On the Quantum Theory of Direct Detection*

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By using the theory of measurements continuous in time in quantum mechanics [1]–[8], a photon detection theory has been formulated [9]–[12]; see Refs. [10]–[12] and [8] for detailed references. A quantum source as an atom, an ion or a more complicated system, eventually placed inside an optical cavity, is stimulated by lasers or by a thermal bath. The emitted light is detected by photon counters (direct detection), possibly after interference with a reference laser beam (heterodyne and homodyne detection). Just to illustrate detection theory, in this paper I shall present only counting processes [1], [3]–[12] (direct detection). Moreover, I shall consider only a concrete example: I shall take as a source a three–level atom in the so called Λ configuration; although simple, such a system shows, when suitably stimulated by lasers, an interesting behaviour: the so called electron–shelving effect (or quantum jumps) [13, 9].

We denote by $|j\rangle$, $j = 0, 1, 2$ the three states; the free atomic Hamiltonian is

$$H_{\mathrm{A}} = -\hbar \sum_{j=1}^{2} \omega_j |j\rangle\langle j| \,, \qquad \omega_j > 0\,; \tag{1}$$

note that $|0\rangle$ is the higher state. Then, we introduce the interaction between the atom and the electromagnetic field in the standard approximations used in quantum optics. The first approximation is to take this interaction linear in the field operators, e.g. we take $\mathbf{p} \cdot \mathbf{A}$ or $\mathbf{d} \cdot \mathbf{E}$. The second step is to take the rotating wave–approximation. We assume the $|1\rangle \leftrightarrow |2\rangle$ transition to be prohibited. By expanding the field in plain waves and by using spherical coordinates for the wave–vector, in the interaction picture with respect to the free–field dynamics we get

$$H_{\mathrm{int}}(t) = \sum_{j=1}^{2} R_j^\dagger \sum_{\lambda=1}^{2} \frac{\hbar}{\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}\omega \int_\Sigma \mathrm{d}_2\sigma \, \overline{g_j(\omega,\sigma,\lambda)} \, \mathrm{e}^{-\mathrm{i}\omega t} \, b(\omega,\sigma,\lambda) + \mathrm{h.c.}\,, \tag{2}$$

where $R_j = |j\rangle\langle 0|$, λ is the polarization index, σ is the direction of propagation, Σ is the full solid angle $\left(\int_\Sigma \mathrm{d}_2\sigma = 4\pi\right)$, $g_j(\omega,\sigma,\lambda)$ is the coupling intensity (it does not depend on σ and λ in the case of spherical symmetry of the atom), $b(\omega,\sigma,\lambda)$ is a Bose field in the

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Fock representation, an overbar means complex conjugation and h.c. means Hermitian conjugate.

A third approximation is to consider the coupling functions g_j flat around the transition frequencies ω_j and zero outside a neighborhood of ω_j:

$$H_{\rm int}(t) = \hbar \sum_{j=1}^{2} R_j^\dagger \sum_{\lambda=1}^{2} \int_\Sigma {\rm d}_2\sigma\, \overline{g_j(\sigma,\lambda)}\, a_j(t,\sigma,\lambda) + \text{h.c.}\,, \tag{3}$$

$$a_j(t,\sigma,\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\omega_j-\theta_j}^{\omega_j+\theta_j} {\rm d}\omega\, {\rm e}^{-{\rm i}\omega t}\, b(\omega,\sigma,\lambda)\,. \tag{4}$$

If the two frequencies ω_1 and ω_2 are well separated (the two frequency intervals do not overlap), we have

$$[a_i(t,\sigma,\lambda)\,,\, a_j^\dagger(t',\sigma',\lambda')] = 0\,, \qquad \text{for } i \neq j\,. \tag{5}$$

The latest approximation is to consider θ_j very large ($\theta_j \to +\infty$: broadband approximation); in order to preserve eq. (5), this approximation has to be realized by adding independent fields for $j = 1, 2$. The final result is that the interaction Hamiltonian is given by eq. (3), where the $a_j(t,\sigma,\lambda)$ are Bose fields in the Fock representation and normalized in such a way that $[a_i(t,\sigma,\lambda)\,,\, a_j^\dagger(t',\sigma',\lambda')] = \delta_{ij}\delta_{\lambda\lambda'}\delta_2(\sigma,\sigma')\delta(t-t')$; $\delta_2(\sigma,\sigma')$ is a spherical Dirac delta with $\int_\Sigma \delta_2(\sigma,\sigma')\,{\rm d}_2\sigma = 1$, $\delta_2(\sigma,\sigma') = \delta_2(\sigma',\sigma)$.

The approximations we have made are a kind of singular coupling limit and it is known that, on the contrary of van Hove scaling (weak coupling limit), singular coupling limit does not give rise to energy shifts; therefore, $H_{\rm A}$ must contain the final physical frequencies.

Let us set now

$$A_j(t) = \frac{\rm i}{\sqrt{\gamma_j}} \sum_{\lambda=1}^{2} \int_0^t {\rm d}t' \int_\Sigma {\rm d}_2\sigma\, \overline{g_j(\sigma,\lambda)}\, a_j(t',\sigma,\lambda)\,; \tag{6}$$

the normalization constants $\sqrt{\gamma_j}$ are chosen in such a way that $[A_j(t)\,,\, A_i(t')] = 0$, $[A_j(t)\,,\, A_i^\dagger(t')] = \delta_{ij}\min(t,t')$. The list of all the derived constants used in the paper is given at the end in eq. (62). By using the fields A_j, we can write the evolution operator, in the interaction picture with respect to the free–field dynamics, as

$$\begin{aligned} U(t) &= \overleftarrow{T} \exp\Big\{ -\frac{\rm i}{\hbar} \int_0^t [H_{\rm A} + H_{\rm int}(t')]{\rm d}t' \Big\} = \\ &= \overleftarrow{T} \exp\Big\{ \int_0^t \Big[-\frac{\rm i}{\hbar}\, H_{\rm A}\, {\rm d}t' + \sum_{j=1}^{2} \sqrt{\gamma_j}\, \Big(R_j\, {\rm d}A_j^\dagger(t') - R_j^\dagger\, {\rm d}A_j(t') \Big) \Big] \Big\}; \end{aligned} \tag{7}$$

