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A CLASS OF P-CONVEX SPACES LACKING NORMAL STRUCTURE

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Abstract. We prove that, for any $\beta > 1$, the space $E_\beta = (l^2, \|\cdot\|_\beta)$ where $\|\cdot\|_\beta = \max\{\|\cdot\|_2, \beta\|\cdot\|_\infty\}$ is P-convex. It is known that, for $\beta \geq \sqrt{2}$, E_β lacks normal structure.

Introduction

The problem whether every superreflexive Banach spaces enjoys the fixed point property (fpp in short) for nonexpansive mappings is a classical open problem in Fixed Point Theory. Two subclasses of superreflexive spaces, both defined by geometric properties of the unit ball, in which the fpp has been widely studied are the class of uniformly nonsquare spaces and the class of P-convex spaces. Each of them includes uniformly convex spaces as well as uniformly smooth spaces.

While the fpp for nonexpansive mappings in uniformly nonsquare spaces has al last been proved by García-Falset, Llorens-Fuster and Mazcuñán-Navarro in [5], the problem is still open in P-convex spaces.

The notion of P-convex space has been introduced by Kottman in [7] as an evaluation of the efficiency of the tightest packing of balls of equal size in the unit ball of X. Kottman proved that the condition is weaker at the same time than uniform convexity and uniform smoothness, but still guarantees reflexivity: moreover he characterized the dual property, called F-convexity.

Naidu and Sastri [11] proved that P- and F-convexity are actually different, that neither one implies uniform nonsquareness nor is implied by uniform nonsquareness, and that both P-convexity and uniform nonsquareness imply a weaker property, that they called O-convexity, which in turn implies superreflexivity.

Recently, the fpp for nonexpansive mappings has been established in the duals of P-convex spaces (F-convex spaces) by Saejung [13], and then in the wider class of duals of O-convex spaces by Dowling, Randrianantoanina and Turett [4]. It is worth noting that Saejung obtained his result proving that duals of P-convex spaces have uniform normal structure, a property which assures the fpp for nonexpansive maps.

Therefore the question naturally arises whether P-convex spaces must have uniformly normal or at least normal structure (it is known that these properties are not self-dual).

Here we show that this is not true, even for normal type properties which, though still assuring the fpp, are weaker than normal structure. Precisely we prove that a family of renormings of l^2, the spaces $E_\beta = (l^2, \|\cdot\|_\beta)$ where $\|\cdot\|_\beta = \max\{\|\cdot\|_2, \beta\|\cdot\|_\infty\}$ are all P-convex. It is known that these spaces have (uniform) normal structure if and only if

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$\beta < \sqrt{2}$. When $\sqrt{2} \leq \beta < 2$ they have asymptotic normal structure, a weaker property still assuring the fpp for nonexpansive mappings, and for $\beta \geq 2$ they lack any kind of normal structure. Therefore the spaces E_β for $\beta \geq \sqrt{2}$ provide, as far as we know, the first examples of P-convex spaces without normal structure.

As it was proved (see [6], [2] for $\sqrt{2} \leq \beta \leq 2$ and [8] for $\beta > 2$) that all the E_β's do have the fpp for nonexpansive mappings, our result does not provide any answer about the fpp for nonexpansive mappings in P-convex spaces.

1. Notation and definitions

Throughout this paper, X denotes an infinite dimensional real Banach space, B_X and S_X its unit ball and unit sphere respectively, $B(x, r)$ the ball centered in x with radius r.

For a set A we denote by $|A|$ the cardinality of the set and by $diam(A)$ its diameter and we denote by $\lfloor a \rfloor$ the integer part of a real number a.

We recall the relevant definitions.

For $\beta > 1$ let $E_\beta = (l^2, \|\cdot\|_\beta)$ be the space l^2 renormed according to

$$\|x\|_\beta = \max\{\|x\|_2, \beta\|x\|_\infty\}$$

where $\|x\|_2, \|x\|_\infty$ denote respectively the l^2 and l^∞ norms of x.

For each cardinal α, let

$$P(\alpha, X) = \sup\{r : there\, exist\; \alpha\; disjoint\, B(x_\alpha, r) \subset B_X\}$$

(set $0 = \sup \emptyset$).

