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Pricing Credit Derivatives in a Wiener-Hopf Framework

Daniele Marazzina, Gianluca Fusai, and Guido Germano

Abstract We present fast and accurate pricing techniques for credit derivative contracts when discrete monitoring is applied and the underlying evolves according to an exponential Lévy process. Our pricing approaches are related to the Wiener-Hopf factorization, and their computational cost is independent of the number of monitoring dates. Numerical results are presented in order to validate the pricing algorithm. Moreover, an analysis on the sensitivity of the probability of default and the credit spread term structures with respect to the process parameters is considered.
Keywords Credit derivative, probability of default, Lévy process, discrete monitoring, Wiener-Hopf factorization, fast Fourier transform, Hilbert transform, Z-transform, Euler summation, Spitzer identity.

1 Introduction

In this article we present fast and accurate pricing techniques for credit derivatives, like defaultable coupon bonds, assuming that default is monitored only on discrete dates and the underlying evolves according to an exponential Lévy process. Our pricing approaches, which are related to the maturity randomization algorithm

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based on the Z-transform presented by Fusai et al. [7, 8], change the problem of computing the default probability into a Wiener-Hopf (WH) factorization problem. This can be solved numerically exploiting the fast Fourier transform (FFT) algorithm. Moreover, the computational cost of our procedures is independent of the number of monitoring dates: indeed exploiting the Euler summation, which is a convergence-acceleration technique for alternating series, we can speed up the inverse Z-transform.

The WH factorization can be dealt with from two different perspectives, analytical and probabilistic. The first approach is based on the solution of integral equations, whilst the second is based on the well known Spitzer identity [16].

The first method, that we call the Z-WH pricing method, is based on a Z-transform approach which converts the usual backward procedure into a set of independent integral equations of WH type. The idea consists in randomizing the contract expiry according to a geometric distribution of a (complex) parameter q [7, 8].[1] The randomization transforms the pricing problem into WH integral equations depending on a parameter q. The solution of these WH equations requires to factorize their kernel into a product of two functions, one analytic in an upper complex half plane, the other in a lower.

The second method, that we call the Z-WH-S method, is related to the Spitzer identity [16]. This identity factorizes the characteristic function of a random walk with iid increments as the product of the characteristic functions of the minimum and of the maximum of the random walk itself, and all random variables are stopped at a geometric random time. The Spitzer identity gives a probabilistic interpretation to these functions in terms of the characteristic function of the extrema of a geometrically stopped random walk. For a recent discussion see Bingham [3] and the references therein. We consider as application the pricing of defaultable bonds following the so called structural approach. This means that the issuer defaults as soon as the firm value falls below a preassigned barrier level. The computation of the default probability is thus computed applying the factorization algorithm to the Spitzer identity and the Z-transform inversion.

Finally, in this paper we show how the factorization and the inversion can be performed numerically with high speed and accuracy for a general Lévy process. In particular, the Z-transform inversion is performed as in [1], and in addition we implement it by using Euler acceleration. The WH factorization is instead based on a idea originally proposed in [9, 10].

Numerical results are presented in order to validate the pricing algorithm. Moreover, an analysis on the sensitivity of the probability of default with respect to the considered Lévy processes' parameters. The problem of computing the term structure of the default probability and of the credit spread is also addressed.

The structure of the paper is the following. In Section 2 we introduce the problem of computing the default probability and the price of a defaultable zero coupon and we describe the necessary recursive formula. In Section 3 we describe how to

[1] The interpretation of transforms as probabilities is also frequently called the method of collective marks and is usually attributed to van Dantzig [17]. See also Resnick [14, page 564].

exploit the FFT to obtain our fast and accurate pricing methods. Finally, in Section 4 we validate our procedures with numerical results, taking into consideration both the accuracy and the computational cost, and we analyze the sensitivity of the probability of default and of the credit spread term structures with respect to the model parameters.

2 Default probability and credit derivatives

We model default adopting a structural approach. The firm value S evolves according to a stochastic process and default will occur as soon as the value of the firm falls below a level L before or at maturity T. The boundary L can be related to bond covenants or to an optimality condition based on the value of equity claims or to the minimal firm value required to operate the company. Instead than with a standard geometric Brownian motion process used by Merton [12] and Black and Cox [4], here we model the firm value with an exponential Lévy process. This fact allows us to generate a term structure of credit spreads that does not tend to zero as the time horizon shortens, which cannot happen if we model the firm value according to a diffusion process. In addition, we assume that the default is monitored only at discrete dates, such as bond coupon payment or balance sheet reporting dates. For aim of simplicity, we assume that the N monitoring dates are equally spaced, so that $\Delta = T/N$ is the time interval between two successive monitoring dates. In the following, we denote with r the risk-free interest rate and we assume that the underlying asset S pays a continuous dividend q.