$\overleftarrow{T}$ is the usual time–ordering prescription. To handle such an evolution operator we need quantum stochastic calculus. The aim of such a calculus is just to define integrals with respect to $A_j(t)$, $A_j^\dagger(t)$ and other related operators and to give the rules to manipulate such integrals. An account of quantum stochastic calculus is given in Ref. [14]; I do not want to present this calculus here, but I shall follow Ref. [12], where the rules of quantum stochastic calculus are recalled and detection theory is developed. The relevance of the flat–spectrum and broadband approximations for the use of quantum stochastic calculus in quantum optics has been pointed out in Ref. [15].

The evolution operator U_t satisfies the quantum stochastic Schrödinger equation

$$\mathrm{d}U_t = \left\{ \left(-\frac{\mathrm{i}}{\hbar}\, H_\mathrm{A} - \frac{1}{2} \sum_{j=1}^{2} \gamma_j R_j^\dagger R_j \right) \mathrm{d}t + \sum_{j=1}^{2} \sqrt{\gamma_j} \left(R_j \mathrm{d}A_j^\dagger(t) - R_j^\dagger \mathrm{d}A_j(t) \right) \right\} U_t \tag{8}$$

(cf Ref. [12], eqs. (3.1) and (3.2)).

Let us call $\mathcal{H}$ the Hilbert space where the emitting system lives; for us $\mathcal{H} = \mathbb{C}^3$. Let us take as initial state $\psi \otimes e(h)$, where $\psi \in \mathcal{H}$, $\|\psi\| = 1$ and $e(h)$ is a (normalized) coherent vector in Fock space: $a_j(t,\sigma,\lambda) e(h) = h_j(t,\sigma,\lambda) e(h)$; to describe nearly monochromatic lasers we take $h_j(t,\sigma,\lambda) \simeq \mathrm{e}^{-\mathrm{i}\alpha_j t}\, l_j(\sigma,\lambda)$, where α_j is near ω_j and $l_j(\sigma,\lambda)$ is different from zero only inside some solid angle S (the direction of the stimulating lasers).

The explicit time dependence due to the lasers can be removed by setting

$$U_\alpha(t) = \exp\left\{ -\mathrm{i} \sum_{j=1}^{2} \alpha_j |j\rangle\langle j| t \right\} U_t\,; \tag{9}$$

$U_\alpha(t)$ satisfies eq. (8) with the substitutions $-\omega_j \to \alpha_j - \omega_j \equiv \Delta_j$, $R_j \to \exp(-\mathrm{i}\alpha_j t) R_j$. Then, the reduced density matrix of the atom, defined by

$$\varrho(t) = \mathrm{Tr}_{\mathrm{Fock}} \{ U_\alpha(t) |\psi \otimes e(h)\rangle\langle \psi \otimes e(h)| U_\alpha^\dagger(t) \}, \tag{10}$$

satisfies the master equation (cf Ref. [12], Sect. 3.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \varrho(t) = \mathcal{L}[\varrho(t)], \tag{11}$$

$$\begin{aligned} \mathcal{L}[\varrho] &= -\mathrm{i} \left[H_\Delta + \sum_{j=1}^{2} \sqrt{\gamma_j} \left(\overline{\lambda_j}\, R_j + \lambda_j R_j^\dagger \right),\ \varrho \right] + \frac{1}{2} \sum_{j=1}^{2} \gamma_j ([R_j, \varrho R_j^\dagger] + [R_j \varrho, R_j^\dagger]) = \\ &= -\mathrm{i} K \varrho + \mathrm{i} \varrho K^\dagger + \sum_{j=1}^{2} \gamma_j |j\rangle\langle 0|\varrho|0\rangle\langle j|\,, \end{aligned} \tag{12}$$

$$H_\Delta = \sum_{j=1}^{2} \Delta_j |j\rangle\langle j|, \qquad K = H_\Delta + \sum_{j=1}^{2} \left[\Omega_j \left(\mathrm{e}^{\mathrm{i}\beta_j} |0\rangle\langle j| + \mathrm{e}^{-\mathrm{i}\beta_j} |j\rangle\langle 0| \right) - \frac{\mathrm{i}}{2}\, \gamma_j |0\rangle\langle 0| \right]. \tag{13}$$

Now we assume to have a detector able to count photons flying through a solid angle S_d; we take $S_\mathrm{d} \cap S = \emptyset$, so the lasers do not send light directly to the counter and only fluorescence light is detected. By neglecting the detector response function and the time of flight from the atom to the detector, we have that the detector performs a continual measurement of the observable

$$Z(t) = \sum_{j=1}^{2} \sum_{\lambda=1}^{2} \int_0^t \mathrm{d}s \int_{S_\mathrm{d}} \mathrm{d}_2\sigma\, a_j^\dagger(s,\sigma,\lambda) a_j(s,\sigma,\lambda); \tag{14}$$

the efficiency of the counter can be taken into account by choosing $S_{\rm d}$ smaller than the geometrical solid angle spanned by the detector. Note that $Z(t)$ is a number operator, with integer eigenvalues.

