dimensional stochastic differential equation for $t > 0$:

$$\begin{cases} dy(s) = (Ay(s) + A_\sharp(s)y(s) + B(s)u(s))\, ds + C(s)y(s)\, dW(s) & s \geq t \\ y(t) = x \end{cases} \tag{2.1}$$

where y is an H valued process and representing the *state* of the system and is our unknown, u is the *control* and takes value in U and the initial data x is in H. To stress its dependence on u, t and x we will denote the mild solution of equation (2.1) by $y^{t,x,u}$ when needed.

Our purpose is to minimize with respect to u the cost functional,

$$J_\infty(0, x, u) = \mathbb{E} \int_0^{+\infty} [\langle S(s)y^{0,x,u}(s), y^{0,x,u}(s)\rangle_H + |u(s)|^2] ds \tag{2.2}$$

We also introduce the following random variables representing, for $t \in [0, +\infty)$ the stochastic value function of the problem:

$$J_\infty(t, x, u) = \mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} [\langle S(s)y^{t,x,u}(s), y^{t,x,u}(s)\rangle_H + |u(s)|^2] ds$$

We will work under the following general assumptions on A, $A\sharp$, B and C that will hold throughout the paper:

Hypothesis 2.1.

A1) *$A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0 semigroup $e^{tA} : H \to H$ with*

$$|e^{tA}|_{L(H)} \leq M_A e^{ta_A}$$

where $M_A \geq 1$ and $a_A \in \mathbb{R}$.

A2) *We assume that* $A_\sharp \in L^\infty_{\mathcal{P},S}((0,+\infty)\times\Omega; L(H))$. *We denote by* $M_{A\sharp}$ *a positive constant such that:* $|A_\sharp(t,\omega)|_{L(H)} < M_{A\sharp}$, $\mathbb{P}$*-a.s. and for a.e.* $t \in (0,+\infty)$.

Moreover we assume that $B \in L^\infty_{\mathcal{P},S}((0,+\infty)\times\Omega; L(U,H))$ *and we denote by* M_B *a positive constant such that:* $|B(t,\omega)|_{L(U,H)} < M_B$, $\mathbb{P}$*-a.s. and for a.e.* $t \in (0,+\infty)$.

A3) *We assume that* C *is of the form:* $C = \sum_{i=1}^\infty C_i(\cdot, f_i)_\Xi$, *where* $\{f_i : i \in \mathbb{N}\}$ *is an orthonormal basis in* Ξ. *Moreover we suppose that* $C_i \in L^\infty_{\mathcal{P},S}((0,+\infty)\times\Omega; L(H))$ *and*

$$\left(\sum_{i=1}^\infty |C_i(t,\omega)|^2_{L(H)}\right)^{1/2} < M_C, \quad \mathbb{P}-a.s.\ \textit{for a.e.}\ t \in (0,+\infty)$$

for a suitable positive constant M_C.

A4) $S \in L^1_{\mathcal{P},S}((0,+\infty); L^\infty(\Omega;\Sigma^+(H)))$.

We recall that under these hypotheses there exists a unique mild solution $y \in C_{\mathcal{P}}([0,T]; L^2(\Omega;H))$ of equation (2.1), see theorem 7.4 in [7].

Results on the finite horizon case

Next we recall some results obtained in [8] for the finite horizon case. We considered there the following finite horizon backward Riccati equation

$$\begin{cases} -dP(t) = (A^*P(t) + P(t)A + A^*_\sharp(t)P(t) + P(t)A_\sharp(t) - P(t)B(t)B^*(t)P(t) + S(t))\,dt + \\ \qquad \mathrm{Tr}[C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t)]\,dt + Q(t)\,dW(t), \qquad t \in [0,T] \\ \\ P(T) = P_T \end{cases} \tag{2.3}$$

Two notions of solutions, were introduced according to different hypotheses on the regularity of the data P_T and S.

Definition 2.2. Assume that $P_T \in L^2(\Omega,\mathcal{F}_T;\Sigma_2^+(H))$ and $S \in L^2_{\mathcal{P}}(\Omega\times[0,T];\Sigma_2^+(H)))$ and fix $T_0 \in [0,T]$. A mild solution for problem (2.3), considered in $[T_0,T]$ is a pair (P,Q) with $P \in L^2_{\mathcal{P}}(\Omega, C([T_0,T];\Sigma_2(H))) \cap L^\infty_{\mathcal{P},S}(\Omega; C([T_0,T];\Sigma^+(H)))$ and $Q \in L^2_{\mathcal{P}}(\Omega\times(T_0,T); L_2(\Xi;\Sigma_2(H)))$ such that for all $t \in [T_0,T]$

$$\begin{aligned} P(t) = &\int_t^T e^{(s-t)A^*}\mathrm{Tr}[C^*(s)P(s)C(s) + C^*(s)Q(s) + Q(s)C(s)]e^{(s-t)A}\,ds \\ &+ e^{(T-t)A^*}P_T e^{(T-t)A} + \int_t^T e^{(s-t)A^*}[S(s) + A^*_\sharp(s)P(s) + P(s)A_\sharp(s)]e^{(s-t)A}\,ds \\ &+ \int_t^T e^{(s-t)A^*}Q(s)e^{(s-t)A}dW(s) - \int_t^T e^{(s-t)A^*}P(s)B(s)B^*(s)P(s)e^{(s-t)A}\,ds \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.4}$$

In the general case, i.e. when $P_T \in L^\infty_S(\Omega,\mathcal{F}_T;\Sigma^+(H))$ and $S \in L^1_{\mathcal{P},S}([0,T]; L^\infty(\Omega;\Sigma^+(H)))$ we have the following:

Definition 2.3. A process $P \in L^\infty_{\mathcal{P},S}(\Omega\times[0,T];\Sigma^+(H))$, is a *generalized solution* of equation (2.3) if there exists a sequence (S^N, P^N, Q^N) where:

(i) $S^N \in L^1_{\mathcal{P},S}([0,T]; L^\infty(\Omega;\Sigma^+(H))) \cap L^2_{\mathcal{P}}(\Omega\times[0,T];\Sigma_2(H))$ and there exists a positive function $c \in L^1([0,T])$ such that $|S^N(t)|_{L(H)} \le c(t)$, for all $N \in \mathbb{N}$, $\mathbb{P}$-a.s. for a.e. $t \in [0,T]$.

(ii) the pair (P^N, Q^N) is a mild solution to the Riccati equation (2.3) in the space of Hilbert Schmidt operators, with forcing term S^N and final data $P^N_T = P^N(T)$. Namely (P^N, Q^N) is the unique mild solution of:

$$\begin{cases} -dP^N(t) = (A^*P^N(t) + P^N(t)A + \mathrm{Tr}[C^*(t)P^N(t)C(t) + C^*(t)Q^N(t) + Q^N(t)C(t)])\,dt \\ \\ +(A^*_\sharp(t)P^N(t) + P^N(t)A_\sharp(t) - P^N(t)B(t)B^*(t)P^N(t) + S^N(t))\,dt + Q^N(t)\,dW(t), \quad t \in [0,T], \\ \\ P^N(T) = P^N_T \end{cases}$$

such that:

(iii) for all $x \in H$:

$$S^N(t,\omega)x \to S(t,\omega)x \text{ in } H \quad \mathbb{P} \text{ a.s. for a.e. } t \in [0,T]$$

(iv) for every $t \in [0,T]$ and for all $x \in H$:

$$P^N(t,\omega)x \to P(t,\omega)x \quad \text{in } H \qquad \mathbb{P} \text{ a.s.}$$

In the next propositions we enumerate the results on equation (2.3) that, proved in [8], will be useful here.

Proposition 2.4. *Assume Hypotheses A1)-A4) and choose P_T in $L^\infty_S(\Omega, \mathcal{F}_T; \Sigma^+(H))$ then*

(1) *The generalized solution P of equation (2.3) exists and is unique (see* [8] *theorem 6.6).*

(2) *There exists a version of P such that for all $x \in H$, $P(t)x \in C_{\mathcal{P}}([0,T]; L^p(\Omega; H))$ and $P(t) \in L(H)$, $\mathbb{P}$-a.s. (see remark 6.2, lemma 6.5 and relation (6.9) in* [8]*).*

(3) *The following "fundamental relation" holds for every $0 \le t \le s \le T$, all $u \in L^2_{\mathcal{P}}(\Omega \times [t,T]; U)$ and all $x \in H$:*

$$\begin{aligned}\langle P(t)x,x\rangle_H &= \mathbb{E}^{\mathcal{F}_t}\langle P(s)y^{t,x,u}(s), y^{t,x,u}(s)\rangle_H + \mathbb{E}^{\mathcal{F}_t}\int_t^s (|\sqrt{S}(r)y^{t,x,u}(r)|^2_H + |u(r)|^2_H)\,dr \\ &\quad -\mathbb{E}^{\mathcal{F}_t}\int_t^s |u(r) + B^*P(r)y^{t,x,u}(r)|^2_H\,dr \qquad \mathbb{P}-a.s.\end{aligned} \tag{2.5}$$

(see lemma 6.4 in [8]*) .*

Then it is possible to perform the synthesis on the optimal control for the finite horizon problem. Setting

$$J^T(t,x,u) \doteq \inf_u \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \left(|\sqrt{S}(s)y(s)|^2_H + |u(s)|^2_H\right)\,ds + \langle P_T y(T), y(T)\rangle_H\right]$$

the main result of [8] reads:

remark 2.5. *Assume that hypotheses A1)-A4) hold true and that P_T in $L^\infty_S(\Omega, \mathcal{F}_T; \Sigma^+(H))$. The following holds for arbitrary $0 < t < T$ and $x \in H$:*