Given a standard filtered probability space, for $0 \leq t \leq T$ the firm value process $S(t)$ is represented as

$$S(t) = S(0)e^{X(t)},$$

where $X(t)$ is a Lévy process, i.e. a stochastically continuous process with independent and stationary increment, uniquely identified by its characteristic function

$$\Psi(\xi;\Delta) = \mathbb{E}_0\left[e^{i\xi X(\Delta)}\right]. \tag{1}$$

2.1 Default probability, defaultable coupon bond and credit spread

If L is the default barrier, the default occurs as soon as $S(t) \leq L$, t being a monitoring date. The quantity

$$z = -\log\frac{S(t)}{L}$$

is commonly known as distance to default. We can also define the default time as the first hitting time of the level L:

$$\tau = \min_{j=0,\dots,N} \{ j\Delta : S(j\Delta) \leq L \}$$

The probability of default $\mathbb{P}(\tau \leq j\Delta)$ is related to the distribution of the minimum value of the underlying asset. Indeed if we define

$$m_i := \min_{j=0,\dots,i} X(j\Delta), \quad i = 0, \cdots, N,$$

we have

$$\mathbb{P}(\tau \leq j\Delta) = \mathbb{P}(m_j \leq \log L).$$

The probability of default $p = \mathbb{P}(m_N \leq \log L)$, as well as its complement, the survival probability $1 - p$, are the key ingredients to price credit derivatives, like, for example, defaultable bonds and credit default swaps (CDS). A defaultable zero-coupon bond written on a risky asset S is a bond which at maturity T pays a unit notional if the asset price stays above the default threshold L, or pays the recovery fraction $R < 1$ of the notional otherwise (R could also be equal to 0).[2] The standard convention among academics and practitioners assumes that the recovery rate is a constant parameter, even if there is some empirical evidence of a negative correlation between default rates and recovery rate, see for example Altman et al. [2]. We will follow the standard industry practice, treating R as a constant parameter. Thus, once the default probability is computed and the recovery rate is assigned, the price $P_{\mathrm{d}}(T)$ of the defaultable zero-coupon with maturity T is

$$P_{\mathrm{d}}(T) = e^{-rT} \left(1 - p + Rp\right).$$

Given that the price of the risk-free zero-coupon is simply $P(T) = e^{-rT}$, the credit spread, i.e. the difference of the yield to maturity of the two zero-coupon bonds, defaultable and non-defaultable, with the same maturity, can be computed as

$$s(T) = -\frac{1}{T} \log \frac{P_{\mathrm{d}}(T)}{P(T)} = -\frac{1}{T} \log \big(1 - p(1 - R)\big).$$

2.2 Recursive valuation of the default probability

In order to compute the default probability at time T when N equally spaced monitoring dates are considered, we can define the function $v(x, j)$ through the recursion

$$\begin{aligned} v(x, j) &= \int_0^{+\infty} f(z - x; \Delta) v(z, j - 1) dz, \quad j = 1, \dots, N, \\ v(x, 0) &= \mathbf{1}_{x>0}, \end{aligned} \tag{2}$$

[2] This is the standard framework of the so called fractional recovery of face value. Another possibility, not considered here, is the fractional recovery of market value at default

where $\mathbf{1}_{x>0}$ is the indicator function which is equal to 1 if $x > 0$, 0 otherwise, and $f(\cdot;\Delta)$ is the transition probability density function of the log-return of the underlying asset. It holds that

$$\mathbb{P}(m_N \leq \log L) = 1 - v(x_0 - \log L, N).$$

See Fusai et al. [8] for details.

3 Fast pricing methods

The recursion in Section 2.2 can be used to compute directly the probability of default. However, faster and more accurate pricing techniques can be considered. More precisely, in Section 3.1 we present the Z-transform approach, that transform the recursive procedure presented in Section 2.2 into the problem of solving independent integral equations. Since this integral equations are of WH type, in Section 3.2 we introduce a fast solution method based on the WH factorization. Finally, in Section 3.3 we present a different, but related, approach for computing the probability of default based on the Spitzer identity. This approach seems to be the fastest and the most accurate. This fact is numerically discussed in Section 4.