The first important point [3] is that

$$[Z(t), Z(s)] = 0, \qquad \forall t, s, \tag{15}$$

so that the family $\{Z(t),\, t \geq 0\}$ of selfadjoint commuting operators has a joint projection valued measure; the Fourier transform of such a measure (up to time t) is

$$F_t(k) = \exp\left\{ {\rm i} \int_0^t k(s)\, {\rm d}Z(s) \right\}, \tag{16}$$

where k varies in a suitable space of real test functions. The second important point [4] is that

$$U_T^\dagger Z(t) U_T = U_t^\dagger Z(t) U_t\,, \qquad \forall T \geq t\,. \tag{17}$$

By introducing the Heisenberg picture $Z_{\rm H}(t) = U_t^\dagger Z(t) U_t \equiv U_\alpha^\dagger(t) Z(t) U_\alpha(t)$, equation (17) implies

$$[Z_{\rm H}(t), Z_{\rm H}(s)] = 0\,, \qquad \forall t, s, \tag{18}$$

$$U_\alpha^\dagger(t) F_t(k) U_\alpha(t) = \exp\left\{ {\rm i} \int_0^t k(s)\, {\rm d}Z_{\rm H}(s) \right\}. \tag{19}$$

Equation (18) says that our observables are continually measurable even when the source is present; moreover, eq. (17) allows us to relate [4, 10] our Heisenberg–picture observables to the output fields of Ref. [15].

The whole information on the counting probabilities is contained in the characteristic functional

$$\Phi_t(k) = \langle u_t | F_t(k) | u_t \rangle, \qquad |u_t\rangle = U_\alpha(t) |\psi \otimes e(h)\rangle. \tag{20}$$

By construction $\Phi_t(k)$ is the Fourier transform of the probability measure; in the case of a (regular) counting process, such a characteristic functional has the structure (cf Ref. [10], Sect. 4)

$$\Phi_t(k) = P_t(0) + \sum_{m=1}^\infty \int_0^t {\rm d}t_m \int_0^{t_m} {\rm d}t_{m-1} \cdots \int_0^{t_2} {\rm d}t_1\, \exp\left\{ {\rm i} \sum_{n=1}^m k(t_n) \right\} p_t(t_m, \ldots, t_1), \tag{21}$$

where $P_t(m)$ is the probability of m counts up to time t and $p_t(t_m, \ldots, t_1)$ is the exclusive probability density of a count around t_1, a count around t_2, ... and no other count up to time t.

On the other side, as proved in Ref. [10], we have

$$\begin{aligned} F_t(k) &= \;:\exp\left\{ \int_0^t \left({\rm e}^{{\rm i}k(s)} - 1 \right) {\rm d}Z(s) \right\}: \;= \qquad (22) \\ &= \;:{\rm e}^{-Z(t)} \left\{ 1 + \sum_{m=1}^\infty \int_0^t {\rm d}Z(t_m) \int_0^{t_m} {\rm d}Z(t_{m-1}) \cdots \int_0^{t_2} {\rm d}Z(t_1)\, \exp\left[{\rm i} \sum_{n=1}^m k(t_n) \right] \right\}:\,, \end{aligned}$$

where the symbol $:\ :$ denotes the normal ordering prescription. Therefore, by comparing eq. (21) with eqs. (20) and (22), we have

$$P_t(0) = \langle u_t | :{\rm e}^{-Z(t)}: | u_t \rangle, \qquad p_t(t_m, \ldots, t_1) = \left\langle u_t \right| :{\rm e}^{-Z(t)}\, \frac{{\rm d}Z(t_1)}{{\rm d}t_1} \cdots \frac{{\rm d}Z(t_m)}{{\rm d}t_m}: \left| u_t \right\rangle. \tag{23}$$

In particular we get

$$
\begin{aligned}
P_t(m) &= \int_0^t \mathrm{d}t_m \int_0^{t_m} \mathrm{d}t_{m-1} \cdots \int_0^{t_2} \mathrm{d}t_1\, p_t(t_m, \ldots, t_1) = \\
&= \Big\langle u_t \Big| :\mathrm{e}^{-Z(t)}\, \frac{1}{m!} \Big(\int_0^t \mathrm{d}Z(s) \Big)^m : \Big| u_t \Big\rangle, \qquad (24)
\end{aligned}
$$

which is Kelley–Kleiner counting formula ([16] Sect. 5.5). Formulae (23) and (24) define consistent probabilities for a counting process, because they are derived from the Fourier transform of a projection valued measure. It is known that Kelley–Kleiner counting formula is not always consistent, e.g. if applied to a discrete–mode field. Our result shows that Kelley–Kleiner formula is consistent at least when applied to the fields involved in quantum stochastic calculus, which correspond ([10] Sect. 3.1) to the electromagnetic field in the quasimonochromatic paraxial approximation [17].

The formulae for the probabilities can be expressed also in terms of atomic quantities only, once the degrees of freedom of the fields have been traced out. Let us define an operator $G_t(k)$ acting on the trace–class operators on $\mathcal{H}$ by

$$
G_t(k)[|\varphi_1\rangle\langle\varphi_2|] = \mathrm{Tr}_{\text{Fock}} \left\{ F_t(k) U_\alpha(t) |\varphi_1 \otimes e(h)\rangle\langle \varphi_2 \otimes e(h)| U_\alpha(t) \right\}, \qquad \forall \varphi_1, \varphi_2 \in \mathcal{H}; \qquad (25)
$$

note that we have $G_t(0) = \exp\{\mathcal{L}t\}$ and

$$
\Phi_t(k) = \mathrm{Tr}_{\mathcal{H}} \{ G_t(k)[|\psi\rangle\langle\psi|] \}. \qquad (26)
$$

