A new approach to numerical solution of defective boundary problems in incompressible fluid dynamics*

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Abstract

We consider the incompressible Navier-Stokes equations with flow rate and mean pressure boundary conditions. There are basically two strategies for solving these defective boundary problems: the *variational approach* (see *J. Heywood, R. Rannacher, S. Turek, Int J Num Meth Fluids 22 (1996), pp. 325-352*) and the *augmented formulation* (see *L. Formaggia, J. F. Gerbeau, F. Nobile, A. Quarteroni, SIAM J Num Anal, 40-1 (2002), pp. 376–401,* and *A. Veneziani, C. Vergara, Int J Num Meth Fluids, 47 (2005), pp. 803–816*). However, these approaches present some drawbacks. The former, for the flow rate problem, resorts to non standard functional spaces, which are quite difficult to discretize. On the other hand, for the mean pressure problem, it yealds exact solutions only in very specific cases. The latter is applicable only to the flow rate problem, since for the mean pressure problem it provides unfeasible boundary conditions.

In this paper, we propose a new strategy, based on a control reformulation of the problems at hand. This approach allows to treat the two problems successfully within the same framework. We carry out the well-posedness analysis of the problems obtained with this approach and we propose some algorithms for their numerical solution. Several numerical results are presented supporting the effectiveness of our approach.

1 Introduction

Having at disposal an incomplete boundary data set is a frequent situation in computational fluid-dynamics, in particular with reference to the *artificial boundaries*, i.e. the boundaries created just to limit the computational domain and not corresponding to a physical interface. On these boundaries often the available measurements only give an incomplete data set from measurements or other computations. This situation has been addressed in [10], while in

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[4, 14, 15, 16] defective boundary problems have been studied in the field of computational haemodynamics, in the context of the so called *geometrical multiscale modeling.* In particular, two conditions have been considered, *the mean pressure problem* in which on the artificial boundary Γ we want to prescribe

$$\frac{1}{|\Gamma|}\int\limits_{\Gamma} p\, d\gamma = P(t),$$

with $p = p(t, \boldsymbol{x})$ the fluid pressure and $P = P(t)$ a given function of time, and the *flow rate problem*, in which we have:

$$\int\limits_{\Gamma} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma = Q(t), \tag{1}$$

where $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x})$ is the fluid velocity, $\boldsymbol{n}$ the outward unit vector and $Q = Q(t)$ is a given function.

In [10] a variational approach is proposed for finding an appropriate numerical solution of these problems. However, in the mean pressure problem this variational approach gives the expected results only for special geometries. Furthermore, a major limitation of this approach concernes the flow rate problems, since the proposed variational formulation resorts to non standard functional spaces. The construction of finite-dimensional subspaces is actually quite problematic. For this reason, a different approach is proposed in [4], based on an augmented formulation, where condition (1) is regarded as a constraint for the solution to be forced by means of a Lagrange multiplier (see also [14]). Unfortunately, the technique cannot treat the mean pressure problem, as it yields inaccurate solutions for the velocity (see [4, 16]).

In this paper, we propose a new general approach for the defective boundary data problems, which does not suffer from the limitations of the previous ones, so that both mean pressure and flow rates problems can be solved successfully within the same framework. This approach relies on the introduction of an appropriate functional which quantifies the difference between the boundary solution and the prescribed data (see, in a different context, [1]). Resorting to a technique similar to that of a control problem, we introduce control variables related to the fluid velocity and pressure on the boundary. Acting on these control variables, we seek the minimum of the functional at hand. This approach is quite general and can be applied virtually to all kind of defective boundary problems. The idea of using boundary data as control variables for forcing some properties of the solution is not new. For instance, it has been used in [9] for an effective splitting of velocity-pressure computation in standard unsteady Stokes problems. In this case, boundary control variables were used to force the incompressibility constraint.

In this paper, we treat both the flow rate and the mean pressure problem. We discuss the effectiveness of this approach in prescribing the defective data in comparison with the variational approach proposed in [10] and with the augmented formulation proposed in [4, 14]. Numerical results presented here confirm the flexibility of our method.

The outline of the paper is as follows. Section 2 is devoted to the flow rate problem. The approach based on the use of normal stresses as control variables is first presented for the generalized Stokes problem and then extended to the nonlinear case. In Section 3 we illustrate the approach for the mean pressure problem. For the sake of simplicity, we limit the analysis to the linear problem. In this latter case, we propose different formulations, featuring different control variables. Section 4 is devoted to the illustration of some algorithmic details, while in Section 5 we present and discuss some numerical results.

1.1 Basic notation

Let us denote by $\Omega \in \mathbb{R}^d$ a domain filled by an incompressible fluid, whose boundary can be split into two parts. The former is denoted by Γ_w and corresponds to a physical wall. The latter is given by the union of the artificial sections Γ_j $(j = 0, \ldots, m)$ which limit the domain of interest, but do not have a direct physical significance. A possible domain of this sort is given in Figure 1. We assume that the walls are rigid, so that the velocity field is zero on Γ_w, and that the fluid is Newtonian. This kind of problems is typical in the haemodynamics of large vessels, which motivates the present study. The extension to the case of compliant vessels, which is of paramount interest for haemodynamics, requires a specific analysis still to be carried out. The Navier-Stokes equations

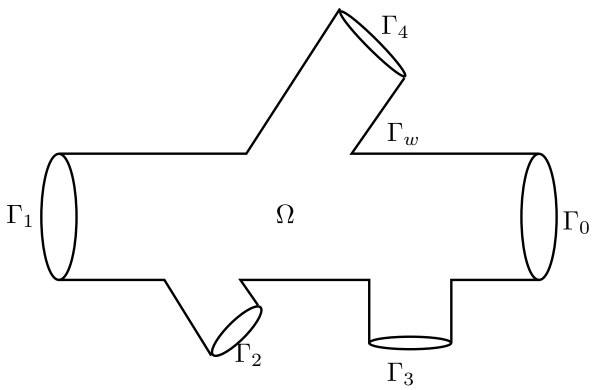

Figure 1: Reference domain Ω

for the problem at hand are:

$$\begin{cases} \rho \dfrac{\partial \boldsymbol{u}}{\partial t} - \mu \triangle \boldsymbol{u} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } (0,T) \times \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & \text{in } (0,T) \times \Omega, \end{cases} \tag{2}$$

where $\boldsymbol{f}(t, \boldsymbol{x})$ is a given forcing term and μ and ρ are the constant fluid viscosity and density, respectively. In the sequel, for the sake of simplicity and without loss of generality we will set $\rho = 1$. Moreover, we have *no-slip* boundary conditions on Γ_w,

$$\boldsymbol{u}|_{\Gamma_w} = \boldsymbol{0} \qquad \text{in } (0,T), \tag{3}$$

as well as the initial condition

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 \qquad \text{in } \Omega, \tag{4}$$

where $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x})$ is a given function, regular enough. Boundary conditions still have to specified on Γ_j. On those boundaries, we will consider two possibilities, concerning the flow rate and the mean pressure, respectively.

In the sequel we will refer to the functional spaces

$$L^2(\Omega) = \Big\{v : \int_\Omega v^2\,d\omega < \infty\Big\}, \qquad H^1(\Omega) = \{v \in L^2(\Omega) : v' \in L^2(\Omega)\},$$
$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v}|_{\Gamma_w} = \boldsymbol{0}\}, \qquad \boldsymbol{V}_{div} = \{\boldsymbol{v} \in \boldsymbol{V} : \nabla\cdot\boldsymbol{v} \in H^1(\Omega)\},$$

where $\boldsymbol{H}^1(\Omega) = (H^1(\Omega))^d$ and use the following notation for scalar functions $s,\ q \in L^2(\Omega)$ and for vector functions $\boldsymbol{v}\,, \boldsymbol{w} \in \mathbf{H}^1(\Omega)$:

$$(s,q) = \int_\Omega s\,q\,d\omega, \qquad (\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}\,d\omega = \int_\Omega \sum_{i=1}^d v_i w_i\,d\omega,$$

$$(\nabla\boldsymbol{v},\nabla\boldsymbol{w}) = \int_\Omega \nabla\boldsymbol{v} : \nabla\boldsymbol{w}\,d\omega = \int_\Omega \sum_{i,j=1}^d \frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\,d\omega,$$

$$(s,q)_{H^1} = (s,q) + (\nabla s,\nabla q), \qquad (\boldsymbol{v},\boldsymbol{w})_{H^1} = (\boldsymbol{v},\boldsymbol{w}) + (\nabla\boldsymbol{v},\nabla\boldsymbol{w}),$$

$$a(\boldsymbol{v},\boldsymbol{w}) = \alpha(\boldsymbol{v},\boldsymbol{w}) + \mu(\nabla\boldsymbol{v},\nabla\boldsymbol{w}), \quad d(q,\boldsymbol{v}) = -(q,\nabla\cdot\boldsymbol{v}), \quad c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}) = ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\boldsymbol{w}),$$

being $\alpha \geq 0$ a given parameter. We point out that $a(\cdot,\cdot)$ is coercive on the space $\boldsymbol{V}$, $\forall\alpha \geq 0$. Moreover, given a vector $\boldsymbol{q} \in \mathbb{R}^m$, we pose

$$\|\boldsymbol{q}\|_p = \Big(\sum_{j=1}^m |q_j|^p\Big)^{1/p}.$$

Finally, given m Hilbert spaces $W_1,\ldots,W_m$, let $\boldsymbol{W} = W_1\times W_2\times\ldots\times W_m$, $N : \boldsymbol{W} \to \mathbb{R}$, such that $(y_1,\ldots,y_m) \in \boldsymbol{W} \to N(y_1,\ldots,y_m) \in \mathbb{R}$ and $< \cdot,\cdot >$ the duality pairing between W' and $\boldsymbol{W}$. We indicate with

$$< dN_{y_j}[w_1,\ldots,w_m], z >=$$
$$\lim_{\varepsilon\to 0}\Big(\frac{N(y_1,\ldots,y_j+\varepsilon z,\ldots,y_m) - N(y_1,\ldots,y_j,\ldots,y_m)}{\varepsilon}\Big)\Big|_{\boldsymbol{y}=\boldsymbol{w}}$$

the Gateaux differential of N, with the respect of y_j computed in the point $\boldsymbol{w} = (w_1,\ldots,w_m) \in \boldsymbol{W}$ and acting along the direction $z \in W_j$.