Following Kottman [7], we say that a space is *P-convex* if $P(n, X) < \frac{1}{2}$ for some positive integer n.

For a set A, the *separation* of A is the number

$$\text{sep}(A) = \inf\{\|x - y\| : x, y \in A\}$$

Considering sequences $\{x_n\} \subset X$, Kottman [7] defined

$$K(X) = \sup\{\text{sep }(\{x_n\}) : \{x_n\} \subset S_X\}.$$

$K(X)$ is called *Kottman's separation constant* of X and it is actually the *separation measure of noncompactness* of S_X.

Clearly $P(n, X) \geq P(n + 1, X) \geq P(\aleph_0, X)$.

It follows from [7] and [12] that $P(\aleph_0, X) = \frac{1}{2}$ if and only if $K(X) = 2$. Therefore P-convexity implies $K(X) < 2$.

For sets $A, B \subset X$ and $x \in X$ we set

$$r(A, x) = \sup\{\|y - x\|; y \in A\} \quad \text{and} \quad r(A, B) = \inf\{r(A, x); x \in B\}$$

$r(A, B)$ is called the *Chebyshev radius* of A with respect to B.

A space X has *normal structure* if for each nonempty, closed, bounded, convex set C $r(C, C) < diam(C)$ and *uniform normal structure* if there exists $N(X) < 1$ such that, for each such C, $r(C, C) < N(X)\, diam(C)$

For $\varepsilon \in [0,2]$ we call *modulus of convexity* of X the function

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \,:\, x, y \in S_X;\ \|x-y\| \geq \varepsilon\right\}.$$

A space X is *uniformly convex* if $\delta_X(\varepsilon) > 0$ for each $\varepsilon > 0$, and *uniformly non square* if $lim_{\varepsilon\to 2}\ \delta_X(\varepsilon) > 0$.

2. Results

In the next two lemmas, we prove that the existence in the unit ball of E_β of sets and sequences with large separation implies the existence of similar sets, with related cardinality, in the unit ball of $(l^2, \|\cdot\|_\infty)$.

Lemma 2.1. *Let $X = E_\beta$ ($\beta > 1$), $0 < \varepsilon < \frac{1}{16\beta^4}$ and $x_1, x_2, ..., x_n \in S_{E_\beta}$ such that*

$$\|x_i - x_j\|_\beta > 2 - \beta\varepsilon \quad \forall i, j = 1, ..., n;\ i \neq j : \tag{1}$$

then there exist $\lfloor \frac{n}{2} \rfloor$ indexes $\{i_j\}$ such that $\|x_{i_j} - x_{i_k}\|_\infty > \frac{2}{\beta} - \varepsilon \quad \forall j, k = 1, 2, ...\lfloor \frac{n}{2} \rfloor;\ j \neq k$.

Proof. As a first step we remark that, in S_{E_β}, (1) implies that for each i there exists at most one index j such that $\|x_i - x_j\|_2 > 2 - \beta\varepsilon$.

In fact, from

$$\|x_i - x_j\|_2^2 + \|x_i + x_j\|_2^2 = 2\|x_i\|_2^2 + 2\|x_j\|_2^2 \leq 4$$

we obtain

$$\|x_i + x_j\|_2^2 < 4 - (2 - \beta\varepsilon)^2 < 4\beta\varepsilon.$$

Assume that there exist two distinct indexes j, k such that

$$\|x_i - x_j\|_2 > 2 - \beta\varepsilon \quad \text{and} \quad \|x_i - x_k\|_2 > 2 - \beta\varepsilon :$$

then

$$\|x_j - x_k\|_2 = \|x_j + x_i - x_i - x_k\|_2 \leq \|x_j + x_i\|_2 + \|x_i + x_k\|_2 \leq 4\sqrt{\beta\varepsilon}$$

and

$$\|x_j - x_k\|_\beta \leq \max\{4\sqrt{\beta\varepsilon}, 4\beta\sqrt{\beta\varepsilon}\} = 4\beta\sqrt{\beta\varepsilon} < 2 - \beta\varepsilon$$

for ε sufficiently small (in particular for $\varepsilon < \frac{1}{16\beta^4}$), a contradiction proving our claim.

