ATTRACTOR FOR A CONSERVED PHASE-FIELD SYSTEM WITH HYPERBOLIC HEAT CONDUCTION

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To Professor Enrico Magenes on his eightieth birthday

Abstract. We consider a conserved phase-field system of Caginalp type, characterized by the assumption that both the internal energy and the heat flux depend on the past history of the temperature and its gradient, respectively. The latter dependence is a law of Gurtin-Pipkin type, so that the equation ruling the temperature evolution is hyperbolic. Thus the model consists of a hyperbolic integrodifferential equation coupled with a fourth-order evolution equation for the phase-field. This model, endowed with suitable boundary conditions, has already been analyzed within the theory of dissipative dynamical systems, and the existence of an absorbing set has been obtained. Here we prove the existence of the universal attractor.

Keywords: Conserved phase-field models, memory effects, infinite-dimensional dissipative dynamical systems, universal attractor.

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1 Introduction

We have recently studied phase-field systems of Caginalp type with memory effects as infinite-dimensional dissipative dynamical systems (see, e.g., [10, 11, 12, 18] and references therein). These models are characterized by constitutive laws for the internal energy and the heat flux which show a dependence on the past history of the (relative) temperature ϑ and its gradient, respectively, through convolution integrals with suitable smooth memory kernels. Thus, via the energy balance, one ends up with an integrodifferential heat equation coupled with an Allen-Cahn or a Cahn-Hilliard type equation governing the order parameter (or phase-field) χ. The former case is usually named *nonconserved*, since the spatial average of χ is not constant in time; while the latter is called the *conserved* case, for the same quantity does not depend on time, provided that no-flux boundary conditions are supposed to hold. The analysis of these models from the point of view of dynamical systems is based on the introduction of an additional variable η, the integrated past history of ϑ, that solves a first-order linear hyperbolic equation. Consequently, we are dealing with a system of three coupled evolution equations governing ϑ, η, and χ. This reformulation of the original model can be interpreted as a dynamical system in a suitable infinite-dimensional phase-space which accounts for the past history of η regarding it as an initial datum in a weighted Hilbert space (on this approach see [15] and its references).

The picture in the nonconserved case is fairly detailed. Indeed, when the heat flux depends both on the past history and on the instantaneous values of $\nabla\vartheta$, i.e., it also contains the term $k\nabla\vartheta$, with $k > 0$, we have proved that the resulting model endowed, e.g., with homogeneous Neumann boundary conditions, generates a strongly continuous semigroup with a universal attractor $\mathcal{A}_k$ of finite fractal dimension (see [11], cf. also [14]). Moreover, the existence of an exponential attractor has been shown in [7]. These results are obtained by assuming that the memory kernels and some of their derivatives satisfy certain monotonicity conditions as well as a suitable behavior at infinity. We recall that these hypotheses do comply with the Second Principle of Thermodynamics (see [9, 10]). Here, the coupling term linking the heat equation with the phase-field equation can have quadratic growth, allowing second order phase transitions (cf. [2]). When $k = 0$, the heat flux law is the linearized version of the Gurtin-Pipkin type (see [19], cf. also [21] and references therein). In this case the mathematical analysis gets much more complicated, since the heat equation becomes hyperbolic and its dissipation features are due to the memory effects only (cf. [13]). Nonetheless, provided that the coupling between the heat equation and the phase-field equation is linear, we can still obtain the existence of the (finite dimensional) universal attractor $\mathcal{A}_0$ which is upper semicontinuous with respect to the family $\{\mathcal{A}_k\}$ (see [12]). The analysis of the exponential attractors will be done in a forthcoming paper.

As far as the conserved model is concerned, some details are still missing. Indeed, we have only analyzed the case $k > 0$ with linear coupling, proving the existence of a finite-dimensional universal attractor (see [18]), as well as the existence of an exponential attractor (cf. [8]). In this paper we want to deal with the apparently most difficult case, that is, $k = 0$. Several results on this model have been obtained, by regarding the past history as an additional source (see [3, 4, 5, 23, 25]). However, concerning the global longterm behavior, it has only been proved that the resulting model is a dynamical system with an absorbing set (see [27]), but the question about the existence of the universal attractor has been left unanswered. Our present goal is to give a positive answer. As we shall see, this result will require some technical efforts. However, an important question remains open; that is, finding some regularity for the universal attractor which would be basic to obtain further results like the upper semicontinuity or the existence of exponential attractors; but this seems a hard task (compare also with [16]).

We can now introduce the problem considered in [27]. Let $\Omega \subset \mathbb{R}^3$ be a bounded connected domain with smooth boundary $\partial\Omega$. The phase-field system we want to analyze is

$$\partial_t \left(\vartheta(t) + \chi(t) + \int_0^\infty a(s)\vartheta(t-s)ds \right) - \int_0^\infty b(s)\Delta\vartheta(t-s)ds = f \tag{1.1}$$

$$\partial_t\chi(t) - \Delta w(t) = 0 \tag{1.2}$$

$$w(t) = -\Delta\chi(t) + \chi^3(t) + \gamma'(\chi(t)) - \vartheta(t), \tag{1.3}$$

in Ω, $t \in \mathbb{R}^+ = (0, \infty)$. Here γ is a smooth function on $\mathbb{R}$ with at most quadratic growth, whereas f is an external source, that for sake of simplicity we assume to be constant in time.

All the physical constants have been taken equal to one. The memory kernels a and b are positive smooth functions on $\mathbb{R}^+$ satisfying suitable properties at infinity (see next section).

We supplement the system with no-flux boundary conditions; namely,

$$\int_0^\infty b(s)\partial_{\boldsymbol{n}}\vartheta(t-s)ds = 0 \qquad \text{on } \partial\Omega,\ t \in \mathbb{R}^+ \tag{1.4}$$

$$\partial_{\boldsymbol{n}}\chi(t) = 0 \qquad \text{on } \partial\Omega,\ t \in \mathbb{R}^+ \tag{1.5}$$

$$\partial_{\boldsymbol{n}}w(t) = 0 \qquad \text{on } \partial\Omega,\ t \in \mathbb{R}^+, \tag{1.6}$$

where $\partial_{\boldsymbol{n}}$ represents the outward normal derivative to $\partial\Omega$.