2 Flow rate boundary conditions

We start by considering the following boundary conditions applied to the artificial boundaries Γ_i:

$$\begin{cases} (p\boldsymbol{n} - \mu\nabla\boldsymbol{u}\,\boldsymbol{n})|_{\Gamma_0} = \boldsymbol{0} & \text{in } (0,T), \\ \displaystyle\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,d\gamma = Q_i, \quad i = 1,2,\ldots,m, & \text{in } (0,T), \end{cases} \tag{5}$$

where $Q_i = Q_i(t)$ are given functions of time. We point out that, since Γ_w is a rigid boundary, the incompressibility constraint implies that the flow rate on

Γ_0 is $Q_0 = -\sum_{i=1}^{m} Q_i$. These conditions are not sufficient to make the problem at hand well posed in the classical setting, where "pointwise" boundary data are needed. The approach proposed in [10] to overcome this situation is based on the introduction of a variational formulation in the "null fluxes" subspace of $\boldsymbol{V}$ and by lifting the solution by means of an appropriate set of functions called *flux-carriers*. In this way, the variational formulation automatically induces the associated natural boundary conditions, ensuring well posedness to the problem. It has been proved in [13] that the boundary conditions implicitely prescribed on the Γ_i by this formulation are constant (in space) Neumann boundary conditions. This procedure solves the flow rate problem, but from the numerical viewpoint it has the drawback of dealing with a non standard functional space, whose finite dimensional discretization is not easy to construct. For this reason, a different approach, based on Lagrange multipliers and on the reformulation of the problem in terms of a constraint minimization, has been investigated and numerically tested in [4] for the steady Stokes problem and then extended to the unsteady Navier-Stokes problem in [14, 15, 16]. The idea is to regard the flow rate boundary condition as a constraint to be forced by means of Lagrange multipliers. The *augmented variational formulation* associated to this approach is given by

Problem 1 *Given* $\boldsymbol{u}_0 \in \boldsymbol{V}$*, with* $\nabla \cdot \boldsymbol{u}_0 = 0$*,* $\boldsymbol{f} \in L^2(0,T;\boldsymbol{L}^2(\Omega))$ *and a vector of functions* $\boldsymbol{Q} \in (C^0([0,T]))^m$*, satisfying the conditions* $\int_{\Gamma_j} \boldsymbol{u}_0 \cdot \boldsymbol{n}\, d\gamma = Q_j(0)$ *for* $j = 1,\dots,m$*, find* $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{L}^2(\Omega))$*,* $p \in L^2(0,T;L^2(\Omega))$ *and* $\boldsymbol{\lambda} \in (L^2(0,T))^m$ *such that at any* $t \in (0,T)$

$$\begin{cases} \left(\dfrac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right) + \mu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \boldsymbol{v}) + d(p,\boldsymbol{v}) + \\ \qquad + \displaystyle\sum_{j=1}^{m} \lambda_j \int_{\Gamma_j} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma = (\boldsymbol{f},\boldsymbol{v}) & \text{in } \Omega, \\ (q, \nabla\cdot\boldsymbol{u}) = 0 & \text{in } \Omega, \\ \displaystyle\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma = Q_i, & i = 1,\dots,m, \end{cases}$$

for all $\boldsymbol{v} \in \boldsymbol{V}$ *and* $q \in L^2(\Omega)$ *and with* $\boldsymbol{u} = \boldsymbol{u}_0$ *for* $t = 0$*.*

In particular, it has been proven in [4, 14] that this approach leads to the same natural boundary conditions on the artficial boundaries as the variational approach presented in [10]. However, in this case we deal with the standard space $\boldsymbol{V}$.

There is however another possible variational formulation of the problem, still prompt to numerical discretization. In the next section we show the main ideas behind it, starting with the simplest case of the generalized Stokes problem. Then, in Section 2.2, we extend the approach to the non linear case.

2.1 The generalized Stokes problem

In order to address conditions $(5)_2$, let us consider the following generalized Stokes problem:

$$\begin{cases} \alpha \boldsymbol{u}(\boldsymbol{k}) - \mu \triangle \boldsymbol{u}(\boldsymbol{k}) + \nabla p(\boldsymbol{k}) = \boldsymbol{f} & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u}(\boldsymbol{k}) = 0 & \text{in } \Omega, \\ \boldsymbol{u}(\boldsymbol{k})|_{\Gamma_w} = \boldsymbol{0}, & \\ (-p(\boldsymbol{k})\boldsymbol{n} + \mu \nabla \boldsymbol{u}(\boldsymbol{k})\,\boldsymbol{n})|_{\Gamma_0} = \boldsymbol{0}, & \\ (-p(\boldsymbol{k})\boldsymbol{n} + \mu \nabla \boldsymbol{u}(\boldsymbol{k})\,\boldsymbol{n})|_{\Gamma_i} = -k_i \boldsymbol{n}, & i = 1, \ldots, m, \end{cases} \tag{6}$$

where $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ is given and $\alpha \geq 0$ is a given parameter. For the solution of unsteady problems, α is related to the time step and to the time advancing scheme (see Remark 1). As our notation suggests, we now regard velocity and pressure fields as a function of the vector $\boldsymbol{k} = (k_1, \ldots, k_m) \in \mathbb{R}^m$. More precisely, for any given $\boldsymbol{k} \in \mathbb{R}^m$, $\boldsymbol{u}(\boldsymbol{k})$ and $p(\boldsymbol{k})$ denote the velocity and the pressure field obtained by solving (6). The key step is to consider $\boldsymbol{k}$ as *control variable*, to be set such that $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{k})$ fulfills the constraint $(5)_2$. To this aim we introduce the functional $J_Q : \boldsymbol{V} \to \mathbb{R}^+$

$$J_Q(\boldsymbol{w}) = \frac{1}{2} \sum_{i=1}^{m} \Big(\int_{\Gamma_i} \boldsymbol{w} \cdot \boldsymbol{n} \, d\gamma - Q_i \Big)^2, \tag{7}$$

which is clearly minimal (and equal to zero) when $(5)_2$ is fulfilled. For each $\boldsymbol{w} \in \boldsymbol{V}$, $s \in L^2(\Omega)$ and $\boldsymbol{\eta} \in \mathbb{R}^m$, exploiting the notation introduced in Section 1.1, we build the following Lagrange functional, where equations (6) play the role of *constraints* for the solution:

$$\mathcal{L}(\boldsymbol{w}, s; \boldsymbol{\lambda}_w, \lambda_s; \boldsymbol{\eta}) = J_Q(\boldsymbol{w}) + a(\boldsymbol{w}, \boldsymbol{\lambda}_w) + d(s, \boldsymbol{\lambda}_w) + \\ + \sum_{i=1}^{m} \int_{\Gamma_i} \eta_i \boldsymbol{\lambda}_w \cdot \boldsymbol{n} \, d\gamma - (\boldsymbol{f}, \boldsymbol{\lambda}_w) + d(\lambda_s, \boldsymbol{w}). \tag{8}$$

Here, $\boldsymbol{\lambda}_w \in \boldsymbol{V}$ and $\lambda_s \in L^2(\Omega)$ are the *adjoint* variables associated to $\boldsymbol{w}$ and s respectively. In order to find the corresponding Euler equations, we impose that in correspondance of the solution $\boldsymbol{s} = [\boldsymbol{u}, p; \boldsymbol{\lambda}_u, \lambda_p; \boldsymbol{k}]$, the Gateaux differentials of $\mathcal{L}$ evaluated for any test function vanish. That is, we will consider the following problem, where for the sake of simplicity we omit to specify that the differentials are computed in $\boldsymbol{s}$:

Problem 2 *Given* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{Q} \in \mathbb{R}^m$, *find* $\boldsymbol{k} \in \mathbb{R}^m$, $\boldsymbol{u}(\boldsymbol{k}) \in \boldsymbol{V}, p(\boldsymbol{k}) \in$

$L^2(\Omega), \boldsymbol{\lambda}_u \in \boldsymbol{V}$ and $\lambda_p \in L^2(\Omega)$, such that, for all $\boldsymbol{v} \in \boldsymbol{V}$, $q \in L^2(\Omega)$ and $\nu \in \mathbb{R}$:

$$
\left\{
\begin{array}{ll}
(P) \left\{
\begin{array}{l}
< d\mathcal{L}_{\boldsymbol{\lambda}_w}, \boldsymbol{v} >= a(\boldsymbol{u},\boldsymbol{v}) + d(p,\boldsymbol{v}) + \displaystyle\sum_{i=1}^{m} \int_{\Gamma_i} k_i \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma - (\boldsymbol{f},\boldsymbol{v}) = 0, \\
< d\mathcal{L}_{\lambda_s}, q >= d(q,\boldsymbol{u}) = 0,
\end{array}\right. \\
(A) \left\{
\begin{array}{l}
< d\mathcal{L}_{\boldsymbol{w}}, \boldsymbol{v} >= a(\boldsymbol{v},\boldsymbol{\lambda}_u) + d(\lambda_p,\boldsymbol{v}) + \\
\qquad\qquad - \displaystyle\sum_{i=1}^{m} \Big(\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma - Q_i \Big) \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma = 0, \\
< d\mathcal{L}_s, q >= d(q,\boldsymbol{\lambda}_u) = 0,
\end{array}\right. \\
(C_j) \quad < d\mathcal{L}_{\eta_j}, \nu >= \displaystyle\int_{\Gamma_j} \nu \boldsymbol{\lambda}_u \cdot \boldsymbol{n}\, d\gamma = 0, \quad j = 1,\ldots,m.
\end{array}\right.
$$

This system couples a generalized Stokes problem (P) with its adjoint (A) and m scalar equations (*optimality conditions*), denoted by (C_j). Observe that the latter force the adjoint variable $\boldsymbol{\lambda}_u$ to have null flux on the artificial boundaries.

By exploiting the symmetry of the bilinear form $a(\cdot,\cdot)$, the strong formulation of the adjoint problem (A) can be readily deduced, giving

$$
\left\{
\begin{array}{ll}
\alpha\boldsymbol{\lambda}_u - \mu\triangle\boldsymbol{\lambda}_u + \nabla\lambda_p = \boldsymbol{0} & \text{in } \Omega, \\
\nabla\cdot\boldsymbol{\lambda}_u = 0 & \text{in } \Omega, \\
\boldsymbol{\lambda}_u|_{\Gamma_w} = \boldsymbol{0}, & \\
(-\lambda_p\boldsymbol{n} + \mu\nabla\boldsymbol{\lambda}_u\ \boldsymbol{n})|_{\Gamma_0} = \boldsymbol{0}, & \\
(-\lambda_p\boldsymbol{n} + \mu\nabla\boldsymbol{\lambda}_u\ \boldsymbol{n})|_{\Gamma_i} = \Big(\displaystyle\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma - Q_i \Big)\boldsymbol{n}, & i = 1,\ldots,m.
\end{array}\right. \tag{9}
$$

To solve Problem 2 numerically, we need to resort to iterative techniques, as we will see in Section 4. It is worth noting that, if the iterative process converges, at the limit (i.e. when $J_Q(\boldsymbol{u}) = 0$), the fulfillment of (A) and (C_j) implies $\boldsymbol{\lambda}_u = \boldsymbol{0}$ and $\lambda_p = 0$. This is promptly verified by selecting $\boldsymbol{v} = \boldsymbol{\lambda}_u$ in (A) and taking into account the (C_j). The adjoint variables are however needed to drive the iterative scheme to the optimal solution, as it will be illustred in Section 4.

