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INTERSECTING POLYTOPES AND TOMOGRAPHIC RECONSTRUCTIONS

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INTERSECTING POLYTOPES AND TOMOGRAPHIC RECONSTRUCTIONS

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Abstract. In this paper we deal with the reconstruction problem in Tomography, focusing on some new classes of subsets of the n-dimensional real space $\mathbb{R}^n$. Such classes are formed by clusters of polytopes mutually intersecting according to a *twisting* notion. The importance for tomography comes from their additivity property, which implies uniqueness of reconstruction. In the case $n = 2$ we give a detailed description of their geometric structure, with some insight in the lattice frame. In particular, for a finite set $\mathcal{D}$ of directions in $\mathbb{Z}^2$, we introduce the class of $\mathcal{D}$-inscribable lattice sets, showing that such sets can be considered as the natural discrete counterpart of the same notion known in the continuous case. Due to their nice tomographic properties, clusters of twisted polytopes might represent good candidates for approximating real shapes, as well as for investigating stability problems.

1. Introduction

In this paper we focus on some classes of sets which provides a nice link among various aspects of tomography. Before presenting our results, we wish to give a detailed description of the background, and provide both theoretical and practical motivations for our work.

Image reconstruction from projections is one of the main inverse problems which appears in several applications. The image is usually represented by an unknown real valued function $f(x, y)$, with bounded support. The values of f are related to physical properties of a two-dimensional section of the object under investigation. Projections are taken with the help of some kind of rays. For instance, in Computerized Tomography (CT), a portion of a human body is reconstructed by measuring the coefficient of linear attenuation of each beam of the X-ray traveling along a line crossing the body. The radiation is produced by photons, issued from a source and collected by a detector, both translating and rotating around the body. The differences between issued and collected photons measure the absorption of radiation by different tissues.

The underlying mathematical approach is known since 1917, and goes back to J. Radon, who described a direct method for inverting the so-called Radon Transform (RT) of f to get the density function $f(x, y)$ of a planar section K of the body. We shall briefly mention the basic ingredients.

To specify a line of photons in the plane we use two coordinates: r, its distance from the origin, and θ, the angle that the line of detectors (orthogonal to the lines of photons) makes with the positive x-axis. Then, a single photon on the line has coordinates

$$\begin{cases} x = r\cos\theta - s\sin\theta \\ y = r\sin\theta + s\cos\theta, \end{cases}$$

where the parameter $s \in \mathbb{R}$ identifies a photon on its line (see Figure 1).
For each $r \in \mathbb{R}$ and $0 \le \theta < \pi$, the collected information is given by the integral of f along a line of photons crossing the body:

$$p_\theta(r)(K) = \int_{-\infty}^{+\infty} f(r\cos\theta - s\sin\theta, r\sin\theta + s\cos\theta)ds.$$

Thus, for each $r \in \mathbb{R}$ the Radon Transform of f is given by

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$$(\mathcal{R}f) = \{p_\theta(r)(K) \ : \ 0 \le \theta < \pi\}.$$

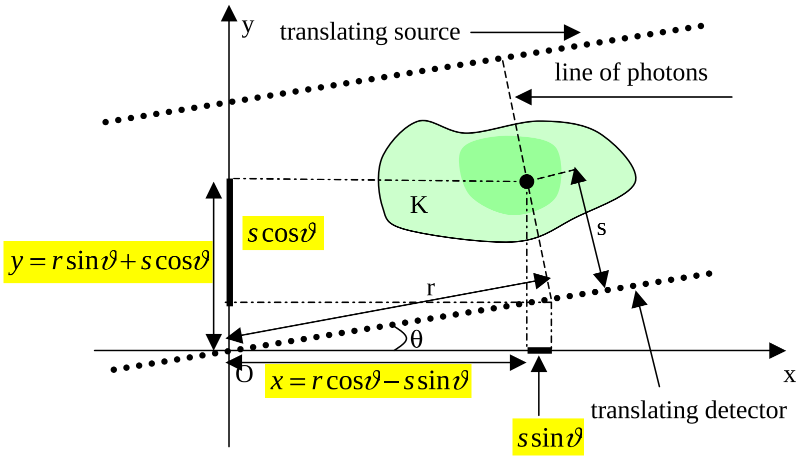

FIGURE 1. Collection of data under a rotation $\theta \in [0, \pi)$.

The main tool for the RT inversion is the so-called *central slice theorem* (or *projection slice theorem*), which says that the restriction to $u = r\cos\theta, v = r\sin\theta$ of the 2-dimensional Fourier transform $\hat{f}(u, v)$ of $f(x, y)$ equals the 1-dimensional Fourier transform of $p_\theta(r)$. From the knowledge of $(\mathcal{R}f)$, $p_\theta(r)$ is known for all θ, and thus $\hat{f}$, so that f can be determined by means of the inverse Fourier Transform [30].

Despite this clear theoretical picture, many problems remain in applications, where the main effort is the discretization of the whole process, due to different reasons. To begin with, only a finite number of projections can be taken, i.e. only a finite number of directions can be considered. Moreover, for each direction the X-ray is not continuous, but actually consists of a finite number of beams. This implies that one can collect only a finite number of data $p_{\theta_i}(r_j)$, $1 \le i \le m$, $1 \le j \le n$. Being the numerical data discrete, the density function f must be redefined on a finite gride, so that the inverse problem itself becomes discrete. Consequently, the implementation of any reconstruction algorithm in CT produces gray-scale digital images, where every gray level has a finite binary representation.

This leads to a natural connection between CT and Discrete Tomography (DT). In DT one aims to reconstruct finite subsets of the integer lattice $\mathbb{Z}^n$ from the knowledge of their line sums in a small number of directions. The beginning of DT goes back to the DIMACS Mini-symposium that Larry Shepp organized in 1994 on this topic. The original motivation came from High-Resolution Transmission Electron Microscopy (HRTEM) which is able to obtain images with atomic resolution and provides quantitative information on the number of atoms that lie in single atomic columns in crystals (see [29, 26, 32]). The high energies required to produce the discrete X-rays of a crystal mean that only a small number of X-rays can be taken before the crystal is damaged. Therefore, DT focuses on the reconstruction of images with few different grey levels, and, in particular, on the reconstruction of binary images from a small number of projections. Here the term *projection* is used with different meaning within different reconstruction problems, but, in general, it refers to a partition of the pixels in the image, such that for each subset in the partition the total number of 1s in the unknown binary image is given (see, for instance, [4, 23, 18]). There are other real applications of discrete tomography where one can assume that the image consists of only two grey values, for instance in connection with the scan of a homogeneous object, where the 1s and the 0s denote, respectively, presence or absence of material in the corresponding pixels. Further applications of DT include quality control in semiconductor industry, image processing, data compression and data security, etc. (see, for example [23, 24, 28, 33]). Detailed accounts of the development and advances that have taken place since 1994 can be found in the two "classical" books on this subject, [24, 25].