The adjoint of $G_t(k)$ acts on $\mathcal{B}(\mathcal{H})$ and is the Fourier transform of an instrument; the notion of instrument generalizes both the usual association of observables with selfadjoint operators and the reduction postulate [1]. Operators like $G_t(k)$ have been introduced in [2] and are at the basis of one of the formulations of continual measurement theory. By using quantum stochastic calculus, we can differentiate the r.h.s. of eq. (25) [3, 10] and we get

$$
\frac{\mathrm{d}}{\mathrm{d}t} G_t(k) = \Big[\widetilde{\mathcal{L}} + \mathrm{e}^{\mathrm{i}k(t)} J \Big] G_t(k), \qquad G_0(t) = 1\!\!1, \qquad (27)
$$

$$
J[\varrho] = \sum_{j=1}^2 \eta_j R_j \varrho R_j^\dagger = \sum_{j=1}^2 \eta_j |j\rangle\langle 0|\varrho|0\rangle\langle j|, \qquad (28)
$$

$$
\widetilde{\mathcal{L}}[\varrho] = \mathcal{L}[\varrho] - J[\varrho] = -\mathrm{i}K\varrho + \mathrm{i}\varrho K^\dagger + \sum_{j=1}^2 (\gamma_j - \eta_j) |j\rangle\langle 0|\varrho|0\rangle\langle j|. \qquad (29)
$$

By expressing the solution of eq. (27) as a Dyson series we have

$$
\begin{aligned}
G_t(k) &= \mathrm{e}^{\widetilde{\mathcal{L}}t} + \sum_{m=1}^\infty \int_0^t \mathrm{d}t_m \int_0^{t_m} \mathrm{d}t_{m-1} \cdots \int_0^{t_2} \mathrm{d}t_1\, \exp\Big\{ \mathrm{i} \sum_{n=1}^m k(t_n) \Big\} \\
&\times\ \mathrm{e}^{\widetilde{\mathcal{L}}(t-t_m)} J \mathrm{e}^{\widetilde{\mathcal{L}}(t_m - t_{m-1})} J \cdots \mathrm{e}^{\widetilde{\mathcal{L}}(t_2 - t_1)} J \mathrm{e}^{\widetilde{\mathcal{L}}t_1}. \qquad (30)
\end{aligned}
$$

By eqs. (22), (26) and (30) we obtain

$$
P_t(0) = \mathrm{Tr}_{\mathcal{H}} \Big\{ \mathrm{e}^{\widetilde{\mathcal{L}}t} [|\psi\rangle\langle\psi|] \Big\}, \qquad (31)
$$

$$p_t(t_m,\dots,t_1) = \mathrm{Tr}_{\mathcal{H}}\Big\{\mathrm{e}^{\widetilde{\mathcal{L}}(t-t_m)} J \cdots \mathrm{e}^{\widetilde{\mathcal{L}}(t_2-t_1)} J \mathrm{e}^{\widetilde{\mathcal{L}}t_1}[|\psi\rangle\langle\psi|]\Big\} =$$

$$= P_{t-t_m}(0|\varrho_0)\, w(t_m-t_{m-1})\cdots w(t_2-t_1)\,\eta\langle 0|\mathrm{e}^{\widetilde{\mathcal{L}}t_1}[|\psi\rangle\langle\psi|]|0\rangle, \tag{32}$$

$$P_\tau(0|\varrho_0) = \mathrm{Tr}_{\mathcal{H}}\Big\{\mathrm{e}^{\widetilde{\mathcal{L}}\tau}[\varrho_0]\Big\}, \qquad \varrho_0 = \sum_{j=1}^{2}\frac{\eta_j}{\eta}\,|j\rangle\langle j|, \qquad w(\tau) = \eta\langle 0|\mathrm{e}^{\widetilde{\mathcal{L}}\tau}[\varrho_0]|0\rangle. \tag{33}$$

Moreover, we have $\mathrm{d}P_t(0|\varrho_0)/\mathrm{d}t = -w(t)$, which says that $w(t)$ is the interarrival waiting–time density. From the structure of the exclusive probability densities, we see that our detection process is a delayed renewal counting process; however, this property is specific of the present simple model.

The case $\Delta_1 = \Delta_2 \equiv \Delta$ is very peculiar, because $K|\varphi_0\rangle = \Delta|\varphi_0\rangle$, $\mathcal{L}[|\varphi_0\rangle\langle\varphi_0|] = 0$, $\widetilde{\mathcal{L}}[|\varphi_0\rangle\langle\varphi_0|] = 0$, where $|\varphi_0\rangle = \frac{1}{\sqrt{\Omega_1^2+\Omega_2^2}}\left(\Omega_2\mathrm{e}^{-\mathrm{i}\beta_1}|1\rangle - \Omega_1\mathrm{e}^{-\mathrm{i}\beta_2}|2\rangle\right)$. Then, $\int_0^{+\infty} w(t)\,\mathrm{d}t < 1$ and there is a non–zero probability that the fluorescence stop. When $\Delta_1 \neq \Delta_2$, $w(t)$ develops more decaying times and the discussion on bright and dark periods goes on in a similar way as in the V-system case [13, 9]. A more realistic model could be obtained by adding a weak $|1\rangle \leftrightarrow |2\rangle$ transition [18].