Proposition 1 *Problem 2 admits a unique solution* $[\boldsymbol{u}(\boldsymbol{k}), p(\boldsymbol{k}); \boldsymbol{\lambda}_u, \lambda_p; \boldsymbol{k}]$.

Proof. Let us set $\|\boldsymbol{v}\|_a = \|\sqrt{a(\boldsymbol{v},\boldsymbol{v})}\|$, $\forall \boldsymbol{v} \in \boldsymbol{V}$, where $\|\cdot\|$ stands for the $L^2(\Omega)$ norm. Thanks to the Poincaré inequality, $\|\cdot\|_a$ is equivalent to $\|\cdot\|_{\boldsymbol{H}^1} = \sqrt{\|\cdot\|^2 + \|\nabla\cdot\|^2}$, i.e.

$$
\frac{1}{C_1}\|\boldsymbol{v}\|_{\boldsymbol{H}^1} \leq \|\boldsymbol{v}\|_a \leq C_1\|\boldsymbol{v}\|_{\boldsymbol{H}^1}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{10}
$$

for a suitable constant $C_1 > 0$. Moreover, we recall the following trace inequality

$$
\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Gamma_j)} \leq C_{T,j}\|\boldsymbol{v}\|_{\boldsymbol{H}^1} \leq C_T\|\boldsymbol{v}\|_{\boldsymbol{H}^1}, \quad j = 1,\ldots,m, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \tag{11}
$$

for suitable constants $C_{T,j} > 0$ and where $C_T = \max_{j=1,\ldots,m} C_{T,j}$.

Let $\tau \in \mathbb{R}$ be given. We introduce the following linear operators:
1) $\boldsymbol{P_f} : \mathbb{R}^m \to \boldsymbol{V}$. It associates to a constant vector $\boldsymbol{h} \in \mathbb{R}^m$ the function $\boldsymbol{w}$, where $[\boldsymbol{w}, s]$ is the solution of the generalized Stokes problem:

$$\begin{cases} a(\boldsymbol{w}, \boldsymbol{v}) + d(s, \boldsymbol{v}) = -\sum_{j=1}^{m} h_j \int_{\Gamma_j} \boldsymbol{v} \cdot \boldsymbol{n}\, d\gamma + (\boldsymbol{f}, \boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{V}, \\ d(q, \boldsymbol{w}) = 0, & \forall q \in L^2(\Omega). \end{cases} \tag{12}$$

In particular, if $\boldsymbol{f} = \boldsymbol{0}$, we will write $\boldsymbol{P_0}$. Now, if we take $\boldsymbol{v} = \boldsymbol{w}$ in (12), thanks to (10), (11) and the Young inequality, we obtain:

$$\|\boldsymbol{w}\|_a^2 \le \sum_{j=1}^{m} |h_j| C_{T,j} \|\boldsymbol{w}\|_{\boldsymbol{H}^1} + \|\boldsymbol{f}\|\,\|\boldsymbol{w}\| \le C_T C_1 \|\boldsymbol{h}\|_1 \|\boldsymbol{w}\|_a + C_1 \|\boldsymbol{f}\|\,\|\boldsymbol{w}\|_a \le$$

$$\le \frac{C_T^2 C_1^2}{2\varepsilon_1} \|\boldsymbol{h}\|_1^2 + \frac{\varepsilon_1}{2} \|\boldsymbol{w}\|_a^2 + \frac{1}{2\varepsilon_2} \|\boldsymbol{f}\|^2 + \frac{C_1^2 \varepsilon_2}{2} \|\boldsymbol{w}\|_a^2,$$

where ε_1 and ε_2 are two arbitrary positive constants. Posing $\varepsilon = \varepsilon_1 = \varepsilon_2$, the following estimate follows:

$$\|\boldsymbol{P_f}(\boldsymbol{h})\|_a = \|\boldsymbol{w}\|_a \le \sqrt{\frac{1}{C_2}\Big(\|\boldsymbol{f}\|^2 + C_3 \|\boldsymbol{h}\|_1^2\Big)} \le \frac{1}{\sqrt{C_2}} \|\boldsymbol{f}\| + \sqrt{\frac{C_3}{C_2}} \|\boldsymbol{h}\|_1, \tag{13}$$

with $C_2 = \varepsilon(2 - \varepsilon(C_1^2 + 1))$ and $C_3 = C_T^2 C_1^2$. The positiveness of C_2 follows from the arbitrariness of ε. For instance, we can take $\varepsilon = 1/(C_1^2 + 1)$ so that $C_2 = 1/(C_1^2 + 1)$.
2) $\boldsymbol{S}_\tau : (\mathbb{R}^m \times \mathbb{R}^m) \to \mathbb{R}^m$, such that

$$\boldsymbol{S}_\tau(\boldsymbol{k}, \boldsymbol{h}) = \boldsymbol{k} + \tau \boldsymbol{h}, \tag{14}$$

3) $\boldsymbol{A} : \boldsymbol{V} \to \mathbb{R}^m$, such that $\boldsymbol{A}(\boldsymbol{w}) = \Big(\int_{\Gamma_1} \boldsymbol{w} \cdot \boldsymbol{n}\, d\gamma, \int_{\Gamma_2} \boldsymbol{w} \cdot \boldsymbol{n}\, d\gamma, \ldots, \int_{\Gamma_m} \boldsymbol{w} \cdot \boldsymbol{n}\, d\gamma\Big)$.
Thanks to the trace inequalities (11) and to (10), we have

$$\|\boldsymbol{A}(\boldsymbol{w})\|_1 = \sum_{j=1}^{m} \Big| \int_{\Gamma_j} \boldsymbol{w} \cdot \boldsymbol{n}\, d\gamma \Big| \le m C_T \|\boldsymbol{w}\|_{\boldsymbol{H}^1} \le m C_T C_1 \|\boldsymbol{w}\|_a. \tag{15}$$

4) Finally, we denote by $\boldsymbol{T} : \mathbb{R}^m \to \mathbb{R}^m$ the operator

$$\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{k}) \equiv \boldsymbol{S}_\tau(\boldsymbol{k}, \boldsymbol{A}(\boldsymbol{P_0}(\boldsymbol{A}(\boldsymbol{P_f}(\boldsymbol{k})) - \boldsymbol{Q}))).$$

Problem 2 can be reformulated as the fixed point problem:

Problem 3 *Find* $\hat{\boldsymbol{k}} \in \mathbb{R}^m$ *such that*

$$\hat{\boldsymbol{k}} = \boldsymbol{T}(\hat{\boldsymbol{k}}). \tag{16}$$

Indeed, let us set $\boldsymbol{\lambda}_u = \boldsymbol{P_0}(\boldsymbol{A}(\boldsymbol{P_f}(\boldsymbol{k})) - \boldsymbol{Q})$. If $\hat{\boldsymbol{k}} = \boldsymbol{T}(\hat{\boldsymbol{k}}) = \hat{\boldsymbol{k}} + \tau\boldsymbol{A}(\boldsymbol{P_0}(\boldsymbol{A}(\boldsymbol{P_f}(\hat{\boldsymbol{k}})) - \boldsymbol{Q}))$, then $\int_{\Gamma_j} \boldsymbol{\lambda}_u \cdot \boldsymbol{n}\, d\gamma = 0$, $\forall j = 1, \dots, m$. Moreover, by definition $\boldsymbol{\lambda}_u$ satisfies system (A) in Problem 2. Therefore, taking $\boldsymbol{v} = \boldsymbol{\lambda}_u$, we obtain $\boldsymbol{\lambda}_u = \boldsymbol{0}$, which implies $(5)_2$, being $\boldsymbol{v}$ arbitrary. On the other hand, if $[\boldsymbol{u}, p, \boldsymbol{\lambda}_u, \lambda_p, \hat{\boldsymbol{k}}]$ is solution of Problem 2, we have $\boldsymbol{A}(\boldsymbol{P_0}(\boldsymbol{A}(\boldsymbol{P_f}(\hat{\boldsymbol{k}})) - \boldsymbol{Q})) = \boldsymbol{0}$ by construction and therefore we obtain trivially $\hat{\boldsymbol{k}} = \hat{\boldsymbol{k}}$.

We can now prove that $\boldsymbol{T}$ is a contraction: thanks to the linearity of the involved operators we have:

$$\boldsymbol{T}(\boldsymbol{k}_1) - \boldsymbol{T}(\boldsymbol{k}_2) = (\boldsymbol{k}_1 - \boldsymbol{k}_2) + \tau\boldsymbol{A}(\boldsymbol{P_0}(\boldsymbol{A}(\boldsymbol{P_0}(\boldsymbol{k}_1 - \boldsymbol{k}_2)))).$$

Ley us pose $\boldsymbol{e} = (1, 1, \dots, 1) \in \mathbb{R}^m$. Therefore, thanks to (13) and (15), we obtain:

$$\begin{aligned}
\|\boldsymbol{T}(\boldsymbol{k}_1) - \boldsymbol{T}(\boldsymbol{k}_2)\|_1 &\le \|(\boldsymbol{k}_1 - \boldsymbol{k}_2) + \tau C_T \|\boldsymbol{P_0}(\boldsymbol{A}(\boldsymbol{P_0}(\boldsymbol{k}_1 - \boldsymbol{k}_2)))\|_{\boldsymbol{H}^1}\, \boldsymbol{e}\|_1 \le \\
&\le \|(\boldsymbol{k}_1 - \boldsymbol{k}_2) + \tau C_T C_1 \|\boldsymbol{P_0}(\boldsymbol{A}(\boldsymbol{P_0}(\boldsymbol{k}_1 - \boldsymbol{k}_2)))\|_a\, \boldsymbol{e}\|_1 \le \\
&\le \|(\boldsymbol{k}_1 - \boldsymbol{k}_2) + \tau C_T C_1 \sqrt{\frac{C_3}{C_2}} \|\boldsymbol{A}(\boldsymbol{P_0}(\boldsymbol{k}_1 - \boldsymbol{k}_2))\|_1\, \boldsymbol{e}\|_1 \le \\
&\le \|(\boldsymbol{k}_1 - \boldsymbol{k}_2) + \tau m C_T^2 C_1^2 \sqrt{\frac{C_3}{C_2}} \|\boldsymbol{P_0}(\boldsymbol{k}_1 - \boldsymbol{k}_2)\|_a\, \boldsymbol{e}\|_1 \le \\
&\le \|(\boldsymbol{k}_1 - \boldsymbol{k}_2) + \tau m C_T^2 C_1^2 \frac{C_3}{C_2} \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|_1\, \boldsymbol{e}\|_1 = \\
&= \sum_{j=1}^{m} \left| k_{1,j} - k_{2,j} + \tau m C_T^2 C_1^2 \frac{C_3}{C_2} \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|_1 \right| \le \\
&\le \sum_{j=1}^{m} \left| \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|_1 + \tau m C_T^2 C_1^2 \frac{C_3}{C_2} \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|_1 \right| \le \\
&\le m \left| 1 + \tau m C_T^2 C_1^2 \frac{C_3}{C_2} \right| \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|_1 = m|1 + \tau m C_T^4 C_1^4 (C_1^2 + 1)| \|\boldsymbol{k}_1 - \boldsymbol{k}_2\|_1. \quad (17)
\end{aligned}$$

Since τ is arbitrary, we select it in such a way that:

$$-\frac{m+1}{m^2 C_T^4 C_1^4 (C_1^2 + 1)} < \tau < -\frac{m-1}{m^2 C_T^4 C_1^4 (C_1^2 + 1)}.$$

The thesis is therefore a consequence of the Banach fixed point theorem. □

In Section 4, the fixed point reformulation of Problem 2 will be considered again for the set up of the numerical solution procedure.