Now start with x_1 and let $x_{\bar{j}_1}$ be the only element (if any exists) such that $\|x_1 - x_{\bar{j}_1}\|_2 > 2 - \beta\varepsilon$; we drop $x_{\bar{j}_1}$ thus obtaining a set containing x_1 and (at least) $n-2$ elements $x_j, j \neq 1$, such that

$$\|x_1 - x_j\|_2 \leq 2 - \beta\varepsilon \quad \forall j \quad \text{and} \quad \|x_i - x_j\|_\beta > 2 - \beta\varepsilon \quad \forall i \neq j.$$

The first inequality implies that

$$\|x_1 - x_j\|_\infty > \frac{2}{\beta} - \varepsilon.$$

Set $x_1 = x_{j_1}$ and let j_2 be the first of the remaining indexes. Drop the element $x_{\bar{j}_2}$, if any, such that $\|x_{j_2} - x_{\bar{j}_2}\|_2 > 2 - \beta\varepsilon$. Iterating the procedure, after K steps, $K \leq \lfloor \frac{n}{2} \rfloor$, we have obtained K elements $x_{j_k} \in S_{E_\beta}$ such that

$$\|x_{j_k} - x_{j_l}\|_\infty > \frac{2}{\beta} - \varepsilon \quad \forall k, l = 1, 2, ..., K$$

and at least $n-2K$ residual elements x_l which satisfy

$$\|x_{j_k}-x_l\|_\infty > \frac{2}{\beta}-\varepsilon \quad \forall k=1,2,...,K \quad \text{and} \quad \forall l > j_K.$$

We can proceed in this way for at least $\lfloor \frac{n}{2} \rfloor$ steps; after $\lfloor \frac{n}{2} \rfloor$ steps drop the remaining elements and consider the set $x_{j_1}, x_{j_2}, ..., x_{j_{\lfloor \frac{n}{2} \rfloor}}$; each element has $\|x_{j_k}\|_\infty \leq \frac{1}{\beta}$ and the set is $(\frac{2}{\beta}-\varepsilon)$-separated with respect to $\|\cdot\|_\infty$. □

Lemma 2.2. *Let* $X = E_\beta \quad (\beta > 1)$*,* $0 < \varepsilon < \frac{1}{16\beta^4}$ *and* $\{x_n\}$ *a* $(2-\beta\varepsilon)$*-separated sequence in* S_{E_β}*; then there exists a subsequence* $\{x_{n_j}\}$ *such that* $\{\beta x_{n_j}\}$ *is* $(2-\beta\varepsilon)$*-separated in* $B_{(l^2, \|\cdot\|_\infty)}$*.*

Proof. Note that, starting with an infinite sequence, we can iterate the process in the proof of lemma 2.1 infinitely many times. □

Now we state our main result, and we prove that, for $\beta > 1$, E_β is P-convex (for $\beta \leq 1$, E_β coincides with l^2).

Theorem 2.3. *E_β is P-convex for each β.*

Proof. By contradiction, assume E_β is not P-convex; then, for each positive integer n, $P(n+1, E_\beta) = \frac{1}{2}$ and from [7], Theorem 1.3, for any $\varepsilon > 0$, there exist n points $x_1, x_2, ..., x_n \in S_{E_\beta}$ such that

$$\|x_i - x_j\|_\beta > 2-\beta\varepsilon \quad \forall i,j = 1,...,n;\ i \neq j.$$

Without loss of generality, we consider an even integer $2n$ and $0 < \varepsilon < \frac{1}{16\beta^4}$. Lemma 2.1 gives us n points that we denote again by $x_1, x_2, ..., x_n \in S_{E_\beta}$ such that

$$\|x_i - x_j\|_\infty > \frac{2}{\beta}-\varepsilon \quad \forall i,j = 1,...,n;\ i \neq j \tag{2}$$