In DT, the usual line integrals are replaced simply by counting the number of points on each line L, which gives a discrete version of RT, the so-called Discrete Radon Transform (DRT). The inversion of DRT aims to deduce the local atomic structure from the collected counting data. It is worth mentioning that this problem was considered in its pure mathematical form even before its connection with electron microscopy ([15]). On this regard, a special class of geometric objects, called *additive* sets, has been studied in considerable depth (see Section 2 for the formal definition). It was shown in [15] that a finite subset F of $\mathbb{Z}^2$ is uniquely determined by its line sums in the coordinate directions if and only if F is additive. The sufficient condition was later extended to any dimension, pointing out that notions of additivity and uniqueness are equivalent when two directions are employed, whereas, for $|\mathcal{D}| \geq 3$ directions, additivity is more demanding than uniqueness. Actually, every additive set is uniquely determined, but there are nonadditive sets of uniqueness [17]. Further generalizations have been considered in [16], where the notion of additivity has been extended to n-dimension, with respect to a set $\mathcal{H}$ of linear manifolds (see also Section 2). This suggests that may be quite difficult to decide whether a lattice set is uniquely determined by its line sums taken in set of more than three directions. In fact, the inversion of DRT is generally NP-hard ([28]), so that any reconstruction algorithm must consist of exponentially many steps in the size of F. However, the problem for additive sets becomes affordable if the discrete approach is relaxed to take advantage of continuous methods. Indeed, the idea is to address the problem by looking for a *fuzzy set* with given line sums, namely a function $f(z)$ such that $0 \leq f(z) \leq 1$ for all lattice points $z \in \mathbb{Z}^2$, and $|\sum_{z\in L} f(z)| = |F \cap L|$ for each lattice line L. All such functions form a convex set, and the functions f such that $f(z) \in \{0, 1\}$ are extreme points. Thus the reconstruction problem can be formulated in terms of linear programming, and there are algorithms for finding f or proving that no such f (and hence no corresponding set F) exists that run in polynomial time (see [17]). One related problem is to find suitable sub-classes of lattice sets that can be reconstructed in polynomial time (see, for instance [5, 8]), or to provide uniqueness results from the analysis of the geometric features of the class (see [21, 22]). This indicates that more significant connections can be made between DT and Geometric Tomography (GT), which is a geometric relative of CT.

In GT the usual density functions appearing in CT are replaced by geometric objects, and one of the main goal is to find conditions which guarantee a faithful reconstruction, possibly unique, within a given geometric class of subsets of $\mathbb{R}^n$. The class of additive sets has been widely considered also in this context. A well-known result states that a $\mathcal{D}$-additive set in $\mathbb{R}^n$ is uniquely determined, among all measurable sets, by its X-rays taken in the directions in $\mathcal{D}$ (in fact a more general result is true, see [20, Theorem 2.3.11]). In the planar case, the additivity property has an interesting interplay with convexity, and, in particular, with the notion of *inscribability.* A convex body $K \subset \mathbb{R}^2$ is said to be inscribable with respect to a finite set $\mathcal{D}$ of directions, or simply $\mathcal{D}$*-inscribable*, if its interior is the union of interiors of convex polygons inscribed in K, each of whose edges is parallel to some direction in $\mathcal{D}$. If $\mathcal{D}$ consists of the set of coordinate directions in the plane, then $\mathcal{D}$-inscribability and uniqueness by means of X-rays in the directions in $\mathcal{D}$ are equivalent (see [27]). Further, in this case, a planar convex body K is $\mathcal{D}$-inscribable if and only if it is $\mathcal{D}$-additive (see [19]). This result partially extends to any finite set $\mathcal{D}$ of directions: every $\mathcal{D}$-inscribable set is $\mathcal{D}$-additive, but the converse is not true (see [19] and also [20, Chapter 1] for an overview on this topic).

The previous discussion shows that uniqueness, additivity, and inscribability are mutually entwining notions, playing an important role in tomography. In particular, a discrete counterpart of inscribable sets could be desirable. In this paper we investigate such problems as follows. First, we derive the additivity property for polytopes in $\mathbb{R}^n$, and extend this result to finite unions of polytopes satisfying a special requirement of mutual intersection (see Section 2). This provides uniqueness results for such *clusters of polytopes*, also holding in the n-dimensional integer lattice $\mathbb{Z}^n$. This can be suitable for DT purposes, when few directions are employed, for instance the set of coordinate directions. In Section 3 we explore in more detail the geometric structure of clusters of polygons, i.e. for $n = 2$. This leads to a discrete notion of inscribable sets, which share several features with their GT analogous. On the other hand, differently from the continuous scenario, such sets need not necessarily be convex, and therefore represent a wider class of lattice sets.

The results presented in this paper could be considered also for inverse problems which are not immediately related to tomography, such as reconstructions of 3D models from image data, typical of computer vision, or the understanding of physical properties of some materials. Let's consider, for instance, a combination of crystals. As it is well known, crystal shapes can be grouped on the basis of their external (i.e. macroscopically visible) symmetry features into seven systems of three dimensional patterns, namely *cubic*, *tetragonal*, *hexagonal*, *trigonal*, *orthorhombic*, *monoclinic*, and *triclinic*. This external shape reflects the characteristic symmetry of the microscopic pattern of atomic arrangement in the various crystals, which, during their growth (for instance into a fluid phase), develop and maintain definite polyhedral forms. Due to this, Theorem 3, and the following Corollary 1 are of special interest. The bounding faces of such polyhedra are perpendicular to the directions of slowest growth, which are determined by several factors such as temperature, pressure, chemical conditions, and amount of available space. In a first-step any such form could be approximated by a polyhedron in $\mathbb{Z}^3$. A combinations of forms belonging to a same crystal class may result in a *crystal cluster*, a formation that consists of a number of single-terminated crystals, each adhering to a common base (see Figg. 2A-D). It turns out that the external structure of a crystal cluster has to do with the different rates of growth. Components having different growing directions can be approximated by different polyhedra. By selecting a set of m different growing directions (different colors in Fig. 2B), the crystal cluster remains approximated by m different intersecting polyhedra (see Figg. 2C-F). Of course, the greater m, the more accurate reconstruction results.

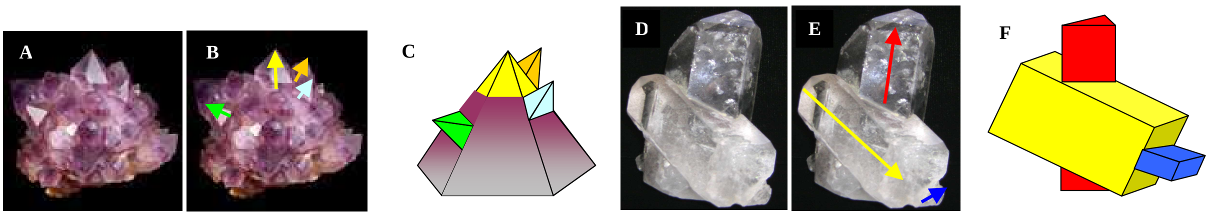

FIGURE 2. **A** An amethyst crystal cluster. **D** A quartz crystal cluster. **B-E** Some meaningful directions selected from the shapes. **C-F** Simple approximations of the crystals by means of clusters of twisted polytopes.

In Section 2 we focus on the tomography of clusters of polytopes formed according to a special assumption of mutual intersection. This is selected in analogy with a set of growing directions in a crystal cluster. The resulting shapes look like that in Figures 2 C-F, so that the study of such a geometric class might be of interest for applications. Theorem 4 and Corollary 2 seem to be of interest in this view.

One related problem is that of finding stability estimates for reconstruction of such sets. In Section 4 we give a brief sketch of further researches in this direction.

2. Additivity of polytopes

As usual, $\mathbb{R}^n$, $n \geq 2$, denotes the Euclidean n dimensional space, and $\{e_1, \dots, e_n\}$ the standard orthonormal basis for $\mathbb{R}^n$. For any subspace $S \subset \mathbb{R}^n$, $H^\perp$ denotes its orthogonal complement.

If A is a measurable set, with respect to the n-dimensional Lebesgue measure λ_n, we denote by $|A|$, int$A$, cl$A$ and conv A the *measure*, *interior*,*closure*, and *convex hull* of A, respectively. For a pair of sets A, B, we write $A \triangle B$ for their symmetric difference. Let $H \subset \mathbb{R}^n$ be a k dimensional subspace, $1 \leq k \leq n-1$, and let

$E \subset \mathbb{R}^n$ be a bounded measurable set. Denote by λ_k the k-dimensional Lebesgue measure. The *k-dimensional X-ray* of E parallel to H is defined for λ_{n-k}-almost all $x \in H^{\perp}$ by $\lambda_k(E \cap (H + x))$, that is the λ_k-measure of the intersection of E with each k-dimensional plane parallel to H (see [20]). Such a definition can easily be assumed also in the integer lattice $\mathbb{Z}^n$, by replacing λ_k with the *counting measure*, namely with the cardinality of intersections.

