On closed boundary value problems for equations of mixed elliptic-hyperbolic type

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Abstract

For partial differential equations of mixed elliptic-hyperbolic type we prove results on existence and existence with uniqueness of weak solutions for "closed" boundary value problems of Dirichlet and mixed Dirichlet-conormal types. These problems are of interest for applications to transonic flow and are over-determined for solutions with classical regularity. The method employed consists in variants of the $a-b-c$ integral method of Friedrichs in Sobolev spaces with suitable weights. Particular attention is paid to the problem of attaining results with a minimum of restrictions on boundary geometry and the form of the type change function.

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1 Introduction.

The purpose of this work is to examine the question of well-posedness for boundary value problems for linear partial differential equations of second order of the form

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$$Lu = K(y)u_{xx} + u_{yy} = f \quad \text{in } \Omega \tag{1.1}$$

$$\mathcal{B}u = g \quad \text{on } \partial\Omega \tag{1.2}$$

where $K \in C^1(\mathbf{R}^2)$ satisfies

$$K(0) = 0 \quad \text{and} \quad yK(y) > 0 \quad \text{for} \quad y \neq 0, \tag{1.3}$$

Ω is a bounded open and connected subset of $\mathbf{R}^2$ with piecewise C^1 boundary, f, g are given functions and $\mathcal{B}$ some given boundary operator. We assume throughout that

$$\Omega^{\pm} := \Omega \cap \mathbf{R}^2_{\pm} \neq \emptyset \tag{1.4}$$

so that (1.1) is of mixed elliptic-hyperbolic type. We will call Ω a *mixed domain* if (1.4) holds. Additional hypotheses on f, g, K and Ω will be given as needed. Such an equation is of Tricomi type and goes by the name of the Chaplygin or Frankl' equation due to its longstanding importance in the problem of transonic fluid flow (see the classic monograph [4] or the modern survey [19], for example). This connection is our principal motivation.

Such a boundary value problem will be called *closed* in the sense that the boundary condition (1.2) is imposed on the entire boundary as opposed to an *open problem* in which (1.2) is imposed on a proper subset $\Gamma \subset \partial\Omega$. Both kinds of problems are interesting for transonic flow; for example, open problems arise in flows in nozzles and closed problems arise in flows about airfoils. Much more is known about open problems, beginning with the work of Tricomi [26] who considered the case $K(y) = y$ with the boundary condition $\mathcal{B}u = u$ on $\Gamma = \sigma \cup AC$ where σ is a simple arc in the elliptic region $y > 0$ and AC is one of two characteristic arcs which form the boundary in the hyperbolic region $y < 0$. Both the equation and this boundary condition now carry his name.

On the other hand, much less is known about closed problems which is due in part to the fact that closed problems for mixed type equations are typically *overdetermined* for classical solutions. This is essentially due to the presence of hyperbolicity in the mixed type problem; for example, it is well known that the Dirichlet problem for the wave equation is not well posed for classical solutions as first noted by Picone [21] and further investigated by Fichera [6, 7]. More precisely, under mild assumptions on the function K and the geometry of the boundary one has a uniqueness theorem for regular solutions to the Tricomi problem of the following form: *Let* $u \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{A, B\}) \cap C^0(\overline{\Omega})$ *solve* $Lu = K(y)u_{xx} + u_{yy} = 0$ *in* Ω *and* $u = 0$ *on* $\sigma \cup AC$*, then* $u = 0$ *in all of* $\overline{\Omega}$. Such uniqueness theorems have been proven by a variety of methods, including energy integrals as in [23] and maximum principles as in [1] and [14]. Such uniqueness theorems imply that the trace of a regular solution u on the boundary arc BC is already determined by the boundary values on the remainder of the boundary and the value of $L(u)$ in Ω. Hence, if one wants to impose the boundary condition on all of the boundary, one must expect in general that some real singularity must be present. Moreover, in order to prove well-posedness, one must make a good guess about where to look for the solution; that is, one must choose some reasonable function space which admits a singularity strong enough to allow for existence but not so strong as to lose uniqueness. This, in practice, has proven to be a difficult problem.

Despite the interest in closed problems for mixed type equations, the literature essentially contains only two results on well-posedness. The first, due to Morawetz [18] concerns the Dirichlet problem for the Tricomi equation ($K(y) = y$) and the second due to her student Pilant [22] concerns the natural analog of the Neumann problem (*conormal boundary conditions*) for the Laurentiev-Bitsadze equation ($K(y) = \mathrm{sgn}(y)$). In both cases, the restrictions on the boundary geometry are quite severe in that the domains must be lens-like and thin in some sense. Such restrictions on boundary geometry and the type change function are not particularly welcome in the transonic flow applications since the boundary geometry reflects profile or nozzle shape and the approximation $K(y) \sim y$ is valid only for nearly sonic speeds. The main purpose of this paper is to show well-posedness continues to hold for classes of type change functions and more general domains.

A brief summary of the results obtained is the following. In section 2, we investigate the notions of *weak* and *strong* solutions for the Dirichlet problem. In a very general setting, we show both the existence of a weak solution and the uniqueness of the strong solution. However, the weak solution may not be unique nor satisfy the boundary condition in a strong sense while the strong solution may not exist. Hence there is a need for a suitable intermediary notion of solution to establish well posedness. In section 3, we deine this suitable intermediary notion and show that under slightly stronger restrictions on the type change function and the domain, one has the existence of a unique *generalized solution* for the Dirichlet problem. Under similar restrictions, in section 4, we show the existence and uniqueness of a generalized solution to a problem with mixed boundary conditions (Dirichlet conditions on the elliptic boundary and conormal conditions on the hyperbolic boundary). The main ingredients in the proofs are variants of the classical $a-b-c$ method of Friedrichs where our multipliers are calibrated to an invariance or almost invariance of the differential operator. In particular, section 2 employs the technique as first used by Protter [23] with a differential multiplier, section 3 uses the idea of Didenko [5] with an integral multiplier, and section 3 uses the technique of Morawetz [17] by first reducing the problem to a first order system and then using a differential multiplier. Attempts to improve Pilant's work on the full conormal problem are in progress.

2 Weak and strong solutions for the Dirichlet problem.

In this section we show that even though closed boundary value problems for mixed type are overdetermined for classical solutions, at least for the Dirichlet boundary condition, one can obtain a general existence result for a very weak formulation of the problem as well a general uniqueness result for a sufficiently strong formulation. In particular, very little is assumed about boundary geometry and only mild and reasonable hypotheses are placed on the type change function K.

In what follows Ω will be a bounded mixed domain (open, connected, satisfying (1.4)) in $\mathbf{R}^2$ with piecewise C^1 boundary so that we may apply the divergence theorem. The function $K \in C^1(\mathbf{R})$ will be taken to satisfy (1.3) and additional assumptions as necessary. We will make use of several natural spaces of functions and distributions; namely for treating the Dirichlet problem

$$Lu = K(y)u_{xx} + u_{yy} = f \quad \text{in} \quad \Omega \tag{2.1}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.2}$$

we define $H_0^1(\Omega; K)$ as the closure of $C_0^\infty(\Omega)$ (smooth functions with compact support) with respect to the weighted Sobolev norm

$$||u||_{H^1(\Omega;K)} := \left[\int_\Omega \left(|K|u_x^2 + u_y^2 + u^2 \right) \, dxdy \right]^{1/2} \tag{2.3}$$

and denote by $H^{-1}(\Omega;K)$ the dual space to $H_0^1(\Omega;K)$ equipped with its negative norm in the sense of Lax

$$||w||_{H^{-1}(\Omega;K)} := \sup_{0 \neq \varphi \in C_0^\infty(\Omega)} \frac{|\langle w, \varphi \rangle|}{||\varphi||_{H_0^1(\Omega;K)}}, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket. One clearly has a rigged triple of Hilbert spaces

$$H_0^1(\Omega;K) \subset L^2(\Omega) \subset H^{-1}(\Omega;K), \tag{2.5}$$

where the scalar product (on L^2 for example) will be denoted by $(\cdot,\cdot)_{L^2(\Omega)}$. Moreover, since $u \in H_0^1(\Omega;K)$ vanishes weakly on the entire boundary, one has a *Poincarè inequality*: there exists $C_P = C_P(\Omega, K)$

$$||u||^2_{L^2(\Omega)} \leq C_P \int_\Omega \left(|K|u_x^2 + u_y^2 \right) \, dxdy, \quad u \in H_0^1(\Omega;K) \tag{2.6}$$