Remark 1 *When one wants to solve the unsteady Stokes problem*

$$\begin{cases}
\dfrac{\partial \boldsymbol{u}(\boldsymbol{k})}{\partial t} - \mu \triangle \boldsymbol{u}(\boldsymbol{k}) + \nabla p(\boldsymbol{k}) = \boldsymbol{f} & \textit{in } (0,T) \times \Omega, \\
\nabla \cdot \boldsymbol{u}(\boldsymbol{k}) = 0 & \textit{in } (0,T) \times \Omega, \\
\boldsymbol{u}(\boldsymbol{k})|_{t=0} = \boldsymbol{u}_0 & \textit{in } \Omega, \\
\boldsymbol{u}(\boldsymbol{k})|_{\Gamma_w} = \boldsymbol{0} & \textit{in } (0,T), \\
(-p(\boldsymbol{k})\boldsymbol{n} + \mu \nabla \boldsymbol{u}(\boldsymbol{k})\, \boldsymbol{n})|_{\Gamma_0} = \boldsymbol{0} & \textit{in } (0,T), \\
(-p(\boldsymbol{k})\boldsymbol{n} + \mu \nabla \boldsymbol{u}(\boldsymbol{k})\, \boldsymbol{n})|_{\Gamma_i} = -k_i(t)\boldsymbol{n}, \quad i = 1, \dots, m, & \textit{in } (0,T),
\end{cases} \quad (18)$$

with $\boldsymbol{f} \in L^\infty(0,T;\boldsymbol{L}^2(\Omega))$, $\boldsymbol{u}_0 \in \boldsymbol{L}^2(\Omega)$ *and* $\nabla \cdot \boldsymbol{u}_0 = 0$, *a possible approach would rely on a Lagrangian funtional for the time dependent problem. However, this implies a time dependent adjoint equation with a final time condition. The solution of the associated control problem in the time interval* $(0,T)$ *would lead to iteratively solve (and store) the solution of the primal problem in order to find the time-reversed adjoint solution. A different approach consists of applying our control technique on the time discretization of (18). In this way, we avoid the solution of the time adjoint problem, since we already know its exact solution, which is zero. In fact, here the adjoint problem is just a tool for enforcing the mean flux boundary conditions. In particular, denoting* $t^n = n\Delta t$, *with* Δt *the time discretization step and referring to a BDF time advancing scheme, we obtain, for each* n, *problem (6) and the Lagrangian (8) with* $\alpha = \beta_0/\Delta t$ *and forcing term equal to* $\boldsymbol{f}^n + \sum_{i=1}^r \beta_i/\Delta t\, \boldsymbol{u}^{n-i}$, *where* $\boldsymbol{k}^n = \boldsymbol{k}(t^n)$, $\boldsymbol{u}^n = \boldsymbol{u}^n(\boldsymbol{k}^n) = \boldsymbol{u}(t^n,\boldsymbol{x})$, $p^n = p^n(\boldsymbol{k}^n) = p(t^n,\boldsymbol{x})$, $\boldsymbol{f}^n = \boldsymbol{f}(t^n,\boldsymbol{x})$, $\boldsymbol{u}^{n-i}$ *is the approximation of* $\boldsymbol{u}(t^{n-i},\boldsymbol{x})$ *and* $\beta_i\,(i = 0,1,\ldots,r \leq n)$ *are the coefficients of the time discretization. Therefore, with this notation, we can interpret Problem 2 as a technique for the solution of an unsteady flow rate Stokes problem discretized in time. In this case the strong formulation of the adjoint problem (A) is given by (9) with* $\alpha = \beta_0/\Delta t$. *It may be noted that in this case system (9) is exactly the BDF discretization at time* $t = t^n$ *of the backward-in-time adjoint problem where the exact solution values* $(\boldsymbol{\lambda}_u^{n+k} = \boldsymbol{0}, \lambda_p^{n+k} = 0)$ *have been used for* $k = 1,2,\ldots,r$.

2.2 The non linear case

We focus now on the system arising when the non-linear convective term is present in the fluid equations. More precisely, we consider the following Navier-Stokes problem:

$$\alpha \boldsymbol{u}(\boldsymbol{k}) - \mu \triangle \boldsymbol{u}(\boldsymbol{k}) + (\boldsymbol{u}(\boldsymbol{k}) \cdot \nabla)\boldsymbol{u}(\boldsymbol{k}) + \nabla p(\boldsymbol{k}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{19}$$

together with $(6)_{2-5}$. To minimize (7) with the constraint given by (19) together with $(6)_{2-5}$, we have to find the stationary point of the following Lagrangian functional:

$$\mathcal{L}(\boldsymbol{w}, s; \boldsymbol{\lambda}_w, \lambda_s; \boldsymbol{\eta}) = J_Q(\boldsymbol{w}) + a(\boldsymbol{w}, \boldsymbol{\lambda}_w) + c(\boldsymbol{w}, \boldsymbol{w}, \boldsymbol{\lambda}_w) + d(s, \boldsymbol{\lambda}_w) +$$
$$+ \sum_{i=1}^m \int_{\Gamma_i} \eta_i \boldsymbol{\lambda}_w \cdot \boldsymbol{n}\, d\gamma - (\boldsymbol{f}, \boldsymbol{\lambda}_w) + d(\lambda_p, \boldsymbol{w}).$$

This leads to the following

Problem 4 *Given* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{Q} \in \mathbb{R}^m$, *find* $\boldsymbol{k} \in \mathbb{R}^m$, $\boldsymbol{u}(\boldsymbol{k}) \in \boldsymbol{V}, p(\boldsymbol{k}) \in$

$L^2(\Omega), \boldsymbol{\lambda}_u \in \boldsymbol{V}$ and $\lambda_p \in L^2(\Omega)$, such that, for all $\boldsymbol{v} \in \boldsymbol{V}$, $q \in L^2(\Omega)$ and $\nu \in \mathbb{R}$:

$$
\begin{cases}
(P)\begin{cases}
< d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u},\boldsymbol{v}) + c(\boldsymbol{u},\boldsymbol{u},\boldsymbol{v}) \\
\qquad\qquad +d(p,\boldsymbol{v}) + \displaystyle\sum_{i=i}^{m} \int_{\Gamma_i} k_i \boldsymbol{v}\cdot\boldsymbol{n}\,d\gamma - (\boldsymbol{f},\boldsymbol{v}) = 0,\\
< d\mathcal{L}_{\lambda_p}, q >= d(q,\boldsymbol{u}) = 0,
\end{cases}\\
(A)\begin{cases}
< d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} > \; = a(\boldsymbol{v},\boldsymbol{\lambda}_u) + c(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\lambda}_u) + c(\boldsymbol{v},\boldsymbol{u},\boldsymbol{\lambda}_u) + \\
\qquad\qquad +d(\lambda_p,\boldsymbol{v}) - \displaystyle\sum_{i=1}^{m} \Big(\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\,d\gamma - Q_i \Big) \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\,d\gamma = 0,\\
< d\mathcal{L}_p, q > \; = d(q,\boldsymbol{\lambda}_u) = 0,
\end{cases}\\
(C_j) \quad < d\mathcal{L}_{k_j}, \nu >= \displaystyle\int_{\Gamma_j} \nu \boldsymbol{\lambda}_u \cdot \boldsymbol{n}\,d\gamma = 0, \quad j = 1,\ldots,m.
\end{cases}
$$

The considerations made for the linear case do extend to this situation, so we omit them for the sake of brevity.

3 Mean pressure boundary conditions

We consider now the second boundary problem illustrated in the Introduction, namely the mean pressure problem. Referring to Figure 1, it is given by (2) together with the boundary condition (3), initial condition (4) and the following defective condition on the artficial sections Γ_j at any $t \in (0,T)$

$$
\frac{1}{|\Gamma_j|} \int_{\Gamma_j} p\,d\gamma = P_j, \quad j = 0,1,\ldots,m, \tag{20}
$$

where the $P_j = P_j(t)$ are given functions of time. We point out that, in order the quantities at left hand side in (20) make sense, we have to provide suitable hypothesis on tha data (see Section 3.1.1). Obviously, also conditions (20) are not sufficient to make the problem at hand well posed.

The variational approach for this type of problem advocated in [10] is based on the forcing of implicit natural homogeneous boundary conditions leading in this case to constant (in space) normal stresses and to zero tangential stresses on each Γ_i. Thus, this formulation is in fact just an approximation of the mean pressure problem and gives the expected results only for specific cases (see Sections 3.1.2 Moreover, differently from the flow rate case, the mean pressure problem cannot be treated in a satisfactory way by forcing the mean pressure conditions as constraints in a Lagrange multipliers framework, as already pointed out in [4]. In fact, referring, for the sake of simplicity, to the linear case, if we write the Lagrangian functional

$$
\mathcal{M}(\boldsymbol{w}, s, \eta_1, \ldots, \eta_m) = \frac{1}{2} a(\boldsymbol{w},\boldsymbol{w}) + \sum_{j=0}^{m} \eta_j \int_{\Gamma_j} (s - P_j)\,d\gamma + (\nabla s, \boldsymbol{w}) - (\boldsymbol{f}, \boldsymbol{w}),
$$

where $\boldsymbol{\eta} \in \mathbb{R}^{m+1}$ is the vector of the Lagrange multipliers related to the boundary conditions (20), it is easy to check that by forcing $(\boldsymbol{u}, p, \boldsymbol{\lambda})$ to be a stationary point for $\mathcal{M}$, we obtain $(\boldsymbol{u}\cdot\boldsymbol{n})|_{\Gamma_j} = \lambda_j$, $\forall j = 0,\ldots,m$. That is the normal component of the velocity is constant on each Γ_j. These conditions are not in general compatible with the no-slip boundary condition (3) on Γ_w.