Concerning the initial conditions, we need to know the value of ϑ and χ at $t = 0$, as well as the values of ϑ for $t < 0$. Hence, we have

$$\vartheta(0) = \vartheta_0 \qquad \text{in } \Omega \tag{1.7}$$

$$\chi(0) = \chi_0 \qquad \text{in } \Omega \tag{1.8}$$

$$\vartheta(-s) = \vartheta_1(s) \qquad \text{in } \Omega \times \mathbb{R}^+, \tag{1.9}$$

where $\vartheta_0, \chi_0 : \Omega \to \mathbb{R}$ and $\vartheta_1 : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are prescribed functions.

In the next section we will introduce some notation as well as the assumptions on the memory kernels and the functions γ and f. Then, in Section 3, we formulate (1.1)-(1.9) as a dynamical system in the history phase-space, by introducing the additional variable η, and we recall the results of [27]. Section 4 is devoted to present the main result on the existence of the universal attractors as well as some technical lemmas, whose proofs are given in Sections 5 and 6.

Remark 1.1. In this paper we will assume that $a(0) > 0$. However, when $a \equiv 0$ our result still holds provided one exploits the fact that the spatial average of the internal energy $\vartheta + \chi$ is conserved. This happens, for instance, when f is independent of time with null average. Of course, the phase-space has a more complicated structure. Things get simpler when, for instance, ϑ satisfies the homogeneous Dirichlet boundary condition on some portion of $\partial\Omega$ with positive surface measure, since this allows the use of the Poincaré inequality. In this case, one can even take $a \equiv 0$, without any significant change in the definition of the phase-space (cf. [13]).

2 Assumptions and notations

We introduce the Hilbert spaces

$$H = L^2(\Omega), \qquad V = H^1(\Omega), \qquad W = \big\{u \in H^2(\Omega) \ : \ \partial_{\boldsymbol{n}}u = 0 \ \text{on} \ \partial\Omega\big\}.$$

We will use the symbols $\langle\cdot,\cdot\rangle_X$ and $\|\cdot\|_X$ to denote the inner product and the norm on a given space X, and we will keep the same notation when the appearing quantities are vectors in X^3. The symbol $\langle\cdot,\cdot\rangle$ will stand for duality pairing between V^* (dual space) and V.

For any $u \in H$, we define the *spatial average* $\overline{u}$ of u to be

$$\overline{u} = \frac{\langle u, 1\rangle_H}{\|1\|_H^2},$$

where 1 denotes the constant-valued function on Ω that equals 1 for every $x \in \Omega$.

The assumptions on the nonlinearity and the source term are[1]

$$\gamma \in C^2(\mathbb{R}) \quad \text{with} \quad \gamma'' \in L^\infty(\mathbb{R}) \tag{2.1}$$

$$f \in H \quad \text{constant in time.} \tag{2.2}$$

Concerning the memory kernels, setting

$$\nu(s) = -a''(s) \qquad \text{and} \qquad \mu(s) = -b'(s),$$

we require (cf. [18, 27])

$$\begin{array}{ll} \text{(K1)} & \nu, \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+) \\ \text{(K2)} & \nu(s) \geq 0, \quad \mu(s) \geq 0, \quad \forall s \in \mathbb{R}^+ \\ \text{(K3)} & \nu'(s) \leq 0, \quad \mu'(s) \leq 0, \quad \forall s \in \mathbb{R}^+ \\ \text{(K4)} & \exists\, \delta > 0 : \ \nu'(s) + \delta \nu(s) \leq 0, \quad \mu'(s) + \delta \mu(s) \leq 0, \quad \forall s \in \mathbb{R}^+. \end{array}$$

Condition (K4) is basic to prove the existence of an absorbing set (see [27]). It is also worth reminding that, alternatively, a can be chosen bounded, nonincreasing, and convex, so that the signs of ν and ν' in (K2) and (K3), respectively, must be reversed. However, in order to get an absorbing set, ν needs to be suitably dominated by μ (see [18]).

In view of (K1)-(K2), we introduce the weighted Hilbert space

$$\mathcal{M} = L^2_\nu(\mathbb{R}^+; H) \cap L^2_\mu(\mathbb{R}^+; V),$$

and we consider the infinitesimal generator of the C_0-semigroup of right-translations on $\mathcal{M}$, that is, the linear operator T on $\mathcal{M}$ with domain

$$\mathcal{D}(T) = \big\{ \eta \in \mathcal{M} \ : \ \partial_s \eta \in \mathcal{M}, \ \eta(0) = 0 \big\}$$

defined by

$$T\eta = -\partial_s \eta, \qquad \eta \in \mathcal{D}(T).$$

Here $\partial_s \eta$ is the distributional derivative of η with respect to the internal variable s. Due to (K1)-(K3), the operator T is dissipative; if we also use (K4), then (see [6])

$$\langle T\eta, \eta \rangle_{\mathcal{M}} \leq -\frac{\delta}{2} \|\eta\|^2_{\mathcal{M}}, \qquad \forall \eta \in \mathcal{D}(T).$$

Finally, we define the product Hilbert space

$$\mathcal{H} = H \times V \times \mathcal{M}.$$

Since from equation (1.2) (provided that, as in the case we will consider, the variables have enough regularity) $\overline{\chi}$ is conserved, we will also need to introduce for every $\alpha \geq 0$, the complete metric space

$$\mathcal{H}_\alpha = \big\{ z = (\vartheta, \chi, \eta) \in \mathcal{H} \ : \ |\overline{\chi}| \leq \alpha \big\},$$

with the metric topology induced by $\mathcal{H}$.

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[1] We take here the occasion to mention that in our joint paper [18] the unnecessary requirement $\gamma' \in L^\infty(\mathbb{R})$ appears in the assumptions. However, it has never been used in the proofs.

Remark 2.1. Even though for better clarity we consider the nonlinearity $\phi(r) = r^3 + \gamma'(r)$, which however includes the physically relevant case $\phi(r) = r^3 - r$, all our results can be extended without substantial changes in the proofs to a more general nonlinear term ϕ of the form $\phi = \phi_0 + \phi_1$, with $\phi_0 \in C^2(\mathbb{R})$ and $\phi_1 \in C^1(\mathbb{R})$ such that

$$\begin{aligned}
&r\phi_0(r) \geq 0, \qquad \forall\, r \in \mathbb{R} \\
&|\phi_0''(r)| \leq k_1(1+|r|), \qquad \forall\, r \in \mathbb{R} \\
&|\phi_1'(r)| \leq k_2(1+|r|^\gamma), \qquad \gamma \in [0,2),\ \forall\, r \in \mathbb{R} \\
&\liminf_{|r|\to\infty} \frac{\phi_1(r)}{r} > -c_P
\end{aligned}$$

for some $k_1, k_2 \geq 0$. Here $c_P > 0$ is the Poincaré-Wirtinger constant for null-average functions of V. The last condition enters only in the proof of the existence of an absorbing set. We also remark that a function $\phi \in C^2(\mathbb{R})$ satisfying the growth condition

$$|\phi''(r)| \leq k(1+|r|), \qquad \forall\, r \in \mathbb{R}$$

for some $k \geq 0$, and the dissipation condition

$$\liminf_{|r|\to\infty} \frac{\phi(r)}{r} > -c_P$$

admits the above decomposition (see [1], cf. also [17]).