In the following, we denote the Fourier transform of a function g with

$$\widehat{g}(\xi) = \mathscr{F}_{x\to\xi}[g(x)](\xi) = \int_{-\infty}^{+\infty} g(x)e^{i\xi x}dx$$

and the inverse transform with

$$g(x) = \mathscr{F}^{-1}_{\xi\to x}[\widehat{g}(\xi)](x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{g}(\xi)e^{-ix\xi}d\xi.$$

We recall that the characteristic function of the Lévy process, which is given in Eq. (1) and known analytically, satisfies

$$\Psi(\xi;\Delta) = \mathscr{F}_{x\to\xi}[f(x;\Delta)](\xi;\Delta).$$

Notice the exception of using Ψ in place of $\widehat{f}$. Furthermore we define the projection operators on the positive ($\mathscr{P}^+_x$) and negative ($\mathscr{P}^-_x$) real axis[3]

$$\mathscr{P}^+_x g(x) = \mathbf{1}_{x>0}\, g(x) \quad \text{and} \quad \mathscr{P}^-_x g(x) = \mathbf{1}_{x<0}\, g(x).$$

Finally, the index $+$ $(-)$ denotes a function analytic on an upper (lower) complex half plane including the real axis. For $\xi \in \mathscr{D}$, $\mathscr{D}$ being a suitable overlapping strip of the two half planes, we set

[3] For numerical purposes it is convenient to use the symmetric Heaviside step function $H(x)$ in place of the indicator function $\mathbf{1}_{x>0}$ and $1-H(x)$ in place of $\mathbf{1}_{x<0}$, the only difference being for $x = 0$, as $H(0) = 1/2 = 1 - H(0)$.

$$\widehat{g_+}(\xi) = \mathscr{F}_{x\to\xi}\left[\mathscr{P}_x^+ g(x)\right](\xi) := \int_0^{+\infty} g(x)e^{i\xi x}dx$$
$$\widehat{g_-}(\xi) = \mathscr{F}_{x\to\xi}\left[\mathscr{P}_x^- g(x)\right](\xi) := \int_{-\infty}^{0} g(x)e^{i\xi x}dx.$$

3.1 The Z-transform approach

Let us consider the recursive evaluation given in Eq. (2) of the function $v(x,j)$ and let us define the (unilateral) Z-transform $V(x,q)$ of $v(x,j)$ as

$$V(x,q) := \sum_{j=0}^{\infty} q^j v(x,j), \quad q \in \mathbb{C}. \tag{3}$$

The Z-transform can be considered a discrete-time relative of the Laplace transform; the reason for its name might be that usually $z = 1/q$ is used in its definition. If we apply the Z-transform to the recursive equation (2), it can be shown that the Z-transform $V(x,q)$ satisfies the integral equation

$$V(x,q) = q\int_0^{+\infty} f(z-x;\Delta)V(z,q)dz + \phi(x), \tag{4}$$

with the "forcing" function $\phi(x) := v(x,0) = \mathbf{1}_{x>0}$. The inverse Z-transform is given by an integral on a closed path around the origin; choosing a circle of radius $\rho < 1$,

$$v(x,N) = \frac{1}{2\pi\rho^N}\int_0^{2\pi} V(x,\rho e^{iu})e^{-iNu}du,$$

that can be approximated by

$$v_h(x,N) = \frac{1}{2N\rho^N}\left[V(x,\rho) + 2\sum_{j=1}^{N-1}(-1)^j \mathrm{Re}\,V(x,\rho e^{ij\pi/N}) + (-1)^N V(x,-\rho)\right]. \tag{5}$$

Setting $\rho = 10^{-\gamma/N}$ leads to an accuracy of $10^{-2\gamma}$ [8]. In our numerical tests we used $\gamma = 6$.

Therefore, in order to obtain $v_h(x,N)$, one must solve $N+1$ independent integral equations given by Eq. (4) with $N+1$ different values of the parameter q. Moreover, Fusai et al. [8] proposed to apply the Euler summation or acceleration [13], which is a convergence-acceleration technique well suited to evaluate alternating series. The idea of the Euler summation is to approximate $v_h(x,N)$ by the binomial average (also called the Euler transform)

$$v_h(x,N) \approx \frac{1}{2^{m_\mathrm{E}}N\rho^N}\sum_{j=0}^{m_\mathrm{E}}\binom{m_\mathrm{E}}{j} b_{n_\mathrm{E}+j}, \tag{6}$$

of the partial sums

$$b_k = \sum_{j=0}^{k} (-1)^j a_j \operatorname{Re} V\left(x, \rho e^{ij\pi/N}\right),$$

ranging from $k = n_{\mathrm{E}}$ to $k = n_{\mathrm{E}} + m_{\mathrm{E}}$, where $a_0 = 1/2$, $a_i = 1,\ i = 1, \ldots,$ and m_{E} and n_{E} are suitably chosen such that $m_{\mathrm{E}} + n_{\mathrm{E}} < N$. Thus the number of integral equations to be solved is $\min(N, m_{\mathrm{E}} + n_{\mathrm{E}}) + 1$. In our numerical experiments we set $m_{\mathrm{E}} = 20$ and $n_{\mathrm{E}} = 12$.