The inequality (2.6) is proven in the standard way by integrating along segments parallel to the coordinate axes for $u \in C_0^1(\Omega)$ and then using continuity. An equivalent norm on $H_0^1(\Omega;K)$ is thus given by

$$||u||_{H_0^1(\Omega;K)} := \left[\int_\Omega \left(|K|u_x^2 + u_y^2 \right) \, dxdy \right]^{1/2}. \tag{2.7}$$

Other notations and function spaces will be introduced as needed.

It is routine to check that the second order operator L in (2.1) is formally self-adjoint when acting on distributions $\mathcal{D}'(\Omega)$ and gives rise to a unique continuous and self-adjoint extension

$$L : H_0^1(\Omega;K) \to H^{-1}(\Omega;K) \tag{2.8}$$

Using standard functional analytic techniques, one can obtain results on weak existence and strong uniqueness for solutions to the Dirichlet problem $(2.1) - (2.2)$. The key point is to obtain a suitable *a priori* estimate by performing an energy integral argument with a well chosen multiplier.

Lemma 2.1. *Let Ω be any bounded region in $\mathbf{R}^2$ with piecewise C^1 boundary. Let $K \in C^1(\mathbf{R})$ be a type change function satisfying* (1.3) *and*

$$K' > 0 \tag{2.9}$$

$$\exists \delta > 0 : \quad 1 + \left(\frac{4K}{K'} \right)' \geq \delta \tag{2.10}$$

Then there exists a constant $C_1(\Omega, K)$ such that

$$||u||_{H_0^1(\Omega;K)} \leq C_1 ||Lu||_{L^2(\Omega)}, \quad u \in C_0^2(\Omega) \tag{2.11}$$

Proof: To obtain the estimate, one considers an arbitrary $u \in C_0^2(\Omega)$ and a triple of (a, b, c) of sufficiently regular functions to be determined. One seeks to estimate the expression $(Mu, Lu)_{L^2}$ from above and below where $Mu = au + bu_x + cu_y$ is the as yet undetermined multiplier. One has that

$$(Mu, Lu)_{L^2} = \int_\Omega [\mathrm{div}(Mu(Ku_x, u_y)) - \nabla(Mu) \cdot (Ku_x, u_y)] \, dxdy \tag{2.12}$$

which can be rewritten with the aid of the divergence theorem as

$$\begin{aligned} (Mu, Lu)_{L^2} \quad &= \quad \frac{1}{2} \int_\Omega \left[\alpha u_x^2 + 2\beta u_x u_y + \gamma u_y^2 + u^2 La \right] \, dxdy \\ &+ \quad \frac{1}{2} \int_{\partial\Omega} \left[2Mu(Ku_x, u_y) - (Ku_x^2 + u_y^2)(b, c) - u^2(Ka_x, a_y) \right] \cdot \nu \, ds \end{aligned} \tag{2.13}$$

where

$$\alpha := -2Ka - Kb_x + (Kc)_y \tag{2.14}$$

$$\beta := -Kc_x - b_y \tag{2.15}$$

$$\gamma := -2a + b_x - c_y \tag{2.16}$$

ν is the unit exterior normal, and ds is the arc length element. For u with compact support, the boundary integrals will vanish and the choices

$$a \equiv -1, \quad b \equiv 0, \quad c = c(y) = \max\{0, -4K/K'\} \tag{2.17}$$

yield

$$\begin{aligned}(Mu, Lu)_{L^2} &= \int_{\Omega^+} \left(Ku_x^2 + u_y^2\right)\, dxdy + \int_{\Omega^-} \left[1 + \left(\frac{4K}{K'}\right)'\right] (-Ku_x^2 + u_y^2)\, dxdy \\ &\geq \min\{1, \delta\} \int_{\Omega} \left(|K|u_x^2 + u_y^2\right)\, dxdy, \quad u \in C_0^2(\Omega)\end{aligned} \tag{2.18}$$

where $\Omega^{\pm} := \Omega \cap \{(x, y) : \pm y \geq 0\}$. Technically, since c is only piecewise $C^1(\Omega)$ one should first cut the integral along $y = 0$ and note that the boundary terms (as given in (2.13)) along the cut will cancel out.

One estimates from above using the Cauchy-Schwartz inequality and the regularity and monotonicity of K to find

$$(Mu, Lu)_{L^2} \leq ||Mu||_{L^2(\Omega)}||Lu||_{L^2(\Omega)} \leq C_K||u||^2_{H_0^1(\Omega;K)}||Lu||_{L^2(\Omega)}, \quad u \in C_0^2(\Omega). \tag{2.19}$$

where C_K depends on $\sup(|K/K'|)$. To complete the estimate, one combines (2.18) and (2.19) with the Poincarè inequality (2.6) to find (2.11) with $C_1(\Omega, K) = [C_K(1 + C_P)/(\min\{1, \delta\})]^{1/2}$.

An existence theorem for weak solutions follows from the a priori estimate (2.11).

Theorem 2.2. *Let Ω be any bounded region in $\mathbf{R}^2$ with piecewise C^1 boundary. Let $K \in C^1(\mathbf{R})$ be a type change function satisfying* (1.3), (2.9) *and* (2.10). *Then for each $f \in H^{-1}(\Omega; K)$ there exists $u \in L^2(\Omega)$ which weakly solves* (2.1) – (2.2) *in the sense that*

$$(u, L\varphi)_{L^2} = \langle f, \varphi\rangle, \quad \varphi \in H_0^1(\Omega; K) : \; L\varphi \in L^2(\Omega). \tag{2.20}$$

Proof: The basic idea is to combine the *a priori* estimate with the Hahn-Banach and Riesz representation theorems. Given any distribution $f \in H^{-1}(\Omega; K)$ one defines the linear functional by $J_f(L\varphi) = \langle f, \varphi\rangle$ for $\varphi \in C_0^\infty(\Omega)$. By the generalized Schwartz inequality and the estimate (2.11) one has

$$|J_f(L\varphi)| \leq C_1||f||_{H^{-1}(\Omega;K)}||L\varphi||_{L^2(\Omega)}, \quad \varphi \in C_0^\infty(\Omega) \tag{2.21}$$

and so this functional is bounded on the subspace $\mathcal{V}$ of $L^2(\Omega)$ of elements of the form $L\varphi$ with $\varphi \in C_0^\infty(\Omega)$. By the Hahn-Banach theorem, J_f extends to the closure of $\mathcal{V}$ in $L^2(\Omega)$ in a bounded way. Extension by zero on the orthogonal complement of $\overline{\mathcal{V}}$ gives a bounded linear functional on all of $L^2(\Omega)$ and so by the Riesz representation theorem there exists $u \in L^2(\Omega)$ so that (2.20) holds.

We note that the Sobolev space $H_0^1(\Omega; K)$ is a *normal space of distributions* and that L is formally self-adjoint and hence the proof of Theorem 2.2 is classical (cf. Lemma B.1 of [25]) where one can show also that the solution map from $H^{-1}(\Omega; K)$ to $L^2(\Omega)$ is continuous with respect to the relevant norms.