(1) *There exists a unique control $\overline{u} \in L^2_{\mathcal{P}}(\Omega \times [t,T]; U)$ such that:*

$$J^T(t,x,\overline{u}) = \inf_{u \in L^2_{\mathcal{P}}(\Omega\times[t,T];U)} J^T(t,x,u)$$

(2) *If $\overline{y}$ is the mild solution of the state equation corresponding to $\overline{u}$ (that is the optimal state), then $\overline{y}$ is the unique mild solution to the* closed loop *equation:*

$$\begin{cases} d\overline{y}(r) = [A\overline{y}(r) + A_\sharp(r)\overline{y}(r) - B(r)B^*(r)P(r)\overline{y}(r)]\,dr + C\overline{y}(r)\,dW(r) \quad r \in [t,T] \\ \overline{y}(t) = x \end{cases} \tag{2.6}$$

and the following feedback law holds $\mathbb{P}$-a.s. for almost every $s \in [t,T]$.

$$\overline{u}(s) = -B^*(s)P(s)\overline{y}(s). \tag{2.7}$$

(3) *The optimal cost is given by $J^T(t,x,\overline{u}) = \inf_{u \in L^2_{\mathcal{P}}(\Omega\times[t,T];U)} J^T(t,x,u) = \langle P(t)x,x\rangle_H$.*

Infinite horizon Riccati equation

Coming to the infinite horizon case, the main object of investigation of the present paper will be the following backward stochastic Riccati equation on $[0,+\infty)$:

$$\begin{aligned}-dP(t) = (&A^*P(t) + P(t)A + A^*_\sharp(t)P(t) + P(t)A_\sharp(t) - P(t)B(t)B^*(t)P(t) + S(t))\,dt + \\ &\mathrm{Tr}[C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t)]\,dt + Q(t)\,dW(t), \qquad t \in [0,+\infty).\end{aligned} \tag{2.8}$$

We notice that in the above equation the final condition has disappeared but we ask that the solution can be extended to the whole positive real half-axis.

Definition 2.6. We say that P is a solution to (2.8) if it is an $\{\mathcal{F}_t\}_{t\geq 0}$- adapted process that takes values in $\Sigma^+(H)$ such that for every fixed $T>0$, $P_{|[0,T]}$ is a generalized solution of the Riccati equation (2.3) in $[0,T]$ with final data $P_T=P(T)$.

3. Synthesis of the optimal control in the infinite horizon case

Definition 3.1. We say that $(A, A_\sharp, B, C)$ is stabilizable relatively to the observation $\sqrt{S}$ (or $\sqrt{S}$ stabilizable) if, for all $t\in[0,+\infty)$ and all $x\in H$ there exists a control $u\in L^2_{\mathcal{P}}([t,+\infty)\times\Omega;U)$ such that

$$\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty}(|\sqrt{S}(s)y^{t,x,u}(s)|^2+|u(s)|^2)\,ds<M_{t,x}\qquad \mathbb{P}-a.s. \tag{3.1}$$

for some positive constant $M_{t,x}$ that may depend on t and x.

We have the following:

Proposition 3.2. *Assume A1) − A4) and that $(A, A_\sharp, B, C)$ is $\sqrt{S}$ stabilizable, then there exists a solution of the Riccati equation in the sense of definition 2.6.*

Proof. We construct a candidate solution using a natural limit argument.

For every positive integer $N>0$ let P^N the generalized solution of the finite horizon Riccati equation (2.3) with final data $P(N)=0$. Each P^N can be extended to the whole $[0,+\infty)$ setting $P^N(t)=0$ for $t>N$.

First of all we notice for each $t>0$ fixed the sequence $P^N(t)$ is increasing. Indeed by theorem 2.5

$$\begin{aligned}\langle P^{N+1}(t)x,x\rangle_H &= \inf_{u\in L^2_{\mathcal{P}}([t,N+1]\times\Omega;U)}\mathbb{E}^{\mathcal{F}_t}\int_t^{N+1}(|\sqrt{S}(r)y^{t,x,u}(r)|^2_H+|u(r)|^2_U)\,dr\\ &\geq \inf_{u\in L^2_{\mathcal{P}}([t,N]\times\Omega;U)}\mathbb{E}^{\mathcal{F}_t}\int_t^{N}(|\sqrt{S}(r)y^{t,x,u}(r)|^2_H+|u(r)|^2_U)\,dr=\langle P^N(t)x,x\rangle_H\end{aligned}$$

Since H is separable and for all $t>0$, $P^N(t)\in L(H)$, $\mathbb{P}$-a.s. the above implies that for all $t>0$

$$\mathbb{P}\left\{\langle P^{N+1}(t)x,x\rangle_H\geq\langle P^N(t)x,x\rangle_H\quad\forall N\in\mathbb{N},\ \forall x\in H\right\}=1 \tag{3.2}$$

Moreover for each t let $\bar{u}$ be the 'stabilizing' control that exists thanks to definition 3.1 then again by theorem 2.5

$$\begin{aligned}\langle P^N(t)x,x\rangle=|\sqrt{P^N(t)}x|^2 &\leq \mathbb{E}^{\mathcal{F}_t}\int_t^N(|\sqrt{S}(r)y^{t,x,\bar{u}}(r)|^2+|\bar{u}(r)|^2)\,dr\\ &\leq \mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty}(|\sqrt{S}(r)y^{t,x,\bar{u}}(r)|^2+|\bar{u}(r)|^2)\,dr\leq M_{t,x},\quad \mathbb{P}-\text{a.s.}\end{aligned}$$

for a suitable constant $M_{t,x}$. If we consider the operator $\sqrt{P^N(t)}$ as a linear operator form H to $L^\infty(\Omega,\mathcal{F}_t,\mathbb{P},H)$ by the Banach-Steinhaus theorem there exists M_t such that

$$|\sqrt{P^N(t)}|_{L(H,L^\infty(\Omega,\mathcal{F}_t,\mathbb{P},H))}\leq M_t.$$

Again since $P^N(t)\in L(H)$ - $\mathbb{P}$-a.s. and H is separable the above implies that

$$\mathbb{P}\left\{\langle P^N(t)x,x\rangle_H\leq M_t|x|^2,\quad\forall N\in\mathbb{N},\forall x\in H\right\}=1 \tag{3.3}$$

or equivalently

$$\mathbb{P}\left\{|P^N(t)|_{L(H)}\leq M_t,\quad\forall N\in\mathbb{N}\right\}=1 \tag{3.4}$$

Define $\phi_N(t,x,y):=\langle P^N(t)x,y\rangle_H$. Thank to (3.4) we know that for all t and for all $x\in H$ the sequence $\{\phi_N(t,x,x):N\in\mathbb{N}\}$ is $\mathbb{P}$-a.s. increasing and bounded. Thus if we define

$$\phi(t,x,y)=\lim_{N\to\infty}\phi_N(t,x,y)=\frac{1}{2}\Big(\lim_{N\to\infty}\phi_N(t,x+y,x+y)-\lim_{N\to\infty}\phi_N(t,x,x)-\lim_{N\to\infty}\phi_N(t,y,y)\Big)$$

then for all t and for all $x,y\in H$ the limit holds $\mathbb{P}$-a.s..

Thanks to (3.4) fixed $t > 0$ there exists Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that on Ω_0 the maps $\{\phi_N(t, x, y) : N \in \mathbb{N}\}$ are continuous in x and y uniformly in N. Consequently

$$\mathbb{P}\{\phi(t, x, y) = \lim_{N\to\infty} \phi_N(t, x, y), \quad \forall x, y \in H\} = 1.$$

Therefore, by the Riesz Representation theorem, there exists an operator $\overline{P}(t)$ linear, bounded, symmetric and non-negative such that with probability 1

$$\phi(t, x, y) = \langle \overline{P}(t)x, y\rangle_H \qquad \forall x, y \in H.$$

and the above relations can be rewritten as:

$$\mathbb{P}\left\{\lim_{N\to+\infty} \langle P^N(t)x, y\rangle = \langle \overline{P}(t)x, y\rangle,\ \forall x, y \in H\right\} = 1$$

and consequently

$$\mathbb{P}\left\{\lim_{N\to+\infty} P^N(t)x = \overline{P}(t)x,\ \forall x \in H\right\} = 1 \tag{3.5}$$

We finally notice that by construction $\overline{P}x$ is, for all $x \in H$, predictable.

We claim that for each $T \geq 0$ there exists a positive constant C_T such that:

$$|\overline{P}(t)|_{L(H)} \leq C_T \qquad \forall\, 0 \leq t \leq T, \mathbb{P} - \text{a.s.}.$$

Indeed, if we fix $N > T$, by theorem 2.5 we have:

$$\langle P^N(t)x, x\rangle \leq \mathbb{E}^{\mathcal{F}_t} \langle P^N(T)y^{t,x,0}(T), y^{t,x,0}(T)\rangle + \mathbb{E}^{\mathcal{F}_t} \int_t^T |\sqrt{S}(r)y^{t,x,0}(r)|^2\, dr. \tag{3.6}$$

where $y^{t,x,0}$ is the solution of equation (2.1) corresponding to a control $u \equiv 0$.