3.2 The Z-WH approach

In the previous section we have shown how the recursive pricing problem presented in Section 2.2 can be transformed into the solution of integral equations of WH type, Eq. (4). In this section we introduce an algorithm by Henery [9] well suited to solve this type of integral equations.

The main steps for solving the WH integral equations (4) are:

1. Assign q, the forcing function $\phi(x)$ and the characteristic function $\Psi(\xi; \Delta)$ and define

$$L(\xi, q) := \frac{1}{1 - q\Psi(\xi; \Delta)}, \quad \widehat{\phi_+}(\xi) := \mathscr{F}_{x \to \xi}\left[\mathscr{P}_x^+ \phi(x)\right](\xi).$$

2. Factorize the function $L(\xi, q)$ into a product of two functions (analytical in an upper or lower complex half plane),

$$L(\xi, q) = L_+(\xi, q) L_-(\xi, q), \tag{7}$$

and thus

$$\log L(\xi, q) = \log L_+(\xi, q) + \log L_-(\xi, q).$$

3. Define $C(\xi, q) := L_-(\xi, q)\widehat{\phi_+}(\xi)$ and decompose it into components analytical in the appropriate complex half plane,

$$C(\xi, q) = C_+(\xi, q) + C_-(\xi, q).$$

4. The solution is now given by the inverse Fourier transform

$$V(x, q) = \mathscr{F}^{-1}_{\xi \to x}[C_+(\xi, q) L_+(\xi, q)](x, q). \tag{8}$$

The conditions under which the factorization or decomposition gives proper results are given by Krein [11]. In fact, the major difficulty in the analytic solution of the WH equation lies in the factorization, that is known only for a few types of analytic functions. The most important condition for the above scheme is that $q\Psi(\xi; \Delta)$ must not be close to 1 anywhere, otherwise the function $L(\xi, q)$ to be factorized diverges. Provided this condition is fulfilled, one benefit of using numerical methods

is that one is no longer restricted to functions for which analytic factorizations are possible. The second advantage of the proposed method is that the basic building block of this numerical solution of the WH equation is the Fourier transform, which can be implemented conveniently via the FFT.

3.2.1 Factorization

As shown above, the main step of the proposed solution algorithm is the computation of the WH factorization. In fact, in order to implement the solution of the transformed equation, we need to factorize the function $L(\xi,q)$, as in Eq. (7), into a product of two functions which are analytic in the overlap of the two half planes. This factorization can be done through the Hilbert transform $\mathscr{H}_\xi$ of $\log L(\xi,q)$ [9],

$$\begin{aligned} L_+(\xi,q) &= \exp\left[\frac{1}{2i}\mathscr{H}_\xi \log L(\xi,q)\right] \\ &:= \exp\left[\frac{1}{2\pi i}\,\text{p.v.}\int_{-\infty}^{+\infty}\frac{\log L(\xi',q)}{\xi'-\xi}d\xi'\right], \qquad \operatorname{Im}\xi' < \operatorname{Im}\xi, \quad \xi' \in \mathscr{D}, \end{aligned}$$

where p.v. denotes the principal value, i.e., the value of a complex function along one chosen branch, in order to make the function single-valued. It can be shown that, since $\operatorname{Im}(\xi-\xi') > 0$,

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\log L(\xi',q)}{i(\xi'-\xi)}d\xi' = \int_{-\infty}^{+\infty}\mathbf{1}_{x>0}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}\log L(\xi',q)e^{-ix\xi'}d\xi'\right)e^{i\xi x}dx.$$

Therefore, we have

$$\log L_+(\xi,q) = \mathscr{F}_{x\to\xi}\left[\mathscr{P}_x^+\,\mathscr{F}^{-1}_{\xi\to x}\log L(\xi,q)\right](\xi,q). \tag{9}$$

This expression suggests the use of the FFT as a numerical tool for computing $L_+(\xi,q)$: indeed, given the function $L(\xi,q)$, the factorization of $\log L(\xi,q)$ can be performed through the sequence inverse Fourier transform, projection on the positive real axis, and Fourier transform. In practice, we have to use twice the FFT algorithm. For details, see Henery [9].