The estimate (2.11) also shows that sufficiently strong solutions must be unique. We say that $u \in H_0^1(\Omega; K)$ is a *strong solution* of the Dirichlet problem $(2.1)-(2.2)$ if there exists an approximating sequence $u_n \in C_0^2(\Omega)$ such that

$$||u_n - u||_{H^1(\Omega;K)} \to 0 \quad \text{and} \quad ||Lu_n - f||_{L^2(\Omega)} \to 0 \quad \text{as} \quad n \to +\infty \tag{2.22}$$

The following theorem is an immediate consequence of the definition.

Theorem 2.3. *Let Ω be any bounded region in $\mathbf{R}^2$ with piecewise C^1 boundary. Let $K \in C^1(\mathbf{R})$ be a type change function satisfying* $(1.3), (2.9)$ *and* (2.10)*. Then any strong solution of the Dirichlet problem* $(2.1)-(2.2)$ *must be unique.*

We note that the class of admissible K is very large and includes the standard models for transonic flow problems such as the Tricomi equation with $K(y) = y$ and the Tomatika-Tamada equation $K(y) = A(1 + e^{2By})$ with A, B constants. Moreover, the result also holds for non strictly monotone functions such as the Gellerstedt equation with $K(y) = y|y|^{m-1}$ where $m > 0$. In this case, one can check that in place of (2.15) it is enough to choose the "dilation multiplier" introduced in [13]

$$a \equiv 0, \quad b = (m+2)x, \quad c = 2y. \tag{2.23}$$

We also note that no boundary geometry hypotheses have been made for the weak existence result above for the Dirichlet problem; in particular, there are no star-like hypotheses on the elliptic part and no sub-characteristic hypotheses on the hyperbolic part. These kinds of hypotheses will enter however later on when we look for existence of solutions in a stronger sense.

On the other hand, it is clear that the existence is in a very weak sense; too weak in fact to be very useful. In particular, the sense in which the solution vanishes at the boundary is only by duality and one may not have uniqueness. In fact, the estimate (2.11) which holds for both L and its formal adjoint $L^* = L$ together with the Poincarè inequality (2.6) gives an $L^2 - L^2$ estimate for both L and L^* for test functions $u, v \in H^2(\Omega) \cap H_0^1(\Omega; K)$. These are the necessary and sufficient conditions of Berezanskii for a problem with *almost correct* boundary condition (cf. Chapter 2 of [3]). In order to be sure that the weak solution which exists is strong and satisfies the boundary condition in the sense of traces, one needs to show that $u \in H^2(\Omega)$, which will not hold in general for $f \in L^2$. In any case, the existence result is a first general indication that while the closed Dirichlet problem is generically over-determined for regular solutions, it is generically not over-determined if one looks for a solution which is taken in a sufficiently weak sense. Moreover, while uniqueness generically holds for strong solutions, one must show that such strong solutions exist.

Example 2.4. Let Ω be any mixed domain containing the origin. Consider the function

$$u(x,y) = \psi E(x,y) = \begin{cases} C_-|9x^2 + 4y^3|^{5/6} & (x,y) \in \Omega \cap D_-(0) \\ 0 & (x,y) \in \Omega \setminus \mathcal{D}_-(0) \end{cases} \tag{2.24}$$

where $\psi(x,y) = 9x^2 + 4y^3$, $\mathcal{D}_- = \{(x,y) \in \mathbf{R}^2 : \ \ 9x^2 + 4y^3 < 0\}$ is the backward light cone from the origin and

$$C_- = \frac{3\Gamma(4/3)}{2^{2/3}\pi^{1/2}\Gamma(5/6)}. \tag{2.25}$$

Then u is a weak solution in the sense (2.20) of the equation $Tu = yu_{xx} + u_{yy} = f$ where

$$f = -18yE\left(\frac{7}{3}E + 3xE_x + 2yE_y\right) \tag{2.26}$$

The distribution E defined by (2.24) is the fundamental solution of Barros-Neto and Gelfand [2] for the Tricomi operator with support in $\mathcal{D}_-(0)$ and one verifies that (2.26) holds in the sense of distributions by repeating their calculations with ψE in the place of E. One has $u \in L^2(\Omega)$ since it is continuous on the closure and that $f \in H^{-1}(\Omega; K)$. The verification of (2.20) then follows. One notes that the restriction of u does not vanish on all of the boundary and moreover that one should then be able to build another weak solution to the equation by adding a suitable smooth function in the kernel of T.

As a final remark in this section we note that the approach of obtaining the a priori estimate (2.11) has its origins in the so-called $a-b-c$ method of Friedrichs which was first used by Protter for showing uniqueness results for classical solutions [23] to open boundary problems. The key difference is that the compact support used here for the weak existence must be replaced with a simple condition of vanishing and hence boundary terms will remain in the expression (2.13) whose signs must be compatible with the sign of the integral over the domain. It is for this reason that technical hypotheses on the boundary will enter.

3 Generalized solutions to the Dirichlet problem

In this section, we show how to steer a course between the weak existence and the strong uniqueness result for the Dirichlet problem by following the path laid out by Didenko [5] for open boundary value problems. More precisely, we will show that the Dirichlet problem is well-posed for *generalized solutions* lying a suitable Hilbert space. Technical hypotheses on the type change function K and the boundary will enter in the process.

To begin, we fix some notation and conventions which will be used throughout this section. We will continue to assume that Ω is a bounded mixed domain with piecewise C^1 boundary with external normal field ν. Since the differential operator (2.1) is invariant with respect to translations in x, we may assume that the origin is the point on the *parabolic line* $AB := \{(x, y) \in \overline{\Omega} : y = 0\}$ with maximal x coordinate; that is, $B = (0, 0)$. This will simplify certain formulas without reducing the generality of the results. We will often require that Ω is *star-shaped with respect to the flow of a given (Lipschitz) continuous vector field* $V = (V_1(x, y), V_2(x, y))$; that is, for every $(x_0, y_0) \in \overline{\Omega}$ one has $\mathcal{F}_t(x_0, y_0) \in \overline{\Omega}$ for each $t \in [0, +\infty]$ where $\mathcal{F}_t(x_0, y_0)$ represents the time-t flow of (x_0, y_0) in the direction of V. We recall that Ω is then simply connected and will have a V-*star-like* boundary in the sense that $V(x, y) \cdot \nu \geq 0$ for each regular point $(x, y) \in \partial\Omega$ (cf. Lemma 2.2 of [13]). We will continue to assume that the type change function K belongs to $C^1(\mathbf{R})$ and satisfies (1.3) where additional hypotheses will be made as needed.

We will also make use of suitably weighted versions of $L^2(\Omega)$ and their properties. In particular, for $K \in C^1(\mathbf{R})$ satisfying (1.3) we define

$$L^2(\Omega; |K|^{-1}) := \{f \in L^2(\Omega) : \quad |K|^{-1/2} f \in L^2(\Omega)\} \tag{3.1}$$

equipped with its natural norm

$$||f||_{L^2(\Omega;|K|^{-1})} = \left[\int_\Omega |K|^{-1} f^2 \, dxdy\right]^{1/2} \tag{3.2}$$

which is the dual space to the weighted space $L^2(\Omega; |K|)$ defined as the equivalence classes of square integrable functions with respect to the measure $|K| \, dxdy$; that is, with finite norm

$$||f||_{L^2(\Omega;|K|)} = \left[\int_\Omega |K| f^2 \, dxdy\right]^{1/2}. \tag{3.3}$$

One has the obvious chain of inclusions

$$L^2(\Omega; |K|^{-1}) \subset L^2(\Omega) \subset L^2(\Omega; |K|), \tag{3.4}$$

where the inclusion maps are continuous and injective (since K vanishes only on the parabolic line, which has zero measure).