Standard estimates and the Gronwall lemma ensure existence of a positive constant K_T such that $\sup_{r\in[t,T]} \mathbb{E}^{\mathcal{F}_t} |y^{t,x,0}(r)|^2 \leq K_T|x|^2$ therefore, thanks also to (3.4), for all $t \in [0, T]$, $x \in H$ and $N \in \mathbb{N}$ one has:

$$\langle P^N(t)x, x\rangle \leq K_T(M_T + |S|_{L^1_{\mathcal{P},S}((0,+\infty);L^\infty(\Omega;\Sigma^+(H)))})|x|^2 := C_T|x|^2 \qquad \mathbb{P} - a.s.$$

Moreover, since $P^N(t) \in L(H)$, $\mathbb{P}$-a.s. we deduce that for all $t \in [0, T]$

$$\mathbb{P}\left\{\langle P^N(t)x, x\rangle \leq C_T|x|^2, \quad \forall N \in \mathbb{N}, \forall x \in H\right\} = 1 \tag{3.7}$$

and by construction

$$\mathbb{P}\left\{\langle \overline{P}(t)x, x\rangle \leq C_T|x|^2, \quad \forall x \in H\right\} = 1 \tag{3.8}$$

We are now in the position to conclude the proof. Let us fix $0 \leq t \leq T$ and chose $N > T$ then, by (2.5), we have for $N > T$:

$$\begin{aligned}\langle P^N(t)x, x\rangle_H \quad &= \quad \mathbb{E}^{\mathcal{F}_t} \langle P^N(T)y^{t,x,u}(T), y^{t,x,u}(T)\rangle_H + \mathbb{E}^{\mathcal{F}_t} \int_t^T (|\sqrt{S}(s)y^{t,x,u}(s)|^2_H + |u(s)|^2_H)\, ds \\ &\quad -\mathbb{E}^{\mathcal{F}_t} \int_t^T |u(s) + B^*P^N(s)y^{t,x,u}(s)|^2_H\, ds \qquad \mathbb{P} - a.s.\end{aligned} \tag{3.9}$$

Thus, thanks to (3.5) and to the uniform bounds (3.7) and (3.8) we can apply the dominated convergence theorem to get that:

$$\begin{aligned}\langle \overline{P}(t)x, x\rangle_H &= \mathbb{E}^{\mathcal{F}_t} \langle \overline{P}(T)y^{t,x,u}(T), y^{t,x,u}(T)\rangle_H + \mathbb{E}^{\mathcal{F}_t} \int_t^T (|\sqrt{S}(s)y^{t,x,u}(s)|^2_H + |u(s)|^2_H)\, ds \\ &\quad - \mathbb{E}^{\mathcal{F}_t} \int_t^T |u(s) + B^*\overline{P}(s)y^{t,x,u}(s)|^2_H\, ds \qquad \mathbb{P} - a.s.\end{aligned} \tag{3.10}$$

Thus if we define

$$\widehat{J}^T(t, x, u) \doteq \mathbb{E}^{\mathcal{F}_t} \left[\int_t^T \left(|\sqrt{S}(s)y(s)|^2_H + |u(s)|^2_H\right)\, ds + \langle \overline{P}(T)y(T), y(T)\rangle_H\right]$$

Relation (3.10) yields (choosing $u = -B^*\overline{P}y$ as in the proof of theorem 6.6 in [8])

$$\langle \overline{P}(t)x, x\rangle_H = \inf_{u\in L^2_{\mathcal{P}}([t,T]\times\Omega;U)} \widehat{J}^T(t,x,u).$$

Consequently by theorem 2.5 $\overline{P}$ is the generalized solution to equation (2.3) with final condition $\overline{P}(T)$ at time T so we can conclude that $\overline{P}$ is the solution to equation (2.8) in the sense of definition 2.6. □

As a by-product of the above construction we have the following corollary:

Corollary 3.3. *$\overline{P}$ is the minimal solution of equation* (2.8)*, in the sense that if there exists another solution $P^\natural$, then $P^\natural(t) \geq \overline{P}(t) \quad \mathbb{P}-a.s.$ for every $t \geq 0$.*

Proof. By (2.5) for $N > t$ we have that:

$$\begin{aligned}
&\langle P^\natural(t)x, x\rangle \\
&= \inf_{u\in L^2_{\mathcal{P}}([t,N]\times\Omega);U)} [\mathbb{E}^{\mathcal{F}_t}\langle P^\natural(N)y^{t,x,u}(N), y^{t,x,u}(N)\rangle_H + \mathbb{E}^{\mathcal{F}_t}\int_t^N (|\sqrt{S}(r)y^{t,x,u}(r)|^2 + |u(r)|^2)\,dr] \\
&\geq \inf_{u\in L^2_{\mathcal{P}}([t,N]\times\Omega);U)} \mathbb{E}^{\mathcal{F}_t}\int_t^N (|\sqrt{S}(r)y^{t,x,u}(r)|^2 + |u(r)|^2)\,dr = \langle P^N(t)x, x\rangle_H,
\end{aligned}$$

thus $\langle P^\natural(t)x, x\rangle \geq \langle P^N(t)x, x\rangle_H$, $\mathbb{P}$-a.s. for every $x \in H$ by (3.5). □

We are in the position to solve the optimal control problem with infinite horizon:

remark 3.4. *Assume $A1) - -A4)$ and that $(A, A_\sharp, B, C)$ is stabilizable relatively to S. Fix $x \in H$, then the following holds:*

1. *There exists a unique control $\overline{u} \in L^2_{\mathcal{P}}(\Omega\times[0,+\infty);U)$ such that:*
$$J_\infty(0,x,\overline{u}) = \inf_{u\in L^2_{\mathcal{P}}(\Omega\times[0,+\infty);U)} J_\infty(0,x,u)$$
2. *If $\overline{y}$ is the mild solution of the state equation corresponding to $\overline{u}$ (that is the optimal state), then $\overline{y}$ is the unique mild solution to the* closed loop *equation:*
$$\begin{cases} d\overline{y}(r) = [A\overline{y}(r) + A_\sharp(r)\overline{y}(r) - B(r)B^*(r)\overline{P}(r)\overline{y}(r)]\,dr + C\overline{y}(r)\,dW(r), \quad t > 0 \\ \overline{y}(0) = x \end{cases} \tag{3.11}$$
and the following feedback law holds $\mathbb{P}$-a.s. for almost every s.
$$\overline{u}(s) = -B^*(s)\overline{P}(s)\overline{y}(s). \tag{3.12}$$
3. *The optimal cost is given by $J_\infty(0,x,\overline{u}) = \langle \overline{P}(0)x, x\rangle_H$.*

Proof. Let us consider the following family of closed loop equations

$$\begin{cases} dy^N(r) = [Ay^N(r) + A_\sharp(r)y^N(r) - B(r)B^*(r)P^N(r)y^N(r)]\,dr + Cy^N(r)\,dW(r), \quad t \in [0,N] \\ y^N(0) = x \end{cases} \tag{3.13}$$

that for any N admits a unique mild solution $y^N \in L^p_{\mathcal{P}}(\Omega; C([0,N];H))$ for every $p \geq 2$ and $u^N(s) = -B^*P^N(s)y^N(s)$. Fix a positive T, then from the construction of $\overline{P}$ and from (2.5), we have that for every $N > T$:

$$\begin{aligned}
\langle \overline{P}(0)x, x\rangle_H \geq \langle P^N(0)x, x\rangle_H &= \mathbb{E}\langle P^N(T)y^N(T), y^N(T)\rangle_H + \mathbb{E}\int_0^T (|\sqrt{S}(r)y^N(r)|^2 + |u^N(r)|^2)\,dr \\
&\geq \mathbb{E}\int_0^T (|\sqrt{S}(r)y^N(r)|^2 + |u^N(r)|^2)\,dr
\end{aligned} \tag{3.14}$$

Let $\bar{y}$ the mild solution to (3.11) considered in $[0,T]$, then the following identity holds

$$\begin{aligned}
y^N(t) - \bar{y}(t) = &\int_0^t e^{A(t-s)}A_\sharp(s)(y^N(s) - \bar{y}(s))\,ds \\
&- \int_0^t e^{A(t-s)}B(s)B^*(s)(P^N(s)y^N(s) - \overline{P}(s)\bar{y}(s))\,ds + \int_0^t e^{A(t-s)}C(y^N(s) - \bar{y}(s))\,dW(s)
\end{aligned}$$

By standard estimates, we get that:

$$\sup_{0\le t\le T}\mathbb{E}|y^N(t)-\bar y(t)|^p \le c_p M_A^p(M_{A_\sharp}^p+M_B^{2p})\mathbb{E}\int_0^T |[P^N(s)-\overline{P}(s)]\bar y(s)|_H^p\,ds$$
$$+c_p M_A^p(M_{A_\sharp}^p+M_B^{2p})\int_0^T |P^N(s)|^p \sup_{0\le r\le s}\mathbb{E}|y^N(r)-\bar y(r)|^p\,ds$$
$$+c_p M_A^p\Big(\sum_{i=1}^{+\infty}|C_i|^2_{L(H)}\Big)^{p/2}\int_0^T \sup_{0\le r\le s}\mathbb{E}|y^N(r)-\bar y(r)|^p\,ds$$

Thus applying the Gronwall lemma to the function $v(s)=\sup_{0\le r\le s}\mathbb{E}|y^N(r)-\bar y(r)|^p$ we obtain:

$$\sup_{0\le t\le T}\mathbb{E}|y^N(t)-\bar y(t)|^p\le C\mathbb{E}\int_0^T |[P^N(s)-\overline{P}(s)]\bar y(s)|_H^p\,ds$$

where C is a positive constant depending only on constants introduced in hypotheses 2.1 and 3.1. Now letting N tend to $+\infty$ we get that, for every fixed $T>0$:

$$\lim_{N\to+\infty}\sup_{0\le t\le T}\mathbb{E}|y^N(t)-\bar y(t)|^p=0 \tag{3.15}$$

Moreover for every $s\in[0,T]$:

$$u_N(s)-\bar u(s)=-B^*P^N(s)[y^N(s)-\bar y(s)]-B^*[P^N(s)-\overline{P}(s)]\bar y(s)\quad \mathbb{P}-a.s.$$

Thus by the dominated convergence theorem we get, thanks to (3.7), (3.8) and (3.15):

$$\lim_{N\to+\infty}E\int_0^T |u_N(s)-\bar u(s)|^p\,ds=0 \tag{3.16}$$

Thanks to (3.15) and (3.16), we can pass to the limit in (3.14) to get:

$$\langle\overline{P}(0)x,x\rangle_H\ge\mathbb{E}\int_0^T(|\sqrt{S}(r)\bar y(r)|^2+|\bar u(r)|^2)\,dr$$