Let H be a subspace of $\mathbb{R}^n$. A *ridge function orthogonal to H* is a function which is constant on each translate of H. Let $\mathcal{H} = \{H_i \ : 1 \leq i \leq m\}$ be a set of subspaces of $\mathbb{R}^n$ of dimension between 1 and $n-1$ inclusive. A bounded set $E \subset \mathbb{R}^n$ is called *$\mathcal{H}$-additive* if

$$E = \{x \in \mathbb{R}^n \ : \sum_i f_i(x) > 0\}, \tag{2.1}$$

where f_i is a ridge function orthogonal to H_i (see [20, Chapter 2], where a slightly more general definition is given). The following theorem is proved in [14].

Theorem 1. *Any $\mathcal{H}$-additive set is uniquely reconstructible by means of X-rays parallel to the subspaces in $\mathcal{H}$.*

The discrete version of this important result can be obtained as follows (see for instance [16]). Let $\mathcal{H} = \{H_1, ..., H_m\}$ be a set of linear subspaces of $\mathbb{R}^n$, of dimension between 1 and $n-1$ inclusive. Then $\mathcal{H}$ is said to be a *discrete Radon base* if, for all $i, j \in \{1, ..., m\}$ the following hold

(1) $\bigcap_{i=1}^m H_i = \{\mathbf{0}\}$;
(2) $i \neq j \ \Rightarrow \ H_i \not\subseteq H_j$;
(3) $|H_i \cap \mathbb{Z}^n| \geq 2$.

For a discrete Radon base $\mathcal{H} = \{H_1, ..., H_m\}$, let $\mathcal{H}_k$ be the family of affine subspaces of $\mathbb{R}^n$ formed by all translates of H_k that have nonempty intersection with $\mathbb{Z}^n$. Hence, $H \in \mathcal{H}_k$ if and only if $H = H_k + a$ for some $a \in \mathbb{R}^n$, i.e. $H = \{x + a \ : \ x \in H_k\}$ for some $a \in \mathbb{R}^n$ and $(H_k + a) \cap \mathbb{Z}^n \neq \emptyset$. Then consider the family $\mathcal{T}$ of all such affine subspaces

$$\mathcal{T} = \bigcup_{k=1}^m \mathcal{H}_k.$$

For any set $E \in \mathbb{Z}^n$, the *discrete Radon transform* (DRT) of E with respect to $\mathcal{T}$ is the function $v_E : \mathcal{T} \to \{0, 1, 2, ...\}$ such that

$$v_E(H) = |E \cap H| \quad \text{for all} \quad H \in \mathcal{T}. \tag{2.2}$$

A finite set $E \subseteq \mathbb{Z}^n$ is a *set of uniqueness with respect to a family $\mathcal{H}$ of linear subspaces*, or simply *$\mathcal{H}$-unique*, if E is uniquely determined by its DRT with respect to $\mathcal{T}$. In other words, if $F \subseteq \mathbb{Z}^n$ verifies $|F \cap H| = |E \cap H|$ for all $H \in \mathcal{T}$, then $F = E$. A finite set $E \subseteq \mathbb{Z}^n$ is a *additive with respect to a discrete Radon base $\mathcal{H}$*, or simply *$\mathcal{H}$-additive* if some mapping $g : \mathcal{T} \to \mathbb{R}$ exists such that

$$E = \left\{ x \in \mathbb{Z}^n, \ \sum_{H \in \mathcal{T}} v_x(H) g(H) > 0 \right\}, \tag{2.3}$$

where v_x is the DRT of $\{x\}$, that is $v_x(H) = 1$ if $x \in H$, and $v_x(H) = 0$ otherwise. In [15] (see also [16]) it is proved the following discrete counterpart of Theorem 1.

Theorem 2. *If a set E is $\mathcal{H}$-additive with respect to a discrete Radon base $\mathcal{H}$, then E is also $\mathcal{H}$-unique.*

Remark 1. Any $x \in \mathbb{Z}^n$ lies in exactly one member of $\mathcal{H}_i$ for each $i \in \{1, ..., m\}$. Therefore, when $H \in \mathcal{H}_i$, $v_x(H)g(H)$ is the discrete analogue of a ridge functions orthogonal to H_i. In this view, the two definitions of additive set coming from (2.1) (continuous case) and (2.3) (discrete case) can be naturally identified.

Let $P \subset \mathbb{R}^n$ be a convex polytope of full dimension n, and denote by $\mathcal{F}^{(k)}(P)$ the set of k-dimensional faces of P, $0 \leq k \leq n$. As usual, the elements of $\mathcal{F}^{(n-1)}(P) = \{F_1, ..., F_m\}$ are called *facets* of P. A *bounding hyperplane* is a hyperplane containing a facet of P. The $n-1$-dimensional subspace parallel to a bounding hyperplane is said to be a *bounding subspace* of the polytope. For $F_i \in \mathcal{F}^{(n-1)}(P)$, $i \in \{1, ..., m\}$, denote by H_i the corresponding bounding subspace, and let $H_i^{\perp}$ be the inner (with respect to P) normal to H_i. The polytope P is the intersection of m half spaces determined by the bounding hyperplanes of P, each of which is defined by a linear inequality. Therefore P can be represented as the solution of a system of inequalities $A^t\mathbf{x} \geq \mathbf{b}$. In particular, each row in the matrix A^t corresponds with a bounding subspace of the polytope P, that is the i-th row of A^t corresponds to the vector $H_i^{\perp}$, $i = 1, ..., m$. Denote by A_i the matrix obtained from A by replacing $H_i^{\perp}$ with $-H_i^{\perp}$, and by keeping each other row. Analogously, let $\mathbf{b}_i$ be the array obtained from $\mathbf{b}$ by changing its i-th entry in the opposite and preserving all the other entries.

Definition 1. Let $P \subset \mathbb{R}^n$ be a convex polytope of full dimension n. Consider the set

$$\mathring{R}(P) = \bigcup_{i=1}^{m} \left\{\mathbf{x} \in \mathbb{R}^n,\ \ A_i^t\mathbf{x} > \mathbf{b}_i\right\}.$$

The *free region* determined by P is the set $R(P)$ obtained as union of $\mathring{R}(P)$ with the portion of its boundary which does not belong to ∂P (see Remark 3).

Remark 2. Since A_i is just A with the i-th row replaced by the opposite, the set $\{\mathbf{x} \in \mathbb{R}^n,\ A_i^t\mathbf{x} > \mathbf{b}_i\}$ corresponds with the (possibly unbounded) open region of $\mathbb{R}^n \setminus P$ delimited by F_i and by all the bounding hyperplanes of P that intersect F_i, $i \in \{1, ..., m\}$. Such open regions are pairwise disjointed.

Due to the introductory discussion concerning the interplay between uniqueness and additivity, the following result is important in view of tomographic approximations of natural shapes.

Theorem 3. *Let P be a non-degenerate convex polytope in $\mathbb{R}^n$, $n \geq 2$. Then P is $\mathcal{H}$-additive with respect to the set $\mathcal{H}$ formed by its bounding subspaces.*

Proof. Let $\mathcal{F}^{(n-1)}(P) = \{F_1, ..., F_m\}$, and let B_j be the bounding hyperplane of P containing F_j. Denote by B_j^+, B_j^- the open half-spaces bounded by B_j, where $P \subset \mathrm{cl}(B_j^+)$. For each $j \in \{1, ..., m\}$, define the following function on $\mathbb{R}^n$

$$f_j(x) = \begin{cases} -(m-1)/m & \text{if } x \in B_j^- \\ \\ 1/m & \text{if } x \in \mathrm{cl}(B_j^+). \end{cases}$$

Note that f_j is a ridge function orthogonal to H_j, for each $j \in \{1, ..., m\}$. Now consider the function

$$f(p) = \sum_{j=1}^{m} f_j(p), \tag{2.4}$$

and compute $f(p)$ for any $p \in \mathbb{R}^n$. To this, let $I(p) \subseteq \{1, ..., m\}$ be the set of indices (depending on p), such that $p \in B_j^-$ for $j \in I(p)$. Since $\mathrm{cl}B_j^+ \cup B_j^- = \mathbb{R}^n$ for all $j \in \{1, ..., m\}$, then $p \in \mathrm{cl}B_j^+$ for $j \in \{1, ..., m\} \setminus I(p)$. From (2.4) we get

$$f(p) = -|I(p)|\frac{m-1}{m} + (m - |I(p)|)\frac{1}{m} = 1 - |I(p)|.$$

If $p \in R(P)$, then, by Remark 2, $|I(p)| = 1$, and $f(p) = 0$. If $p \in \mathbb{R}^n \setminus (P \cup R(P))$, then $|I(p)| \geq 2$, so that $f(p) \leq -1$. If $p \in P$, then $p \in \mathrm{cl}B_j^+$ for all $j = 1, ..., m$, and $|I(p)| = \emptyset$, which implies $f(p) = m(1/m) = 1$. Therefore we have

$$P = \left\{ p \in \mathbb{R}^n : \ f(p) = \sum_{j=1}^{m} f_j(p) > 0 \right\},$$

and consequently P is $\mathcal{H}$-additive. □

From [14] we have the following result.