3 The dissipative dynamical system $S(t)$

As in [18, 27], we want to rewrite equations (1.1)-(1.3) in order to obtain a solution semigroup $S(t)$. Therefore, as we mentioned in the introduction, we define the *summed past history*

$$\eta^t(x,s) = \int_0^s \vartheta(x, t-y)dy, \qquad (x,t,s) \in \Omega \times \overline{\mathbb{R}^+} \times \mathbb{R}^+,$$

and we perform a (formal) integration by parts in the convolution terms. This procedure allows us to translate (1.1)-(1.9) into the following system (see [18] for more details):

$$\partial_t\big(\vartheta + \chi\big) + \vartheta + \int_0^\infty \nu(s)\eta(s)ds - \int_0^\infty \mu(s)\Delta\eta(s)ds = f \tag{3.1}$$

$$\partial_t\chi - \Delta\big(-\Delta\chi + \chi^3 + \gamma'(\chi) - \vartheta\big) = 0 \tag{3.2}$$

$$\partial_t\eta = T\eta + \vartheta, \tag{3.3}$$

along with the boundary conditions

$$\int_0^\infty \mu(s)\partial_{\boldsymbol{n}}\eta(s)ds = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+ \tag{3.4}$$

$$\partial_{\boldsymbol{n}}\chi = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+ \tag{3.5}$$

$$\partial_{\boldsymbol{n}}\big(-\Delta\chi + \chi^3 + \gamma'(\chi) - \vartheta\big) = 0 \qquad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{3.6}$$

and the initial conditions

$$\vartheta(0) = \vartheta_0 \qquad \text{in } \partial\Omega \tag{3.7}$$

$$\chi(0) = \chi_0 \qquad \text{in } \partial\Omega \tag{3.8}$$

$$\eta^0 = \eta_0 \qquad \text{in } \partial\Omega \times \mathbb{R}^+, \tag{3.9}$$

where we have set $a(0) = 1$ and

$$\eta_0(x, s) = \int_0^s \vartheta_1(x, y)dy.$$

Then we have

Theorem 3.1. *Let conditions* (2.1)-(2.2) *and* (K1)-(K3) *hold. Then for every* $\alpha \geq 0$*, every* $T > 0$*, and every* $z_0 = (\vartheta_0, \chi_0, \eta_0) \in \mathcal{H}_\alpha$*, system* (3.1)-(3.9) *admits a unique solution*

$$z(t) = (\vartheta(t), \chi(t), \eta^t) \in C([0, T], \mathcal{H}_\alpha).$$

Moreover, the solution continuously depends on the initial data.

We recall that the proof of Theorem 3.1 is basically obtained by a vanishing viscosity argument applied to the corresponding parabolic problem studied in [18]; that is, the problem in which a term of the form $-\varepsilon\Delta\vartheta$ appears in the right-hand side of (3.1) (see [27]).

Remark 3.2. By means of Theorem 3.1, the solutions $z(t)$ to (3.1)-(3.9) can be expressed in terms of a C_0-semigroup $S(t)$ of (nonlinear) operators, namely,

$$z(t) = S(t)z_0.$$

Notice that $S(t)$ is a C_0-semigroup both on the phase-space $\mathcal{H}$ and on the phase-space $\mathcal{H}_\alpha$, for all $\alpha \geq 0$.

Remark 3.3. We point out that the translation of (1.1)-(1.9) into (3.1)-(3.9) is not only formal. Indeed, provided that the initial data are smooth enough, it is possible to show that a triplet (ϑ, χ, η) is a solution to (3.1)-(3.9) if and only if the corresponding functions ϑ and χ solve (1.1)-(1.9). In fact, (3.1)-(3.9) actually generalize (1.1)-(1.9), since they support less regular initial data. This equivalence is analyzed in detail for a general class of differential systems with memory in the paper [15].

When the memory kernels exhibit the decay property (K4), the semigroup $S(t)$ is shown to be dissipative; more precisely, it possesses a bounded (connected) invariant absorbing set. This is the main result of [27]:

Theorem 3.4. *Let conditions* (2.1)-(2.2) *and* (K1)-(K4) *hold. Then for every* $\alpha \geq 0$*, there exists a closed ball* $\mathcal{B}_0 = \mathcal{B}_0(\alpha) \subset \mathcal{H}_\alpha$ *such that, for every bounded set* $\mathcal{B} \subset \mathcal{H}_\alpha$*, there exists a time* $t_0 = t_0(\mathcal{B}) \geq 0$ *such that*

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \qquad \forall t \geq t_0.$$

Moreover, $S(t)\mathcal{B}_0 \subset \mathcal{B}_0$ *for every* $t \geq 0$*. The same result holds true with* $\mathcal{H}$ *in place of* $\mathcal{H}_\alpha$*.*

4 The universal attractor

The aim of the present work is to show that the semigroup $S(t)$ acting on the phase-space $\mathcal{H}_\alpha$ possesses a universal attractor, that is, a compact fully invariant subset of $\mathcal{H}_\alpha$ which attracts bounded sets with respect to the Hausdorff semidistance (cf. [20, 26]).

Indeed, we state

Theorem 4.1. *Let* (2.1)-(2.2) *and* (K1)-(K4) *hold. Then for every $\alpha \geq 0$, the semigroup $S(t)$ acting on $\mathcal{H}_\alpha$ possesses a connected universal attractor $\mathcal{A} = \mathcal{A}(\alpha)$.*

To prove this theorem, we proceed with a decomposition of the semigroup, in order to apply the techniques of the theory of the attractors for dynamical systems.