Now letting $T\to+\infty$ we get:

$$\langle\overline{P}(0)x,x\rangle_H\ge\mathbb{E}\int_0^{+\infty}(|\sqrt{S}(r)\bar y(r)|^2+|\bar u(r)|^2)\,dr=J_\infty(0,x,\bar u)$$

On the other hand, since P^N is a generalized solution to (2.3) in $[0,N]$ with $P^N(N)=0$, we get by theorem 2.5 that for every control $u\in L^2_{\mathcal{P}}((0,N)\times\Omega;U)$ and every $x\in H$:

$$\langle P^N(0)x,x\rangle_H\le\mathbb{E}\int_0^N(|\sqrt{S}(r)y^{t,x,u}(r)|^2+|u(r)|^2)\,dr,$$

in particular for every $u\in L^2_{\mathcal{P}}((0,+\infty)\times\Omega;U)$ we get

$$\langle P^N(0)x,x\rangle_H\le\mathbb{E}\int_0^{+\infty}(|\sqrt{S}(r)y^{t,x,u}(r)|^2+|u(r)|^2)\,dr=J_\infty(0,x,u) \tag{3.17}$$

This concludes the proof of the theorem. □

Remark 3.5. Arguing exactly as in the proof of the previous theorem we infer that for each $t\ge0$ and every $x\in H$ the following identity holds:

$$\begin{aligned}\langle\overline{P}(t)x,x\rangle_H &= \inf_{u\in L^2_{\mathcal{P}}([t,+\infty)\times\Omega);U)}\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty}(|\sqrt{S}(r)y^{t,x,u}(r)|^2+|u(r)|^2)\,dr\\ &= \inf_{u\in L^2_{\mathcal{P}}([t,+\infty)\times\Omega);U)} J(t,x,u)\qquad \mathbb{P}-\text{a.s.}.\end{aligned}$$

In the next result we show that the stabilizability of $(A,A_\sharp,B,C)$ is also a necessary condition for the existence of a solution of equation (2.8).

Corollary 3.6. *Assume* A1) $--$A4). *The following two assertions are equivalent:*

(i) $(A,A_\sharp,B,C)$ *is stabilizable relatively to* $\sqrt{S}$;

(ii) *there exists a solution of* (2.8) *in the sense of definition 2.6.*

Proof. It lasts to prove (ii)$\Rightarrow$ (i). Let P^* be a solution to (2.8). Then arguing as in corollary 3.3 we obtain $P^*(t) \geq P^N(t)$. Thus (3.3) holds and, proceeding as in the proof of proposition 3.2, we obtain the existence of a minimal solution $\overline{P}$. Hence we can find an optimal control $\bar{u} = -B^*\overline{P}(t)\bar{y}(t)$ (where $\bar{y}$ is the solution to (3.11)) that stabilize the coefficients $(A, A_\sharp, B, C)$. Indeed from (3.10) we have that for every $t \in [0, +\infty)$ and every $x \in H$,

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} (|\sqrt{S}\bar{y}^{t,x}(s)|^2 + |-B^*\overline{P}(s)\bar{y}^{t,x}(s)|^2)\, dt \leq \langle \overline{P}(t)x, x\rangle < +\infty, \qquad \mathbb{P}-a.s..$$

□

Remark 3.7. Notice that as in the deterministic case, if the $(A, A_\sharp, B, C)$ is stabilizable relatively to $\sqrt{S}$, then it is stabilizable by a (stochastic) linear feedback.

We end this section with a standard sufficient condition for stabilizability that will be used in the example, see Section 6.

Proposition 3.8. *Assume that there exists $X \in \Sigma^+(H)$ such that $\forall t \geq 0$, $\forall x \in D(A)$ and for some $c > 0$*

$$\left\{\ 2\langle Ax, Xx\rangle + 2\langle A_\sharp(t)x, Xx\rangle - |B^*(t)Xx|^2 + \sum_{i=1}^\infty \langle C_i(t)x, XC_i(t)x\rangle \leq -c\langle S(t)x, x\rangle, \quad \mathbb{P}\text{ -a.s.,} \right. \tag{3.18}$$

then $(A, A_\sharp, B, C)$ is $\sqrt{S}$-stabilizable.

Proof. Let us fix $t \geq 0$ and $x \in H$ and consider the following equation associated to X:

$$\begin{cases} dy(s) = (Ay(s) + A_\sharp(s)y(s) - B(s)B^*(s)Xy(s))\, ds + C(s)y(s)\, dW(s) \quad s \geq t \\ y(t) = x \end{cases} \tag{3.19}$$

This equation has a unique mild solution $y \in L^p_{\mathcal{P}}(\Omega; C([t, +\infty); H))$ for any $p \geq 2$, see for instance theorem 7.4 of [7]. In order to apply condition (3.18) we introduce the operators $J_n = n(nI - A)^{-1}$ for every $n > a_A$ and consider $y^n(s) : J^n y(s)$ that has the following representation

$$\begin{aligned} y^n(s) = e^{(s-t)A}J^n x + \int_t^s e^{(s-r)A}J^n A_\sharp(r)y(r)\, dr - \int_t^s e^{(s-r)A}J^n B(r)B^*(r)Xy(r)\, dr \\ + \int_t^s e^{(s-r)A}J^n C(r)y(r)\, dW(r) \end{aligned}$$

Since $y^n(t) \in D(A)$ we can apply the Itô formula to $\langle Xy^n(t), y^n(t)\rangle$ to get the following identities $\mathbb{P}$- a.s.:

$$\begin{aligned} &\mathbb{E}^{\mathcal{F}_t}\langle Xy^n(T), y^n(T)\rangle - \langle XJ^n x, J^n x\rangle \\ &= \mathbb{E}^{\mathcal{F}_t}\int_t^T [\langle XA^*y^n(s), y^n(s)\rangle + \langle Xy^n(s), Ay^n(s)\rangle]\, ds \\ &\quad + \mathbb{E}^{\mathcal{F}_t}\int_t^T [\langle XJ^n A_\sharp(s)y(s), y^n(s)\rangle + \langle Xy^n(s), J^n A_\sharp(s)y(s)\rangle]\, ds \\ &\quad - \mathbb{E}^{\mathcal{F}_t}\int_t^T [\langle XJ^n B(s)B^*(s)Xy(s), y^n(s)\rangle + \langle Xy^n(s), J^n B(s)B^*(s)Xy(s)\rangle]\, ds \\ &\quad + \mathbb{E}^{\mathcal{F}_t}\int_t^T \langle Tr[(J^n C(s))^* XJ^n C(s)]y(s), y^n(s)\rangle\, ds \\ &= \mathbb{E}^{\mathcal{F}_t}\int_t^T [\langle (XA + A^*X)y^n(s), y^n(s)\rangle + \langle (XA_\sharp(s) + A_\sharp^*(s)X)y^n(s), y^n(s)\rangle\rangle]\, ds \\ &\quad - 2\mathbb{E}^{\mathcal{F}_t}\int_t^T \langle XB(s)B^*(s)Xy^n(s), y^n(s)\rangle]\, ds + \mathbb{E}^{\mathcal{F}_t}\int_t^T \langle Tr[C(s)^*XC(s)]y^n(s), y^n(s)\rangle\, ds + M_n \end{aligned}$$

where

$$M_n = \mathbb{E}^{\mathcal{F}_t} \int_t^T \langle XJ^n A_\sharp(s)y(s) - XA_\sharp(s)y^n(s)), y^n(s)\rangle\, ds$$
$$+ \mathbb{E}^{\mathcal{F}_t} \int_t^T \langle XJ^n A_\sharp(s)y(s) - XA_\sharp(s)y^n(s)), y^n(s)\rangle\, ds$$
$$+ 2\mathbb{E}^{\mathcal{F}_t} \int_t^T \langle XB(s)B^*(s)Xy^n(s) - XJ^n B(s)B^*(s)Xy(s), y^n(s)\rangle\, ds$$
$$+ \mathbb{E}^{\mathcal{F}_t} \int_t^T \langle Tr[(J^n C(s))^* XJ^n C(s)]y(s) - Tr[C(s)^* XC(s)]y^n(s), y^n(s)\rangle\, ds$$

Taking into account (3.18) we obtain:

$$\mathbb{E}^{\mathcal{F}_t}\langle Xy^n(T), y^n(T)\rangle - \langle XJ^n x, J^n x\rangle$$
$$\leq -c\,\mathbb{E}^{\mathcal{F}_t} \int_t^T \langle S(s)y^n(s), y^n(s)\rangle - \mathbb{E}^{\mathcal{F}_t} \int_t^T |B^*(s)Xy^n(s)|^2\, ds + \mathbb{E}^{\mathcal{F}_t} M_n \qquad \mathbb{P}-\text{a.s.}$$

By standard estimates since $|J_n|$ is uniformly bounded and $J_n x \to x$ as n tends to $+\infty$, we get $\mathbb{E}^{\mathcal{F}_t} M_n \to 0$, $\mathbb{P}-$a.s. as $n \to +\infty$. Thus letting n tend to $+\infty$:

$$\mathbb{E}^{\mathcal{F}_t}\langle Xy(T), y(T)\rangle - \langle Xx, x\rangle \leq -c\,\mathbb{E}^{\mathcal{F}_t} \int_t^T \langle S(s)y(s), y(s)\rangle - \mathbb{E}^{\mathcal{F}_t} \int_t^T |B^*(s)Xy(s)|^2\, ds \qquad \mathbb{P}-\text{a.s.} \tag{3.20}$$

Notice that equation (3.19) corresponds to equation (2.1) with $u(s) = -B(s)Xy(s)$. The claim is proved.