Therefore, for any i, j, $i \neq j$, there exists k_{ij} such that

$$|x_i^{k_{ij}} - x_j^{k_{ij}}| > \frac{2}{\beta} - 2\varepsilon. \tag{3}$$

From $\|x_i\|_\beta \leq 1$ we have, for all $k \in N$, $|x_i^k| \leq \frac{1}{\beta}$ hence $|x_j^{k_{ij}}| > \frac{1}{\beta} - 2\varepsilon$ and, for ε small, $sign \quad x_i^{k_{ij}} \neq sign \quad x_j^{k_{ij}}$. Reasoning is symmetric in i and j hence we have proved that $\forall i, j, \quad |x_i^{k_{ij}} - x_j^{k_{ij}}| > \frac{2}{\beta} - 2\varepsilon$ implies

$$|x_i^{k_{ij}}| > \frac{1}{\beta} - 2\varepsilon \wedge |x_j^{k_{ij}}| > \frac{1}{\beta} - 2\varepsilon \wedge sign\, x_i^{k_{ij}} \neq sign\, x_j^{k_{ij}}. \tag{4}$$

Remark that for ε small and for any i, $\|x_i\|_2 \leq \|x_i\|_\beta \leq 1$ implies $|x_i^k| > \frac{1}{\beta} - 2\varepsilon$ for at most M distinct indexes k, and having taken $\varepsilon < \frac{1}{16\beta^4}$ we may choose an M which depends only on β.

Now fix i; we claim that for no pairs of points x_p and x_q there exists an index k such that

$$|x_i^k - x_p^k| > \frac{2}{\beta} - 2\varepsilon \wedge |x_i^k - x_q^k| > \frac{2}{\beta} - 2\varepsilon \wedge |x_p^k - x_q^k| > \frac{2}{\beta} - 2\varepsilon \tag{5}$$

i.e.

$$|x_i^k - x_p^k| > \frac{2}{\beta} - 2\varepsilon \wedge |x_i^k - x_q^k| > \frac{2}{\beta} - 2\varepsilon \implies |x_p^k - x_q^k| \leq \frac{2}{\beta} - 2\varepsilon \tag{6}$$

In fact, by (4), the first two inequalities in (5) imply

$$|x_p^k| > \frac{1}{\beta} - 2\varepsilon \wedge |x_q^k| > \frac{1}{\beta} - 2\varepsilon \wedge sign\, x_p^k = sign\, x_q^k$$

which together with $|x_p^k| \leq \frac{1}{\beta}$ and $|x_q^k| \leq \frac{1}{\beta}$ give

$$|x_p^k - x_q^k| = \max\{|x_p^k|, |x_q^k|\} - \min\{|x_p^k|, |x_q^k|\} < \frac{1}{\beta} - (\frac{1}{\beta} - 2\varepsilon) = 2\varepsilon$$

contradicting

$$|x_p^k - x_q^k| > \frac{2}{\beta} - 2\varepsilon \quad \text{if} \quad \varepsilon < \frac{1}{2\beta}.$$

Now start with x_1 and for x_j with $j = 2, ..., n$ let k_{1j} be as in (3); then by (4) we have $|x_1^{k_{1j}}| > \frac{1}{\beta} - 2\varepsilon$ for any j. This can be true only for at most M distinct k_{1j}'s, therefore there exist one index, which we call k_1, and a set R_1, with cardinality at least $\lfloor \frac{n-1}{M} \rfloor$, of indexes $\tilde{j}$ such that $|x_1^{k_1} - x_{\tilde{j}}^{k_1}| > \frac{2}{\beta} - 2\varepsilon$. By (6) for each couple $\tilde{j}, \tilde{k} \in R_1$ we have

$$|x_{\tilde{j}}^{k_1} - x_{\tilde{k}}^{k_1}| \leq \frac{2}{\beta} - 2\varepsilon. \tag{7}$$

Let j_2 be the first index in R_1 and, to simplify notation, set $x_1 = z_1$ and $x_{j_2} = z_2$. For any of the $\lfloor \frac{n-1}{M} \rfloor - 1$ remaining $\tilde{j}$'s in R_1, let $k_{2\tilde{j}}$ the index associated as in (3) to the couple z_2 and $x_{\tilde{j}}$. Note that, by (7), $k_{2\tilde{j}} \neq k_1$ for all $\tilde{j} \in R_1$.