To introduce the stochastic representation of the measurement process [6, 7, 11], we need some new objects: the Weyl operators

$$W_1(t) = \exp\Big\{\sum_{j,\lambda=1}^{2}\int_0^t \mathrm{d}s\int_\Sigma \mathrm{d}_2\sigma\,\mathrm{e}^{-\mathrm{i}\alpha_j s}\, l_j(\sigma,\lambda)a_j^\dagger(s,\sigma,\lambda) - \mathrm{h.c.}\Big\}, \tag{34}$$

$$W_2(t) = \exp\Big\{\sum_{j,\lambda=1}^{2}\int_t^{+\infty} \mathrm{d}s\int_\Sigma \mathrm{d}_2\sigma\, h_j(s,\sigma,\lambda)a_j^\dagger(s,\sigma,\lambda) - \mathrm{h.c.}\Big\}, \tag{35}$$

$$W_3(t) = \exp\Big\{-\mathrm{i}\sum_{j,\lambda=1}^{2}\int_0^t \mathrm{d}s\int_\Sigma \mathrm{d}_2\sigma\,\Big[\mathrm{e}^{-\mathrm{i}\alpha_j s}\, g_j(\sigma,\lambda)a_j^\dagger(s,\sigma,\lambda) + \mathrm{h.c.}\Big]\Big\}, \tag{36}$$

and the "quantum Poisson processes"

$$N_j^i(t) = \sum_{\lambda=1}^{2}\int_0^t \mathrm{d}s\int_{C_i}\mathrm{d}_2\sigma\,\Big[a_j^\dagger(s,\lambda,\sigma) + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha_j s}\,\overline{g_j(\sigma,\lambda)}\Big]\Big[a_j(s,\lambda,\sigma) - \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\alpha_j s}\, g_j(\sigma,\lambda)\Big], \tag{37}$$

$$C_1 = S_\mathrm{d}\,, \quad C_2 = \Sigma\backslash S_\mathrm{d}\,, \qquad N_\mathrm{d}(t) = \sum_{j=1}^{2} N_j^1(t)\,. \tag{38}$$

Let us stress again that the whole physical information is contained in $G_t(k)$ given in eq. (25). The strategy is to rewrite the r.h.s. of eq. (25) in such a way that the only Fock space operators involved are commuting selfadjoint operators, which can be simultaneously diagonalized and so can be interpreted as classical random variables [12]. First, by using the quantities (34)–(38), eq. (25) can be written as

$$G_t(k)[|\psi\rangle\langle\psi|] = \mathrm{Tr}_{\mathrm{Fock}}\Big\{\widetilde{F}_t(k)|\psi_t\rangle\langle\psi_t|\Big\}\,, \qquad |\psi_t\rangle = \widetilde{U}_t|\psi\otimes e(0)\rangle, \tag{39}$$

$$\widetilde{U}_t = W_3^\dagger(t)W_2^\dagger(t)W_1^\dagger(t)U_\alpha(t)W_1(t)W_2(t) = W_3^\dagger(t)W_1^\dagger(t)U_\alpha(t)W_1(t)\,, \tag{40}$$

$$\begin{aligned}\widetilde{F}_t(k) &= W_3^\dagger(t)W_2^\dagger(t)W_1^\dagger(t)F_t(k)W_1(t)W_2(t)W_3(t) = W_3^\dagger(t)F_t(k)W_3(t) = \\ &= \exp\left\{\mathrm{i}\int_0^t k(s)\,\mathrm{d}N_{\mathrm{d}}(s)\right\}. \end{aligned} \tag{41}$$

In getting eqs. (39)–(41) we have used the relation $W_1(t)W_2(t)e(0) = e(h)$ and the fact that $W_2(t)$ commutes with all the operators involved and $W_1(t)$ commutes with $F_t(k)$. Moreover, quantum stochastic calculus gives

$$\mathrm{d}\widetilde{U}_t = \left\{\left[-\mathrm{i}K + \sum_{j=1}^{2}\gamma_j\left(R_j - \tfrac{1}{2}\right)\right]\mathrm{d}t + \sum_{j=1}^{2}\sqrt{\gamma_j}\left[\mathrm{e}^{-\mathrm{i}\alpha_j t}\left(R_j - 1\right)\mathrm{d}A_j^\dagger(t) - \mathrm{h.c.}\right]\right\}\widetilde{U}_t\,. \tag{42}$$

The second step is to note that the increments of the various quantum processes commute with $\widetilde{U}_t$ and that $a_j(s,\lambda,\sigma)$ annihilates the vacuum, so that we can write

$$\sum_{j=1}^{2}\sqrt{\gamma_j}\left[\mathrm{e}^{-\mathrm{i}\alpha_j t}\left(R_j - 1\right)\mathrm{d}A_j^\dagger(t) - \mathrm{h.c.}\right]\widetilde{U}_t|\psi\otimes e(0)\rangle = \tag{43}$$

$$= \sum_{j=1}^{2}\sqrt{\gamma_j}\,\mathrm{e}^{-\mathrm{i}\alpha_j t}\left(R_j - 1\right)\widetilde{U}_t\,\mathrm{d}A_j^\dagger(t)|\psi\otimes e(0)\rangle = \sum_{j=1}^{2}\left(R_j - 1\right)\left(\sum_{i=1}^{2}\mathrm{d}N_j^i(t) - \gamma_j\,\mathrm{d}t\right)|\psi_t\rangle.$$

This allows us to write the evolution equation for ψ_t as

$$\mathrm{d}\psi_t = \left\{\sum_{i,j=1}^{2}\left(R_j - 1\right)\mathrm{d}N_j^i(t) + \left[-\mathrm{i}K + \tfrac{1}{2}\left(\gamma_1 + \gamma_2\right)\right]\mathrm{d}t\right\}\psi_t\,. \tag{44}$$