The semigroup decomposition. For a fixed $\alpha \geq 0$, we consider the solution $z(t) = S(t)z_0$ with

$$z_0 \in \mathcal{B}_0 \subset \mathcal{H}_\alpha,$$

where $\mathcal{B}_0$ is the connected, invariant and bounded absorbing set, whose existence is given by Theorem 3.4. Then we choose a function $g \in V$, and we write $z(t)$ as the sum

$$z(t) = z_d(t) + z_c(t),$$

where $z_d = (\vartheta_d, \chi_d, \eta_d)$ and $z_c = (\vartheta_c, \chi_c, \eta_c)$ are the solutions to the systems

$$\begin{aligned}
\partial_t \vartheta_d &= -\vartheta_d - \partial_t \chi_d - \int_0^\infty \nu(s)\eta_d(s)ds + \int_0^\infty \mu(s)\Delta\eta_d(s)ds + f - g && (4.1)\\
\partial_t \chi_d &= \Delta(-\Delta\chi_d + \chi_d^3 - \vartheta_d) && (4.2)\\
\partial_t \eta_d &= T\eta_d + \vartheta_d && (4.3)\\
z_d(0) &= (\vartheta_0, \chi_0 - \overline{\chi_0}, \eta_0) && (4.4)
\end{aligned}$$

and

$$\begin{aligned}
\partial_t \vartheta_c &= -\vartheta_c - \partial_t \chi_c - \int_0^\infty \nu(s)\eta_c(s)ds + \int_0^\infty \mu(s)\Delta\eta_c(s)ds + g && (4.5)\\
\partial_t \chi_c &= \Delta(-\Delta\chi_c + \chi^3 - \chi_d^3 + \gamma'(\chi) - \vartheta_c) && (4.6)\\
\partial_t \eta_c &= T\eta_c + \vartheta_c && (4.7)\\
z_c(0) &= (0, \overline{\chi_0}, 0). && (4.8)
\end{aligned}$$

Systems (4.1)-(4.4) and (4.5)-(4.8) admit unique solutions; moreover $z_d(t) \in C(\mathcal{H}_0, \mathcal{H}_0)$ and $z_d(t) \in C(\mathcal{H}_\alpha, \mathcal{H}_\alpha)$ for every fixed time $t \geq 0$.

The longterm properties of z_d and z_c are subsumed in the next two lemmas, whose proofs will be given in the last sections.

Lemma 4.2. *For every $\omega > 0$ there exist $t_\omega > 0$ and $g = g_\omega \in V$, both independent of $z_0 \in \mathcal{B}_0$, such that*

$$\|z_d(t)\|_{\mathcal{H}} \leq \omega, \qquad \forall t \geq t_\omega \tag{4.9}$$

for all $z_0 \in \mathcal{B}_0$.

Lemma 4.3. *For every $t \geq 0$ and every $g \in V$ there exists a compact set $\mathcal{K}(t, g, \alpha) \subset \mathcal{H}_\alpha$ such that*

$$z_c(t) \in \mathcal{K}(t, g) \tag{4.10}$$

for all $z_0 \in \mathcal{B}_0$.

Collecting (4.9) and (4.10), it is straightforward to see that

$$\lim_{t \to \infty} \alpha_{\mathcal{H}_\alpha}[S(t)\mathcal{B}_0] = 0,$$

$\alpha_{\mathcal{H}_\alpha}$ being the Kuratowski measure of noncompactness in $\mathcal{H}_\alpha$. This fact, on account of standard arguments of the theory of dynamical systems (cf. [20]), yields the thesis of Theorem 4.1.

It is also worth noticing that the flow on $\mathcal{H}$ is injective. On account of the full invariance of $\mathcal{A}$ this means that the C_0-semigroup $S(t)$ restricted to $\mathcal{A}$ is in fact a C_0-group. This is a consequence of the following backward uniqueness property

Proposition 4.4. *Let* (2.1)-(2.2) *and* (K1)-(K4) *hold. Consider $z_{01}, z_{02} \in \mathcal{H}$ and assume that for some $\tau > 0$ the equality $S(\tau)z_{01} = S(\tau)z_{02}$ holds. Then $z_{01} = z_{02}$.*

Proof. For $i = 1, 2$, denote

$$\begin{aligned} z_0 = (\vartheta_0, \chi_0, \eta_0) &= z_{01} - z_{02} \\ (\vartheta_i(t), \chi_i(t), \eta_i^t) &= S(t)z_{0i} \\ z(t) = (\vartheta(t), \chi(t), \eta^t) &= S(t)z_{01} - S(t)z_{02}. \end{aligned}$$

Since $\eta^\tau = 0$, we get directly from the representation formula

$$\eta^t(s) = \begin{cases} \int_0^s \theta(t-y)dy, & 0 < s \leq t \\ \eta_0(s-t) + \int_0^t \theta(t-y)dy, & s > t \end{cases}$$

that $\eta_0 = 0$ and so $\eta^t = 0$ in $[0, \tau]$. Therefore, from equation (3.3) we deduce $\vartheta = 0$ in $[0, \tau]$ so that, owing to the linear coupling, equation (3.1) gives $\partial_t \chi = 0$ almost everywhere in $[0, \tau]$. Being $\chi(\tau) = 0$ we infer $\chi = 0$ in $[0, \tau]$. Therefore $z_0 = (0, 0, 0)$, as desired. □

Remark 4.5. It is worth mentioning that, although we studied the stationary case, with minor efforts the results can be generalized for a time-dependent translation-compact source term f in a suitable space (cf. [18]).

Remark 4.6. At the beginning, we put $a(0) = 1$ (i.e., the coefficient of ϑ in (3.1)). However, the greater is $a(0)$, the higher is the dissipation of the system. In fact, if one takes $a(0)$ large enough, it is possible to say more about the attractor, for instance, to prove some regularity results, and, possibly, to investigate the existence of exponential attractors, along the lines of [7, 8].

5 Proof of Lemma 4.2

We just give a sketch the proof, since it is very similar to the one of Lemma 7.8 in [18]. Introducing the null-average spaces

$$V_A = \big\{ v \in V \;:\; \langle v, 1\rangle_H = 0 \big\} \qquad \text{and} \qquad V_A' = \big\{ \xi \in V^* \;:\; \langle \xi, 1\rangle = 0 \big\},$$

we can define the Riesz map $\mathcal{T} : V_A' \to V_A$ as

$$-\Delta \mathcal{T}\xi = \xi, \qquad \forall \xi \in V_A'.$$

In particular, there holds

$$\langle \zeta, \mathcal{T}\xi \rangle = \langle \zeta, \xi \rangle_{V^*}, \qquad \forall \zeta, \xi \in V_A'.$$

We also recall that

$$\|\nabla v\|_H^2 \le \|v\|_V^2 \le (1 + c_P)\|\nabla v\|_H^2, \qquad \forall v \in V_A,$$

where c_P is the Poincaré-Wirtinger constant.