Reasoning as above, we find an index k_2 and a set $R_2 \subset R_1$ with cardinality at least $\lfloor \left(\lfloor \frac{n-1}{M} \rfloor - 1\right) \frac{1}{M-1} \rfloor$ of indexes $\tilde{j}$ such that $|z_2^{k_2} - x_{\tilde{j}}^{k_2}| > \frac{2}{\beta} - 2\varepsilon$ for all $\tilde{j} \in R_2$. Again, by (6), for each couple $\tilde{j}, \tilde{k} \in R_2$ we have $|x_{\tilde{j}}^{k_2} - x_{\tilde{k}}^{k_2}| \leq \frac{2}{\beta} - 2\varepsilon$, hence

$$|x_{\tilde{j}}^{k_h} - x_{\tilde{k}}^{k_h}| \leq \frac{2}{\beta} - 2\varepsilon \quad h = 1, 2. \tag{8}$$

Iterating this procedure, after K steps we have selected K elements $z_1, z_2, ..., z_K$ among the $x_1, x_2, ..., x_n$ and K distinct (by (8)) indexes $k_1, k_2, ..., k_K$ such that

$$|z_i^{k_i} - z_k^{k_i}| > \frac{2}{\beta} - 2\varepsilon \quad \forall k > i. \tag{9}$$

In particular, for z_K we have $K - 1$ distinct indexes for which, by (4), $|z_K^{k_i}| > \frac{1}{\beta} - 2\varepsilon$.

Moreover we are left with a set R_K of indexes $\tilde{j}$ such that, by (6), for all $\tilde{j}, \tilde{k} \in R_K$

$$|x_{\tilde{j}}^{k_h} - x_{\tilde{k}}^{k_h}| \leq \frac{2}{\beta} - 2\varepsilon \quad h = 1, 2, ..., K. \tag{10}$$

R_K has cardinality

$$|R_K| \geq \left\lfloor (|R_{K-1}| - 1) \frac{1}{M - K + 1} \right\rfloor \tag{11}$$

and it can be easily verified by induction on K that

$$|R_K| \geq \Big\lfloor \frac{n-1}{M(M-1)...(M-K+1)} - \tag{12}$$

$$-2\left(\frac{1}{(M-1)...(M-K+1)} + \frac{1}{(M-2)...(M-K+1)} + \frac{1}{M-K+1}\right)\Big\rfloor$$

Now take $K = M$; by(12),

$$|R_M| \geq \Big\lfloor \frac{n-1}{M!} - 2\left(\sum_{m=1}^{M-1} \frac{1}{m!}\right)\Big\rfloor.$$

Remark that, since n can be taken arbitrarily large while M is fixed, depending only on β, cardinality of R_M can be assumed as big as we need. Actually it is enough that $|R_M| \geq 2$.

We know that, for all $i = 1, 2, ..., M$ and any $\tilde{j}$ in R_M, $|z_i^{k_i} - x_{\tilde{j}}^{k_i}| > \frac{2}{\beta} - 2\varepsilon$ hence, by (4), $|x_{\tilde{j}}^{k_i}| > \frac{1}{\beta} - 2\varepsilon$. We have remarked that for each $x_{\tilde{j}}$ this can be true for at most M indexes therefore $|x_{\tilde{j}}^k| \leq \frac{1}{\beta} - 2\varepsilon$ for all indexes k different from $k_1, k_2, ..., k_M$.

Call j_{M+1} the first of the $\tilde{j}$'s in R_M and set $z_{M+1} = x_{j_{M+1}}$. Pick any other $\tilde{j} \in R_M$.

$|z_{M+1}^k| \leq \frac{1}{\beta} - 2\varepsilon$ together with $|x_{\tilde{j}}^k| \leq \frac{1}{\beta} - 2\varepsilon$ for all indexes k different from $k_1, k_2, ..., k_M$ implies that

$$|z_{M+1}^k - x_{\tilde{j}}^k| < \frac{2}{\beta} - 2\varepsilon \quad \forall k \neq k_h, h = 1, 2, ..., M.$$

while at the same time, from (10),

$$|z_{M+1}^{k_h} - x_{\tilde{j}}^{k_h}| \leq \frac{2}{\beta} - 2\varepsilon \quad h = 1, 2, ..., M.$$