To diagonalize the operators $N_j^i(t)$ appearing in $\widetilde{F}_t(k)$ and ψ_t, let us consider the trajectory space Ω of a Poisson point process of intensity $|g_j(\sigma,\lambda)|^2\,\mathrm{d}_2\sigma\mathrm{d}t$ with its Poisson probability measure P. Fock space is isomorphic to $L^2(\Omega,P)$ (where the inner product is the mathematical expectation of the product); under this isomorphism, the operators $N_j^i(t)$ become multiplication operators by independent Poisson processes: $N_j^1(t)$ has intensity $\eta_j\mathrm{d}t$ and $N_j^2(t)$ has intensity $(\gamma_j - \eta_j)\mathrm{d}t$. We take this isomorphic transformation and, without changing notation, we interpret eq. (44) as a classical stochastic differential equation for a process ψ_t with values in $\mathcal{H}$; such an equation enjoyes remarkable properties, which we shall discuss in the following. From now on, only classical stochastic calculus for counting processes is involved; the formal rules of this calculus are summarized by $(\mathrm{d}t)^2 = 0$, $\mathrm{d}t\mathrm{d}N_j^i(t) = 0$, $\mathrm{d}N_j^i(t)\mathrm{d}N_k^r(t) = \delta_{jk}\delta_{ir}\mathrm{d}N_j^i(t)$. Also the notion of conditional expectation will be essential. Norms and inner products will refer to $\mathcal{H}$.

Let $\mathcal{F}_t$ be the σ-algebra in Ω generated by the process $N_j^i(s)$, $0 \le s \le t$, $i,j = 1,2$; $\mathcal{F}_t$, $t \ge 0$, is a filtration ($\mathcal{F}_s \subset \mathcal{F}_t$ for $s \le t$) and, for a fixed t, $\mathcal{F}_t$ contains the events up to time t. The increments $\mathrm{d}N_j^i(t)$ "point into the future", they are independent of $\mathcal{F}_t$ and satisfy

$$\mathbb{E}_P\left[\mathrm{d}N_j^1\middle|\,\mathcal{F}_t\right] = \eta_j\,\mathrm{d}t\,, \qquad \mathbb{E}_P\left[\mathrm{d}N_j^2\middle|\,\mathcal{F}_t\right] = (\gamma_j - \eta_j)\,\mathrm{d}t\,. \tag{45}$$

The process ψ_t is $\mathcal{F}_t$–adapted (non–anticipating), i.e. $\mathbb{E}_P[\psi_t|\mathcal{F}_t] = \psi_t$. By the rules of stochastic calculus, we obtain easily from eq. (44)

$$\mathrm{d}|\psi_t\rangle\langle\psi_t| = \sum_{j=1}^{2}\left(R_j|\psi_t\rangle\langle\psi_t|R_j^\dagger - |\psi_t\rangle\langle\psi_t|\right)\left[\sum_{i=1}^{2}\mathrm{d}N_j^i(t) - \gamma_j\mathrm{d}t\right] + \mathcal{L}\left[|\psi_t\rangle\langle\psi_t|\right]\mathrm{d}t \tag{46}$$

and, by taking the trace,

$$\mathrm{d}\|\psi_t\|^2 = \|\psi_t\|^2 \sum_{j=1}^{2} \left(\left\| R_j \widehat{\psi}_t \right\|^2 - 1 \right) \left[\sum_{i=1}^{2} \mathrm{d}N_j^i(t) - \gamma_j \mathrm{d}t \right], \qquad \widehat{\psi}_t = \psi_t / \|\psi_t\| \,. \tag{47}$$

By taking the expectation of eq. (46) and taking into account eq. (45), we obtain that $\mathbb{E}_P[|\psi_t\rangle\langle\psi_t|]$ satisfies the same master equation as $\varrho(t)$; because they coincide at time zero, we have the following stochastic representation of the reduced density matrix:

$$\varrho(t) = \mathbb{E}_P[|\psi_t\rangle\langle\psi_t|]. \tag{48}$$

By taking the trace of eq. (48) we have also $\mathbb{E}_P[\|\psi_t\|^2] = 1$. Moreover, from eq. (47), one has that $\|\psi_t\|^2$ is a martingale, i.e. $\mathbb{E}_P\left[\|\psi_t\|^2 | \mathcal{F}_s\right] = \|\psi_s\|^2$, $s \leq t$. A mean–one and positive martingale can be used as a density with respect to P; we define a new probability measure $\widehat{P}$ by

$$\widehat{P}(F) = \mathbb{E}_P\left[1_F \|\psi_t\|^2\right], \qquad \forall F \in \mathcal{F}_t\,, \quad \forall t \geq 0\,. \tag{49}$$

Note that $\mathbb{E}_P\left[1_F \|\psi_T\|^2\right] = \mathbb{E}_P\left[1_F \|\psi_t\|^2\right]$, $\forall T \geq t$, $\forall F \in \mathcal{F}_t$, because $\|\psi_t\|^2$ is a martingale, and this implies that eq. (49) is a consistent definition of a unique probability measure $\widehat{P}$. Moreover, by eqs. (47)–(49), we have another stochastic representation of $\varrho(t)$:

$$\varrho(t) = \mathbb{E}_{\widehat{P}}\left[\left| \widehat{\psi}_t \right\rangle \left\langle \widehat{\psi}_t \right| \right]. \tag{50}$$

Under the new probability law $\widehat{P}$, the processes $N_j^i(t)$ are counting processes with stochastic intensities

$$\mathbb{E}_{\widehat{P}}\left[\mathrm{d}N_j^1(t) \middle| \mathcal{F}_t \right] = \eta_j \left\| R_j \widehat{\psi}_t \right\|^2 \mathrm{d}t\,, \qquad \mathbb{E}_{\widehat{P}}\left[\mathrm{d}N_j^2(t) \middle| \mathcal{F}_t \right] = (\gamma_j - \eta_j) \left\| R_j \widehat{\psi}_t \right\|^2 \mathrm{d}t\,. \tag{51}$$

Note that $\left\| R_j \widehat{\psi}_t \right\|^2 \equiv \left| \left\langle 0 \middle| \widehat{\psi}_t \right\rangle \right|^2$ is a random quantity, so that the $N_j^i(t)$ are no more Poisson processes. The proof of eq. (51) needs some properties of conditional expectations under a change of measure; by such properties and eq. (47) one has

$$\mathbb{E}_{\widehat{P}}\left[\mathrm{d}N_j^i(t) \middle| \mathcal{F}_t \right] = \frac{1}{\|\psi_t\|^2} \mathbb{E}_P\left[\|\psi_{t+\mathrm{d}t}\|^2 \mathrm{d}N_j^i(t) \middle| \mathcal{F}_t \right] = \left\| R_j \widehat{\psi}_t \right\|^2 \mathbb{E}_P\left[\mathrm{d}N_j^i(t) \middle| \mathcal{F}_t \right],$$

from which eq. (51) follows.