Since $\chi_d(t) \in V_A$ for all $t \ge 0$, we take the products in H of (4.1) and ϑ_d, of (4.2) and $\mathcal{T}\partial_t \chi_d$, and of (4.2) and $\kappa \mathcal{T}\chi_d$, for some suitably small $\kappa > 0$. Next we take the product in $L^2_\nu(\mathbb{R}^+, H)$ of (4.3) and η_d, and the product in $L^2_\mu(\mathbb{R}^+, H)$ of the gradient of (4.3) and $\nabla \eta_d$. Following [18], we then integrate by parts in ds, and we make use of (K4). Collecting all the above estimates, and setting

$$\rho(\eta_d) = \int_0^\infty \nu(s)\|\eta_d(s)\|_H^2 ds + \int_0^\infty \mu(s)\|\nabla \eta_d(s)\|_H^2 ds$$

and

$$\Phi_d(t) = \|\vartheta_d(t)\|_H^2 + \kappa\|\chi_d(t)\|_{V^*}^2 + \|\nabla\chi_d(t)\|_H^2 + \|\chi_d(t)\|_{L^4}^4 + \rho(\eta_d^t),$$

it is possible to find $\varepsilon > 0$ such that

$$\frac{d}{dt}\Phi_d + 2\varepsilon\Phi_d + \|\partial_t\chi_d\|_{V^*}^2 \le \|f - g\|_H^2. \tag{5.1}$$

Thus by the Gronwall Lemma we end up with

$$\Phi_d(t) \le \Phi_d(0)e^{-2\varepsilon t} + \frac{1}{2\varepsilon}\|f - g\|_H^2, \qquad \forall t \ge 0. \tag{5.2}$$

Set now

$$\Psi_d(t) = \|\vartheta_d(t)\|_H^2 + \kappa\|\chi_d(t)\|_{V^*}^2 + \|\nabla\chi_d(t)\|_H^2 + \|\chi_d(t)\|_{L^4}^4 + \|\eta_d^t\|_{\mathcal{M}}^2,$$

and repeat the same argument leading to (5.1) with $\|\eta_d\|_{\mathcal{M}}^2$ in place of $\rho(\eta_d)$, to obtain, for some $c = c(\varepsilon) > 0$,

$$\begin{aligned}
\frac{d}{dt}\Psi_d + 2\varepsilon\Psi_d &\le \|f - g\|_H^2 + 2\int_0^\infty \mu(s)\langle \eta_d(s), \vartheta_d\rangle_H ds \\
&\le \|f - g\|_H^2 + \varepsilon\|\eta_d\|_{\mathcal{M}}^2 + c\|\vartheta_d\|_H^2 \\
&\le \|f - g\|_H^2 + \varepsilon\Psi_d + c\Phi_d
\end{aligned}$$

In light of (5.2), the above inequality turns into

$$\frac{d}{dt}\Psi_d(t) + \varepsilon\Psi_d(t) \leq c\Phi_d(0)e^{-2\varepsilon t} + \frac{c+2\varepsilon}{2\varepsilon}\|f-g\|_H^2,$$

and a further application of the Gronwall Lemma yields

$$\Psi_d(t) \leq \Big(\Psi_d(0) + \frac{c}{\varepsilon}\Phi_d(0)\Big)e^{-\varepsilon t} + \frac{c+2\varepsilon}{2\varepsilon^2}\|f-g\|_H^2.$$

The thesis then follows quite directly, upon choosing g properly close in the H-norm to f, and t enough large. □

Remark 5.1. Notice that integrating (5.1), it is also possible to find an L^2-estimate for $\|\partial_t \chi_d\|_{V^*}$, which will be useful later.

6 Proof of Lemma 4.3

Let A be the strictly positive operator on H defined by

$$A = \mathbb{I} - \Delta \qquad \text{with domain} \qquad \mathcal{D}(A) = W.$$

Then, for $s \in \mathbb{R}$, we introduce the Hilbert spaces

$$V_s = \mathcal{D}(A^{s/2})$$

endowed with the inner products

$$\langle\cdot,\cdot\rangle_{V_s} = \langle A^{s/2}\cdot, A^{s/2}\cdot\rangle_H.$$

Also, we set

$$\mathcal{M}_s = L^2_\nu(\mathbb{R}^+, V_s) \cap L^2_\mu(\mathbb{R}^+, V_{1+s}).$$

We recall for $s \in [0, 3/2)$ we have the continuous embeddings

$$V_{1-s} \hookrightarrow L^{6/(1+2s)}(\Omega) \tag{6.1}$$

and

$$V_{1+s} \hookrightarrow L^{6/(1-2s)}(\Omega). \tag{6.2}$$

Moreover (see [22]),

$$\nabla : V_s \to (V_{s-1})^3 \ \text{ is a continuous linear operator } \forall s \geq 0,\ s \neq 1/2. \tag{6.3}$$

Since we work in a regularization scheme, it is convenient to rewrite equations (4.5)-(4.6) in terms of the operator A; namely, as

$$\begin{aligned}
\partial_t\vartheta_c &= -\vartheta_c - \partial_t\chi_c + \int_0^\infty \big[\mu(s)-\nu(s)\big]\eta_c(s)ds - \int_0^\infty \mu(s)A\eta_c(s)ds + g && (6.4)\\
\partial_t\chi_c &= -A^2\chi_c + 2A\chi_c - \chi_c - A(\chi^3-\chi_d^3) + \chi^3 - \chi_d^3 \\
&\quad -A\gamma'(\chi) + \gamma'(\chi) + A\vartheta_c - \vartheta_c. && (6.5)
\end{aligned}$$

Through the end of the proof, we will consider fixed $t \geq 0$ and $\alpha \geq 0$, and we will denote by c a generic positive constant depending only on t, α and $g \in V$, but independent of $z_0 \in \mathcal{B}_0$. Also, we select

$$\sigma \in (0, \tfrac{1}{2}).$$

In view of the results of Section 3 and Section 5, we know in particular that

$$\sup_{\tau\in[0,t]} \Big[\|\vartheta(\tau)\|_H + \|\vartheta_d(\tau)\|_H + \|\chi(\tau)\|_V + \|\chi_d(\tau)\|_V \Big] \leq c \tag{6.6}$$

and

$$\int_0^t \big(\|\chi(y)\|_W^2 + \|\chi_d(y)\|_W^2 + \|\partial_t\chi(y)\|_{V^*}^2 + \|\partial_t\chi_d(y)\|_{V^*}^2 \big) dy \leq c. \tag{6.7}$$

It seems convenient to break the proof in some lemmas.