3.2.2 The algorithm and its computational cost

Summarizing, the solution of the WH equation (4) can be computed as in Eq. (8),

$$V(x,q) = \mathscr{F}^{-1}_{\xi\to x}[C_+(\xi,q)L_+(\xi,q)](x,q), \tag{10}$$

where, using the factorization formula (9), C_+ and L_+ can be computed as follows:

1. $L(\xi,q) := 1/(1-q\Psi(\xi;\Delta))$;

2. $L_+(\xi,q) = \exp\left\{\mathscr{F}_{x\to\xi}[\mathscr{P}_x^+ \mathscr{F}_{\xi\to x}^{-1} \log L(\xi,q)](\xi,q)\right\}$;
3. $L_-(\xi,q) = \exp\left\{\mathscr{F}_{x\to\xi}[\mathscr{P}_x^- \mathscr{F}_{\xi\to x}^{-1} \log L(\xi,q)](\xi,q)\right\} = L(\xi,q)/L_+(\xi,q)$;
4. $\widehat{\phi_+}(\xi) := \mathscr{F}_{x\to\xi}[\mathscr{P}_x^+ \phi(x)](\xi)$;
5. $C_+(\xi,q) = \mathscr{F}_{x\to\xi}[\mathscr{P}_x^+ \mathscr{F}_{\xi\to x}^{-1}[L_-(\xi,q)\widehat{\phi_+}(\xi)]](\xi,q)$.

The computational cost of this Z-WH pricing procedure is of 6 Fourier transforms for each integral equation, i.e., $6(\min(N, n_{\mathrm{E}}+m_{\mathrm{E}})+1)\, m \log m$ operations. However, we can easily decrease the computational cost applying the Z-transform inversion directly in the Fourier space, i.e., considering Eqs. (5) and (10), if we define

$$\widehat{V}(\xi,q) := C_+(\xi,q)L_+(\xi,q),$$

it holds that

$$v_h(x,N) = \frac{1}{2N\rho^N}\mathrm{Re}\,\mathscr{F}_{\xi\to x}^{-1}\left[\widehat{V}(\xi,\rho) + 2\sum_{j=1}^{N-1}(-1)^j \widehat{V}(\xi,\rho e^{ij\pi/N}) + (-1)^N \widehat{V}(\xi,-\rho)\right](x,N),$$

or, using the Euler summation (6)

$$v_h(x,N) \approx \frac{1}{2^{m_{\mathrm{E}}}N\rho^N}\mathrm{Re}\,\mathscr{F}_{\xi\to x}^{-1}\left[\sum_{j=0}^{m_{\mathrm{E}}}\binom{m_{\mathrm{E}}}{j} b_{n_{\mathrm{E}}+j}(\xi)\right],$$

where

$$b_k(\xi) = \sum_{j=0}^{k}(-1)^j a_j\, \widehat{V}\left(\xi,\rho e^{ij\pi/N}\right).$$

Moreover, the computation of $\widehat{\phi_+}(\xi) = \mathscr{F}_{x\to\xi}[\mathscr{P}_x^+\phi(x)](\xi)$ can be performed only once. Thus to compute the default probability $4(\min(N, n_{\mathrm{E}}+m_{\mathrm{E}})+1)+2$ FFTs are necessary.

3.3 The Z-WH-S approach

In this section we discuss how to compute the probability of default exploiting the Spitzer identity. The Spitzer identity [16] is strictly related to the solution of WH integral equations. Indeed, in the Z-WH approach integral equations are solved computing a decomposition into two functions, one analytic in an upper complex half plane, the second one in a lower complex half plane; the two half planes overlap in a strip that includes the real axis. The Spitzer identity consists in providing a probabilistic interpretation of these two functions in terms of the characteristic function of the minimum of a geometrically stopped random walk. More precisely, the Spitzer

identity states that

$$\sum_{j=0}^{+\infty} q^j \mathbb{E}_0\left[e^{i\xi m_j}\right] = L_+(0,q)L_-(\xi,q),$$

where L_+ and L_- have been defined previously in Section 3.2. Thus, once the WH factorization is computed as above, we can invert the Z-transform as in Eq. (6) exploiting the Euler acceleration, obtaining an approximation of the characteristic function of the discrete minimum m_N. Finally, applying an inverse Fourier transform, we obtain the probability density function of the minimum $p_{m_N}(x;\Delta)$, i.e., if $N > n_\mathrm{E} + m_\mathrm{E}$,

$$p_{m_N}(x;\Delta) \approx \frac{1}{2^{m_\mathrm{E}} N \rho^N} \mathrm{Re}\, \mathscr{F}^{-1}_{\xi\to x}\left[\sum_{j=0}^{m_\mathrm{E}} \binom{m_\mathrm{E}}{j} b_{n_\mathrm{E}+j}(\xi)\right],$$

where

$$b_k(\xi) = \sum_{j=0}^{k} (-1)^j a_j\, L_+\left(0, \rho e^{ij\pi/N}\right) L_-\left(\xi, \rho e^{ij\pi/N}\right).$$

Summarizing, the Z-WH-S algorithm is the following.