Remark 3.9. Notice that when condition (3.18) applies, the constant $M_{t,x}$ that appears in definition 3.1 can be chosen to be independent of t.

4. Attractivity and Maximality properties of the solution of the Riccati equation

In the whole section we will assume $A1) - -A4)$ and that $(A, A_\sharp, B, C)$ is stabilizable relatively to $\sqrt{S}$. We introduce the following definitions:

Definition 4.1. We say that a solution P of equation (2.8) is bounded, if there exists a constant $M > 0$ such that for every $t \geq 0$

$$|P(t)|_{L(H)} \leq M \qquad \mathbb{P}-a.s.$$

Remark 4.2. Whenever the constant $M_{t,x}$ that appears in definition 3.1 can be chosen independently of t, for instance when condition (3.18) is verified, then the minimal solution $\overline{P}$ is automatically bounded, see remark 3.5.

Moreover we introduce the following definition:

Definition 4.3. Let P be a solution to (2.8). We say that P stabilize $(A, A_\sharp, B, C)$ relatively to I uniformly in time if for every $t > 0$ and $x \in H$ there exists a positive constant M, independent of t, such that

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} |y^{t,x}(r)|_H^2\, dr \leq M \qquad \mathbb{P}-a.s., \tag{4.1}$$

where $y^{t,x}$ is the mild solution to:

$$\begin{cases} dy^{t,x}(s) = [A - B(s)B^*(s)P(s)]y^{t,x}(s)\, ds + C(s)y^{t,x}(s)\, dW(s), \quad s \geq t \\ y^{t,x}(t) = x \end{cases} \tag{4.2}$$

We can prove the following result

Proposition 4.4. *Let P_1 and P_2 be two solutions of equation (2.8) that are bounded. If P_1 stabilizes $(A, A_\sharp, B, C)$ relatively to I uniformly in time then for every $t \geq 0$:*

$$P_1(t) \geq P_2(t), \quad \mathbb{P}-a.s.$$

Proof. Let us set $Z(t) = P_1(t) - P_2(t)$, since both P_1 and P_2 are generalized solutions in any $[0,T]$, from (2.5) we get for every $t \in [0,T], x \in H$:

$$\langle Z(t)x, x\rangle = \mathbb{E}^{\mathcal{F}_t}\langle Z(T)y_1^{t,x}(T), y_1^{t,x}(T)\rangle + \mathbb{E}^{\mathcal{F}_t}\int_t^T |B^* P_2(s)y_1^{t,x}(s) - B^* P_1(s)y_1^{t,x}(s)|^2\, ds \quad \mathbb{P}\text{- a.s.},$$

where $y_1^{t,x}$ is the solution to (4.2) with $P = P_1$. Thus

$$\langle Z(t)x, x\rangle \geq \mathbb{E}^{\mathcal{F}_t}\langle Z(T)y_1^{t,x}(T), y_1^{t,x}(T)\rangle \qquad \mathbb{P}\text{- a.s. and } \forall x \in H$$

Thanks to (4.1) one can find a sequence $\{T_n\}_{n\in\mathbb{N}}$, with $\lim_{n\to+\infty} T_n = \infty$ such that

$$\liminf_{n\to+\infty} \mathbb{E}^{\mathcal{F}_t}|y_1^{t,x}(T_n)|^2 = 0$$

Since Z is bounded

$$\langle Z(t)x, x\rangle \geq \liminf_{n\to+\infty} \mathbb{E}^{\mathcal{F}_t}\langle Z(T_n)y_1^{t,x}(T_n), y_1^{t,x}(T_n)\rangle = 0$$

□

From corollary 3.3 and the last proposition we deduce the following:

Corollary 4.5. *If $\overline{P}$ stabilize the coefficients $(A, A_\sharp, B, C)$ with respect to I uniformly in time and it is bounded, then $\overline{P}$ is the unique bounded solution of equation* (2.8).

Now we study the attractivity property of the minimal solution $\overline{P}$. To this purpose we introduce the following notion. Let us define a two parameter linear semigroup, $T_P(u,t) : L^2(\Omega, \mathcal{F}_t, \mathbb{P}; H) \to L^2(\Omega, \mathcal{F}_u, \mathbb{P}; H)$, $0 \leq t \leq u < +\infty$, $T_P(t,u)\eta = y^{t,\eta}(u)$, where $y^{t,\eta}(u)$ is the mild solution of equation (4.2) with initial condition η. Condition (4.1) is equivalent to the existence of a constant $M > 0$ such that for every $t \geq 0$:

$$\mathbb{P}\Big(\mathbb{E}^{\mathcal{F}_t}\int_t^{+\infty} |T_P(r,t)x|_H^2\, dr \leq M|x|^2,\ \forall x \in H\Big) = 1 \tag{4.3}$$

Indeed it is possible to consider (4.2) as a state equation with $B = 0$. Thus (4.3) is the analogous of (3.8). Notice that since the constant M in (4.1) does not depend on t the same holds for the constant M in (4.3).

We can prove that a version of the stochastic Datko theorem, see [4] and [15], holds in our situation:

Proposition 4.6. *Let us assume that a solution P of equation* (2.8) *stabilizes the coefficients $(A, A_\sharp, B, C)$ relatively to I uniformly in time and is bounded. Then there exist $C > 0$ and $a > 0$ such that, for every $x \in H$ and for every $t \geq 0$,*

$$\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2 \leq Ce^{-a(u-t)}|x|^2, \qquad \mathbb{P} - a.s. \text{ and } \forall u \geq t. \tag{4.4}$$

Proof. First of all we notice that applying the Gronwall lemma we find that for every $t \geq 0$ and every η in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}; H)$ there exists a constant $C > 0$ (independent of t and of η) such that,

$$\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)\eta|^2 \leq Ce^{C(u-t)}|\eta|^2 \qquad \forall u \geq t \qquad \mathbb{P} - a.s. \tag{4.5}$$

Thus by (4.3) we have, for every $x \in H$ and every $t \geq 0$:

$$\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2 \int_t^u C^{-1}e^{-C(u-r)}\, dr = \int_t^u C^{-1}e^{-C(u-r)}\mathbb{E}^{\mathcal{F}_t}\mathbb{E}^{\mathcal{F}_r}|T_P(u,r)T_P(r,t)x|^2\, dr$$

$$\leq \int_t^u C^{-1}e^{-C(u-r)}Ce^{C(u-r)}\mathbb{E}^{\mathcal{F}_t}|T_P(r,t)x|^2\, dr = \int_t^u \mathbb{E}^{\mathcal{F}_t}|T_P(r,t)x|^2\, dr \leq M|x|^2, \mathbb{P} - a.s. \text{ and } \forall u \geq t.$$

Note that the exceptional set may depend on r, unless we choose the continuous version of T_P. Now fix $L > 0$ such that:

$$\int_0^L C^{-1}e^{-Cs}\, ds = K$$

thus for every $x \in H$ and every $t \geq 0$:

$$\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2 \leq \frac{M}{K}|x|^2 \qquad \mathbb{P} - a.s. \text{ and } \forall u \geq t \geq 0 : u - t \geq L. \tag{4.6}$$

Combining (4.6) and (4.5) for every $x \in H$ and every $t \geq 0$ we find an $R > 0$ (independent of t and of x) such that:

$$\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2 \leq R|x|^2 \qquad \forall u \geq t \geq 0 \tag{4.7}$$

Therefore:

$$(u-t)\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2 = \int_t^u \mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2\,dr = \int_t^u \mathbb{E}^{\mathcal{F}_t}\mathbb{E}^{\mathcal{F}_r}|T_P(u,r)T_P(r,t)x|^2\,dr \tag{4.8}$$
$$\leq R\int_t^{+\infty} \mathbb{E}^{\mathcal{F}_t}|T_P(r,t)x|^2\,dr \leq R\,M\,|x|^2$$

Thus for any $r \in]0,1[$ there exists $L = L(r)$ such that, for every $t \geq 0$ and $x \in H$:

$$\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2 \leq \frac{RM}{(u-t)}|x|^2 \leq r|x|^2 \qquad \mathbb{P}-a.s. \text{ and } \forall u \geq t \geq 0 : u-t \geq L \tag{4.9}$$

Finally for any $u - t > L$ let n be an integer that verify $nL \leq u - t < (n+1)L$. Therefore one can find a partition $u = t_{n+1} \geq t_n > \cdots > t_1 \geq t_0 = t$ such that $t_{i+1} - t_i \geq L$, $i = 0, \ldots, n-1$. Hence for every $t \geq 0$ and $x \in H$ we get:

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2 &\leq \mathbb{E}^{\mathcal{F}_t}\ldots\mathbb{E}^{\mathcal{F}_{t_{n-1}}}\mathbb{E}^{\mathcal{F}_{t_n}}|T_P(u,t_n)\cdot\dots\cdot T_P(t_1,t)x|^2\\ &\leq R\mathbb{E}^{\mathcal{F}_t}\ldots\mathbb{E}^{\mathcal{F}_{t_{n-2}}}\mathbb{E}^{\mathcal{F}_{t_{n-1}}}|T_P(t_n,t_{n-1})\cdot\dots\cdot T_P(t_1,t)x|^2\\ &\leq r^n R|x|^2 \leq \frac{R}{r}e^{\frac{\ln r}{L}(u-t)} \qquad \mathbb{P}-a.s. \text{ and } \forall u : u-t \geq L\end{aligned}$$

Therefore setting $a = -\frac{\ln r}{L}$, one gets for every $t \geq 0$ and $x \in H$:

$$\mathbb{E}^{\mathcal{F}_t}|T_P(u,t)x|^2 \leq Me^{-a(u-t)}|x|^2, \quad \mathbb{P}-a.s. \text{ and } \forall u \geq t \geq 0$$

where $M = \max\{\frac{R}{r}, Re^{aL}\}$. □

We will use previous result in the following:

Proposition 4.7. *Assume that $S \geq \beta I$ for some $\beta > 0$ and that the minimal solution $\overline{P}$ to equation* (2.8) *is bounded. Let $\{\hat{P}^N\}_{N\in\mathbb{N}}$ be any sequence of processes in $L^\infty_{\mathcal{P},S}(\Omega \times [0,T];\Sigma^+(H))$ such that for each N, $\hat{P}^N$ is a generalized solution of* (2.3) *in $[0,N]$. If $|\hat{P}^N(N)| \leq c$, $\forall N \in \mathbb{N}$ $\mathbb{P}-a.s.$ for some positive c then there exist two constants $M > 0$, $a > 0$ such that $\forall t \in [0,N]$,*

$$|\overline{P}(t) - \hat{P}^N(t)|_{L(H)} \leq Me^{-a(N-t)} \qquad \mathbb{P}-a.s.$$

Proof. We divide the proof in three steps.
Attractivity from above. Assume that $\hat{P}^N(N) \geq \overline{P}(N)$, $\mathbb{P}-a.s.$, replacing in (2.5) first P by $\hat{P}^N$ and then P by $\overline{P}$ and in both cases using $\bar{u} = -B^*\overline{P}\bar{y}^{t,x}$, one deduces for every $t \in [0,N]$ and every $x \in H$:

$$\begin{aligned}0 \leq \langle(\hat{P}^N(t) - \overline{P}(t))x,x\rangle_H &= \mathbb{E}^{\mathcal{F}_t}\langle(\hat{P}^N(N) - \overline{P}(N))\bar{y}^{t,x}(N),\bar{y}^{t,x}(N)\rangle_H\\ &- \mathbb{E}^{\mathcal{F}_t}\int_t^N |B^*\hat{P}^N(s)\bar{y}^{t,x}(s) - B^*\overline{P}(s)\bar{y}^{t,x}(s)|^2\,ds \quad \mathbb{P}-a.s.\end{aligned}$$

Thanks to the hypotheses on $\hat{P}^N(N)$ there exists a constant $C > 0$ such that:

$$\langle(\hat{P}^N(t) - \overline{P}(t))x,x\rangle_H \leq \mathbb{E}^{\mathcal{F}_t}\langle(\hat{P}^N(N) - \overline{P}(N))\bar{y}^{t,x}(N),\bar{y}^{t,x}(N)\rangle_H \leq C\mathbb{E}^{\mathcal{F}_t}|\bar{y}^{t,x}(N)|^2$$

Now notice that $S \geq \beta I$ and $\overline{P}$ is bounded, so $\overline{P}$ stabilize $(A, A_\sharp, B, C)$ relatively to the identity uniformly in time. Thus, applying proposition 4.6 to $\bar{y}^{t,x}$, one gets that for every $t \in [0,N]$ and every $x \in H$ there exist positive constants C and a such that the following holds:

$$|(\hat{P}^N(t) - \overline{P}(t))x|^2_H \leq Ce^{-a(N-t)}|x|^2 \qquad \mathbb{P}-a.s. \tag{4.10}$$

Attractivity from below. Let P^N the approximating sequence introduced in proposition 3.2. We recall that $P^N(N) = 0$. Again relation (2.5) yields that for every $t \in [0, N]$ and every $x \in H$:

$$\langle (\overline{P}(t) - P^N(t))x, x\rangle_H = \mathbb{E}^{\mathcal{F}_t} \langle \overline{P}(N) y_N^{t,x}(N), y_N^{t,x}(N)\rangle_H \\ - \mathbb{E}^{\mathcal{F}_t} \int_t^N |B^* P^N(s) y_N^{t,x}(s) - B^* \overline{P}(s) y_N^{t,x}(s)|^2 \, ds \quad \mathbb{P} - a.s. \quad (4.11)$$

where $y_N^{t,x}$ is the solution to

$$\begin{cases} dy^{t,x}(s) = [A - B(s)B^*(s)P^N(s)]y^{t,x}(s)\, ds + C(s)y^{t,x}(s)\, dW(s), \quad s \in [t, N] \\ \\ y^{t,x}(t) = x \end{cases} \quad (4.12)$$

Let us extend the process $y_N^{t,x}$ outside $[0, N]$ in the usual way by setting $y_N^{t,x}(s) = 0$, $\mathbb{P}-a.s.$ and $\forall s \geq N$. Moreover since $y_N^{t,x}$ and $-B^* P^N(s) y_N^{t,x}$ are respectively the optimal state and the optimal control (relatively to S) and $P^N(N) = 0$ one has for every $t \geq 0$ and $x \in H$:

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} (|y_N^{t,x}(s)|^2 + |B^* P^N(s) y_N^{t,x}|^2)\, ds = \mathbb{E}^{\mathcal{F}_t} \int_t^N (|y_N^{t,x}(s)|^2 + |B^* P^N(s) y_N^{t,x}|^2)\, ds \quad (4.13)$$

$$\leq \frac{1}{\beta} \mathbb{E}^{\mathcal{F}_t} \int_t^N (|\sqrt{S}(s) y_N^{t,x}(s)|^2 + |B^* P^N(s) y_N^{t,x}|^2)\, ds \leq \frac{1}{\beta} \mathbb{E}^{\mathcal{F}_t} \int_t^N (|\sqrt{S}(s) y^{t,x}(s)|^2 + |B^* \overline{P}(s) y^{t,x}|^2)\, ds$$

$$= \frac{1}{\beta} \mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} (|\sqrt{S}(s) y^{t,x}(s)|^2 + |B^* \overline{P}(s) y^{t,x}(s)|^2)\, ds = \langle \overline{P}(t)x, x\rangle_H \leq C|x|^2 \quad \mathbb{P} - a.s.$$

where C, thanks to the hypothesis on $\overline{P}$, depends neither on t nor on x. Moreover by standard estimates and the Gronwall lemma one can find a constant C, independent on N (since all P^N are bounded by $\overline{P}$) such that for every $t \geq 0$ and $x \in H$:

$$\mathbb{E}^{\mathcal{F}_t} |y_N^{t,x}(\sigma)|^2 \leq Ce^{C(\sigma - t)} |x|^2 \qquad \mathbb{P}\text{-a.s. and } \forall \sigma \geq t. \quad (4.14)$$

Arguing as in the proof of proposition 4.6 relations (4.13) and (4.14) (which are the analogous of (4.3) and (4.5)) imply that we can find positive constants M and a, independent of N such that for every $t \geq 0$ and every $x \in H$:

$$\mathbb{E}^{\mathcal{F}_t} |y_N^{t,x}(N)|^2 \leq Me^{-a(N-t)} |x|^2 \qquad \mathbb{P}\text{-a.s..}$$

Therefore coming back to (4.11) one has that for every $t \geq 0$ and for every $x \in H$:

$$\langle (\overline{P}(t) - P^N(t))x, x\rangle_H \leq \mathbb{E}^{\mathcal{F}_t} \langle \overline{P}(N) y_N^{t,x}(N), y_N^{t,x}(N)\rangle_H \\ \leq C\mathbb{E}^{\mathcal{F}_t} |y_N^{t,x}(N)|^2 \leq \tilde{M} e^{-a(N-t)} |x|^2, \mathbb{P} - a.s.$$

Conclusion. If $\hat{P}^N$ is a general sequence then we let $\hat{\hat{P}}^N(N) = \hat{P}^N(N) + dI$ where $d \geq |\overline{P}(t)|_{L(H)}$, $\mathbb{P}-a.s.$. Clearly, for every $t \in [0, N]$

$$P^N(t) \leq \hat{P}^N(t) \leq \hat{\hat{P}}^N(t) \qquad \qquad \mathbb{P} - a.s.$$

and

$$P^N(t) \leq \overline{P}(t) \leq \hat{\hat{P}}^N(t) \qquad \qquad \mathbb{P} - a.s.$$

Thus, by the previous steps for every $t \in [0, N]$

$$|\hat{P}^N(t) - \overline{P}(t)|_{L(H)} \leq |\hat{\hat{P}}^N(t) - P^N(t)|_{L(H)} \leq Ce^{-a(N-t)}, \qquad \mathbb{P} - a.s..$$

□

5. Stationary Case

We set a stationary framework as in [5] and [22]. Namely $(\Omega, \mathcal{E}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic base verifying the usual conditions. Moreover $\{W_t : t \geq 0\}$ is a Ξ-valued, $(\mathcal{F}_t)_{t\geq 0}$-Wiener process and we assume that $\{W_t : t \geq 0\}$ is independent of $\mathcal{F}_0$ and that $\mathcal{F}_t = \sigma\{\mathcal{F}_0; W_s, s \in [0,t]\}$. Finally we introduce the semigroup $(\theta_t)_{t\geq 0}$ of measurable mappings $\theta_t : (\Omega, \mathcal{E}) \to (\Omega, \mathcal{E})$ verifying

(1) $\theta_0 = \mathrm{Id}$, $\theta_t \circ \theta_s = \theta_{t+s}$, for all $t, s \geq 0$
(2) θ_t is measurable: $(\Omega, \mathcal{F}_t) \to (\Omega, \mathcal{F}_0)$ and $\{\{\theta_t \in A\} : A \in \mathcal{F}_0\} = \mathcal{F}_t$
(3) $\mathbb{P}\{\theta_t \in A\} = \mathbb{P}(A)$ for all $A \in \mathcal{F}_0$
(4) $W_t \circ \theta_s = W_{t+s} - W_s$

We notice that here we no longer have $\mathcal{F}_t = \sigma(W_s, s \in [0,t])$ as assumed in the previous sections. On the other hand, since $\mathcal{F}_0$ has been chosen to be independent from $\{W_t : t \geq 0\}$, nothing changes in relation to the martingale representation theorem, see [5].