Lemma 6.1. *The following estimate holds:*

$$\begin{aligned} \langle \partial_t(\chi^3 - \chi_d^3), A^\sigma\chi_c\rangle_H &\leq c\big(\|\chi\|_W^2 + \|\chi_d\|_W^2 + \|\partial_t\chi_d\|_{V^*}^2\big)\|\chi_c\|_{V_{1+\sigma}}^2 \\ &\quad + c\|\chi_c\|_{V_{2+\sigma}}^2 + \frac{1}{2}\|\partial_t\chi_c\|_{V_{-1+\sigma}}^2. \end{aligned}$$

Proof. Let us rewrite the left-hand side of the above inequality as

$$\begin{aligned} \langle \partial_t(\chi^3 - \chi_d^3), A^\sigma\chi_c\rangle_H &= 3\langle \chi^2\partial_t\chi_c, A^\sigma\chi_c\rangle_H + 3\langle \chi_c\chi\partial_t\chi_d, A^\sigma\chi_c\rangle_H \\ &\quad + 3\langle \chi_c\chi_d\partial_t\chi_d, A^\sigma\chi_c\rangle_H. \end{aligned} \tag{6.8}$$

We now examine the three pieces separately.

For the first one, we have

$$3\langle \chi^2\partial_t\chi_c, A^\sigma\chi_c\rangle_H \leq 3\|\partial_t\chi_c\|_{V_{-1+\sigma}}\|\chi^2 A^\sigma\chi_c\|_{V_{1-\sigma}}.$$

Exploiting the continuous Sobolev embedding

$$W^{1,6/(3+2\sigma)}(\Omega) \hookrightarrow V_{1-\sigma},$$

we obtain

$$\|\chi^2 A^\sigma\chi_c\|_{V_{1-\sigma}} \leq c\|\chi^2 A^\sigma\chi_c\|_{L^{6/(3+2\sigma)}} + c\|\chi\nabla\chi A^\sigma\chi_c\|_{L^{6/(3+2\sigma)}} + c\|\chi^2\nabla A^\sigma\chi_c\|_{L^{6/(3+2\sigma)}}.$$

By means of the Hölder inequality, (6.2)-(6.3) and (6.6), we get the estimates

$$\begin{aligned} c\|\chi^2 A^\sigma\chi_c\|_{L^{6/(3+2\sigma)}} &\leq c\|\chi\|_{L^6}^2\|A^\sigma\chi_c\|_{L^{6/(1+2\sigma)}} \\ &\leq c\|\chi\|_V\|A^\sigma\chi_c\|_{V_{1-\sigma}} \\ &\leq c\|\chi\|_W\|\chi_c\|_{V_{1+\sigma}}, \end{aligned}$$

$$\begin{aligned} c\|\chi\nabla\chi A^\sigma\chi_c\|_{L^{6/(3+2\sigma)}} &\leq c\|\chi\|_{L^6}\|\nabla\chi\|_{L^6}\|A^\sigma\chi_c\|_{L^{6/(1+2\sigma)}} \\ &\leq c\|\chi\|_V\|\chi\|_W\|A^\sigma\chi_c\|_{V_{1-\sigma}} \\ &\leq c\|\chi\|_W\|\chi_c\|_{V_{1+\sigma}}, \end{aligned}$$

and

$$
\begin{aligned}
c\|\chi^2\nabla A^\sigma\chi_c\|_{L^{6/(3+2\sigma)}} &\le c\|\chi\|_{L^6}^2\|\nabla A^\sigma\chi_c\|_{L^{6/(1+2\sigma)}}\\
&\le c\|\chi\|_V^2\|\nabla A^\sigma\chi_c\|_{V_{1-\sigma}}\\
&\le c\|A^\sigma\chi_c\|_{V_{2-\sigma}}\\
&\le c\|\chi_c\|_{V_{2+\sigma}}.
\end{aligned}
$$

Hence, from the Young inequality,

$$
\begin{aligned}
3\langle\chi^2\partial_t\chi_c, A^\sigma\chi_c\rangle_H &\le c\|\partial_t\chi_c\|_{V_{-1+\sigma}}\big(\|\chi\|_W\|\chi_c\|_{V_{1+\sigma}}+\|\chi_c\|_{V_{2+\sigma}}\big)\\
&\le c\|\chi\|_W^2\|\chi_c\|_{V_{1+\sigma}}^2+c\|\chi_c\|_{V_{2+\sigma}}^2+\frac{1}{2}\|\partial_t\chi_c\|_{V_{-1+\sigma}}^2. \qquad (6.9)
\end{aligned}
$$

Concerning the second term, we get

$$
3\langle\chi_c\chi\partial_t\chi_d, A^\sigma\chi_c\rangle_H\le 3\|\partial_t\chi_d\|_{V^*}\|\chi_c\chi A^\sigma\chi_c\|_V,
$$

with

$$
\|\chi_c\chi A^\sigma\chi_c\|_V\le c\|\chi_c\chi A^\sigma\chi_c\|_H+c\|\nabla\chi_c\chi A^\sigma\chi_c\|_H+c\|\chi_c\nabla\chi A^\sigma\chi_c\|_H+c\|\chi_c\chi\nabla A^\sigma\chi_c\|_H.
$$

The Hölder inequality, (6.1)-(6.3) and (6.6), lead to

$$
\begin{aligned}
c\|\chi_c\chi A^\sigma\chi_c\|_H &\le c\|\chi_c\|_{L^{6/(1-2\sigma)}}\|\chi\|_{L^6}\|A^\sigma\chi_c\|_{L^{6/(1+2\sigma)}}\\
&\le c\|\chi_c\|_{V_{1+\sigma}}\|\chi\|_V\|A^\sigma\chi_c\|_{V_{1-\sigma}}\\
&\le c\|\chi\|_W\|\chi_c\|_{V_{1+\sigma}}^2,
\end{aligned}
$$

$$
\begin{aligned}
c\|\nabla\chi_c\chi A^\sigma\chi_c\|_H &\le c\|\nabla\chi_c\|_{L^{6/(1-2\sigma)}}\|\chi\|_{L^6}\|A^\sigma\chi_c\|_{L^{6/(1+2\sigma)}}\\
&\le c\|\nabla\chi_c\|_{V_{1+\sigma}}\|\chi\|_V\|A^\sigma\chi_c\|_{V_{1-\sigma}}\\
&\le c\|\chi_c\|_{V_{2+\sigma}}\|\chi_c\|_{V_{1+\sigma}},
\end{aligned}
$$