1. Compute the WH factorization as suggested above:
   a. $L(\xi,q) := 1/(1-q\Psi(\xi;\Delta))$;
   b. $L_-(\xi,q) = \exp\left\{\mathscr{F}_{x\to\xi}[\mathscr{P}^-_x \mathscr{F}^{-1}_{\xi\to x} \log L(\xi,q)](\xi,q)\right\}$;
   c. $L_+(0,q) = L(0,q)/L_-(0,q)$.
2. Compute the inverse of the Z-transform exploiting the Euler acceleration, obtaining the characteristic function of the minimum m_N.
3. Compute the inverse Fourier transform of the characteristic function of the minimum to obtain its distribution $p_{m_N}(\cdot;\Delta)$.
4. Compute the cumulative density function $l \to \mathbb{P}(m_N \le l)$ and thus the default probability.

In this case we have to compute only $2(\min(N, n_\mathrm{E}+m_\mathrm{E})+1)+1$ Fourier transforms.

4 Numerical experiments

In this section we present numerical results in order to validate the pricing algorithm. More precisely, in Section 4.1 we price a defaultable bond with the Z-WH and the Z-WH-S methods, for different maturities, comparing both the accuracy and the computational costs. As a benchmark, we consider the method of Feng and Linetsky [6]. Moreover, an analysis on the sensitivity of the probability of default and the credit spread with respect to the model's parameters is considered in Section 4.2. In our numerical experiments we set the risk-free rate $r = 0.05$ and we assume that

the underlying asset pays a dividend equal to $q = 0.02$. The asset price at time 0 is $S(0) = 1$, the default barrier is $L = 0.3$ and the recovery is $R = 0.5$. All the numerical experiments have been performed with Matlab R2009b running under Windows 7 on a personal computer equipped with an Intel Dual-Core 2.70 GHz processor and 4 GB of RAM. The characteristic exponents of the considered Lévy processes are reported in Table 1.

Table 1 Characteristic exponents of some parametric Lévy processes.

Model	Characteristic Exponent
G	$-\frac{1}{2}\sigma^2\omega^2$
NIG	$-\delta\left(\sqrt{\alpha^2-(\beta+i\omega)^2}-\sqrt{\alpha^2-\beta^2}\right)$
CGMY	$C\Gamma(-Y)\left((M-i\omega)^Y-M^Y+(G+i\omega)^Y-G^Y\right)$
DE	$-\frac{1}{2}\sigma^2\omega^2+\lambda\left(\frac{(1-p)\eta_2}{\eta_2+i\omega}+\frac{p\eta_1}{\eta_1+i\omega}-1\right)$
JD	$-\frac{1}{2}\sigma^2\omega^2+\lambda\left(e^{i\omega\alpha-\frac{1}{2}\omega^2\delta^2}-1\right)$

4.1 Numerical validation

In Table 2 we report the default probabilities p and the defaultable zero-coupon bond prices considering different maturities T and 52 monitoring dates a year. We assume that the underlying asset evolves as an exponential Normal-Inverse Gaussian (NIG) process with the same parameters as those used by Feng and Linetsky [6], i.e., $\alpha = 5,\ \beta = -1,\ \gamma = 0.75$. Finally, we denote with m the number of grid points used to compute the FFT. Our results are tested considering as a benchmark the method of Feng and Linetsky [6] based on the Hilbert transform (HILB).

Table 2 Defaultable zero-coupon bonds with weekly monitoring: price and CPUtime (in seconds).

		Z-WH			Z-WH-S			HILB		
T	m	p (%)	price	CPUt	p (%)	Price	CPUt	p (%)	price	CPUt
1	2^{12}	0.9335	0.946789	0.17	0.9329	0.946792	0.13	0.9330	0.946791	0.29
1	2^{14}	0.9330	0.946791	0.66	0.9330	0.946791	0.48	0.9330	0.946791	0.58
2	2^{12}	4.5656	0.884181	0.17	4.5650	0.884184	0.14	4.5652	0.884183	0.35
2	2^{14}	4.5652	0.884183	0.68	4.5652	0.884183	0.46	4.5652	0.884183	0.88
5	2^{12}	21.5713	0.694801	0.18	21.5702	0.694806	0.13	21.5703	0.694805	0.58
5	2^{14}	21.5703	0.694805	0.69	21.5703	0.694805	0.43	21.5703	0.694805	1.99
10	2^{12}	42.9653	0.476321	0.17	42.9663	0.476228	0.14	42.9670	0.476226	0.73
10	2^{14}	42.9624	0.476240	0.70	42.9670	0.476226	0.44	42.9670	0.476226	3.45