Corollary 1. *A convex polytope of full dimension is uniquely determined by the X-rays parallel to its bounding subspaces.*

Theorem 3 can be generalized to special union of polytopes. To this we need two more definitions.

Definition 2. Let $P_1, P_2 \subset \mathbb{R}^n$ be convex polytopes of full dimension n, and $R(P_1), R(P_2)$ be the free regions determined by P_1, P_2 respectively. Then P_1, P_2 are said to be *twisted polytopes* if each one is contained in the union of the other with the corresponding free region, namely $P_1 \subset P_2 \cup R(P_2)$ and $P_2 \subset P_1 \cup R(P_1)$.

Definition 3. A set $C \subset \mathbb{R}^n$ is said to be a *cluster of twisted polytopes* if C is the finite union of pairwise twisted polytopes.

Theorem 4. *Let $C \subset \mathbb{R}^n$ be a cluster of twisted polytopes $P_1, ..., P_r$. Denote by m_i the number of facets of P_i, and let $\mathcal{H}$ be the set of all the bounding subspaces of $P_1, ..., P_r$. Then C is $\mathcal{H}$-additive.*

Proof. Denote by m_i the number of facets of P_i, $i = 1, ..., r$, and let $\{B_{ij} \ : j = 1, ..., m_i\}$ be the set of the bounding hyperplanes of P_i. Consider the functions

$$f_j^{(P_i)}(x) = \begin{cases} -(m_i - 1)/m_i & \text{if } \ x \in B_{ij}^- \\ \\ 1/m_i & \text{if } \ x \in \mathrm{cl}B_{ij}^+, \end{cases}$$

for all $i = 1, ..., r$ and for all $j = 1, ..., m_i$. Let $f^{(P_i)}$ be the sum of all $f_j^{(P_i)}$, namely

$$f^{(P_i)}(p) = \sum_{j=1}^{m_i} f_j^{(P_i)}(p).$$

By Theorem 3, $f^{(P_i)}(p) > 0$ if $p \in P_i$, $f^{(P_i)} = 0$ if $p \in R(P_i)$, and $f^{(P_i)}(p) < 0$ if $p \notin P_i \cup R(P_i)$. Now define

$$f(p) = \sum_{i=1}^{r} f^{(P_i)}(p) = \sum_{i=1}^{r} \sum_{j=1}^{m_i} f_j^{(P_i)}(p).$$

If $p \in C$, then there exists a set of indices (depending on p) $I(p) \subseteq \{1, ..., r\}$ such that $p \in P_k$, for $k \in I(p)$. Therefore, $f^{(P_k)}(p) > 0$ for $k \in I(p)$. Suppose now $h \notin I(p)$. Since P_h and P_k are twisted polytopes, $P_k \triangle P_h \subset R(P_h)$ for all $k \in I(p)$. Consequently, $f^{(P_h)}(p) = 0$, for all $h \notin I(p)$, so that $f(p) > 0$. If $p \notin C$, then $p \notin P_i$, and $f^{(P_i)}(p) \leq 0$ for all $i = 1, ..., r$. Therefore $f(p) \leq 0$, so that

$$C = \left\{ p \in \mathbb{R}^n : \ f(p) = \sum_{i=1}^{r} f^{(P_i)}(p) > 0 \right\}. \tag{2.5}$$

Let's label the members of $\mathcal{H}$ in some order, so that $\mathcal{H} = \{H_1, ..., H_m\}$, $m = m_1 + ... + m_r$, where possible repetitions might occur if different polytopes have the same bounding subspaces at some of their facets. For

$H_h \in \mathcal{H}$, denote by g_h the corresponding ridge function $f_j^{(P_i)}$, or the sum of all the corresponding ridge functions if H_h is repeated. In any case, g_h is a ridge function orthogonal to H_h. Consequently we can write

$$f(p) = \sum_{i=1}^{r} \sum_{j=1}^{m_i} f_j^{(P_i)}(p) = \sum_{h=1}^{m} g_h(p). \tag{2.6}$$

Therefore f is the sum of ridge functions, and by (2.5), the cluster C is $\mathcal{H}$-additive. □

Remark 3. The crucial step in proving Theorem 4 relies on the fact that, for a single polytope P, Theorem 3 gives $f(p) = 0$ if and only if p belongs to $R(P)$. Definition 2 motivates and explains the term *free region* associated to P, since it represents the set of points where we are allowed to place further components of the resulting cluster.

Example 1. Let $\mathcal{C} = B_1 \cup B_2$ be a cluster of a square B_1 and a rectangle B_2. In Figure 3 is represented the computation of $f(p) = f^{(B_1)}(p) + f^{(B_2)}(p)$ as in the proof of Theorem 4. Each region is labeled with two numbers, corresponding to the values of $f^{(B_1)}(p)$, $f^{(B_2)}(p)$ (the first and the second one, respectively) for p in that region, showing that $C = \left\{p \in \mathbb{R}^n : \ f(p) = \sum_{i=1}^{r} f^{(P_i)}(p) > 0\right\}$, as required. The 00 labeled regions allow the displacement of further vertices for possible additions of twisted polygons to form a new cluster.

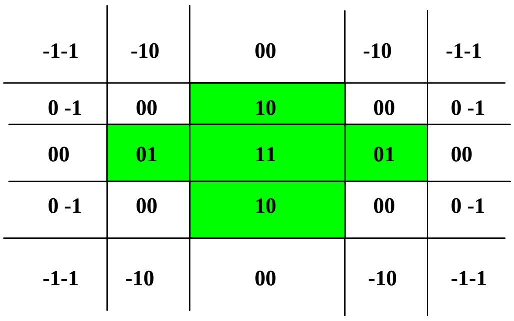

FIGURE 3. A two-dimensional cluster formed with a pair of twisted boxes.

From Theorem 1 we immediately get the following corollary.

Corollary 2. *A cluster of twisted polytopes of full dimension is uniquely reconstructible, among all measurable sets, by X-rays parallel to the family of its bounding subspaces.*

Corollary 2 shows that approximating a natural shape with a cluster of polytopes could be tomographically meaningful, since the two-dimensional sections with planes orthogonal to the specified directions uniquely determine the cluster. See, for instance, Figure 2 in the Introduction.

Since a convex lattice polytope (i.e. a convex polytope whose vertices belong to $\mathbb{Z}^n$) can still be determined in $\mathbb{Z}^n$ as the solution of a system of inequalities, all the previous definitions concerning free-regions, twisted polytopes and clusters of polytopes can be considered also in the lattice $\mathbb{Z}^n$, $n \geq 2$. It is then natural to ask whether the above results extend to this discrete setting.