Under suitable hypotheses, we will show the existence of a unique $H^1_0(\Omega; K)$ solution to the Dirichlet problem $(2.1) - (2.2)$ for each $f \in L^2(\Omega; |K|^{-1})$ The sense in which we will find a solution is contained in the following definition.

Definition 3.1. *We say that $u \in H^1_0(\Omega; K)$ is a generalized solution of the Dirichlet problem* $(2.1) - (2.2)$ *if there exists a sequence $u_n \in C^\infty_0(\Omega)$ such that*

$$||u_n - u||_{H^1_0(\Omega;K)} \to 0 \;\text{ and }\; ||Lu_n - f||_{H^{-1}(\Omega;K)} \to 0, \;\text{ for }\; n \to +\infty \tag{3.5}$$

or equivalently

$$\langle Lu, \varphi \rangle = \int_\Omega (K u_x \varphi_x + u_y \varphi_y)\, dxdy = \langle f, \varphi \rangle, \quad \varphi \in H^1_0(\Omega, K), \tag{3.6}$$

where $\langle \cdot, \cdot \rangle$ *is the duality paring between* $H^1_0(\Omega; K)$ *and* $H^{-1}(\Omega; K)$ *and* L *is the continuous extension defined in* (2.8).

The equivalence of (3.5) and (3.6) follows by integration by parts on $C^\infty_0(\Omega)$ which is dense in $H^1_0(\Omega; K)$. We note that since $L^2(\Omega; |K|^{-1})$ is a subspace of $L^2(\Omega) \subset H^{-1}(\Omega; K)$, (3.5) and (3.6) make sense for $f \in L^2(\Omega; |K|^{-1})$. Generalized solutions give an intermediate notion between the weak solutions (2.20) and the strong solutions (2.22) where the approximation property (3.5) is compatible with the continuity property (2.8).

Our first result concerns the Gellerstedt operator; that is, with K of pure power type

$$K(y) = y|y|^{m-1}, \quad m > 0. \tag{3.7}$$

Theorem 3.2. *Let* Ω *be a bounded mixed domain with piecewise* C^1 *boundary and parabolic segment* AB *with* $B = 0$*. Let* K *be of pure power form* (3.7)*. Assume that* Ω *is star-shaped with respect to the vector field* $V = (-(m+2)x, -\mu y)$ *where* $\mu = 2$ *for* $y > 0$ *and* $\mu = 1$ *for* $y < 0$*. Then, for each* $f \in L^2(\Omega; |K|^{-1})$ *there exists a unique generalized solution* $u \in H^1_0(\Omega; K)$ *in the sense of Definition 3.1 to the Dirichlet problem* (2.1) – (2.2).

Proof: For the existence, the basic idea is, as in the proof of existence for weak solutions (Theorem 2.2), to obtain an *a priori* estimate for setting up a Hahn-Banach and Riesz representation argument. The needed estimate involves one derivative less than in the weak existence proof (cf. Lemma 2.2) and is contained in the following lemma.

Lemma 3.3. *Under the hypotheses of Theorem 3.2, one has the a priori estimate: there exists* $C_1 > 0$ *such that*

$$||u||_{L^2(\Omega; |K|)} \leq C_1 ||Lu||_{H^{-1}(\Omega; K)}. \tag{3.8}$$

We will call a domain Ω *admissible* for generalized solutions to the Dirichlet problem if the estimate (3.8) holds. Assuming that Ω is admissible, one defines a linear functional J_f for $\varphi \in C_0^\infty(\Omega)$ by the formula $J_f(L\varphi) = (f,\varphi)_{L^2(\Omega)}$ and the estimate (3.8) together with the Cauchy-Schwartz inequality yields

$$|J_f(L\varphi)| \leq ||f||_{L^2(\Omega;K^{-1})}||\varphi||_{L^2(\Omega;K)} \leq C_1||f||_{L^2(\Omega;K^{-1})}||L\varphi||_{H_0^1(\Omega;K)}, \quad \varphi \in C_0^\infty(\Omega). \tag{3.9}$$

and so J_f is bounded on the subspace $\mathcal{V}$ of $H^{-1}(\Omega;K)$ of elements of the form $L\varphi$ with $\varphi \in C_0^\infty(\Omega)$. Now, as in the proof of Theorem 2.2, one obtains the existence of $u \in H_0^1(\Omega;K)$ such that

$$\langle u, L\varphi\rangle = (f,\varphi)_{L^2(\Omega)}, \quad \varphi \in H_0^1(\Omega;K), \tag{3.10}$$

where L is the self-adjoint extension defined in (2.8).

This weak solution is a generalized solution in the sense of Definition 3.1. In fact, given a sequence $u_n \in C_0^\infty(\Omega)$ which approximates u in the norm (2.3) the continuity property (2.8) shows that $f_n := Lu_n$ is norm convergent to some element $\tilde{f} \in H^{-1}(\Omega : K)$. One has also

$$\langle u_n, L\varphi\rangle = (f_n,\varphi)_{L^2(\Omega)}, \quad \varphi \in H_0^1(\Omega;K). \tag{3.11}$$

Taking the difference between (3.10) and (3.11) and passing to the limit shows that $\tilde{f} = f$ and hence (3.5) holds.

For the uniqueness claim, again we use the estimate (3.8). In fact, for fixed f, let $u,v \in H_0^1(\Omega;K)$ be two generalized solutions with approximating sequences $\{u_n\},\{v_n\}$ satisfying (3.5). Using the linearity of L and (3.8) one has that $u_n - v_n$ tends to zero in $L^2(\Omega;K)$. Hence $u_n - v_n$ also tends to zero in $L^2(\Omega)$ and in $H_0^1(\Omega;K)$ by the injectivity of the inclusion (3.4) and the Poincarè inequality (2.6). Thus $u = v$ in $H_0^1(\Omega;K)$ and finishes the proof of the theorem, modulo the proof of the lemma.

Proof of Lemma 3.3. The basic idea is to estimate from above and below the expression $(Iu, Lu)_{L^2(\Omega)}$ for each $u \in C_0^\infty(\Omega)$ where $v = Iu$ is the solution to the following auxiliary Cauchy problem

$$\begin{cases} Mv := av + bv_x + cv_y = u \ \text{ in } \ \Omega \\ v = 0 \ \text{ on } \ \partial\Omega \setminus B \end{cases} \tag{3.12}$$

where $B = (0,0)$ is the righthand endpoint of the parabolic line and

$$a \equiv -1/4, \quad (b,c) = -V = ((m+2)x, \mu y) \tag{3.13}$$

where we recall $\mu = 2$ in Ω^+ and $\mu = 1$ in Ω^-. The choice of the coefficients (a,b,c) will ensure the positivity of a certain quadratic form used later in the estimate. We note the difference with the proof of Lemma 2.1 in which an integral expression $v = Iu$ replaces the differential expression Mu in (2.12). Since

$$(Iu, Lu)_{L^2(\Omega)} = (v, Lu)_{L^2(\Omega)} = (v, LMv)_{L^2(\Omega)}, \tag{3.14}$$

one actually performs an a priori estimate for the third order operator LM acting on the auxiliary function v. We divide the rest of the proof into 4 steps.