It follows that

$$\|z_{M+1} - x_{\tilde{j}}\|_\infty \leq \frac{2}{\beta} - 2\varepsilon \quad \forall \tilde{j} \in R_M, \tilde{j} \neq j_{M+1} \tag{13}$$

contradicting our separation condition (2). ☐

Corollary 2.4. *Kottman's separation constant $K(E_\beta) < 2$ for any β.*

Proof. Clearly $P(n, X) \geq P(n+1, X) \geq P(\aleph_0, X)$. It follows from [7] and [12] that $P(\aleph_0, X) = \frac{1}{2}$ if and only if $K(X) = 2$. Therefore P-convexity of E_β implies $K(E_\beta) < 2$. ☐

Remark 2.5. *It is known that E_β has normal structure (also uniform normal structure) if and only if $\beta < \sqrt{2}$ ([2], [3]). As far as we know, the E_β's for $\beta \geq \sqrt{2}$ provide the first examples of P-convex spaces without normal structure.*

As about the converse problem, i.e.whether some kind of normal structure must imply P-convexity, the answer is obviously negative for normal structure, which does not even imply reflexivity. For uniform normal structure, which does imply reflexivity (see [1],[9]), the answer is nevertheless negative. An example is provided by Bynum's space $l_{2,1}$.

Example 2.6. *Let $l_{2,1} = (l^2, \|\cdot\|_{2,1})$ where $\|x\|_{2,1} = \|x^+\|_2 + \|x^-\|_2$. Smith and Turett* [14] *proved that $l_{2,1}$ has uniform normal structure. It is easy to see that the canonical basis $\{e_n\}$ is a 2-separated sequence in $S_{l_{2,1}}$, hence $K(l_{2,1}) = 2$ and $l_{2,1}$ cannot be P-convex.*

Remark 2.7. *When $\delta_X(1) > 0$ P-convexity follows from Theorem 1.9 in* [10]. *Corollary 4.1 in* [3] *shows that $\delta_{E_\beta}(1) > 0$ if and only if $\beta < \frac{\sqrt{5}}{2}$ hence for these values of β the result in Theorem 2.3 follows from* [10]. *The assumption $\delta_X(1) > 0$ implies at the same time that X possesses uniform normal structure, therefore* [10] *does not provide an example of a P-convex space without normal structure.*

References

[1] Bae, J.S., *Reflexivity of a Banach space with a uniformly normal structure*, Proc. Amer. Math. Soc., **90**, 1984, 269-270

[2] Baillon, J.B. and Schöneberg, R., *Asymptotic normal structure and fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **81**, 1981, 257-264

[3] Casini, E. and Maluta, E., *Fixed points of Uniformly Lipschitzian Mappings in Spaces with Uniformly Normal Structure*, Nonlinear Analysis TMA, **9**, 1985, 103-108

[4] Dowling, P.N., Randrianantoanina, B. and Turett, B., *The fixed point property via dual space properties*, J.Funct.Anal., **255**, 2008, 768-775

[5] García-Falset, J., Llorens-Fuster, E. and Mazcuñán-Navarro, E., *Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings*, J.Funct.Anal., **233**, 2006, 494-514

[6] Karlovitz, L.A., *Existence of fixed points for nonexpansive mappings in a space without normal structure*, Pacific J. Math., **90**, 1980, 153-159

[7] Kottman, C.A., *Packing and Reflexivity in Banach Spaces*, Trans.Amer.Math.Soc., **150**, 1970, 565-576

[8] Lin,P.K., *Unconditional bases and fixed points of nonexpansive mappings*, Pacific J.Math., **116**, 1985, 69-76

[9] Maluta, E., *Uniformly normal structure and related coefficients*, Pacific J. Math., **111**, 1984, 357-369

[10] Muñiz-Pérez, O., *On P and p-convexity of Banach spaces*, Abstr. Appl. Anal, **2010**, 2010, 1-14

[11] Naidu, S.V.R. and Sastry, K.P.R., *Convexity conditions in normed linear spaces*, J.Reine Angew. Math., **297**, 1978, 35-53

[12] Papini, P.L., *Some parameters of Banach spaces*, Rend.Sem.Mat.Fis. Milano, **53**, 1983, 131-148

[13] Saejung,S., *Convexity conditions and normal structure of Banach spaces*, J.Math.Anal.Appl., **344**, 2008, 851-856

[14] Smith, M.A. and Turett, B., *Some examples concerning normal and uniform normal structure in Banach spaces*, J. Austral. Math. Soc. (A), **48**, 1990, 223-234

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