Theorem 5. *Let $\mathcal{C} \subset \mathbb{Z}^n$, $n \geq 2$, be a cluster of twisted lattice polytopes $P_1, ..., P_r$, $r \geq 1$, and let $\mathcal{H}$ be the family of its bounding subspaces. Then the following hold.*

(1) *$\mathcal{C}$ is $\mathcal{H}$-additive.*
(2) *$\mathcal{C}$ is uniquely reconstructible from the X-rays parallel to the hyperplanes in $\mathcal{H}$.*

Proof. (1) The proof of Theorem 3 does not depend on the continuous structure of $\mathbb{R}^n$, but just on the set of ridge functions orthogonal to the bounding subspaces of the convex polytope P. Therefore we can reproduce precisely the same proof even in the case when $P \subset \mathbb{Z}^n$ (see also Remark 1). Moreover, it is easy to see that the family $\mathcal{H}$ formed by the bounding subspaces of a lattice convex polytope P is a discrete Radon base. Therefore P is $\mathcal{H}$-additive, which proves the statement for $r = 1$. If $r > 1$, the proof of Theorem 4 can be reproduced without changes, but in the case of $\mathbb{Z}^n$ we need to show that the set $\mathcal{H} = \{H_1, ..., H_m\}$ of bounding subspaces is a Radon base. Of course conditions (1) and (3) in the definition of a Radon base trivially hold, since these are already fulfilled by the bounding subspaces of each P_i forming the cluster. In the case where different polytopes in $\mathcal{C}$ have common bounding subspaces to some of their facets, we have $H_i = H_j$ for different indices i, j, which would contradict condition (2). However a Radon base can be determined from $\mathcal{H}$ simply by including just once such hyperplanes. This implies the replacing of the corresponding ridge functions with their sum. Consequently, formula (2.6) still implies that f in (2.5) is a sum of ridge functions, so proving that $\mathcal{C}$ is $\mathcal{H}$-additive.

(2) The statement immediately follows by (1) and by Theorem 2. □

Remark 4. We are not aware of other uniqueness results for higher dimensional X-rays different from that presented in Theorem 5-(2) (see also the comment in [22]). On the other hand such a result can be restated in terms of lower dimensional X-rays as follows. Let $E \subset \mathbb{R}^n$ be $\mathcal{H}$-additive with respect to a set $\mathcal{H}$ of k-dimensional subspaces. Let $j < k$, and assume we know the j-dimensional X-rays of E with respect to some j dimensional subspace S_H of H, for each $H \in \mathcal{H}$. Let $\mathcal{S}$ be the family of these j-dimensional X-rays. Suppose that, for all $H \in \mathcal{H}$, F has the same X-rays of E for all $S_H \in \mathcal{S}$. Then, by the Cavalieri principle, $\lambda_k(F \cap H) = \lambda_k(E \cap H)$ for all $H \in \mathcal{H}$. Since E is $\mathcal{H}$-additive, then E is $\mathcal{H}$-unique, and consequently $F = E$. In other words, we know that E is the set of points where the sum of the given ridge functions orthogonal to H, for all $H \in \mathcal{H}$, is positive. Such ridge functions can also be regarded as ridge functions orthogonal to j dimensional subspaces of $\mathcal{H}$, so that E is $\mathcal{S}$-additive with respect to such a family $\mathcal{S}$, and therefore is $\mathcal{S}$-unique. In particular, from data concerning 1-dimensional X-rays, Corollary 2 (or Theorem 5-(2) in the lattice case) guarantee uniqueness of reconstruction for cluster of twisted polytopes having facets parallel to the given directions.

The results in Theorem 5 appear to be useful in view of applications. However, as explained in the Introduction, in DT it is also important to employ a limited number of directions. This suggests to look for clusters where all the twisted polytopes share a common discrete Radon base. The simplest way to get such a cluster is to employ the discrete Radon base formed by the coordinate hyperplanes, so that the employed polytopes are *coordinate boxes*, namely parallelepipedes with sides parallel to the coordinate directions. In this case the above results have an interesting further connection with the reconstruction algorithms of CT. As well known, these are mainly based on the *back-projection method*, consisting in the smearing of the various projections back onto each other, along the X-rays directions. Let $B \subset \mathbb{Z}^n$ be a coordinate box. If $F \in \mathcal{F}^{(k)}(B)$, then F is the intersection of $n-k$ bounding hyperplanes. The outer normal unit vectors to the bounding subspaces at F span a closed cone Γ_F^{n-k} of dimension $n-k$. For each point x_0 in the relative interior of F, let $\Gamma_F^{n-k}(x_0)$ be the *normal cone* to F at x_0, that is the image of Γ_F^{n-k} under the translation of the origin to x_0.

Definition 4. The *back-projection* of F in $\mathbb{R}^n$ is the set

$$b(F) = \bigcup_{x_0 \in \mathrm{relint} F} int\, \Gamma_F^{n-k}(x_0), \tag{2.7}$$

where relint$F$ means the relative interior of F.

□

If F is a facet of B, then $b(F)$ corresponds with the unbounded open region of $\mathbb{R}^n \setminus B$ delimited by F and by the bounding hyperplanes intersecting F. Therefore, by Remark 2, and according to Definition 1, we have

$$R(B) = \bigcup_{F \in \mathcal{F}^{(n-1)}(B)} (\mathrm{cl}\, b(F)) \setminus F.$$

Example 2. In Figure 4 the computation of $f(p)$ as in Theorem 3 is shown when P is a cube.

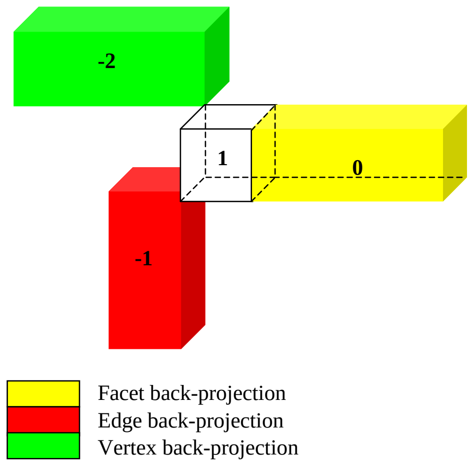

FIGURE 4. The sum of the ridge functions in a cube and in the back-projection of its faces.

Theorem 4 and Corollary 2 can be reformulated in the case of clusters of twisted coordinate boxes, so matching the requirement of few directions, typical of DT, with the back-projection algorithm of CT, which allows easy constructions of such clusters (see for instance Figure 5)

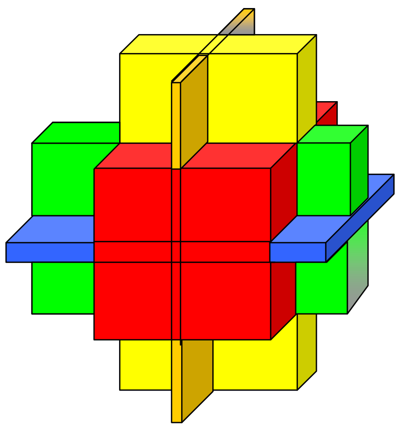

FIGURE 5. An example of cluster of twisted boxes in $\mathbb{Z}^3$

Approximations of 2-dimensional sections of natural shapes, also with curvilinear boundary, can also be considered by means of clusters of twisted polygons, and interesting related geometric problems could be formulated concerning special construction producing such clusters. See, for instance Figure 6, where different copies of a square, rotated about its center, have been mutually intersected. By employing more and more rotations the resulting cluster might be used to approximate a disc.

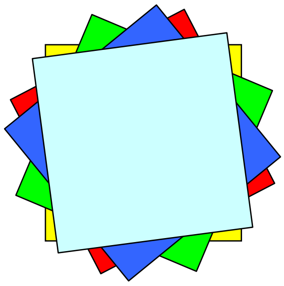

Figure 6. Cluster of five mutually intersecting squares.

3. Discrete $\mathcal{D}$–inscribable sets

The aim of this section is to investigate the geometric structure of clusters of polytopes in the planar case and show that our definition provides a discrete counterpart of the existing notion of $\mathcal{D}$-inscribable sets in the continuous case. We start recalling the original definition and some peculiar properties of this class of sets [20, Chapter 2].