Step 1. Existence and properties of the solution v to (3.12) : We claim that for every $u \in C_0^\infty(\Omega)$ there exists $v \in C^\infty(\Omega^\pm) \cap C^0(\overline{\Omega} \setminus B)$ solving (3.12) with the additional properties

$$\lim_{(x,y)\to B} v(x,y) = 0 \tag{3.15}$$

$$\int_{\Omega^\pm} \left(|K| v_x^2 + v_y^2\right) \, dxdy < +\infty \tag{3.16}$$

and hence by defining $v(B) = 0$ one has that

$$v \in C^0(\overline{\Omega}) \cap H_0^1(\Omega; K). \tag{3.17}$$

To prove these claims, one first notes that since Ω is V star-shaped, each flow line of V starting from the boundary and entering the interior will stay in the closure and that the only singular point of V is in the origin B. Hence the method of characteristics will give the existence of a unique $v \in C^\infty(\Omega^\pm) \cap C^0(\overline{\Omega} \setminus B)$ solving (3.12). We recall that the coefficients a and b are C^∞ while c is globally Lipschitz.

In order to verify the claims (3.14) and (3.15), we appeal to the explicit nature of $(b,c) = -V$ and the compact support of u. We parameterize the integral curves of (b,c) by $\gamma(t) = (x(t), y(t)) = (x_0 e^{(m+2)t}, y_0 e^{\mu t})$ for $(x_0, y_0) \in \partial\Omega$ and $t \in [-\infty, 0]$. We parameterize the boundary by $\Gamma(s)$ where

$s \in [0, S]$ is the arc length parameter, $\Gamma(0) = \Gamma(S) = B$, and we use the positive orientation an Γ. The support $\mathrm{supp}(u)$ of u, which is compact and fixed, will be disjoint from each ϵ neighborhood of the boundary $N_\epsilon(\partial\Omega) = \{(x, y) \in \overline{\Omega} : \mathrm{dist}((x, y), \partial\Omega) \leq \epsilon\}$ for each ϵ sufficiently small. Moreover, there will be two critical values $\underline{s}$ (resp. $\overline{s}$) of the arc length defined as the infimum (resp. supremum) of the values of s for which γ arrives at $(x_0, y_0) = \Gamma(s)$ and γ intersects $\mathrm{supp}(u)$. In this way, the flow of $-V$ divides Ω into three regions $\Omega_1, \Omega_2, \Omega_3$ corresponding to the intervals $[0, \underline{s}], [\underline{s}, \overline{s}], [\overline{s}, S]$.

For each $(x, y) \in \Omega_1 \cup \Omega_3$, the unique C^1 solution satisfying (3.12) must vanish as $u \equiv 0$ there and v starts from zero. Similarly, for (x, y) in the "wedge" $B_\epsilon((0, 0)) \cap \Omega_2$, we can re-initialize the Cauchy problem at time $t = T < 0$ by starting from points (x_T, y_T) on the "outer boundary" of the wedge. The unique C^1 solution along each flow line is

$$\psi(t) = v(x(t), y(t)) = v(x_T, y_T)e^{t/4}, \quad t \in (-\infty, T]. \tag{3.18}$$

Since v is C^1 in Ω, the variation in $v(x_T, y_T)$ is bounded for (x_T, y_T) on the initial data surface. The claims (3.15) and (3.16) are now easily verified.

Step 2. Estimate 3.14 *from below:* Since $v \in C^0(\Omega)$ and $Lu \in C_0^\infty(\Omega)$, the integrand is compactly supported and the expression (3.14) is finite. Splitting along the parabolic line (since v and c are only piecewise smooth)

$$(v, Lu)_{L^2(\Omega)} = \int_{\Omega^+} vLu\, dxdy + \int_{\Omega^-} vLu\, dxdy \tag{3.19}$$

and applying the divergence theorem yields

$$\begin{aligned}(v, Lu)_{L^2(\Omega)} &= \frac{1}{2}\int_{\Omega^+\cup\Omega^-} \left[\alpha v_x^2 + 2\beta v_x v_y + \gamma v_y^2 + v^2 La\right] dxdy \\ &+ \frac{1}{2}\int_{\partial\Omega^+\cup\partial\Omega^-} \left[2v(Ku_x, u_y) - (Kv_x^2 + v_y^2)(b, c) - v^2(Ka_x, a_y)\right] \cdot \nu\, ds\end{aligned} \tag{3.20}$$

where

$$\alpha := -2Ka - Kb_x + (Kc)_y = (2m + 1)K/2 \ \text{ in } \ \Omega^+ \ \text{ and } \ \alpha = -K/2 \ \text{ in } \ \Omega^- \tag{3.21}$$

$$\beta := -Kc_x - b_y = 0 \tag{3.22}$$

$$\gamma := -2a + b_x - c_y = (2m+1)/2 \;\text{ in }\; \Omega^+ \;\text{ and }\; \alpha = (2m+3)/2 \;\text{ in }\; \Omega^- \tag{3.23}$$

Using the compact support of u and that $a \equiv -1/4$, (3.20) – (3.23) yield

$$(v, Lu)_{L^2(\Omega)} \geq \frac{1}{2}\int_\Omega \left(|K|v_x^2 + v_y^2\right)\, dxdy - \frac{1}{2}\int_{\partial\Omega^+ \cup \partial\Omega^-} (Kv_x^2 + v_y^2)(b,c)\cdot \nu\, ds. \tag{3.24}$$

The first integral is finite by (3.15) and so is the second, where v vanishes identically in a sufficiently small neighborhood of each point on $\partial\Omega \setminus B$. Moreover the contributions along the cut $y = 0$ will cancel out and so the boundary integral vanishes. Hence one obtains the estimate

$$(v, Lu)_{L^2(\Omega)} \geq \frac{1}{2}\, ||v||^2_{H^1_0(\Omega;K)}, \quad u \in C_0^\infty(\Omega) \tag{3.25}$$

Step 3. Estimate (3.14) from above: We use the generalized Schwartz inequality on the non-negative quantity (3.18) to obtain

$$(v, Lu)_{L^2(\Omega)} \leq ||v||_{H^1_0(\Omega;K)} ||Lu||_{H^{-1}(\Omega;K)}, \quad u \in C_0^\infty(\Omega). \tag{3.26}$$

Step 4. Completing the estimate (3.8): Combining (3.25) and (3.26) one has

$$||v||_{H^1_0(\Omega;K)} \leq 2\, ||Lu||_{H^{-1}(\Omega;K)}, \quad u \in C_0^\infty(\Omega). \tag{3.27}$$

Finally, it is easy to check that the first order differential operator M is continuous from $H^1_0(\Omega;K)$ into $L^2(\Omega;|K|)$; that is, there exists $C_M > 0$ such that

$$||u||_{L^2(\Omega;|K|)} = ||Mv||_{L^2(\Omega;|K|)} \leq C_M ||v||_{H^1_0(\Omega;K)}, \quad v \in H^1_0(\Omega;K). \tag{3.28}$$

Combining (3.27) and (3.28) gives the desired estimate (3.8) with $C_1 = 2C_M$. This completes Lemma 3.3 and hence Theorem 3.2 as well.

We complete the results of this section by showing how to eliminate almost entirely the boundary geometry restrictions in the elliptic part of the domain and indicate how to replace the type change functions K of pure power type to with more general forms. We denote by

$$\Omega_\delta := \{(x,y) \in \Omega : \quad y < \delta\} \tag{3.29}$$

in the following result, which for the Gellerstedt equation shows that the Dirichlet problem is well-posed so long as the domain is suitably star-shaped for $y < \delta$ with $\delta > 0$ arbitrarily small.