On the other hand, it is possible to formulate the mean pressure problem in terms of a control problem, by extending the approach introduced in the previous Section. To this aim, we need to select an appropriate control variable. This step is crucial, as we will see. For the moment, the (constant) normal stresses $\boldsymbol{k}$ will be retained as control variables. We start again by considering the generalized Stokes case.

3.1 The generalized Stokes problem

3.1.1 Basic approach

We consider the generalized Stokes problem (6) where also at Γ_0 we prescribe a condition like $(6)_5$. The control variabes $\boldsymbol{k}$ are determined so that the following functional is minimized:

$$J_P(s) = \frac{1}{2}\Big(\sum_{i=0}^{m} \frac{1}{|\Gamma_i|}\int_{\Gamma_i} s\, d\gamma - P_i\Big)^2. \tag{21}$$

We refer to the following Lagrangian functional:

$$\begin{aligned}\mathcal{L}(\boldsymbol{w}, s; \boldsymbol{\lambda}_w, \lambda_s; \boldsymbol{\eta}) = \quad & J_P(s) + a(\boldsymbol{w}, \boldsymbol{\lambda}_w) + d(s, \boldsymbol{\lambda}_w) + \\ & + \sum_{i=0}^{m} \int_{\Gamma_i} \eta_i \boldsymbol{\lambda}_w \cdot \boldsymbol{n}\, d\gamma - (\boldsymbol{f}, \boldsymbol{\lambda}_w) + d(\lambda_s, \boldsymbol{w}),\end{aligned} \tag{22}$$

for all $\boldsymbol{w} \in \boldsymbol{V}_{div}$, $s \in H^1(\Omega)$, $\boldsymbol{\lambda}_w \in \boldsymbol{V}_{div}$, $\lambda_s \in H^1(\Omega)$ and $\boldsymbol{\eta} \in \mathbb{R}^{m+1}$. We point out that we take $s \in H^1(\Omega)$ so that its trace on Γ_i, $i = 0, \ldots, m$, is meaningful. This choice forces the regularity of the other functions, as it will be clear in Proposition 2. Of course, we assume that Ω is sufficient regular. The stationary point of $\mathcal{L}$ fulfills the following

Problem 5 *Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{P} \in \mathbb{R}^{m+1}$, find $\boldsymbol{k} \in \mathbb{R}^{m+1}$, $\boldsymbol{u}(\boldsymbol{k}) \in \boldsymbol{V}_{div}$, $p(\boldsymbol{k}) \in H^1(\Omega), \boldsymbol{\lambda}_u \in \boldsymbol{V}_{div}$ and $\lambda_p \in H^1(\Omega)$, such that, for all $\boldsymbol{v} \in \boldsymbol{V}_{div}$, $q \in H^1(\Omega)$ and $\nu \in \mathbb{R}$,*

$$\left\{\begin{array}{ll} (P) \left\{\begin{array}{l} < d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u}, \boldsymbol{v}) + d(p, \boldsymbol{v}) + \displaystyle\sum_{i=0}^{m} \int_{\Gamma_i} k_i \boldsymbol{v} \cdot \boldsymbol{n}\, d\gamma - (\boldsymbol{f}, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_{\lambda_p}, q >= d(q, \boldsymbol{u}) = 0, \end{array}\right. \\ (A) \left\{\begin{array}{l} < d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} >= a(\boldsymbol{v}, \boldsymbol{\lambda}_u) + d(\lambda_p, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_p, q >= \displaystyle\sum_{i=0}^{m} \Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p\, d\gamma - P_i\Big) \frac{1}{|\Gamma_i|}\int_{\Gamma_i} q\, d\gamma + d(q, \boldsymbol{\lambda}_u) = 0, \end{array}\right. \\ (C_j) \quad < d\mathcal{L}_{k_i}, \nu >= \displaystyle\int_{\Gamma_i} \nu \boldsymbol{\lambda}_u \cdot \boldsymbol{n}\, d\gamma = 0, \quad i = 0, \ldots, m. \end{array}\right.$$

In this case, we obtain a system coupling a generalized Stokes (P) and a fluid problem (A) featuring a non zero divergence velocity. These problems are well posed in the spaces $[\boldsymbol{V}_{div}, H^1(\Omega)]$. Indeed we have the following result.

Proposition 2 *Let us consider the following generalized Stokes system*

$$\begin{cases} a(\boldsymbol{w},\boldsymbol{v}) + d(s,\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}), & \forall \boldsymbol{v} \in \boldsymbol{V}_{div}, \\ d(q,\boldsymbol{w}) = (g,q)_{H^1}, & \forall q \in H^1(\Omega). \end{cases} \tag{23}$$

Then, if $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $g \in H^1(\Omega)$*, system (23) admits an unique solution* $[\boldsymbol{w}, s] \in [\boldsymbol{V}_{div}, H^1(\Omega)]$*. Moreover, the following estimate holds*

$$\|s\|_{H^1} \leq C_4\|\boldsymbol{f}\| + C_5\|g\|_{H^1}. \tag{24}$$

Proof. Let us start recalling the following result (see [3, 8])

Theorem 1 *Given two Hilbert spaces* X *and* M*, two functionals* $l \in X'$ *and* $\chi \in M'$ *and two bilinear forms* $f : X \times X \to \mathbb{R}$ *and* $b : X \times M \to \mathbb{R}$*, we consider the problem of finding* $u \in X$ *and* $\lambda \in M$ *such that*

$$\begin{cases} f(u,v) + b(v,\lambda) =< l, v >, & \forall v \in X \\ b(u,\mu) =< \chi, \mu > & \forall \mu \in M. \end{cases} \tag{25}$$

If f *is continue and coercive on* X *and* b *satisfies an inf-sup condition, then there exists an unique solution* $[u, \lambda]$ *of system (25). Moreover, the following estimates hold*

$$\begin{cases} \|u\|_X \leq C_6(\|l\|_{X'} + \|\chi\|_{M'}) \\ \|\lambda\|_M \leq C_4\|l\|_{X'} + C_5\|\chi\|_{M'} \end{cases} \tag{26}$$

The coercivity of $a(\cdot,\cdot)$ on $\boldsymbol{V}_{div}$ follows form the coercivity on $\boldsymbol{V}$ and by noting that $\boldsymbol{V}_{div} \subset \boldsymbol{V}$. Moreover, the bilinear form $d(\cdot,\cdot)$ satisfies the inf-sup condition if

$$\exists \beta > 0 : \forall q \in H^1(\Omega), \exists \boldsymbol{w} \in \boldsymbol{V}_{div} \text{ such that } d(q,\boldsymbol{w}) \geq \beta\|q\|_{H^1}\|\boldsymbol{w}\|_{\boldsymbol{V}_{div}}.$$

To show the latter, we prove that the Fortin criterion is satisfied (see [2]). Bearing in mind that the couple $[\boldsymbol{V}, L^2(\Omega)]$ is conform to the inf-sup condition, we have to prove that there exists a linear projector $\Pi : \boldsymbol{V} \to \boldsymbol{V}_{div}$, such that for any $\boldsymbol{w} \in \boldsymbol{V}$

$$d(q, \boldsymbol{w} - \Pi\boldsymbol{w}) = 0, \quad \forall q \in H^1(\Omega) \qquad \text{and} \qquad \|\Pi\|_{\mathcal{L}(\boldsymbol{V},\boldsymbol{V}_{div})} \leq C, \tag{27}$$

for a suitable constant $C > 0$. To this aim, let us introduce the following bilinear form $\forall \boldsymbol{v} \in \boldsymbol{V}_{div}$ and $\forall q \in H^1(\Omega)$:

$$e(q,\boldsymbol{v}) = -(q, \nabla\cdot\boldsymbol{v}) - (\nabla q, \nabla(\nabla\cdot\boldsymbol{v})).$$

Let us consider for any $\boldsymbol{w} \in \boldsymbol{V}$ the following problem in the unknowns $[\boldsymbol{z}, r]$

$$\begin{cases} (\boldsymbol{z},\boldsymbol{v})_{\boldsymbol{H}^1} + (\nabla\cdot\boldsymbol{z}, \nabla\cdot\boldsymbol{v})_{\boldsymbol{H}^1} + e(r,\boldsymbol{v}) = 0 & \forall \boldsymbol{v} \in \boldsymbol{V}_{div} \\ e(q,\boldsymbol{z}) = d(q,\boldsymbol{w}) - (\nabla q, \nabla\Pi_1(\nabla\cdot\boldsymbol{w})) & \forall q \in H^1(\Omega), \end{cases} \tag{28}$$

where $\Pi_1 : L^2 \to H^1(\Omega)$, is definied by $(\Pi_1 f, g)_{H^1} = (f, g)$, $\forall g \in H^1(\Omega)$. Taking $g = \Pi_1 f$, we obtain

$$\|\Pi_1 f\|_{H^1} \leq \|f\|. \tag{29}$$

In [8] it is shown that $\forall q \in L^2(\Omega)$, there exists $\widetilde{\boldsymbol{v}} \in \boldsymbol{V}$ such that $\nabla \cdot \widetilde{\boldsymbol{v}} = q$ and

$$\|\widetilde{\boldsymbol{v}}\|_{\boldsymbol{H}^1} \leq C_7 \|q\|. \tag{30}$$

In particular, if $q \in H^1(\Omega)$, then there exists $\widetilde{\boldsymbol{v}} \in \boldsymbol{V}_{div}$ such that $\nabla \cdot \widetilde{\boldsymbol{v}} = q$. Adding $\|q\|_{H^1}$ to both sides of (30) and setting $\|\boldsymbol{v}\|^2_{\boldsymbol{V}_{div}} = \|\boldsymbol{v}\|^2_{\boldsymbol{H}^1} + \|\nabla \cdot \boldsymbol{v}\|^2_{H^1}$, we obtain

$$\|\widetilde{\boldsymbol{v}}\|_{\boldsymbol{V}_{div}} \leq C_8 \|q\|_{H^1}.$$

Therefore, we obtain:

$$\frac{e(q, -\widetilde{\boldsymbol{v}})}{\|\widetilde{\boldsymbol{v}}\|_{\boldsymbol{V}_{div}}} \geq \frac{\|q\|^2_{H^1}}{C_8 \|q\|_{H^1}} = \frac{1}{C_8} \|q\|_{H^1},$$

i.e. an inf-sup condition for the bilinear form $e(\cdot, \cdot)$ in the spaces $\boldsymbol{V}_{div}$ and $H^1(\Omega)$. Since form $(\boldsymbol{z}, \boldsymbol{v})_{\boldsymbol{H}^1} + (\nabla \cdot \boldsymbol{z}, \nabla \cdot \boldsymbol{v})_{\boldsymbol{H}^1}$ is continue and coercive on $\boldsymbol{V}_{div}$, we conclude from Theorem 1 that system (28) admits an unique solution $[\boldsymbol{w}, r]$. Thanks to $(26)_1$ and (29), we have the following estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{V}_{div}} \leq C_6 (\|\nabla \cdot \boldsymbol{w}\| + \|\nabla \Pi_1 (\nabla \cdot \boldsymbol{w})\|) \leq C_6 (1+1) \|\nabla \cdot \boldsymbol{w}\| \leq 2C_6 \|\boldsymbol{w}\|_{\boldsymbol{H}^1}. \tag{31}$$