Let $\mathcal{D}$ be a finite set of at least two distinct directions in $\mathbb{R}^2$. A convex body $K \subset \mathbb{R}^2$ (i.e. a compact convex set with non-empty interior) is $\mathcal{D}$-*inscribable* if each point p in the boundary of K is the vertex of a polygon inscribed in K (i.e. having all its vertices in the boundary of K) whose edges are parallel to some directions in $\mathcal{D}$. Such a polygon may be degenerate if it contains the intersection with K of each line through its vertices parallel to a direction in $\mathcal{D}$. In [19] the geometric structure of $\mathcal{D}$-inscribable sets was analyzed in order to show that $\mathcal{D}$-inscribable sets are $\mathcal{D}$-additive and hence $\mathcal{D}$-unique. For earlier results concerning the case where $\mathcal{D}$ consists of the coordinate directions, and additional information see [20, Note 2.3].

We now reinterpret the previous notion in a discrete setting, namely in the integer lattice $\mathbb{Z}^2$. We need some preliminaries. A *lattice line* is a line containing at least two points in $\mathbb{Z}^2$. A *lattice direction* is a lattice vector in $\mathbb{Z}^2 \setminus \{o\}$ whose coordinates are relatively prime. We refer to a finite subset of the integer lattice $\mathbb{Z}^2$ as a *lattice set*. A *convex lattice set* is a finite subset F of $\mathbb{Z}^2$ such that $F = (\operatorname{conv} F) \cap \mathbb{Z}^2$. A lattice set F is *line-convex* along a lattice direction $u \in \mathbb{Z}^n \setminus \{o\}$ if the intersection of any line parallel to u and F is a convex lattice set, possibly empty.

Given two distinct lattice directions $u, v \in \mathbb{Z}^2 \setminus \{o\}$, we denote by Z_k^{uv}, where $k \in \{0, 1, 2, 3\}$, the closed cones determined by the pairs of vectors $\{-u, -v\}$, $\{u, -v\}$, $\{u, v\}$, $\{-u, v\}$, respectively. For any point $p \in \mathbb{Z}^2$, the translation of Z_k^{uv} by p is denote by $Z_k^{uv}(p)$, and called a *quadrant* centred at p.

Let $\mathcal{D}$ be a finite set of at least two distinct directions and let F be a lattice set in $\mathbb{Z}^2$. The $\mathcal{D}$-*boundary* of F is the set of points $p \in F$ such that there exists a quadrant $Z_k^{uv}(p)$ centred at p, with $u, v \in \mathcal{D}$ and $k \in \{0, 1, 2, 3\}$, such that $(\operatorname{int} Z_k^{uv}(p)) \cap F = \emptyset$, see Figure 7. Let $p \in F$ be a point in the $\mathcal{D}$-boundary of F, a quadrant $Z_k^{uv}(p)$ such that $(\operatorname{int} Z_k^{uv}(p)) \cap F = \emptyset$ is called a *support quadrant* of F at p.

Definition 5. Let $\mathcal{D}$ be a finite set of at least two distinct directions and let F be a lattice set in $\mathbb{Z}^2$. A non-degenerate convex lattice polygon $P \subset F$, whose edges are parallel to some directions in $\mathcal{D}$, is called $\mathcal{D}$-*inscribed* if the following conditions hold:

(i) each vertex of P belongs to the $\mathcal{D}$-boundary of F;
(ii) the support quadrant of P at a vertex $p \in P$ bounded by the two semi-lines parallel to the edges of P issuing from p is a support quadrant of F at p.

A $\mathcal{D}$-inscribed polygon may degenerate to a segment parallel to a direction in $\mathcal{D}$. In this case it must contain the intersection with F of each line through its endpoints parallel to a direction in $\mathcal{D}$.

Notice that condition (ii) corresponds to the usual local condition for the tangent cones of two planar convex bodies touching each other and with one contained in the other one. Actually, (i) easily follows from (ii), but we prefer to give this extended definition in order to emphasize the analogy with the continuous case.

We now present the discrete notion of inscribable sets.

Definition 6. Let $\mathcal{D} \subset \mathbb{Z}^2$ be a finite set of at least two distinct lattice directions. A lattice set $F \subset \mathbb{Z}^2$ is said to be $\mathcal{D}$-*inscribable* if each point of F is contained in a $\mathcal{D}$-inscribed polygon.

Examples of $\mathcal{D}$-inscribable sets are shown in Figures 7, 8, 9, and 13.

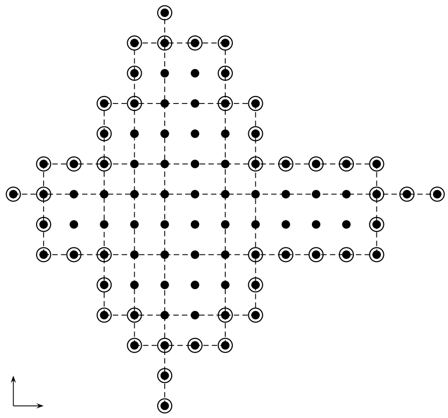

FIGURE 7. A $\mathcal{D}$-inscribable set, for $\mathcal{D} = \{e_1, e_2\}$ (the $\mathcal{D}$-boundary of the set consists of the circled points).

If all the $\mathcal{D}$-inscribed polygons of a $\mathcal{D}$-inscribable set F degenerate to segments parallel to some directions in $\mathcal{D}$ (see Figure 8), we say that F is a *degenerate* $\mathcal{D}$-inscribable set. A $\mathcal{D}$-inscribable set which contains a non-degenerate $\mathcal{D}$-inscribed polygon is called *non-degenerate*.

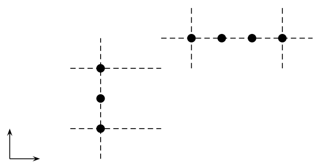

FIGURE 8. A degenerate $\mathcal{D}$-inscribable set, for $\mathcal{D} = \{e_1, e_2\}$.

It is worth remarking that our notion of $\mathcal{D}$-inscribable sets does not require any convexity property for the set. In fact, Figure 7 shows an example of a non convex lattice set which is $\mathcal{D}$-inscribable. There are also weaker convexity conditions that are relevant in discrete tomography, e.g. Q-convexity. A lattice set F is Q-*convex (quadrant-convex)*, with respect to a set $D = \{u, v\}$ of two distinct lattice directions, if $Z_k^{uv}(p) \cap F \neq \emptyset$, for all $k \in \{0, 1, 2, 3\}$, implies $p \in F$. A lattice set F is Q-convex with respect to a finite set D of directions if it is Q-convex with respect to every pair of directions in D. Notice that if F is Q-convex with respect to a finite set D of directions, then F is line-convex along any direction in D. For an introduction to the known results on Q-convex sets and their applications, see [7, 12, 13] . The example in Figure 7 is Q-convex with respect to $\mathcal{D} = \{e_1, e_2\}$, without being convex. Figure 9 shows an example of $\mathcal{D}$-inscribable set which is not Q-convex with respect to $\mathcal{D} = \{e_1, e_2, u = (1, 1)\}$, since it is not line-convex in the vertical direction.

On the other hand, if F is assumed to be convex, then Definition 6 is equivalent to require that each point in the boundary of F is vertex of an inscribed polygon, so matching the definition of the continuous case.

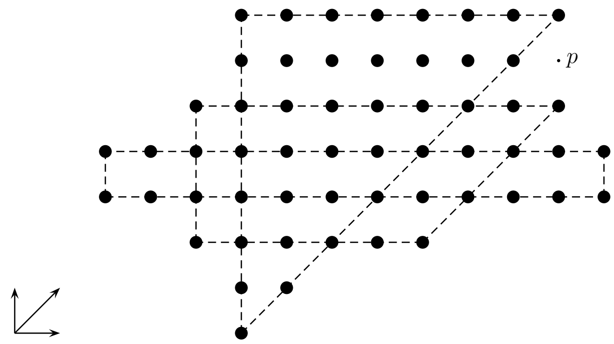

FIGURE 9. A $\mathcal{D}$-inscribable set which is not line-convex along the directions in $\mathcal{D} = \{e_1, e_2, u = (1, 1)\}$ (p does not belong to the set).