Theorem 3.4. *Let $\widehat{\Omega}$ be a bounded mixed domain which "caps off" an admissible domain Ω in the sense that there exists $\delta > 0$ such that $\widehat{\Omega}_\delta = \Omega_\delta$. Then $\widehat{\Omega}$ is also admissible and the conclusion of Theorem 3.2 is valid also for $\widehat{\Omega}$.*

Proof: The idea is originally due to Didenko [5] for open problems and has been used by the authors [12] for the Tricomi problem. For completeness, we sketch the main points of the proof. We need to show that there exists $C > 0$ such that

$$||u||_{L^2(\widehat{\Omega};|K|)} \leq C||Lu||_{H^{-1}(\Omega;K)}, \quad u \in C_0^\infty(\widehat{\Omega};K). \tag{3.30}$$

For an arbitrary mixed domain Ω, one has the easy *a priori* estimate

$$||u||_{H_0^1(\Omega^+;K)} \leq ||Lu||_{H^{-1}(\Omega^+;K)}, \quad u \in C_0^\infty(\Omega^+) \tag{3.31}$$

for the degenerate elliptic operator L on Ω^+. In fact, one merely takes $(a,b,c) = (-1,0,0)$ in the way of section 2 (Lemma 2.1). Next, one picks a cut-off function $\phi \in C^\infty(\widehat{\Omega})$ such that

$$\phi(x,y) = \begin{cases} 1 & y \geq 2\delta/3 \\ \chi(y) & \delta/3 \leq y \leq 2\delta/3 \\ 0 & y \leq \delta/3 \end{cases} \tag{3.32}$$

where χ is any smooth transition function. One has that $u = \phi u + (1-\phi)u$ with $\phi u \in C_0^\infty(\widehat{\Omega}^+)$ and $(1-\phi)u \in C_0^\infty(\Omega)$. Applying (3.31) with $\widehat{\Omega}$ in place of Ω to the first term and (3.8) to the second term yields

$$||u||_{L^2(\widehat{\Omega};|K|)} \leq ||L(\phi u)||_{H^{-1}(\widehat{\Omega}^+;K)} + C_1||L((1-\phi)u)||_{H^{-1}(\Omega;K)}. \tag{3.33}$$

To complete the estimate (3.30), one uses the Leibniz formula, the definition of the norm in $H^{-1}(\widehat{\Omega}^+;K)$, the generalized Schwarz inequality and the continuity of $\partial_y : H_0^1(\widehat{\Omega}^+;K) \to L^2(\widehat{\Omega}^+) \subset L^2(\widehat{\Omega}^+;|K|)$ to control the first term of (3.33) and a similar argument to control the second term (see the proof of Theorem 2.2 in [12] for details).

As a final improvement, one need not choose a type change function of pure power type in order to have a well-posed problem. An example is contained in the following proposition in which the function $\mu = \mu(y)$ defined by

$$\mu = (K/K')' \tag{3.34}$$

plays a key role. Both K and μ will be thought of as functions defined on Ω which are independent of x.

Proposition 3.5. *Let $K \in C^2(\mathbf{R})$ be a type change function and Ω a bounded mixed domain with $B = (0,0)$ such that*

$$K'(y) > 0 \quad \text{in } \overline{\Omega} \tag{3.35}$$

$$2 \sup_{\Omega^+} \mu < 3 + 2 \inf_{\Omega^+} \mu \tag{3.36}$$

where μ is defined by (3.34) *and Ω is star-shaped with respect to the vector field $V = (-b_0 x, -2K/K')$ with*

$$b_0 = \begin{cases} 1 + 2\sup_{\Omega^+} \mu & \text{in } \Omega^+ \\ 3 + 2\inf_{\Omega^+} \mu & \text{in } \Omega^- \end{cases} \tag{3.37}$$

Then Ω is admissible and the Dirichlet problem is well posed in the sense of Theorem 3.2.

The proof follows precisely the argument used in the proof of Lemma 3.3, where we note only that the multiplier used is again of the form (3.13) with $a \equiv -1/4$ and $(b,c) = -V = (b_0 x, 2K/K')$. We remark that the vector field V here merely generalizes the one used in (3.13) for a pure power K with the associated anisotropic dilation invariance. The technical hypothesis (3.36) is always satisfied in the pure power case, but in general asks that Ω be thin in y direction if K has a large variation. Of course, this thinness condition can be removed in the elliptic region by applying Theorem 3.4. As a result, one obtains well posedness for the Dirichlet problem for the Tomatika-Tamada equation, for example, for any mixed domain which has a sufficiently thin and V-star-shaped hyperbolic part.

Finally, it should be noted that the norm $H^1(\Omega; K)$ employed here has a weight K which vanishes on the entire parabolic line and hence one might worry that the solution is not locally H^1 due to the term $|K|u_x^2$. In fact, the solution does lie in $H^1_{\mathrm{loc}}(\Omega)$ as follows from a microlocal analysis argument (cf. [20]). On the other hand, the norms used in [18] for treating the equation by way of a first order system were carefully constructed so as not to have weights vanishing on the interior. This gives a direct proof of the lack of H^1 singularities in the interior for the corresponding solution to the scalar problem in special cases.

4 Well posedness for mixed boundary conditions.

Having treated the well posedness for generalized solutions to the Dirichlet problem, we turn our attention to the other natural boundary condition for the transonic flow applications, namely the conormal boundary condition. Here, we follow the path laid out first by Morawetz [17] for open boundary value problems and then later adapted to very special cases for closed boundary value problems of Dirichlet type in [18] and conormal type in [22]. The basic idea is to rewrite the second order scalar equation for u as a first order system for the gradient of u, which has the effect of "eliminating a derivative" and allowing for a unified treatment of both natural boundary conditions. For general classes of type change functions and for fairly general domains, we can obtain results for the Dirichlet problem (and hence an alternative treatment to the method of section 3) as well as for mixed boundary conditions. For now, we are unable to make the technique work directly on the full conormal problem and hence the result of Pilant [22] (in the special case $K(y) = \mathrm{sgn}(y)$) remains the only one.

As a first step, we recall why boundary conditions of *Dirichlet* and *conormal* type are natural for the class of equations (2.1). As already noted, the operator L is formally self-adjoint and one has the following *Green's identity* for sufficiently regular functions u, v:

$$\int_{\Omega} (uLv - vLu)\; dxdy = \int_{\partial\Omega} (vu_{\nu} - uv_{\nu})\; ds \tag{4.1}$$

where $u_{\nu} = (Ku_x, u_y) \cdot \nu$ is the so-called *conormal derivative* with respect to the divergence form operator $L = \mathrm{div} \circ (K\partial_x, \partial_y)$. In this way, one expects that representation formulas for solutions to (2.1) – (2.2) might be available using as data the "forcing" function f in Ω as well as the values of u, u_ν on the boundary as happens for the the Laplace equation in terms of the Dirichlet and Neumann data. This approach has been used for open boundary value problems with some success (cf. [9], [24] and the references there in).

We note that we will continue to use much of the notation and conventions which were introduced in section 3. In particular: Ω is a bounded domain with piecewise C^1 bounndary; the parabolic line $AB = \{(x,y) \in \overline{\Omega} : y = 0\}$ has its maximal x-coordinate in $B = (0,0)$; the boundary $\partial\Omega$ will often be star-shaped with respect to a given (Lipschitz) continuous vector field V; the type change function $K \in C^1(\mathbf{R})$ satisfies (1.3) and additional conditions as needed.