$$
\begin{aligned}
c\|\chi_c\nabla\chi A^\sigma\chi_c\|_H &\le c\|\chi_c\|_{L^{6/(1-2\sigma)}}\|\nabla\chi\|_{L^6}\|A^\sigma\chi_c\|_{L^{6/(1+2\sigma)}}\\
&\le c\|\chi_c\|_{V_{1+\sigma}}\|\chi\|_W\|A^\sigma\chi_c\|_{V_{1-\sigma}}\\
&\le c\|\chi\|_W\|\chi_c\|_{V_{1+\sigma}}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
c\|\chi_c\chi\nabla A^\sigma\chi_c\|_H &\le c\|\chi_c\|_{L^{6/(1-2\sigma)}}\|\chi\|_{L^6}\|\nabla A^\sigma\chi_c\|_{L^{6/(1+2\sigma)}}\\
&\le c\|\chi_c\|_{V_{1+\sigma}}\|\chi\|_V\|\nabla A^\sigma\chi_c\|_{V_{1-\sigma}}\\
&\le c\|\chi_c\|_{V_{1+\sigma}}\|A^\sigma\chi_c\|_{V_{2-\sigma}}\\
&\le c\|\chi_c\|_{V_{1+\sigma}}\|\chi_c\|_{V_{2+\sigma}}.
\end{aligned}
$$

The Young inequality then yields

$$
\begin{aligned}
3\langle\chi_c\chi\partial_t\chi_d, A^\sigma\chi_c\rangle_H &\le c\|\partial_t\chi_d\|_{V^*}\big(\|\chi_c\|_{V_{1+\sigma}}\|\chi_c\|_{V_{2+\sigma}}+\|\chi\|_W\|\chi_c\|_{V_{1+\sigma}}^2\big)\\
&\le c\|\partial_t\chi_d\|_{V^*}^2\|\chi_c\|_{V_{1+\sigma}}^2+c\|\chi_c\|_{V_{2+\sigma}}^2+c\|\partial_t\chi_d\|_{V^*}\|\chi\|_W\|\chi_c\|_{V_{1+\sigma}}^2. \quad (6.10)
\end{aligned}
$$

Finally, the third term is treated exactly as the second one with χ_d in place of χ, so giving

$$3\langle \chi_c \chi_d \partial_t \chi_d, A^\sigma \chi_c \rangle_H \leq c\|\partial_t \chi_d\|^2_{V^*} \|\chi_c\|^2_{V_{1+\sigma}} + c\|\chi_c\|^2_{V_{2+\sigma}} + c\|\partial_t \chi_d\|_{V^*} \|\chi_d\|_W \|\chi_c\|^2_{V_{1+\sigma}}. \quad (6.11)$$

Collecting (6.8)-(6.11), and using once more the Young inequality, the proof is done. □

Lemma 6.2. *The differential inequality*

$$\begin{aligned}
&\frac{d}{dt}\Big(\|\vartheta_c\|^2_{V_\sigma} + \|\chi_c\|^2_{V_{1+\sigma}} + \|\eta_c\|^2_{\mathcal{M}_\sigma} + 2\langle \chi^3 - \chi_d^3, A^\sigma \chi_c \rangle_H + 2\langle \gamma'(\chi), A^\sigma \chi_c \rangle_H \Big) \\
&\leq c + c\|\chi_c\|^2_{V_{1+\sigma}} + c\|\eta_c\|^2_{\mathcal{M}_\sigma} + c\|\partial_t \chi\|^2_{V^*} + c\|\partial_t \chi_d\|^2_{V^*} + k\|\chi_c\|^2_{V_{2+\sigma}} \\
&\quad + c\big(\|\chi\|^2_W + \|\chi_d\|^2_W + \|\partial_t \chi_d\|^2_{V^*}\big)\|\chi_c\|^2_{V_{1+\sigma}} \qquad (6.12)
\end{aligned}$$

holds for some $k > 0$.

Proof. The reason why we single out the constant k will be clear in a while. Taking the product in H of (6.4) and $A^\sigma \vartheta_c$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\vartheta_c\|^2_{V_\sigma} + \|\vartheta_c\|^2_{V_\sigma} = -\langle \partial_t \chi_c, A^\sigma \vartheta_c \rangle_H - \langle \eta_c, \vartheta_c \rangle_{\mathcal{M}_\sigma} + \langle \eta_c, \vartheta_c \rangle_{L^2_\mu(\mathbb{R}^+, V_\sigma)} + \langle g, A^\sigma \vartheta_c \rangle_H. \quad (6.13)$$

The product in H of (6.5) and $A^{-1+\sigma}\partial_t \chi_c$ furnishes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\chi_c\|^2_{V_{1+\sigma}} + 2\langle \chi^3 - \chi_d^3, A^\sigma \chi_c \rangle_H + 2\langle \gamma'(\chi), A^\sigma \chi_c \rangle_H \Big) + \|\partial_t \chi_c\|^2_{V_{-1+\sigma}} \\
&= \langle \partial_t(\chi^3 - \chi_d^3), A^\sigma \chi_c \rangle_H + \langle \gamma''(\chi)\partial_t \chi, A^\sigma \chi_c \rangle_H \\
&\quad + \langle \vartheta_c, A^\sigma \partial_t \chi_c \rangle_H + \langle \mathcal{F}, A^{-1+\sigma}\partial_t \chi_c \rangle_H, \qquad (6.14)
\end{aligned}$$

where we set

$$\mathcal{F} = 2A\chi_c - \chi_c + \chi^3 - \chi_d^3 + \gamma'(\chi) - \vartheta_c. \quad (6.15)$$

Finally, the product in $\mathcal{M}_\sigma$ of (4.7) and η_c, on account of (K3) and an integration by parts in ds, bears

$$\frac{1}{2}\frac{d}{dt}\|\eta_c\|^2_{\mathcal{M}_\sigma} \leq \langle \vartheta_c, \eta_c \rangle_{\mathcal{M}_\sigma}. \quad (6.16)$$

With regard to the right-hand sides of (6.13)-(6.14), the following estimates are easily seen to hold:

$$\begin{aligned}
\langle \eta_c, \vartheta_c \rangle_{L^2_\mu(\mathbb{R}^+, V_\sigma)} &\leq \frac{1}{2}\|\vartheta_c\|^2_{V_\sigma} + c\|\eta_c\|^2_{\mathcal{M}_\sigma} && (6.17) \\
\langle g, A^\sigma \vartheta_c \rangle_H &\leq c + \frac{1}{2}\|\vartheta_c\|^2_{V_\sigma} && (6.18) \\
\langle \gamma''(\chi)\partial_t \chi, A^\sigma \chi_c \rangle_H &\leq c\|\partial_t \chi\|^2_{V^*} + c\|\chi_c\|^2_{V_{2+\sigma}} && (6.19) \\
\langle \mathcal{F}, A^{-1+\sigma}\partial_t \chi_c \rangle_H &\leq c + c\|\chi_c\|^2_{V_{1+\sigma}} + c\|\partial_t \chi_c\|^2_{V^*} + \frac{1}{2}\|\partial_t \chi_c\|^2_{V_{-1+\sigma}}. && (6.20)
\end{aligned}$$