We now examine the link between the discrete notion of $\mathcal{D}$-inscribable sets and that of clusters of twisted polygons introduced in the previous section. We shall show that these two classes are the same, provided we extend the definition of clusters of twisted polygons by including degenerate polygons. In particular, let C be a finite union of polygons $P_1, ..., P_r$, not all degenerate, whose edges are parallel to some directions in $\mathcal{D}$. We say that C is a *$\mathcal{D}$-cluster of twisted polygons* if any two full dimensional polygons P_i, P_j are twisted polygons, according to Definition 2, and each degenerate polygon P_k of the union is a segment, parallel to some direction in $\mathcal{D}$, which contains the intersection with C of each line through its vertices parallel to a direction in $\mathcal{D}$.

Proposition 1. *Let $\mathcal{D} \subset \mathbb{Z}^2$ be a finite set of at least two distinct lattice directions and let $F \subset \mathbb{Z}^2$ be a lattice set. The following statements are equivalent*

(1) *F is a $\mathcal{D}$-cluster of twisted polygons.*
(2) *F is a non degenerate $\mathcal{D}$-inscribable set.*

Proof. Both these classes of sets consist of finite unions of polygons whose edges are parallel to some direction in $\mathcal{D}$. Moreover, the requirement for degenerate polygons to belong to one set in each class is the same. Thus the statements follows from the fact that condition (ii) in Definition 5, applied to any pair of non degenerate $\mathcal{D}$-inscribed polygons, is equivalent to the requirement of being a pair of twisted polygons. □

We now turn to the question of reconstruction of lattice sets from their X-rays. We have seen in the previous section that clusters of polytopes are $\mathcal{H}$-additive and hence also $\mathcal{H}$-unique. Since we use here a slightly more general notion we have to prove that such uniqueness result holds even in the case of $\mathcal{D}$-cluster of twisted polygons. To this end, we notice that in the planar case the additivity proof in the case of cluster of full dimensional polygons can be completed by adding also the degenerate case as follows.

Lemma 1. *Let $\mathcal{D}$ be a finite set of at least two distinct directions in $\mathbb{R}^2$. Each segment parallel to some direction in $\mathcal{D}$ is $\mathcal{D}$-additive.*

Proof. Consider a segment $[a,b]$, with end points $a,b\in\mathbb{R}^2$, parallel to a direction $u_k\in\mathcal{D}$. Let $L[a,b]$ denote the line through a and b. For each $u_j\in D$, where $j\neq k$, we denote by L_j the closed strip containing the segment $[a,b]$ and bounded by the lines through a,b, respectively, parallel to u_j.
For $p\in\mathbb{Z}^2$, we define

$$f_j(p)=\begin{cases}0 & \text{if } p\in L_j\\ -1 & \text{if } p\notin L_j\end{cases}\quad \text{for } j\neq k, \qquad \text{and} \qquad f_k(p)=\begin{cases}1 & \text{if } p\in L[a,b]\\ 0 & \text{if } p\notin L[a,b]\end{cases}$$

Notice that the function f_j, f_k are ridge functions, i.e. they are constant on each line parallel to u_j, u_k, respectively.
Then we define

$$f(p)=\sum_{j=1}^{m} f_j(p)$$

where m is the number of directions in D.

If $p\in[a,b]$ then $p\in L_j$ for all j, so that

$$f(p)=\sum_{j=1}^{m} f_j(x)=f_k(x)=1.$$

If $p\in L[a,b]\setminus[a,b]$ then $p\notin L_j$ for all j, so that

$$f(p)=\sum_{j\neq k} f_j(p)+f_k(p)=-(m-1)+1\leq 0,$$

as $m\geq 2$.

If $p\in(\bigcap_{j\neq k}L_j)\setminus[a,b]$ then $f(p)=0$. If $p\in\mathbb{R}^2\setminus(\bigcap_{j\neq k}L_j)$ then there exists $j_0\neq k$ such that $p\in\mathbb{R}^2\setminus L_{j_0}$, so that $f_{j_0}(p)=-1$ and

$$f(p)=f_{j_0}(p)+f_k(p)+\sum_{j\neq k,j_0} f_j(p)\leq -1+1+\sum_{j\neq k,j_0} f_j(p)\leq 0.$$

Therefore, it follows that

$$[a,b]=\left\{p\in\mathbb{R}^2: f(p)=\sum_{j=1}^{m} f_j(p)>0\right\},$$

and consequently the segment $[a,b]$ is $\mathcal{D}$-additive. □

Notice that if $p\in[a,b]$ then $f(p)=1$, and if $p\in(\bigcap_{j\neq k}L_j)\setminus[a,b]$ then $f(p)=0$. Therefore, in the planar case the proof of Theorem 4 can be easily extended to the case of cluster containing degenerate segments to get the following result.

Theorem 6. *Let $\mathcal{D}$ be a finite set of at least two distinct directions in $\mathbb{R}^2$. Every $\mathcal{D}$-cluster of twisted polygon is $\mathcal{D}$-additive.*

This results, together with Theorem 1 enable us to state the following uniqueness result.

Theorem 7. *Let $\mathcal{D}$ be a finite set of at least two nonparallel lattice directions. Then the class of non-degenerate $\mathcal{D}$-inscribable sets is $\mathcal{D}$-unique.*

Notice that Theorem 7 does not hold for degenerate $\mathcal{D}$-inscribable sets, as it is shown in Figure 10.
Figure 10 shows a typical situation of sets which are not $\mathcal{D}$-unique and form a so-called *switching component*. Switching components play a key role in the study of non $\mathcal{D}$-unique sets both in the continuous and discrete setting (see [20, Chapter 2] and [16]). When $\mathcal{D}=\{e_1,e_2\}$, non degenerate discrete $\mathcal{D}$-inscribable sets contain all the lattice sets corresponding to *maximal binary matrices* as characterized by Ryser [31].

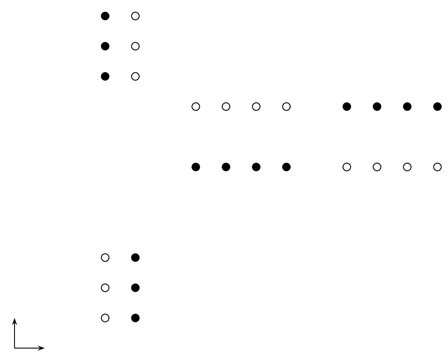

FIGURE 10. Two degenerate $\mathcal{D}$-inscribable sets $E = \{\bullet\}, F = \{\circ\}$ with the same X-rays in the directions in $\mathcal{D} = \{e_1, e_2\}$.

We wish to point out that the proof of $\mathcal{D}$-additivity for $\mathcal{D}$-inscribable sets given in the continuous case (see [19]) is more involved, with respect to the one presented here, and it is based on an interesting geometric property which we illustrate in what follows.
Let $\mathcal{D} = \{u_1, ..., u_m\}$ be a finite set of distinct lattice directions, where $m \geq 2$, arranged in order of increasing angle with the positive x-axis. Let P be a convex lattice polygon whose edges a_j, where $1 \leq j \leq n$, are parallel to some directions in $\mathcal{D}$. We label the edges a_j anticlockwise around the boundary of P, from the edge making the smallest angle with the positive x-axis. Let $u_{i_j} \in \mathcal{D}$ denote the direction of the edge a_j, we call the n-tuple $(i_1, ..., i_n)$ the *$\mathcal{D}$-type* of P. Two $\mathcal{D}$-inscribed polygons with same $\mathcal{D}$-type and an even number of edges have *interlacing boundaries* if all the edges of one polygon whose labels have same parity meet the boundary of the second polygon, and the remaining edges all lie outside. In other words, the consecutive edges of one polygon have alternatingly an empty and a non-empty intersection with the second polygon. Examples of hexagons with interlacing boundaries for $\mathcal{D} = \{e_1, e_2, u = (1,1)\}$ are illustrated in Figure 13. In [19] it was shown that if a planar convex body K is strictly convex, i.e. its boundary does not contain segments, then any two inscribed polygons with same $\mathcal{D}$-type and an even number of edges have interlacing boundaries. This properties holds even in the discrete setting, with strictly convexity assumption replaced by the following property, which represents a more restrictive notion of twisted polygons.