Before giving the main results, we briefly recall the formulation of the boundary value problems via first order systems as done in [18, 22]. In place of the second order differential equation for the scalar valued u

$$Lu = Ku_{xx} + u_{yy} = f \tag{4.2}$$

with boundary conditions of Dirichlet

$$u = 0 \quad \text{on} \quad \Gamma \subseteq \partial\Omega \tag{4.3}$$

or conormal type

$$u_\nu = 0 \quad \text{on} \quad \partial\Omega \setminus \Gamma \tag{4.4}$$

one considers the system $\mathcal{L}v = g$

$$\mathcal{L}v = \begin{bmatrix} K\partial_x & \partial_y \\ \partial_y & -\partial_x \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \tag{4.5}$$

In the case that $(g_1, g_2) = (f, 0)$ and $v = \nabla u$, then a solution v to (4.5) gives rise to a solution u of (4.2). In order to give a suitable formulation of weak solutions to (4.5) and in order to build in the analogous boundary conditions to (4.3) – (4.4), one exploits the following *integration by parts formula* valid for every v, φ regular enough in a domain Ω:

$$(\mathcal{L}v, \varphi) = -(v, \mathcal{L}\varphi) + \int_{\partial\Omega} \varphi_1 \left(Kv_1\, dy - v_2\, dx\right) - \int_{\partial\Omega} \varphi_2 \left(v_1\, dx + v_2\, dy\right), \tag{4.6}$$

where in this section, we will denote by

$$(v, \varphi) = \int_\Omega (v_1\varphi_1 + v_2\varphi_2)\; dxdy \tag{4.7}$$

the scalar product on $\mathcal{H}(\Omega) = L^2(\Omega; \mathbf{R}^2)$. Keeping in mind that v should be thought of ∇u, the Dirichlet condition (4.3) for u becomes the condition

$$v_1\, dx + v_2\, dy = 0 \;\text{ on } \Gamma \subseteq \partial\Omega \tag{4.8}$$

on v, which of course corresponds to u identically constant on Γ. As for the conormal condition (4.4), one places the following condition on v:

$$Kv_1\, dy - v_2\, dx = 0 \;\text{ on }\; \partial\Omega \setminus \Gamma. \tag{4.9}$$

We will make use of several spaces of vector valued functions analogous to those used in the solvability discussions of sections 2 and 3. We will find solutions in the Hilbert space $\mathcal{H}_\mathcal{K}(\Omega)$ of all measurable functions $v = (v_1, v_2)$ on Ω for which the norm

$$||v||_{\mathcal{H}_\mathcal{K}} = \left[\int_\Omega \left(|K|v_1^2 + v_2^2\right)\, dxdy\right]^{1/2} \tag{4.10}$$

is finite, where we denote the scalar product by

$$(v, \varphi)_\mathcal{K} = \int_\Omega (|K|v_1\varphi_1 + v_2\varphi_2)\; dxdy. \tag{4.11}$$

Using the inner product (4.7), the dual space to $\mathcal{H}_\mathcal{K}(\Omega)$ can be identified with the space $\mathcal{H}_{\mathcal{K}^{-1}}(\Omega)$ in which $|K|^{-1}$ replaces $|K|$ in (4.10) and (4.11). These spaces are special cases of the Hilbert space

$$\mathcal{H}_\mathcal{A}(\Omega) = \{v : \mathcal{A}v \in \mathcal{H}(\Omega)\} \tag{4.12}$$

where $\mathcal{A}$ some piecewise continuous matrix value function on Ω, often invertible almost everywhere. In particular, the spaces $\mathcal{H}_\mathcal{K}(\Omega), \mathcal{H}_{\mathcal{K}^{-1}}(\Omega)$ are of this general form with the diagonal matrix $\mathcal{K}$ having entries $|K|^{\pm 1}$ and 1. We will denote by $\mathcal{A}^*$ the transpose of the matrix $\mathcal{A}$.

Using the formulas (4.6), (4.8), and (4.9) we have the following notion of a weak solutions.

Definition 4.1. *Let Ω be a bounded mixed domain and g such that $\mathcal{K}^{-1}\mathcal{M}^*g \in \mathcal{H}(\Omega)$ for a fixed matrix valued function $\mathcal{M}$ which is piecewise continuous and invertible almost everywhere. Let $\Gamma \subseteq \partial\Omega$ be a possibly empty subset. A weak solution to the system* (4.5) *with the Dirichlet condition on $\Gamma \subseteq \partial\Omega$ and conormal condition on $\partial\Omega \setminus \Gamma$ is function $v \in \mathcal{H}_K(\Omega)$ such that*

$$-(v, \mathcal{L}\varphi) = (g, \varphi) \tag{4.13}$$

for each test function φ such that

$$\varphi \in C^1(\overline{\Omega}; \mathbf{R}^2) \text{ and } \mathcal{K}^{-1}\mathcal{L}\varphi \in \mathcal{H}(\Omega) \tag{4.14}$$

and

$$\varphi_1 = 0 \text{ on } \Gamma \tag{4.15}$$

$$\varphi_2 = 0 \text{ on } \partial\Omega \setminus \Gamma \tag{4.16}$$

A weak existence theorem for the system follows directly from a suitable a priori estimate. We record the following general result, which has its origins in [17] for open problems.

Lemma 4.2: *Let Ω be a bounded mixed domain and Γ a potentially empty subset of $\partial\Omega$. Assume that there exists a matrix valued function $\mathcal{M}$ for which the following a priori estimate holds: there exists $C > 0$ such that*

$$||\mathcal{K}\mathcal{M}^{-1}\varphi||_{\mathcal{H}(\Omega)} \leq C||\mathcal{K}^{-1}\mathcal{L}\varphi||_{\mathcal{H}(\Omega)} \tag{4.17}$$

holds for each test function satisfying (4.14) − (4.16)*. Then there exists a weak solution u in the sense of Definition 4.1.*

Proof: The proof is classical (cf. example the proof of Theorem 3 in [18] or section 2 of [22]). For completeness, we give a sketch. One defines the linear functional $J_f(\mathcal{L}\varphi) = (g, \varphi)$ which by the Cauchy-Scwartz inequality and the a priori estimate (4.17) yields

$$\begin{aligned} |J_f(\mathcal{L}\varphi| &= |(\mathcal{K}^{-1}\mathcal{M}^*g, \mathcal{K}\mathcal{M}^{-1}\varphi)| \\ &\leq ||\mathcal{K}^{-1}\mathcal{M}^*g||_{\mathcal{H}(\Omega)}||\mathcal{K}\mathcal{M}^{-1}\varphi||_{\mathcal{H}(\Omega)} \\ &\leq C_g C||\mathcal{K}^{-1}\mathcal{L}\varphi||_{\mathcal{H}(\Omega)} \end{aligned}$$

where we note $\mathcal{K}$ is diagonal and hence self-adjoint and C_g is a constant which depends on the g which is fixed. Hence J_f gives a bounded linear functional on the subspace $\mathcal{W}$ of $\mathcal{H}_{\mathcal{K}^{-1}}(\Omega)$ of elements of the form $\mathcal{L}\varphi$. One then extends J_f to whole of $\mathcal{H}_{\mathcal{K}^{-1}}(\Omega)$ in the standard way (first by continuity on $\overline{\mathcal{W}}$ and then by zero on its orthogonal complement). The Riesz representation theorem then gives $w \in \mathcal{H}_{\mathcal{K}^{-1}}(\Omega)$ such that

$$(g, \varphi) = (w, \mathcal{L}\varphi)_{\mathcal{K}^{-1}} = (\mathcal{K}^{-2}w, \mathcal{L}\varphi)$$

and hence $v = -\mathcal{K}^{-2}w$ is the desired solution.

The theorem shows that for a given domain Ω and a given type change function K, it is then enough to find a suitable multiplier matrix $\mathcal{M}$ for which the estimate (4.15) holds. One again, a suitable version of the $a-b-c$ method as developed in [17] is the key.