As expected, the HILB method rapidly reaches a six decimal digits accuracy due to its exponential convergence [6]. However our methods show a similar accuracy, and have a computational cost independent of the number of monitoring dates, as shown in Table 2. Thus the Z-WH and Z-WH-S algorithms appear to be the best methods when more than 2 years with weekly monitoring (i.e., 104 monitoring dates) are considered. Finally, the Z-WH-S method appears to outperform the Z-WH algorithm in both accuracy (especially with a small number of grid points) and speed. We recall that the Z-WH-S method requires approximately $2\min(N, m_{\mathrm{E}} + n_{\mathrm{E}})\, m \log m$ operations, while the Z-WH $4\min(N, m_{\mathrm{E}} + n_{\mathrm{E}})\, m \log m$.

4.2 Credit spread term structure

In Figure 1 we show the default probability and the credit spread term structures. We consider a weekly monitoring and 30 different maturities from 2 weeks to 30 years, comparing the HILB method with the Z-WH-S with $m = 2^{14}$ grid points. The two methods provide the same term structure, however the CPU time necessary to obtain the term structures is 9.35 seconds with the Z-WH-S algorithm and 47.82 seconds with the HILB method.

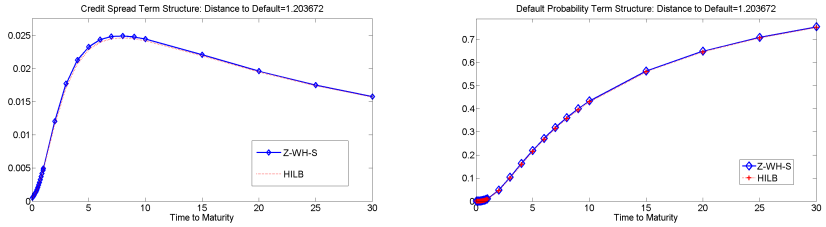

Fig. 1 Credit spread (left) and default probability (right) term structure.

In Figure 2 we show the default probability and the credit spread term structures, assuming that the underlying asset evolves according to a Gaussian process with different values of the volatility σ. The term structures are computed with the Z-WH-S method with $m = 2^{14}$ grid points. In this figure we can see how both the default probability and the credit spread increase as the parameter σ increases, as we should expect.

In Figure 3 we report similar plots considering the NIG process and four different sets of parameters:

1. $\alpha = 5, \quad \beta = -1, \quad \gamma = 0.75;$
2. $\alpha = 2, \quad \beta = -1, \quad \gamma = 0.75;$
3. $\alpha = 5, \quad \beta = -0.5, \quad \gamma = 0.75;$
4. $\alpha = 5, \quad \beta = -1, \quad \gamma = 1.$

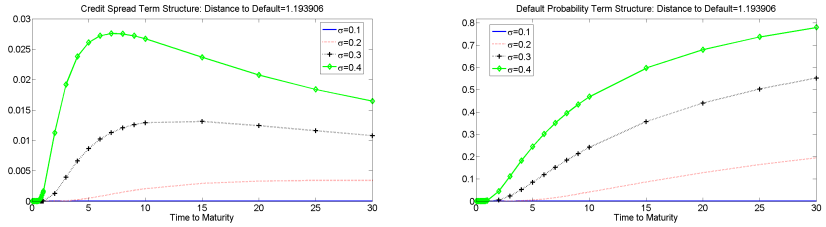

Fig. 2 Term structure of the credit spread (left) and of the default probability (right) when the log-firm value evolves according to a Gaussian process.

We recall that for the NIG process the variance is given by $\alpha^2\delta(\alpha^2-\beta^2)^{-1.5}$. Thus the four sets of parameters have variance equal to 0.1599, 0.5774, 0.1523 and 0.2126, respectively. In Figure 3 we see a behavior similar to the one in Figure 2: increasing the variance, both term structures increase.

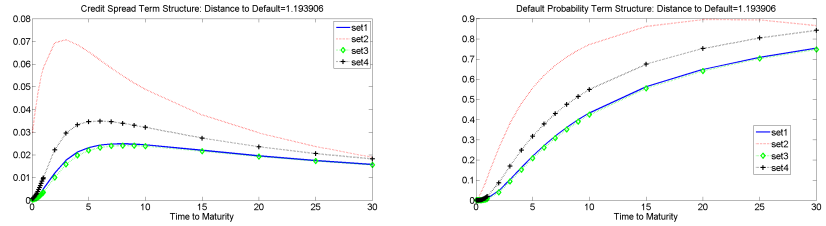

Fig. 3 Term structure of the credit spread (left) and of the default probability (right) when the log-firm value evolves according to a NIG process.