Setting $\Pi \boldsymbol{w} = \boldsymbol{z}$, from $(28)_2$ we obtain the first of (27). Moreover, using (31), we have

$$\|\Pi\|^2_{\mathcal{L}(\boldsymbol{V}, \boldsymbol{V}_{div})} = \sup_{\|\boldsymbol{w}\|_{\boldsymbol{H}^1}=1} \|\Pi \boldsymbol{w}\|^2_{\boldsymbol{V}_{div}} = \sup_{\|\boldsymbol{w}\|_{\boldsymbol{H}^1}=1} \{\|\Pi \boldsymbol{w}\|^2_{\boldsymbol{H}^1} + \|\nabla \cdot \Pi \boldsymbol{w}\|^2_{H^1}\} \leq$$

$$\leq 2C_6 \|\boldsymbol{w}\|_{\boldsymbol{H}^1} = 2C_6,$$

showing that the projector Π is bounded. Finally, we notice that estimate (24) follows from $(26)_2$. □

We point out that systems (P) and (A) in Problem 5 can be formulated in the form (23), thanks to the Riesz representation theorem (see next Proposition).

Proposition 3 *Problem 5 admits a unique solution* $[\boldsymbol{u}(\boldsymbol{k}), p(\boldsymbol{k}); \boldsymbol{\lambda}_u, \lambda_p; \boldsymbol{k}]$.

Proof. Let $\boldsymbol{h} \in \mathbb{R}^{m+1}$ be a given vector. We introduce the following linear operators:
1) $M_{\boldsymbol{f}} : \mathbb{R}^{m+1} \to H^1(\Omega)$. It associates to a constant vector $\boldsymbol{h} \in \mathbb{R}^{m+1}$ the function s, where $[\boldsymbol{w}, s]$ is the solution of system (12), tested against functions $\boldsymbol{v} \in \boldsymbol{V}_{div}$ and $q \in H^1(\Omega)$. From Riesz representation theorem, there exists a function $\boldsymbol{\zeta} \in \boldsymbol{H}^1(\Omega)$ such that

$$(\boldsymbol{\zeta}, \boldsymbol{v})_{\boldsymbol{H}^1} = -\sum_{i=0}^{m} h_i \int_{\Gamma_i} \boldsymbol{v} \cdot \boldsymbol{n} \, d\gamma + (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega). \tag{32}$$

Moreover, setting $\boldsymbol{v} = \boldsymbol{\zeta}$ in (32), we have

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{H}^1} \leq C_T \|\boldsymbol{h}\|_1 + \|\boldsymbol{f}\|.$$

Therefore, thanks to (24), the following estimate holds:

$$\|M_{\boldsymbol{f}}(\boldsymbol{h})\|_{H^1} = \|s\|_{H^1} \leq C_4 \|\boldsymbol{\zeta}\|_{\boldsymbol{H}^1} \leq C_4 (C_T \|\boldsymbol{h}\|_1 + \|\boldsymbol{f}\|). \tag{33}$$

2) $\boldsymbol{N} : \mathbb{R}^{m+1} \to \boldsymbol{V}$. It associates to a constant vector $\boldsymbol{h} \in \mathbb{R}^{m+1}$ the function $\boldsymbol{w}$, where $[\boldsymbol{w}, s]$ is the solution of the generalized Stokes problem:

$$\begin{cases} a(\boldsymbol{w}, \boldsymbol{v}) + d(s, \boldsymbol{v}) = 0 & \forall \boldsymbol{v} \in \boldsymbol{V}_{div}, \\ d(q, \boldsymbol{w}) = -\sum_{i=0}^{m} h_i \int_{\Gamma_i} q \, d\gamma, & \forall q \in H^1(\Omega). \end{cases} \tag{34}$$

From Riesz representation theorem, it follows that there exists a function $\xi \in H^1(\Omega)$ such that

$$(\xi, q)_{H^1} = -\sum_{i=0}^{m} h_i \int_{\Gamma_i} q \, d\gamma, \quad \forall q \in H^1(\Omega). \tag{35}$$

Moreover, setting $q = \xi$ in (35), we have $\|\xi\|_{H^1} \le C_T \|\boldsymbol{h}\|_1$. Therefore, thanks to (24), the following estimate for s in problem (34) holds:

$$\|s\|_{H^1} \le C_5 \|\xi\|_{H^1} \le C_T C_5 \|\boldsymbol{h}\|_1. \tag{36}$$

Moreover, setting $\boldsymbol{v} = \boldsymbol{w}$ in $(34)_1$, we obtain thanks to (36)

$$\|\boldsymbol{N}(\boldsymbol{h})\|_a^2 = \|\boldsymbol{w}\|_a^2 \le \sum_{i=0}^{m} |h_j| \int_{\Gamma_i} s \, d\gamma \le C_T \|\boldsymbol{h}\|_1 \|s\|_{H^1} \le C_T^2 C_5 \|\boldsymbol{h}\|_1^2. \tag{37}$$

3) $\boldsymbol{B} : H^1(\Omega) \to \mathbb{R}^{m+1}$, such that

$$\boldsymbol{B}(s) = \Big(\frac{1}{|\Gamma_0|^2} \int_{\Gamma_0} s \, d\gamma, \frac{1}{|\Gamma_1|^2} \int_{\Gamma_1} s \, d\gamma, \ldots, \frac{1}{|\Gamma_m|^2} \int_{\Gamma_m} s \, d\gamma \Big).$$

Thanks to the trace inequality (11), we have

$$\|\boldsymbol{B}(s)\|_1 = \sum_{j=0}^{m} \Big| \frac{1}{|\Gamma_j|^2} \int_{\Gamma_j} s \, d\gamma \Big| \le \Big(\sum_{j=0}^{m} \frac{1}{|\Gamma_j|^2} \Big) C_T \|s\|_{H^1}. \tag{38}$$

4) Finally, we denote by $\boldsymbol{Z} : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ the operator

$$\boldsymbol{Z}(\boldsymbol{k}) \equiv \boldsymbol{S}_\tau(\boldsymbol{k}, \boldsymbol{A}(\boldsymbol{N}(\boldsymbol{B}(M_{\boldsymbol{f}}(\boldsymbol{k})) - \boldsymbol{P}))),$$

where $\boldsymbol{S}_\tau$ is given by (14). Problem 5 can be reformulated as the fixed point problem:

Problem 6 *Find* $\hat{\boldsymbol{k}} \in \mathbb{R}^{m+1}$ *such that* $\hat{\boldsymbol{k}} = \boldsymbol{Z}(\hat{\boldsymbol{k}})$.

The equivalence follows the same arguments used in Proposition 1 to show that Problem 2 is equivalent to Problem 3.

Thanks to (15), (33), (37) and (38) and following the same arguments of Proposition 1, it is possible to show that there exist $\widetilde{\tau_1} < \widetilde{\tau_2} \le 0$ such that $\boldsymbol{Z}$ is a contraction for $\widetilde{\tau_1} < \tau < \widetilde{\tau_2}$. We omit the details for the sake of brevity. □

Also in this case, we observe that, in correspondence of the solution of Problem 5 the only solution of the adjoint problem is the trivial one. The extension of the previous formulation to the non-linear case follows the same route illustrated for the flow rate problem.

3.1.2 More complex functionals

In this "defective boundary" problem often we want to introduce in the solution *a priori* informations on the flow field, often driven by experimental observations or physical considerations. This requirement could drive the choice of the functional to minimize and, consequently, of the set of control variables in the optimization process. For instance, suppose that the domain is a pipe in which we expect that the axial component is the only non null component of the velocity and let assume that the artificial boundary where the mean pressure is prescribed is not normal to the flow, like in the example depicted in Figure 2. In this case,

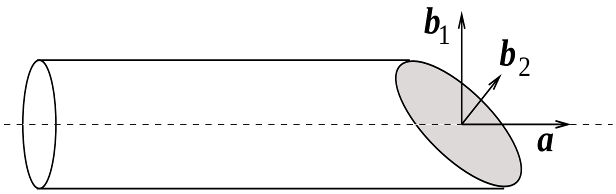

Figure 2: Artificial section Γ_i, axial direction $\boldsymbol{a}$ and orthogonal directions $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$.

Problem 5 would provide a velocity on Γ_i that is not axially oriented, contrary to what expected. This is a consequence to the fact that Problem 5 forces zero tangential stresses on Γ_i as natural boundary condition. To obtain a better control on the boundary velocity in this situation, we need to augment the set of control variables. More precisely, we assume that the whole normal stresses vector has to be determined to force the desired solution features. That is, we set

$$(-p\boldsymbol{n} + \mu\nabla\boldsymbol{u}\,\boldsymbol{n})|_{\Gamma_j} = -\boldsymbol{K}_j(\boldsymbol{x}), \quad j = 0, \dots, m, \tag{39}$$

where, for instance, the $\boldsymbol{K}_j \in (L^2(\Gamma_j))^d$, $j = 0, \dots, m$, are chosen to minimize

$$J_P(\boldsymbol{w}, s) = \frac{1}{2}\sum_{i=0}^{m}\Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} s\,d\gamma - P_i\Big)^2 + \mathcal{S}(\boldsymbol{w}, s, \Omega), \tag{40}$$

where $\mathcal{S}$ depends also on the velocity field and on the domain. For example, for the case at hand a possible expression for $\mathcal{S}$ is

$$\mathcal{S}_1(\boldsymbol{w}, \boldsymbol{b}_1, \dots, \boldsymbol{b}_{d-1}) = \frac{1}{2}\sum_{l=1}^{d-1}\sum_{i=0}^{m}\int_{\Gamma_i} |\boldsymbol{w}\cdot\boldsymbol{b}_l|^2\,d\gamma, \tag{41}$$

where $\boldsymbol{b}_l$, $l = 1, \dots, d-1$, are the tangential unit vector to $\partial\Omega$. With this choice, we look for the conditions (39) that minimize the tangential velocity. Building the Lagrangian functional obtained from (40) with (41), constrained by $(6)_1$, $(6)_2$, $(6)_3$ and (39), we obtain the following