Definition 5.1. We say that a stochastic process $X : [0,\infty[\times\Omega \to E$, (where E is an arbitrary Banach space) is stationary if for all $s > 0$

$$X_t \circ \theta_s = X_{t+s} \qquad \mathbb{P}\text{-a.s. for a.e. } t \geq 0$$

In the rest of this section we will assume stationarity of the data of the problem, namely:

Hypothesis 5.2. *$A_\sharp : [0,+\infty[\times\Omega \to L(H)$, $B : [0,+\infty[\times\Omega \to L(U,H)$, $S : [0,+\infty[\times\Omega \to L(H)$, $C_i : [0,+\infty[\times\Omega \to L(H)$, $i = 1, 2, ...$ are stationary processes.*

We start from the following consequence of our definition of stationarity which comes immediately from the construction of the Itô integral.

Lemma 5.3. *Assume that $X : [0,\infty[\times\Omega \to L_2(\Xi, H)$ is a predictable stationary process such that $\mathbb{E}\int_0^T |X_t|^2_{L_2(\Xi,H)} dt < +\infty$ for some (and consequently for all) $T > 0$ then fixed any $s > 0$ it holds with probability one:*

$$\int_{a+s}^{b+s} X_t dW_t = \left(\int_a^b X_t dW_t\right) \circ \theta_s \qquad \forall a, b \geq 0,\ a < b.$$

Due to the uniqueness of the solution of the finite horizon BSREs in the Hilbert-Schmidt case (see [8] Section 5 and in particular theorem 5.13) the above immediately yields:

Lemma 5.4. *Fix $T > 0$ and beside Hypothesis 2.1 and 5.2 assume that $S \in L^2_{\mathcal{P}}(\Omega \times (0,T); \Sigma_2^+(H))$ and $P_T \in L^2(\Omega, \mathcal{F}_T; \Sigma_2^+(H))$ (where $\Sigma_2^+(H)$ denotes the cone of the non-negative symmetric operators in the Hilbert space of Hilbert-Schmidt operators $H \to H$).*

Let (P, Q) with $P \in L^2_{\mathcal{P}}(\Omega; C([0,T]; \Sigma_2(H))) \cap L^\infty_{\mathcal{P},S}(\Omega; C([0,T]; \Sigma^+(H)))$ and $Q \in L^2_{\mathcal{P}}(\Omega \times [0,T]; L_2(\Xi; \Sigma_2(H)))$ be the mild solution of the finite horizon BSRE see theorem 5.13 in [8]*:*

$$\begin{cases} -dP(t) = (A^*P(t) + P(t)A + A_\sharp^*(t)P(t) + P(t)A_\sharp(t) - P(t)B(t)B^*(t)P(t) + S(t))\,dt + \\ \qquad\qquad \mathrm{Tr}[C^*(t)P(t)C(t) + C^*(t)Q(t) + Q(t)C(t)]\,dt + Q(t)\,dW(t), \qquad t \in [0,T] \\ \\ P(T) = P_T \end{cases} \tag{5.1}$$

For fixed $s > 0$ we define $\widehat{P}(t+s) = P(t)\theta_s$, $\widehat{Q}(t+s) = Q(t)\theta_s$ then $(\widehat{P}, \widehat{Q})$ is the (unique) mild solution in $[s, T+s]$ of the equation

$$\begin{cases} -d\widehat{P}(t) = (A^*\widehat{P}(t) + \widehat{P}(t)A + A_\sharp^*(t)\widehat{P}(t) + \widehat{P}(t)A_\sharp(t) - \widehat{P}(t)B(t)B^*(t)\widehat{P}(t) + S(t))\,dt + \\ \qquad\qquad \mathrm{Tr}[C^*(t)\widehat{P}(t)C(t) + C^*(t)\widehat{Q}(t) + \widehat{Q}(t)C(t)]\,dt + \widehat{Q}(t)\,dW(t), \qquad t \in [s, T+s] \\ \\ P(T+s) = P_T \circ \theta_s \end{cases} \tag{5.2}$$

The above result can be extended to generalized solution.

Lemma 5.5. *Fix $T > 0$ and assume Hypothesis 2.1, 5.2 and that P_T in $L^\infty_S(\Omega, \mathcal{F}_T; \Sigma^+(H))$. Let P be the generalized solution of equation 5.1. Fixed $s > 0$ define $\widehat{P}(t+s) = P(t)\theta_s$, then $\widehat{P}$ is the (unique) generalized solution in $[s, T+s]$ of the equation* 5.2.

Proof. Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis in H and let Π_M be the orthogonal projection on the space generated by $e_1, ..., e_M$. Let $S^M(t) = \Pi_M S(t)\Pi_M$, $P^M_T = \Pi_M P_T \Pi_M$. Notice that $S^M \in L^2_{\mathcal{P}}(\Omega \times (0,T); \Sigma^+_2(H))$ and $P^M_T \in L^2(\Omega, \mathcal{F}_T; \Sigma^+_2(H))$. Moreover let (P^M, Q^M) be the mild solution of equation:

$$\begin{cases} -dP^M(t) = (A^* P^M(t) + P^M(t)A + A^*_\sharp(t)P^M(t) + P^M(t)A_\sharp(t) - P^M(t)B(t)B^*(t)P^M(t))dt \\ \qquad\qquad +S^M(t)dt + \mathrm{Tr}[C^*(t)P^M(t)C(t) + C^*(t)Q^M(t) + Q^M(t)C(t)]\, dt + Q^M(t)\, dW(t), \\ \\ P^M(T) = P^M_T \end{cases}$$

and let $(\widehat{P}^M, \widehat{Q}^M)$ be the mild solution of equation:

$$\begin{cases} -d\widehat{P}^M(t) = (A^* \widehat{P}^M(t) + \widehat{P}^M(t)A + A^*_\sharp(t)\widehat{P}^M(t) + \widehat{P}^M(t)A_\sharp(t) - \widehat{P}^M(t)B(t)B^*(t)\widehat{P}^M(t))\, dt + \\ \qquad\qquad +S^M(t) + \mathrm{Tr}[C^*(t)\widehat{P}^M(t)C(t) + C^*(t)\widehat{Q}^M(t) + \widehat{Q}^M(t)C(t)]\, dt + \widehat{Q}^M(t)\, dW(t), \\ \\ P^M(T+s) = P^M_T \circ \theta_s \end{cases}$$

Then by lemma 5.4, $\forall t \geq 0$, $\widehat{P}^M(t+s) = P^M(t) \circ \theta_s$, $\mathbb{P}$-a.s..

Relation (6.10) in [8] says that for all $t \geq 0$, $\mathbb{P}\{\widehat{P}^M(t+s)x \to \widehat{P}(t+s)x \ \forall x \in H\} = 1$ and $\mathbb{P}\{P^M(t)x \to P(t)x \ \forall x \in H\} = 1$. Moreover being θ_s measure preserving we have $\mathbb{P}\{(P^M(t) \circ \theta_s)x \to (P(t) \circ \theta_s)x \ \forall x \in H\} = 1$ and the conclusion follows. □

Proposition 5.6. *Beside Hypothesis 2.1 and 5.2 assume that $(A, A_\sharp, B, C)$ is $\sqrt{S}$ stabilizable, then the minimal solution $\overline{P}$ of the infinite horizon stochastic Riccati equation (2.8) is stationary.*

Proof. Extending the notation of the proof of proposition 3.2 for all $\rho > 0$ we denote by P^ρ the mild solution of equation (2.3) with final condition $P^\rho(\rho) = 0$. Exactly as in the proof of proposition 3.2 we can show that, for all $t \in [0, \rho]$, $P^{\rho+s}(t) \geq P^\rho(t)$, $\mathbb{P}$-a.s.. Thus in particular, denoting by $\lfloor \rho \rfloor$ the integer part of ρ, for all N for all $t \in [0, \lfloor N+s \rfloor]$, $P^{\lfloor N+s \rfloor}(t) \leq P^{N+s}(t) \leq P^{\lfloor N+s \rfloor + 1}(t)$, $\mathbb{P}$-a.s..

Recalling that, for all $t \geq 0$, $\mathbb{P}$-a.s. $\overline{P}(t)x = \lim_{N \to \infty} \overline{P}^N(t)x$ for all $x \in H$ the above monotonicity immediately yields that for all $s > 0$, $t \geq 0$, $\mathbb{P}$-a.s. $\overline{P}(t)x = \lim_{N \to \infty} \overline{P}^{N+s}(t)x$ for all $x \in H$.

Now we can conclude noticing that by lemma 5.5

$$P^{N+s}(t+s) = P^N(t) \circ \theta_s.$$

Thus letting $N \to +\infty$ we obtain that for all $t \geq 0$, and $s > 0$:

$$\mathbb{P}\left\{\overline{P}(t+s)x = \overline{P}(t)x \circ \theta_s, \quad \forall x \in H\right\} = 1.$$

□

Remark 5.7. Under the assumption 5.2, to verify that $(A, A_\sharp, B, C)$ is $\sqrt{S}$ stabilizable, it is enough to check condition 3.1 for $t = 0$. Namely if for all $x \in H$ there exists a control $u \in L^2_{\mathcal{P}}([0, +\infty) \times \Omega; U)$ such that

$$\mathbb{E}^{\mathcal{F}_0} \int_0^{+\infty} (|\sqrt{S}(s)y^{0,x,u}(s)|^2 + |u(s)|^2)\, ds < M_x \qquad \mathbb{P} - a.s. \tag{5.3}$$

then $(A, A_\sharp, B, C)$ is stabilizable. Indeed using the same notation as above and proceeding as in the proof of proposition 3.2 we see the condition (5.3) implies that for all ρ:

$$\mathbb{P}\left\{\langle P^\rho(0)x, x\rangle_H \leq M|x|^2, \quad \forall x \in H\right\} = 1$$

But for arbitrary $t > 0$ lemma 5.5 yields (for $\rho > t$) $P^\rho(t) = P^{\rho - t}(0) \circ \theta_t$ and consequently

$$\mathbb{P}\left\{\langle P^\rho(t)x, x\rangle_H \leq M|x|^2, \quad \forall x \in H\right\} = 1$$

Thus relation (3.3) holds and the rest of the proof of proposition 3.2 can be repeated to obtain existence of a solution to the Riccati equation (2.8) then by corollary 3.6 we can conclude that $(A, A_\sharp, B, C)$ is $\sqrt{S}$ stabilizable.