From eqs. (44) and (47) one obtains, under the law $\widehat{P}$, a stochastic equation for the normalized vector $\widehat{\psi}_t$:

$$\mathrm{d}\widehat{\psi}_t = \sum_{i,j=1}^{2} \left(\frac{R_j}{\left\| R_j \widehat{\psi}_t \right\|} - 1 \right) \widehat{\psi}_t \, \mathrm{d}N_j^i(t) + \left[-\mathrm{i}K + \frac{1}{2} \sum_{j=1}^{2} \gamma_j \left\| R_j \widehat{\psi}_t \right\|^2 \right] \widehat{\psi}_t \, \mathrm{d}t\,. \tag{52}$$

The meaning of this equation is very simple. If at time t there is a jump of the process $N_j^i(t)$, then the wave–vector changes according to the rule

$$\widehat{\psi}_{t-} \to \widehat{\psi}_{t+} = \frac{R_j \widehat{\psi}_{t-}}{\left\| R_j \widehat{\psi}_{t-} \right\|} = |j\rangle \text{ (up to a phase)}. \tag{53}$$

Between two jumps $\widehat{\psi}_t$ satisfies eq. (52) without the term containing $\mathrm{d}N_j^i(t)$, which is equivalent to

$$\widehat{\psi}_t = \frac{\varphi_t}{\|\varphi_t\|}\,, \qquad \frac{\mathrm{d}\varphi_t}{\mathrm{d}t} = -\mathrm{i}K\varphi_t\,. \tag{54}$$

The Monte–Carlo wavefunction method [19] is based on eqs. (50), (51), (53), (54).

Up to now, inside the stochastic formulation, we have not taken into account the fact that we observe only the process $N_\mathrm{d}(t) = N_1^1(t) + N_2^1(t)$. Let $\mathcal{E}_t$ be the σ-algebra generated by $N_\mathrm{d}(s)$, $0 \leq s \leq t$, and set $\mathcal{E}_\infty = \bigvee_{t\geq 0}\mathcal{E}_t$. By elementary properties of independent Poisson processes and conditional expectations, we have

$$\mathbb{E}_P\left[\mathrm{d}N_j^1(t)\middle|\,\mathcal{E}_\infty\right] = \frac{\eta_j}{\eta}\,\mathrm{d}N_\mathrm{d}(t)\,, \qquad \mathbb{E}_P\left[\mathrm{d}N_j^2(t)\middle|\,\mathcal{E}_\infty\right] = (\gamma_j - \eta_j)\,\mathrm{d}t\,. \tag{55}$$

By taking the conditional expectation with respect to $\mathcal{E}_\infty$ in eq. (46), we obtain

$$\mathrm{d}\kappa_t = \Big(\sum_{j=1}^2 \frac{\eta_j}{\eta}\,R_j\kappa_t R_j^\dagger - \kappa_t\Big)\left(\mathrm{d}N_\mathrm{d}(t) - \eta\mathrm{d}t\right) + \mathcal{L}[\kappa_t]\mathrm{d}t\,, \tag{56}$$

$$\kappa_t = \mathbb{E}_P[|\psi_t\rangle\langle\psi_t||\mathcal{E}_\infty] = \mathbb{E}_P[|\psi_t\rangle\langle\psi_t||\mathcal{E}_t] \tag{57}$$

By normalizing the positive trace–class operator κ_t we get the random density matrix

$$\widehat{\kappa}_t = \kappa_t/\mathrm{Tr}_\mathcal{H}\{\kappa_t\} = \mathbb{E}_{\widehat{P}}\left[\left|\widehat{\psi}_t\right\rangle\left\langle\widehat{\psi}_t\right|\middle|\,\mathcal{E}_t\right]. \tag{58}$$

Note that $\mathrm{Tr}_\mathcal{H}\{\kappa_t\} = \mathbb{E}_P[\|\psi_t\|^2|\mathcal{E}_t]$, so that eqs. (48), (50), (58) give

$$\varrho(t) = \mathbb{E}_P[\kappa_t] = \mathbb{E}_{\widehat{P}}[\widehat{\kappa}_t]\,. \tag{59}$$

Let us denote by $\widehat{P}_\mathcal{E}$ the probability measure $\widehat{P}$ restricted to $\mathcal{E}_\infty$. From eq. (56) one can derive an equation for $1/\mathrm{Tr}_\mathcal{H}\{\kappa_t\}$ and, then, for $\widehat{\kappa}_t$; the final result is that the random state $\widehat{\kappa}_t$, under the law $\widehat{P}_\mathcal{E}$, satisfies the non–linear stochastic equation

$$\begin{aligned}\mathrm{d}\widehat{\kappa}_t &= \left(\frac{\sum_j \eta_j R_j\widehat{\kappa}_t R_j^\dagger}{\sum_r \eta_r \mathrm{Tr}_\mathcal{H}\left\{R_r\widehat{\kappa}_t R_r^\dagger\right\}} - \widehat{\kappa}_t\right)\left(\mathrm{d}N_\mathrm{d}(t) - \sum_l \eta_l \mathrm{Tr}_\mathcal{H}\left\{R_l\widehat{\kappa}_t R_l^\dagger\right\}\mathrm{d}t\right) + \mathcal{L}\left[\widehat{\kappa}_t\right] \\ &= \left(\varrho_0 - \widehat{\kappa}_t\right)\mathrm{d}N_\mathrm{d}(t) + \widetilde{\mathcal{L}}\left[\widehat{\kappa}_t\right]\mathrm{d}t + \eta\,\langle 0\,|\widehat{\kappa}_t|\,0\rangle\,\widehat{\kappa}_t\,\mathrm{d}t\,. \end{aligned} \tag{60}$$