Problem 7 *Given* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{P} \in \mathbb{R}^{m+1}$, *find* $\boldsymbol{K}_j \in (L^2(\Gamma_j))^d$, $j = 0, \dots, m$, $\boldsymbol{u}(\boldsymbol{K}_0, \dots, \boldsymbol{K}_m) \in \boldsymbol{V}_{div}$, $p(\boldsymbol{K}_0, \dots, \boldsymbol{K}_m) \in H^1(\Omega)$, $\boldsymbol{\lambda}_u \in \boldsymbol{V}_{div}$ *and*

$\lambda_p \in H^1(\Omega)$, such that, $\forall \boldsymbol{v} \in \boldsymbol{V}_{div}$, $q \in H^1(\Omega)$ and $\boldsymbol{\nu} \in (L^2(\Gamma_i))^d$, $i = 0, \dots, m$:

$$\begin{cases} (P) \begin{cases} < d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u}, \boldsymbol{v}) + d(p, \boldsymbol{v}) + \displaystyle\sum_{i=0}^{m} \int_{\Gamma_i} \boldsymbol{K}_i \cdot \boldsymbol{v} \, d\gamma - (\boldsymbol{f}, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_{\lambda_p}, q >= d(q, \boldsymbol{u}) = 0, \end{cases} \\ (A) \begin{cases} < d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} >= \displaystyle\sum_{l=1}^{d-1} \sum_{i=0}^{m} \int_{\Gamma_i} (\boldsymbol{u} \cdot \boldsymbol{b}_l)(\boldsymbol{v} \cdot \boldsymbol{b}_l) \, d\gamma + a(\boldsymbol{v}, \boldsymbol{\lambda}_u) + d(\lambda_p, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_p, q >= \displaystyle\sum_{i=0}^{m} \Big(\frac{1}{|\Gamma_i|} \int_{\Gamma_i} p \, d\gamma - P_i \Big) \frac{1}{|\Gamma_i|} \int_{\Gamma_i} q \, d\gamma + d(q, \boldsymbol{\lambda}_u) = 0, \end{cases} \\ (C_i) \quad < d\mathcal{L}_{\boldsymbol{k}_i}, \boldsymbol{\nu} >= \displaystyle\int_{\Gamma_i} \boldsymbol{\nu} \cdot \boldsymbol{\lambda}_u \, d\gamma = 0, \quad i = 0, \dots, m. \end{cases}$$

Let us notice that the optimality conditions (C_i) imply that $\boldsymbol{\lambda}_u|_{\Gamma_i} = \boldsymbol{0}$, $i = 0, \dots, m$. Therefore, the conditions of fulfilment of the optimal state are more restrictive of the ones in Problem 5.

Alternatively, we could consider the following functional:

$$\mathcal{S}_2(\boldsymbol{w}, \boldsymbol{a}) = \frac{1}{2} \sum_{i=0}^{m} \int_{\Gamma_i} \|\nabla \boldsymbol{w} \, \boldsymbol{a}\|_2^2 \, d\gamma. \tag{42}$$

Here $\| \cdot \|_2$ is the Euclidean norm made on the components of the vector $\nabla \boldsymbol{w} \, \boldsymbol{a}$, where $\boldsymbol{a}$ is the axial direction over $\cup_i \Gamma_i$ (see Figure 2). The idea is that in this way the variation of the velocity along $\boldsymbol{a}$ on the artificial boundary is minimized. Due to the incompressibility constraint, a variation of the velocity along $\boldsymbol{a}$ will be compensated by variations of the components orthogonal to $\boldsymbol{a}$. For this reason, this form of $\mathcal{S}$ can be considered as an indirect way of forcing null tangential velocity. This leads to the following

Problem 8 *Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{P} \in \mathbb{R}^{m+1}$, find $\boldsymbol{K}_j \in (L^2(\Gamma_j))^d$, $j = 0, \dots, m$, $\boldsymbol{u}(\boldsymbol{K}_0, \dots, \boldsymbol{K}_m) \in \boldsymbol{V}_{div}$, $p(\boldsymbol{K}_0, \dots, \boldsymbol{K}_m) \in H^1(\Omega)$, $\boldsymbol{\lambda}_u \in \boldsymbol{V}_{div}$ and $\lambda_p \in H^1(\Omega)$, such that, for all $\boldsymbol{v} \in \boldsymbol{V}_{div}$, $q \in H^1(\Omega)$ and $\boldsymbol{\nu} \in (L^2(\Gamma_i))^d$, $i = 0, \dots, m$:*

$$\begin{cases} (P) \begin{cases} < d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u}, \boldsymbol{v}) + d(p, \boldsymbol{v}) + \displaystyle\sum_{i=0}^{m} \int_{\Gamma_i} \boldsymbol{K}_i \cdot \boldsymbol{v} \, d\gamma - (\boldsymbol{f}, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_{\lambda_p}, q >= d(q, \boldsymbol{u}) = 0, \end{cases} \\ (A) \begin{cases} < d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} >= \displaystyle\sum_{i=0}^{m} \int_{\Gamma_i} (\nabla \boldsymbol{u} \, \boldsymbol{a}) \cdot (\nabla \boldsymbol{v} \; \boldsymbol{a}) \, d\gamma + a(\boldsymbol{v}, \boldsymbol{\lambda}_u) + d(\lambda_p, \boldsymbol{v}) = 0, \\ < d\mathcal{L}_p, q >= \displaystyle\sum_{i=0}^{m} \Big(\frac{1}{|\Gamma_i|} \int_{\Gamma_i} p \, d\gamma - P_i \Big) \frac{1}{|\Gamma_i|} \int_{\Gamma_i} q \, d\gamma + d(q, \boldsymbol{\lambda}_u) = 0, \end{cases} \\ (C_i) \quad < d\mathcal{L}_{\boldsymbol{K}_i}, \boldsymbol{\nu} >= \displaystyle\int_{\Gamma_i} \boldsymbol{\nu} \cdot \boldsymbol{\lambda}_u \, d\gamma = 0, \quad i = 0, \dots, m \end{cases}$$

We observe that, in correspondence of the solution of Problem 7 and of Problem 8 the only solution of the adjoint problems is the trivial one. Moreover, it is possible to extend the previous control problems to the non linear case.

Remark 2 *The minimization of functional (40) with (41) or (42) is suitable also for reducing the boundary effects due to the complete formulation of the Cauchy stress, i.e. replacing* $\mu\triangle \boldsymbol{u}$ *by* $\mu\nabla\cdot(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ *in* $(6)_1$.

3.2 Using the flow rates as control variables

For solving the mean pressure problem we can also pursue a sort of "dual" approach to the one proposed in Section 2 for the flow rate problem. Here, the control variables are given by the flow rates Q_j on Γ_j, $j = 1, \ldots, m$. More precisely, we consider the generalized Stokes problem

$$\begin{cases} \alpha \boldsymbol{u}(\boldsymbol{Q}) - \mu\triangle \boldsymbol{u}(\boldsymbol{Q}) + \nabla p(\boldsymbol{Q}) = \boldsymbol{f} & \text{in } \Omega, \\ \nabla\cdot \boldsymbol{u}(\boldsymbol{Q}) = 0 & \text{in } \Omega, \\ \boldsymbol{u}(\boldsymbol{Q})|_{\Gamma_w} = \boldsymbol{0}, & \\ (-p(\boldsymbol{Q})\boldsymbol{n} + \mu\nabla \boldsymbol{u}(\boldsymbol{Q})\,\boldsymbol{n})|_{\Gamma_0} = -P_0\boldsymbol{n}, & \\ \displaystyle\int_{\Gamma_j} \boldsymbol{u}(\boldsymbol{Q})\cdot\boldsymbol{n}\,d\gamma = Q_j, & j = 1,\ldots,m. \end{cases} \tag{43}$$

A Neumann condition (constant in space) on Γ_0 is prescribed to avoid compatibility conditions on the flow rate data. When we now formulate the corresponding control problem where (21) is the functional to be minimized, while (43) are the state equations, we need to reminder that the latter are solved via the augmented Lagrangian formulation illustrated in Problem 1. Therefore, we need to introduce the following Lagrangian functional

$$\begin{aligned} \mathcal{L}(\boldsymbol{w}, s, \boldsymbol{\xi}; \boldsymbol{\lambda}_w, \lambda_s, \boldsymbol{\lambda}_\xi; \boldsymbol{\eta}) = \quad & J_P(s) + a(\boldsymbol{w}, \boldsymbol{\lambda}_w) + d(s, \boldsymbol{\lambda}_w) + \sum_{i=1}^m \xi_i \int_{\Gamma_i} \boldsymbol{\lambda}_w\cdot\boldsymbol{n}\,d\gamma + \\ & -(\boldsymbol{f}, \boldsymbol{\lambda}_w) + d(\lambda_s, \boldsymbol{w}) + \sum_{i=1}^m \lambda_{\xi_i}\Big(\int_{\Gamma_i} \boldsymbol{w}\cdot\boldsymbol{n}\,d\gamma - \eta_i\Big), \end{aligned}$$

where the $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ are the Lagrange multipliers of the augmented formulation (see [4, 14]). The quantities $\boldsymbol{\lambda}_w, \lambda_s$ and $\lambda_{\xi_1}, \ldots, \lambda_{\xi_m}$ are the adjoint variables related to $\boldsymbol{w}, s$ and $\xi_1, \ldots, \xi_m$ respectively. By forcing the derivatives of $\mathcal{L}$ with respect all the variables in correspondance of the solution $[\boldsymbol{u}, p, \boldsymbol{\zeta}; \boldsymbol{\lambda}_u, \lambda_p, \boldsymbol{\lambda}_\zeta; \boldsymbol{Q}]$ to vanish, we obtain

Problem 9 *Given* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ *and* $\boldsymbol{P} \in \mathbb{R}^m$, *find* $\boldsymbol{Q} \in \mathbb{R}^m$, $\boldsymbol{u}(\boldsymbol{Q}) \in \boldsymbol{V}_{div}, p(\boldsymbol{Q}) \in H^1(\Omega), \zeta_j(\boldsymbol{Q}) \in \mathbb{R}$, $j = 1, \ldots, m$, $\boldsymbol{\lambda}_u \in \boldsymbol{V}_{div}, \lambda_p \in H^1(\Omega)$ *and* $\lambda_{\zeta_j} \in \mathbb{R}$, $j =$