Definition 7. Let $P_1, P_2 \subset \mathbb{R}^2$ be non degenerate twisted polygons, and $R(P_1), R(P_2)$ be the free regions determined by P_1, P_2 respectively. Then P_1, P_2 are said to be *strictly twisted polygons* if the vertices of P_1, P_2 are contained in $\mathrm{int}R(P_2), \mathrm{int}R(P_1)$, respectively.

A *$\mathcal{D}$-cluster of strictly twisted polygons* is a cluster of polygons such that any two non degenerate polygons are strictly twisted.

Proposition 2. *Let $\mathcal{D} \subset \mathbb{Z}^2$ be a finite set of at least two distinct lattice directions and let F be a $\mathcal{D}$-cluster of strictly twisted polygons. Then*

(i) *two polygons of the cluster with same $\mathcal{D}$-type and an even number of edges have interlacing boundaries;*
(ii) *there exists at most one polygon in the cluster with a given $\mathcal{D}$-type and an odd number of edges.*

Proof. (i) Let P and Q be two strictly twisted polygons with same $\mathcal{D}$-type $(i_1, ..., i_n)$ and an even number n of edges. We label the vertices p_j, q_j of P, Q, respectively, anticlockwise around the boundary of P, Q, where $1 \leq j \leq n$. By assumption, either $q_j \in \mathrm{int}R(P)$ or $p_j \subset \mathrm{int}R(Q)$. We may assume $q_j \in \mathrm{int}R(P)$, up to exchanging the role of P and Q. Denote by A_j, where $j \in \{1, ..., n\}$, the component of the free region $\mathrm{int}R(P)$ bounded by the edge $a_j = [p_j, p_{j+1}]$ of P (see Figure 11).

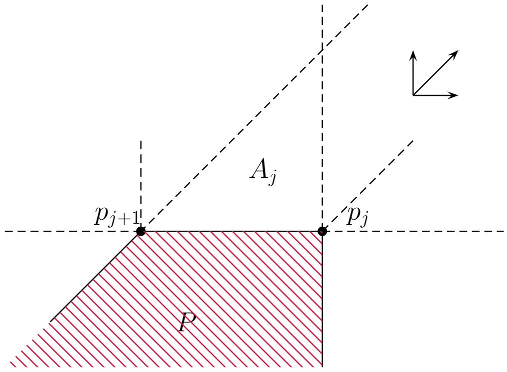

FIGURE 11

Let $b_j = [q_j, q_{j+1}]$ denote the edge of Q parallel to a_j. Since P and Q have same $\mathcal{D}$-type, then b_j is parallel to a_j, so that $b_j \subset A_j$. We have to show that both the edges b_{j-1}, b_{j+1} have nonempty intersection with a_j. Let us assume that one of the edges b_{j-1}, b_{j+1} does not intersect a_j, say b_{j+1}. Then the support quadrant of Q centered at q_{j+2}, bounded by the semi-line parallel to the edges b_{j+1}, b_{j+2}, contains the vertex p_{j+2}, which contradicts the assumption that P and Q are twisted (see Figure 12). In the same way we can show that b_{j-1} intersects a_j.

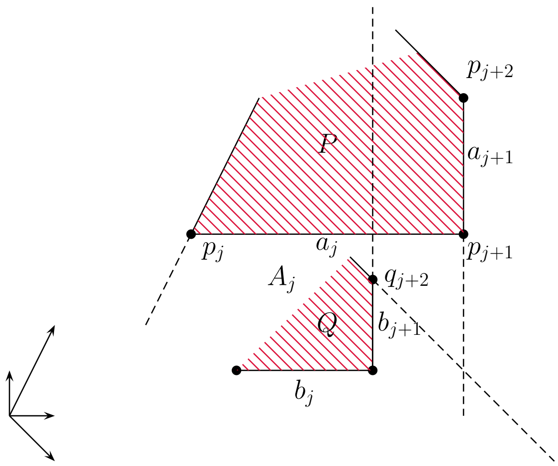

FIGURE 12

(ii) We first consider the non-degenerate case. Let P and Q be two distinct strictly twisted polygon with same $\mathcal{D}$-type and an odd number of edges. Then we can prove, as in the previous case, that two parallel edges $a_j \in P$ and $b_j \in Q$ do not intersect, whereas a_j meets b_{j+1} or b_j meets a_{j+1}, for each j. Since P and Q have an odd number of edges this yields a contradiction.
Let us now consider an inscribed segment Q parallel to $u \in \mathcal{D}$. Since F is a $\mathcal{D}$-cluster of twisted polygons, it contains a non-degenerate $\mathcal{D}$-inscribed polygon P. Since Q contains the intersection with F of each line through its endpoints parallel to a direction in $\mathcal{D}$, the endpoints of Q belong to different regions A_j, associated

to the polygon P as in (i). This also implies that there exists at most one inscribed segment parallel to $u \in \mathcal{D}$. $\square$

This proposition shows that the class of clusters of twisted polytopes represents a natural extension of the class of $\mathcal{D}$-inscribable sets.

Notice that two $\mathcal{D}$-inscribed polygons with different $\mathcal{D}$-types may have empty intersection, as it is shown in Figure 13

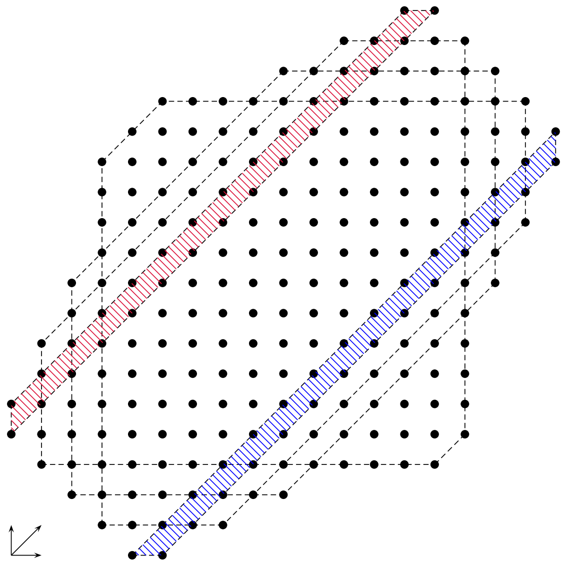

FIGURE 13. A $\mathcal{D}$-inscribable set, for $\mathcal{D} = \{e_1, e_2, u = (1,1)\}$, with two disjoint non-degenerate $\mathcal{D}$-inscribed polygons (the shaded ones).

4. Conclusion and perspectives

We have introduced a new class of discrete lattice sets, namely cluster of twisted polytopes, giving motivations to their importance in image reconstruction. It is worth remarking that questions of stability are also relevant for any inverse problem, since noise in the data cannot be avoided for all practical applications. Let us briefly mention some important results in DT on this topic. The papers [2, 3] show that the reconstruction of lattice sets from X-rays taken along more than two directions is highly unstable. This instability persists even when the X-rays uniquely determine the object. In [6], S. Brunetti and A. Daurat proved that if the sets are additive then a stability result holds when the error on the data is "small". In particular, they obtained an upper bound for the symmetric difference of two lattice sets depending on the distance of their X-rays and the maximal size of the sets. However, the additivity assumption is not enough to get general stability results if the number of directions is larger than two, as it is shown in [3], where the constructed counterexamples are additive. Experimental results in [6] suggest the conjecture that convex lattice sets are additive. This would imply a stability result which is in agreement with the continuous case, where the reconstruction of convex bodies is well posed [34]. Positive stability results when the X-rays are taken in two directions were obtained in [1], under the assumption that the error on the data is small. Recently, these results have been generalized and sharpened in [9, 10, 11]. In a second paper, which continues the present one, we show that the geometric structure of cluster of twisted polytopes leads to positive stability results which depend only on the data error, differently from all the positive known results, where the sizes of the sets are also involved.

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