Moreover, we have

$$\begin{aligned}&\mathbb{E}_{\widehat{P}}\left[\mathrm{d}N_\mathrm{d}(t)|\,\mathcal{E}_t\right] = \mathbb{E}_{\widehat{P}}\left[\mathbb{E}_{\widehat{P}}\left[\mathrm{d}N_\mathrm{d}(t)|\,\mathcal{F}_t\right]\middle|\,\mathcal{E}_t\right] = \\ &= \sum_{j=1}^2 \eta_j \mathbb{E}_{\widehat{P}}\left[\left\|R_j\widehat{\psi}_t\right\|^2\middle|\,\mathcal{E}_t\right]\mathrm{d}t = \sum_{j=1}^2 \eta_j \mathrm{Tr}_\mathcal{H}\left\{R_j\widehat{\kappa}_t R_j^\dagger\right\}\mathrm{d}t = \eta\,\langle 0\,|\widehat{\kappa}_t|\,0\rangle\,\mathrm{d}t\,,\end{aligned} \tag{61}$$

which says that $N_\mathrm{d}(t)$ is a counting process of stochastic intensity $\eta\,\langle 0\,|\widehat{\kappa}_t|\,0\rangle\,\mathrm{d}t$. Together with eq. (60), this implies that, under the law $\widehat{P}_\mathcal{E}$, $N_\mathrm{d}(t)$ is just the counting process described by the probabilities (31)–(33); moreover, $\widehat{\kappa}_t$ is a conditional state (*a posteriori* state): the state of the source system at time t, having observed a certain trajectory for N_d (a sequence of counts) up to time t.

Let us end by the list of all the derived constants introduced in the paper:

$$\gamma_j = \sum_{\lambda=1}^{2} \int_\Sigma |g_j(\sigma,\lambda)|^2 \mathrm{d}_2\sigma \,, \qquad \eta_j = \sum_{\lambda=1}^{2} \int_{S_\mathrm{d}} |g_j(\sigma,\lambda)|^2 \mathrm{d}_2\sigma \,,$$

$$\lambda_j = \frac{1}{\sqrt{\gamma_j}} \sum_{\lambda=1}^{2} \int_\Sigma \overline{g_j(\sigma,\lambda)}\, l_j(\sigma,\lambda)\, \mathrm{d}_2\sigma \,, \tag{62}$$

$$\Delta_j = \alpha_j - \omega_j \,, \quad \Omega_j = \sqrt{\gamma_j}\, |\lambda_j| \,, \quad \beta_j = \arg\left(\lambda_j \sqrt{\gamma_j}\right) , \quad \eta = \eta_1 + \eta_2 \,.$$

References

[1] E. B. Davies, *Quantum Theory of Open Systems* (Academic, London, 1976).

[2] A. Barchielli, L. Lanz, G. M. Prosperi, Found. Phys. **13** (1983) 779–812.

[3] A. Barchielli, G. Lupieri, J. Math. Phys. **26** (1985) 2222–2230.

[4] A. Barchielli, Phys. Rev. A **34** (1986) 1642–1649.

[5] A. S. Holevo, in *Advances in Statistical Signal Processing*, vol. 1 (JAI Press, 1987) pp. 157–202.

[6] V. P. Belavkin, in *Modelling and Control of Systems*, edited by A. Blaquière, Lect. Notes Control Inform. Sciences **121** (Springer, Berlin, 1988) pp. 245–265.

[7] A. Barchielli, V. P. Belavkin, J. Phys. A: Math. Gen. **24** (1991) 1495–1514.

[8] P. Staszewski, *Quantum Mechanics of Continuously Observed Systems* (Nicholas Copernicus University Press, Toruń, 1993).

[9] A. Barchielli, J. Phys. A: Math. Gen. **20** (1987) 6341–6355.

[10] A. Barchielli, Quantum Opt. **2** (1990) 423–441.

[11] A. Barchielli, in *Stochastic Evolution of Quantum States in Open Systems and in Measurement Processes*, edited by L. Diósi *et al.* (World Scientific, Singapore, 1994) pp. 1–14.

[12] A. Barchielli, A. M. Paganoni, Quantum Semiclass. Opt. **8** (1996) 133–156.

[13] C. Cohen–Tannoudji, J. Dalibard, Europhys. Lett. **1** (1986) 441–448.

[14] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus* (Birkhäuser, Basel, 1992).

[15] C. W. Gardiner, M. J. Collet, Phys. Rev. A **31** (1985) 3761–3774.

[16] H. J. Carmichael, *An Open System Approach to Quantum Optics*, Lect. Notes Phys. **m18** (Springer, Berlin, 1993).

[17] H. P. Yuen, J. H. Shapiro, IEEE Trans. Inform. Theory **IT-24** (1978) 657–668.

[18] P. Zoller, M. Marte, D. F. Walls, Phys. Rev. A **35** (1987) 198–207.

[19] K. Mølmer, Y. Castin, J, Dalibard, J. Opt. Soc. Am. B **10** (1993) 524–538.