We finally notice that any stationary solution of equation (2.8) is automatically bounded as stated in definition 4.1. Consequently, in the stationary case, all the results in Section 4 assume a simpler aspect. For instance we have:

Proposition 5.8. *Let P_1 and P_2 be two stationary solutions of equation* (2.8)*. If P_1 stabilizes $(A, A_\sharp, B, C)$ relatively to I then for every $t \geq 0$:*

$$P_1(t) \geq P_2(t), \quad \mathbb{P} - a.s.$$

Consequently if the minimal solution $\overline{P}$ of equation (2.8) stabilizes $(A, A_\sharp, B, C)$ relatively to I then it is the unique solution of equation (2.8).

6. Wave equation in random media; stabilization and optimal control

We assume that the system is evolving in a random media and this influences its evolution in two ways: through a stochastic force of elastic type (the term $\sum_{i=1}^{\infty} c_i(t,\zeta)\xi(t,\zeta)d\beta_i(t)$ below) and through a stochastic damping (the term $\mu(t,\zeta)\partial_t\xi(t,\zeta)$ below).

We consider the state equation

$$\begin{cases} d_t\partial_t\xi(t,\zeta) = \Big[\Delta_\zeta\xi(t,\zeta) + \mu(t,\zeta)\partial_t\xi(t,\zeta) + b(t,\zeta)u(t,\zeta)\Big]dt + \displaystyle\sum_{i=1}^{\infty} c_i(t,\zeta)\xi(t,\zeta)d\beta_i(t), \quad \zeta \in \mathcal{D},\ t \geq 0, \\ \xi(t,\zeta) = 0, \quad \zeta \in \partial\mathcal{D},\ t \geq 0, \\ \xi(0,\zeta) = x_0(\zeta),\ \partial_t\xi(0,\zeta) = v_0(\zeta) \qquad \zeta \in \mathcal{D}, \end{cases} \tag{6.1}$$

and the cost functional

$$J(0,x,u) \;=\; \mathbb{E}\int_0^{+\infty}\int_{\mathcal{D}}\left[\xi^2(t,\zeta) + \left|\frac{\partial\xi}{\partial\zeta}(t,\zeta)\right|^2_{\mathbb{R}^d} + \left(\frac{\partial\xi}{\partial t}(t,\zeta)\right)^2\right]d\zeta\,dt + \mathbb{E}\int_0^{+\infty}\int_{\mathcal{D}} u^2(t,\zeta)d\zeta\,dt \tag{6.2}$$

In the above formulae $\mathcal{D} \subset \mathbb{R}^d$ is a bounded domain with regular boundary. By $\mathcal{B}(\mathcal{D})$ we denote the Borel σ-field in $\mathcal{D}$.

Moreover $\{\beta_i : i = 1, 2...\}$ are independent standard (real valued) brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We set $\mathcal{F}_t = \sigma\{\beta_i(s) : s \in [0,t],\ i = 1, 2...\}$ and denote by $\mathcal{P}$ the predictable σ-field in $\Omega \times [0,T]$.

On the coefficients we assume the following:

1. μ is a bounded measurable process defined on $([0,T] \times \Omega) \times \mathcal{D}$ endowed with the σ-field $\mathcal{P} \otimes \mathcal{B}(\mathcal{D})$ with values in $\mathbb{R}^+$ (with Borel σ-field).
2. b and c_i, $i = 1, 2, ..$ are bounded measurable maps $[0,T] \times \mathcal{D} \to \mathbb{R}$. We assume that there exists a constant $\alpha > 0$ such that:
$$|b(t,\zeta)| \geq \alpha \text{ for a.e. } t \in [0,+\infty) \text{ and a.e. } \zeta \in \mathcal{D}.$$
3. There exists a constant $M > 0$ such that $\sum_{i=1}^{\infty} |c_i(t,\zeta)|^2 \leq M$ for a.e. $t \in [0,T]$ and a.e. $\zeta \in \mathcal{D}$.

We set:

1. $H = H_0^1(\mathcal{D}) \times L^2(\mathcal{D})$, $U = L^2(\mathcal{D})$
2. $W(t) = \sum_{i=1}^{\infty} f_i\beta_i(t)$ where $\{f_i : i = 1, 2, ...\}$ is an orthonormal basis in an arbitrary separable real Hilbert space Ξ
3. $\mathcal{D}(A) = \left[H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})\right] \times H_0^1(\mathcal{D})$ and
$$\left(A\begin{pmatrix}\xi\\ v\end{pmatrix}\right)(\zeta) = \begin{pmatrix} v(\zeta) \\ \Delta_\zeta\xi(\zeta)\end{pmatrix}, \quad \begin{pmatrix}\xi\\ v\end{pmatrix} \in \mathcal{D}(A)$$
$$\left(A_\sharp(t)\begin{pmatrix}\xi\\ v\end{pmatrix}\right)(\zeta) = \begin{pmatrix} 0 \\ \mu(t,\zeta)v(\zeta)\end{pmatrix}, \quad \begin{pmatrix}\xi\\ v\end{pmatrix} \in H$$
4. $(B(t)u)(\zeta) = \begin{pmatrix} 0 \\ b(t,\zeta)u(\zeta)\end{pmatrix}, \quad \left(C_i(t)\begin{pmatrix}\xi\\ v\end{pmatrix}\right)(\zeta) = \begin{pmatrix} 0 \\ c_i(t,\zeta)\xi(\zeta)\end{pmatrix}$

(5) $S(t) = I_H$, $x = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$

With this setting the state equation (6.1) is equivalent to (2.1) and the cost (6.2) is equivalent to (2.2), see [1].

In order to prove stabilizability of $(A, A_\sharp, B, C)$ with respect to I we choose the control

$$\bar{u}(t) = b^{-1}(t)[-m\partial_t\bar{\xi}(t) - \mu(t)\partial_t\bar{\xi}(t)],$$

where $\bar{\xi}$ solves the following equation:

$$\begin{cases} d_t\partial_t\bar{\xi}(t,\zeta) = \Delta_\zeta\bar{\xi}(t,\zeta)dt - m\partial_t\bar{\xi}(t,\zeta)dt + \sum_{i=1}^{\infty} c_i(t,\zeta)\xi(t,\zeta)d\beta_i(t), & \zeta \in \mathcal{D}, t \geq 0, \\ \xi(t,\zeta) = 0, \qquad \zeta \in \partial\mathcal{D}, t \geq 0, \\ \xi(0,\zeta) = x_0(\zeta),\ \partial_t\xi(0,\zeta) = v_0(\zeta) \qquad \zeta \in \mathcal{D}, \end{cases} \tag{6.3}$$

(clearly the solution of (6.1) corresponding to $\overline{u}$ is $\overline{\xi}$).

If we set

$$\left(A_m \begin{pmatrix} \xi \\ v \end{pmatrix}\right)(\zeta) = \begin{pmatrix} v(\zeta) \\ \Delta_\zeta\xi(\zeta) - mv(\zeta) \end{pmatrix}, \quad \begin{pmatrix} \xi \\ v \end{pmatrix} \in \mathcal{D}(A_m) = \mathcal{D}(A),$$

then equation (6.3) can be rewritten in abstract form as:

$$\begin{cases} d\tilde{y}(t) = A_m\tilde{y}(t)\,dt + C\tilde{y}(t)\,dW(t), \quad t \geq 0, \\ \tilde{y}(0) = x, \end{cases} \tag{6.4}$$

Notice that equation (6.4) is an abstract stochastic evolution equation as (2.1) in the special case $B = 0$, consequently the sufficient condition (3.18) for the stabilization of the coefficients relatively to the identity reads:

$$2\langle A_m x, Xx\rangle + \sum_{i=1}^{\infty}\langle C_i(t)x, XC_i(t)x\rangle \leq -c|x|^2, \quad \text{for some } c > 0,\ \mathbb{P}\text{ -a.s., } \forall t \geq 0,\ \forall x \in D(A_m). \tag{6.5}$$

From proposition 3.5 in [19] we know that setting

$$X_m := \begin{pmatrix} I - \frac{m^2}{2}A^{-1} & -\frac{m}{2}A^{-1} \\ \frac{m}{2}I & I \end{pmatrix}$$

it holds $\langle X_m A_m x, x\rangle_H \leq -m|x|_H^2$. Moreover by direct computation

$$|\sum_{i=1}^{\infty}\langle C_i(t)x, X_m C_i(t)x\rangle|_H \leq M|x|_H^2.$$

Thus for m large enough relation (6.5) is verified and by proposition 3.8 we get that $(A, A_\sharp, B, C)$ is stabilizable relatively to I uniformly in time. Taking into account remarks 3.9 and 4.2, by proposition 3.2 and corollary 4.5 we can conclude that there exists a unique solution to the corresponding Riccati equation (2.8) which is bounded in time and globally attractive as specified in proposition 4.7. Finally theorem 3.4 can be applied to obtain the synthesis of the optimal control.

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Giuseppina Guatteri, Dipartimento di Matematica, Politecnico di Milano, piazza Leonardo da Vinci 32, 20133 Milano, e-mail: guatteri@mate.polimi.it

Gianmario Tessitore, Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, via R. Cozzi 53 - Edificio U5, 20125 Milano, e-mail: gianmario.tessitore@unimib.it