Putting together (6.13)-(6.14) and (6.16), and exploiting (6.17)-(6.20) and Lemma 6.1, we find the desired inequality (6.12). □

Lemma 6.3. *There holds:*

$$\frac{d}{dt}\|\chi_c\|^2_{V_\sigma} + \|\chi_c\|^2_{V_{2+\sigma}} \le c + c\|\vartheta_c\|^2_{V_\sigma} + c\big(\|\chi\|^2_W + \|\chi_d\|^2_W\big). \tag{6.21}$$

Proof. We take the product in H of (6.5) and $A^\sigma\chi_c$. This yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\chi_c\|^2_{V_\sigma} + \|\chi_c\|^2_{V_{2+\sigma}} \quad &= \quad -\langle A(\chi^3-\chi_d^3), A^\sigma\chi_c\rangle_H - \langle A\gamma'(\chi), A^\sigma\chi_c\rangle_H \\ &\qquad +\langle A\vartheta_c, A^\sigma\chi_c\rangle_H + \langle \mathcal{F}, A^\sigma\chi_c\rangle_H,\end{aligned} \tag{6.22}$$

with $\mathcal{F}$ given by (6.15). Due to (6.6) and the Hölder and the Young inequalities,

$$-\langle A(\chi^3-\chi_d^3), A^\sigma\chi_c\rangle_H \le \|\chi^3-\chi_d^3\|_V\|\chi_c\|_{V_{2+\sigma}} \le c\big(\|\chi\|^2_W + \|\chi_d\|^2_W\big) + \frac{1}{6}\|\chi_c\|^2_{V_{2+\sigma}}$$

and

$$\langle A\vartheta_c, A^\sigma\chi_c\rangle_H \le \|\vartheta_c\|_{V_\sigma}\|\chi_c\|_{V_{2+\sigma}} \le c\|\vartheta_c\|^2_{V_\sigma} + \frac{1}{6}\|\chi_c\|^2_{V_{2+\sigma}}.$$

Finally, we leave to the reader the easy check that

$$-\langle A\gamma'(\chi), A^\sigma\chi_c\rangle_H + \langle \mathcal{F}, A^\sigma\chi_c\rangle_H \le c + \frac{1}{6}\|\chi_c\|^2_{V_{2+\sigma}}$$

Plugging the above inequalities into (6.22) we get the thesis. $\square$

We are now ready to find an estimate for the solution z_c at time t in a more regular space.

Lemma 6.4. *There exists a constant $K = K(t, g, \alpha) > 0$ such that*

$$\|\vartheta_c(t)\|^2_{V_\sigma} + \|\chi_c(t)\|^2_{V_{1+\sigma}} + \|\eta_c(t)\|^2_{\mathcal{M}_\sigma} \le K \tag{6.23}$$

for all $z_0 \in \mathcal{B}_0$.

Proof. For k as in Lemma 6.2, let us define

$$\mathcal{E}(t) = \|\vartheta_c(t)\|^2_{V_\sigma} + \|\chi_c(t)\|^2_{V_{1+\sigma}} + \|\eta_c(t)\|^2_{\mathcal{M}_\sigma}$$

and

$$\Psi(t) = c + k\|\chi_c(t)\|^2_{V_\sigma} + 2\langle \chi^3(t) - \chi_d^3(t), A^\sigma\chi_c(t)\rangle_H + 2\langle \gamma'(\chi(t)), A^\sigma\chi_c(t)\rangle_H.$$

By virtue of (2.1) and (6.6), it is a standard matter to verify that, upon choosing $c = c(k) > 0$ large enough,

$$\frac{1}{\beta}\mathcal{E}(t) \le \mathcal{E}(t) + \Psi(t) \le \beta\mathcal{E}(t) + c \tag{6.24}$$

for some $\beta = \beta(k) > 1$. Addition of (6.12) and k-times (6.21), along with (6.24), entail

$$\frac{d}{dt}\big(\mathcal{E} + \Psi\big) \le h\mathcal{E} + h \le h\big(\mathcal{E} + \Psi\big) + h,$$

where we set

$$h(t) = c\big(1 + \|\chi(t)\|_W^2 + \|\chi_d(t)\|_W^2 + \|\partial_t\chi(t)\|_{V^*}^2 + \|\partial_t\chi_d(t)\|_{V^*}^2\big).$$

Recall that, by (6.7),

$$\int_0^t h(y)dy \le c.$$

Moreover, from (4.8),

$$\mathcal{E}(0) + \Psi(0) \le c.$$

Therefore, the integral Gronwall lemma yields

$$\mathcal{E}(t) + \Psi(t) \le e^{\int_0^t h(y)dy}\bigg\{c + \int_0^t h(y)dy\bigg\} \le c,$$

which, using again (6.24), concludes the proof. □

In order to gain the required compactness, we have to take care of the third component η_c of z_c. This because the embedding $\mathcal{M}_\sigma \hookrightarrow \mathcal{M}$, in general, lacks of compactness. However, we have

Lemma 6.5. *The set*

$$\mathcal{C} = \bigcup_{z_0 \in \mathcal{B}_0} \eta_N^t \subset \mathcal{M}$$

is relatively compact in $\mathcal{M}$.

For the proof of the lemma, which is based on a compactness result from [24], we address the reader to [18], where basically the same situation is encountered.

Conclusion of the proof. With K given by Lemma 6.4, denote by $\mathcal{B}_K$ the ball of radius K in $V_\sigma \times V_{1+\sigma}$ centered at zero. Since the embedding $V_\sigma \times V_{1+\sigma} \hookrightarrow H \times V$ is compact, and since $\mathcal{H}_\alpha$ is a closed subset of $\mathcal{H}$, from Lemma 6.5 we learn that the set

$$\mathcal{K} = \overline{(\mathcal{B}_K \times \mathcal{E})} \cap \mathcal{H}_\alpha \subset \mathcal{H}_\alpha$$

is compact. After Lemma 6.4 and Lemma 6.5, it is apparent that $z_c(t) \in \mathcal{K}$.

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