Finally in Figures 4-5 we compare the term structures for different Lévy processes, assuming again a weekly monitoring. More precisely, in Figure 4, in order to make the term structures comparable across models, the parameters (see Table 1) have been chosen assuming that the CGMY model, as estimated by Schoutens [15], is the true one:

$$C = 0.0244,\ G = 0.0765,\ M = 7.5515,\ Y = 1.2945.$$

Therefore, we calibrate the other models by minimizing the square integrated difference between the real part of the characteristic functions of the CGMY and the other models. Thus the calibrated parameters for the Merton jump diffusion (JD) model are $\sigma = 0.126349,\ \alpha = -0.390078,\ \lambda = 0.174814$ and $\delta = 0.338796$, while the calibrated parameters for the Kou double exponential (DE) model are $\sigma = 0.120381,\ \lambda = 0.330966,\ p = 0.20761,\ \eta_1 = 9.65997$ and $\eta_2 = 3.13868$. The risk-free rate is 3.67% per year, and the dividend yield is set equal to zero.
We recall that considering non-Gaussian processes allows us to generate term structures of credit spreads that do not tend to zero as the time horizon shortens: this can be easily seen in Figure 4.

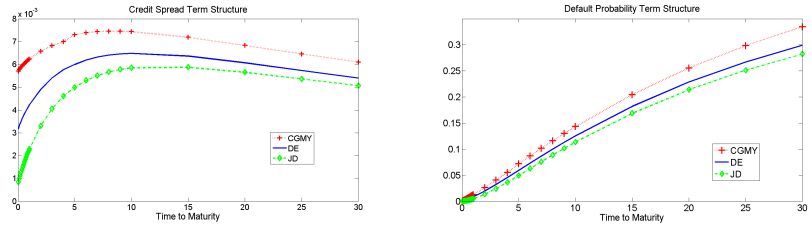

Fig. 4 Term structure of the credit spread (left) and of the default probability (right) when the log-firm value evolves according to a CGMY, DE or JD process.

In Figure 5 we consider the parameters used by Černý and Kyriakou [5], where the models are calibrated in order to achieve volatility 0.3, and, for the non-Gaussian distributions, skewness -0.5 and kurtosis 3.7; more precisely, for the Gaussian process (G) we set $\sigma = 0.3$, for the NIG model $\alpha = 12.34$, $\beta = -5.8831$, $\gamma = 0.7543$, and finally for the CGMY model $C = 0.6509$, $G = 5.853$, $M = 18.27$, $Y = 0.8$. The risk-free rate is 4% per year, and the dividend yield is set equal to zero.

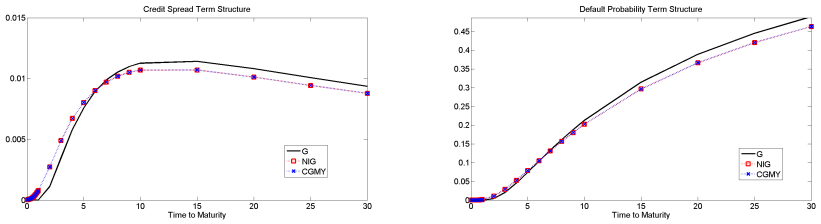

Fig. 5 Term structure of the credit spread (left) and of the default probability (right) when the log-firm value evolves according to a CGMY, NIG or Gaussian (G) process.

Figure 5 shows that, once the different Lévy processes are calibrated in order to exhibit the same variance, skewness and kurtosis, the term structures of credit spreads and default probabilities are very similar. Therefore, the model risk related to the choice of different exponential Lévy processes is quite limited. We also notice a difference with respect to the Gaussian case. For example, from Figure 5 we observe a higher (lower) probability of default and a higher (lower) credit spread for small (large) maturities. This behavior is still related to the presence of jumps that generate a different behavior as the time horizon shortens with respect to the Gaussian case.

5 Conclusions

In this article, adopting a structural approach and assuming that the firm value evolves according to an exponential Lévy process, we have proposed two algorithms

for computing the default probability based on the Wiener-Hopf factorization: the Z-WH method, in which a set of integral equations has to be solved, and the Z-WH-S ones, which is based on the Spitzer identity. We have shown that the two methods are accurate and fast, and that they are convenient when a large number of monitoring dates is considered, since the computational cost is independent of this number thanks to the Euler acceleration which improves the Z-transform inversion. Moreover, we have provided numerical experiments to validate the algorithms. Finally, we have shown how to compute the probability of default and the price of defaultable zero-coupon bonds and the corresponding term structure of credit spreads, performing also some comparative analysis across different Lévy processes.

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