Lemma 4.3. *Let Ω be a bounded mixed domain and Γ a potentially empty subset of $\partial\Omega$. Let $K \in C^1(\mathbf{R})$ be a type change function satisfying* (1.3). *Assume that there exists a pair (b, c) of continuous and piecewise C^1 functions on Ω such that:*

$$b^2 + Kc^2 > 0 \quad \text{on } \Omega, \tag{4.18}$$

there exists a constant $\delta > 0$ such that for each $(x, y) \in \Omega$ and each $\Phi \in \mathbf{R}^2$ one has

$$Q(\Phi) := \alpha(x,y)\Phi_1^2 + 2\beta(x,y)\Phi_1\Phi_2 + \gamma(x,y)\Phi_2^2 \geq \delta\left(|K(y)|\Phi_1^2 + \Phi_2^2\right), \tag{4.19}$$

where

$$\alpha = Kb_x - (Kc)_y, \quad \beta = Kc_x + b_y, \quad \gamma = c_y - b_x \tag{4.20}$$

and

$$b\,dy - c\,dx \leq 0 \quad \text{on } \Gamma \tag{4.21}$$

$$K(y)(b\,dy - c\,dx) \geq 0 \quad \text{on } \partial\Omega \setminus \Gamma. \tag{4.22}$$

Then there exists a matrix $\mathcal{M}$ for which the a priori estimate (4.17) *holds for each test function satisfying* (4.14) − (4.16).

Before giving the proof, we make a few remarks. Lemma 4.3 together with Lemma 4.2 reduces the weak existence to the art of finding a suitable multiplier pair (b, c). Notice also that in this system approach the multiplier a plays no role since one is working on $v = \nabla u$ in the place of u. The conditions (4.21) and (4.22) say that Γ and $\partial\Omega$ are suitably star-like with respect to the vector field $V = (b, c)$. In terms of the flow of V one is asking for "inflow" in the case of Dirichlet conditions and "inflow/outflow" in the case of conormal conditions depending on whether y is negative/positive. This need of reversing orientation of the flow across $y = 0$ makes the problem of finding a multiplier for the conormal problem particularly difficult. The proof of Lemma 4.3 is essentially classical; as done in [18], [22]. The minor difference here is the possible mixture of boundary conditions and the a priori choice of the norm in which to find solutions, as governed by the right hand side of (4.19). More generally, one could replace (4.19) by asking only that $Q > 0$ on Ω and use this quadratic form to define a norm in a different way.

Proof of Lemma 4.3: Given Ω and (b, c) satisfying $(4.18) - (4.22)$, one defines a multiplier matrix $\mathcal{M}$ by the recipe of [17]:

$$\mathcal{M} = \begin{bmatrix} b & c \\ -Kc & b \end{bmatrix} \tag{4.23}$$

which is invertible except possibly at $B = (0, 0)$ by (4.18). Given any test function φ satisfying $(4.14) - (4.16)$, one defines

$$\Phi = \mathcal{M}^{-1}\varphi \quad \text{or} \quad \varphi = \mathcal{M}\Phi \tag{4.24}$$

and tries to estimate $(\mathcal{L}\varphi, \mathcal{M}^{-1}\varphi) = (\mathcal{L}\mathcal{M}\Phi, \Phi)$ from above and below. Notice that the conditions (4.15) and (4.16) on φ become

$$(\mathcal{M}\Phi)_1 := b\Phi_1 + c\Phi_2 = 0 \quad \text{on } \Gamma \tag{4.25}$$

$$(\mathcal{M}\Phi)_2 := -Kc\Phi_1 + b\Phi_2 = 0 \quad \text{on } \partial\Omega \setminus \Gamma \tag{4.26}$$

One easily checks that by integrating by parts as in (4.6) and using (4.25) and (4.26) one has

$$\begin{aligned} 2(\mathcal{L}\mathcal{M}\Phi, \Phi) \quad = \quad & \int_\Omega Q(\Phi)\, dxdy + \int_\Gamma -\frac{1}{c^2}\Phi_1^2(b^2 + Kc^2)(b\, dy - c\, dx) \\ & + \int_{\partial\Omega\setminus\Gamma} \frac{1}{b^2}\Phi_1^2(b^2 + Kc^2)K(y)(b\, dy - c\, dx) \end{aligned}$$

$$\geq \int_\Omega Q(\Phi)\, dxdy \tag{4.27}$$

where we have used (4.18), (4.21) and (4.22) and Q is defined in (4.19). Using the lower bound in (4.19) one has

$$2(\mathcal{L}\mathcal{M}\Phi, \Phi) \geq \delta \int_\Omega \left(|K|\Phi_1^2 + \Phi_2^2\right)\, dxdy = \delta ||\mathcal{K}\mathcal{M}^{-1}\varphi||^2_{\mathcal{H}(\Omega)}. \tag{4.28}$$

Then estimating from above, one has

$$\begin{aligned} 2(\mathcal{L}\mathcal{M}\Phi, \Phi) &= 2(\mathcal{K}^{-1}\mathcal{L}\varphi, \mathcal{K}\mathcal{M}^{-1}\varphi) \\ &\leq 2||\mathcal{K}^{-1}\mathcal{L}\varphi||_{\mathcal{H}(\Omega)} ||\mathcal{K}\mathcal{M}^{-1}\varphi||_{\mathcal{H}(\Omega)}, \end{aligned} \tag{4.29}$$

by the Cauchy-Scwartz inequality since φ satisfies (4.14). Combining (4.28) and (4.29) yields

$$||\mathcal{K}\mathcal{M}^{-1}\varphi||_{\mathcal{H}(\Omega)} \leq \frac{2}{\delta} ||\mathcal{K}^{-1}\mathcal{L}\varphi||_{\mathcal{H}(\Omega)} \tag{4.30}$$

as claimed.

Our first existence result for the system (4.5) with Dirichlet or mixed boundary conditions is the following analog of Theorem 3.2 for type change functions of pure power type K.

Theorem 4.4. *Let $K = y|y|^{m-1}$ with $m > 0$ be a pure power type change function. Let Ω be a mixed domain and let Γ be either the entire boundary $\partial\Omega$ or the elliptic boundary $(\partial\Omega)^+$. Assume that* (4.18), (4.21) *and* (4.22) *hold with respect to the multiplier pair*

$$(b, c) = (-(m + 2)x), -\mu y) \tag{4.31}$$

where $\mu = 2$ in Ω^+ and $\mu = 1$ in Ω^-. Then, for each g such that $\mathcal{K}^{-1}\mathcal{M}^* g \in \mathcal{H}(\Omega)$, there exists a weak solution $v \in \mathcal{H}_{\mathcal{K}}(\Omega)$ in the sense of Definition 4.1 to the system (4.5) *with Dirichlet conditions on Γ and conormal conditions on $\partial\Omega \setminus \Gamma$.*

Proof: By the Lemmas 4.2 and 4.3 one needs only to check that the multiplier pair (4.31) gives rise to the inequality (4.19) for some $\delta > 0$. One easily checks that $\delta = \min\{1, m\}$ will do.

We remark that the multiplier pair (4.31) is almost the dilation multiplier; it has been fudged a bit in the hyperbolic region in order to get the correct sign for the quadratic form Q. In fact, one can slightly relax the hypotheses on Ω by choosing μ to satisfy $\mu(m+1) > m+2$ in Ω^+ and $\mu(m+1) < m+2$ in Ω^-.

Using the same proof, one can generalize Theorem 4.4 by replacing the pure power type change function K by an "almost pure power" K which satisfies the hypotheses of Proposition 3.5. Choosing again $(b, c) = (-b_0 x, -2K/K')$ as the multiplier pair defines a class of mixed domains Ω for which one has a weak solution to the system for the Dirichlet or the mixed boundary value problem.

We conclude with a few additional remarks. In order to get uniqueness of the weak solutions, one needs to show that there is sufficient regularity. This can be accomplishes by following the techniques laid out in [18] or [22], which in turn use ideas from [11] and [8] in which mollifying in the x-direction plays a key role. In a similar way, if there is enough regularity then in the case $(g_1, g_2) = (f, 0)$ one can use the second equation in the system (4.5) in order to define a potential function u for which $v = \nabla u$ and this u will solve the scalar equation (4.2) with the associated boundary conditions (4.3) – (4.4).

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