$1,\dots,m$, *such that, for all* $\boldsymbol{v}\in\boldsymbol{V}_{div}$, $q\in H^1(\Omega)$ *and* $\nu\in\mathbb{R}$*:*

$$
\begin{cases}
(P) \begin{cases}
< d\mathcal{L}_{\boldsymbol{\lambda}_u}, \boldsymbol{v} >= a(\boldsymbol{u},\boldsymbol{v}) + d(p,\boldsymbol{v}) + \displaystyle\sum_{i=1}^{m} \zeta_i \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma - (\boldsymbol{f},\boldsymbol{v}) = 0,\\
< d\mathcal{L}_{\lambda_p}, q >= d(q,\boldsymbol{u}) = 0,\\
< d\mathcal{L}_{\lambda_{\zeta_i}}, \nu >= \Big(\displaystyle\int_{\Gamma_i} \boldsymbol{u}\cdot\boldsymbol{n}\, d\gamma - Q_i\Big)\nu = 0,
\end{cases}\\
(A) \begin{cases}
< d\mathcal{L}_{\boldsymbol{u}}, \boldsymbol{v} >= a(\boldsymbol{v},\boldsymbol{\lambda}_u) + d(\lambda_p,\boldsymbol{v}) + \displaystyle\sum_{i=1}^{m} \lambda_{\zeta_i} \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\, d\gamma = 0,\\
< d\mathcal{L}_{p}, q >= \displaystyle\sum_{i=1}^{m} \Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p\, d\gamma - P_i\Big)\frac{1}{|\Gamma_i|}\int_{\Gamma_i} q\, d\gamma + d(q,\boldsymbol{\lambda}_u) = 0,\\
< d\mathcal{L}_{\zeta_i}, \nu >= \Big(\displaystyle\int_{\Gamma_i} \boldsymbol{\lambda}_u\cdot\boldsymbol{n}\, d\gamma\Big)\nu = 0,
\end{cases}\\
(C_i) \quad < d\mathcal{L}_{Q_i}, \nu >= -\lambda_{\zeta_i}\nu = 0, \quad i=1,\dots,m
\end{cases}
$$

Conditions (C_i) are equivalent to set $\lambda_{\zeta_i}=0$, $i=1,\dots,m$, i.e. to force that the normal stress in the adjoint problem is zero on each artificial section (see [4]). This is in perfect analogy with the problems proposed in Section 2, for which we have that the fluxes of the adjoint problem were zero. Also in this case, in correspondence of the solution of Problem 9 the adjoint problem features the (unique) zero solution.

The previous approach can be extended to the non-linear case, in analogy with Section 2.

As for the other problems, also for Problem 9 we resort to an iterative numerical solution, where the optimality conditions (C_i), $\lambda_{\zeta_i}=0$, drive the convergence of the scheme.

4 Numerical algorithms

We present now some numerical procedures for the solution of the problems introduced in the previous sections. In particular, in Section 4.1 we investigate the flow rate problem, introducing firstly a general strategy applied, by way of example, to Problem 2 and then a special algorithm applicable for the imposition of just one flow rate condition. In Section 4.2 we illustrate numerical algorithms for solving Problem 5 as well, as an example of mean pressure problem. The extension to all the other problems is straightforward.

4.1 Flow rate problems

Let us consider Problem 2. For its numerical solution, we can resort to an iterative method such that at each iteration we solve separately problems (P) and (A) and check condition (C_j) until convergence.

Let us notice that the adjoint problem (A) in Problem 2 depends linearly on the values of the natural boundary condition on Γ_j. Therefore, if we denote by

$(\boldsymbol{\lambda}_{u,i},\ \boldsymbol{\lambda}_{p,i})$ the solutions of:

$$\begin{cases} a(\boldsymbol{v},\boldsymbol{\lambda}_{u,i}) + d(\lambda_{p,i},\boldsymbol{v}) - \int_{\Gamma_i} \boldsymbol{v}\cdot\boldsymbol{n}\,d\gamma = 0, & \forall \boldsymbol{v}\in\boldsymbol{V}, \\ d(q,\boldsymbol{\lambda}_{u,i}) = 0, & \forall q\in L^2(\Omega), \end{cases} \tag{44}$$

for all $i = 1,\ldots,m$, then the solution of (A) in Problem 2 can be written as

$$\boldsymbol{\lambda}_u = \sum_{i=1}^{m}\Big(\int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,d\gamma - Q_i\Big)\boldsymbol{\lambda}_{u,i}, \qquad \lambda_p = \sum_{i=1}^{m}\Big(\int_{\Gamma_i}\boldsymbol{u}\cdot\boldsymbol{n}\,d\gamma - Q_i\Big)\lambda_{p,i}.$$

Hence, the computational cost of the adjoint problem reduces effectively to the m problems (44), which can be precomputed.

The algorithm we propose to solve Problem 2 coincides with the *steepest descend method* (see [11]) applied to the minimization of the Lagrange functional (8), leading to the fixed point reinterpretation (16) used in proving Proposition 1. Let us introduce two inf-sup compatible subspaces $\boldsymbol{V}_h$ and Q_h of $\boldsymbol{V}$ and $L^2(\Omega)$, respectively (see [12]). Then, starting from some initial guesses $\boldsymbol{k}_h^1$, we use the residuals of equations (C_j) to update the values of $\boldsymbol{k}_h$. We have the following iterative procedure (we have denoted with l the subiteration index):

Algorithm 1

1. `Solve for` $j = 1,\ldots,m$ `problems (44) discretized in space, giving solutions` $\boldsymbol{\lambda}_{u,j,h}$ `and` $\lambda_{p,j,h}$`, for` $j = 1,\ldots,m$.

2. `Loop: given` $k_{j,h}^1$, $j = 1,\ldots,m$, `and` ε, `set` $l = 1$ `and do until convergence`

   - `Solve` $\forall \boldsymbol{v}_h\in\boldsymbol{V}_h$ `and` $\forall q_h\in Q_h$

$$\begin{cases} a(\boldsymbol{u}_h^l,\boldsymbol{v}_h) + d(p_h^l,\boldsymbol{v}_h) + \sum_{i=1}^{m}\int_{\Gamma_i} k_{i,h}^l \boldsymbol{v}_h\cdot\boldsymbol{n}\,d\gamma - (\boldsymbol{f},\boldsymbol{v}_h) = 0, \\ d(q_h,\boldsymbol{u}_h^l) = 0, \end{cases}$$

   - `Compute the adjoint solutions`

$$\begin{cases} \boldsymbol{\lambda}_{u,h}^l = \sum_{i=1}^{m}\Big(\int_{\Gamma_i}\boldsymbol{u}_h^l\cdot\boldsymbol{n}\,d\gamma - Q_i\Big)\boldsymbol{\lambda}_{u,i,h}^l, \\ \lambda_{p,h}^l = \sum_{i=1}^{m}\Big(\int_{\Gamma_i}\boldsymbol{u}_h^l\cdot\boldsymbol{n}\,d\gamma - Q_i\Big)\lambda_{p,i,h}^i. \end{cases}$$

   - `Convergence test: if` $\Big|\int_{\Gamma_j}\boldsymbol{\lambda}_{u,h}^l\cdot\boldsymbol{n}\,d\gamma\Big| < \varepsilon$, $\forall j = 1,\ldots,m$

     `then break`

     `else` $k_{j,h}^{l+1} = k_{j,h}^l + \tau^l\int_{\Gamma_j}\boldsymbol{\lambda}_{u,h}^l\cdot\boldsymbol{n}\,d\gamma, \quad \forall j = 1,\ldots,m,$

     `and set` $l = l+1$.

`end`

From the proof of Proposition 1 we know that coefficient τ is negative and, in principle, we know its optimal value which is $\tau = -1/(mC_T^4 C_1^4(C_1^2+1))$. Actually, for this value the coefficient of $\|\boldsymbol{k}_1 - \boldsymbol{k}_2\|_1$ in (17) vanish, so that the convergence of the algorithm is reached after one iteration. However, in practice this cannot be pursued due to the difficulties in estimating C_T and C_1. A possible, dynamics strategy to determine the τ is to resort to the following expression:

$$\tau^l = \tau_N^l = -\frac{J_Q(\boldsymbol{u}_h^l)}{\|\mathcal{L}_{\boldsymbol{k}}(\boldsymbol{s}_h^l)\|_2^2}, \tag{45}$$

where with $\boldsymbol{s}_h^l$ we mean the numerical solution at iterate l. This leads to the update rule:

$$k_{j,h}^{l+1} = k_{j,h}^l - \frac{\frac{1}{2}\sum_{j=1}^m \left(\int_{\Gamma_j} \boldsymbol{u}_h^l \cdot \boldsymbol{n}\, d\gamma - Q_j\right)^2 \int_{\Gamma_j} \boldsymbol{\lambda}_{u,h}^l \cdot \boldsymbol{n}\, d\gamma}{\sum_{j=1}^m |\int_{\Gamma_j} \boldsymbol{\lambda}_{u,h}^l \cdot \boldsymbol{n}\, d\gamma|^2},$$

that coincides with the *Newton method* applied to the equation $J_Q(\boldsymbol{k}) = 0$. Indeed, it can be shown that $< J_Q'(\boldsymbol{k}), \boldsymbol{\nu} > = < \mathcal{L}_{\boldsymbol{k}}(\boldsymbol{s}), \boldsymbol{\nu} >$, $\forall \boldsymbol{\nu} \in \mathbb{R}^m$. Therefore, since J_Q is a quadratic function, an improvement of the rate of convergence is done by posing $\tau^l = 2\tau_N^l$.

We point out that Algorithm 1 could be easily extended to the non-linear Problem 4.

4.1.1 The imposition of a single flow rate

In the particular case of only one prescribed flux, we can resort to a different algorithm, based on the variational formulation of (6) (or (19) for the non linear case). For instance, considering equation (19) together with $(6)_{2-5}$ and $(5)_2$ and with $m = 1$, setting $\boldsymbol{v} = \boldsymbol{u}$ we obtain:

$$a(\boldsymbol{u}, \boldsymbol{u}) + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{u}) + \int_\Gamma k\boldsymbol{u} \cdot \boldsymbol{n}\, d\gamma = (\boldsymbol{f}, \boldsymbol{u}),$$

and then, exploting the fact that k is constant, we can calculate directly

$$k = \frac{(\boldsymbol{f}, \boldsymbol{u}(k)) - \|\boldsymbol{u}(k)\|_a^2 - ((\boldsymbol{u}(k) \cdot \nabla)\boldsymbol{u}(k), \boldsymbol{u}(k))}{Q} = \phi(k). \tag{46}$$

A fixed point iterations may then be set up as follows:

Algorithm 2

`Given` k_h^1, $\boldsymbol{u}_h^0$ `and` ε, `set` $l = 1$ `and do until convergence`

1. `Solve` $\forall \boldsymbol{v}_h \in \boldsymbol{V}_h$, `and` $\forall q_h \in Q_h$

$$\begin{cases} a(\boldsymbol{u}_h^l, \boldsymbol{v}_h) + ((\boldsymbol{u}_h^{l-1} \cdot \nabla)\boldsymbol{u}_h^l, \boldsymbol{v}_h) + d(p_h^l, \boldsymbol{v}_h) + \int_\Gamma k_h^l \boldsymbol{v}_h \cdot \boldsymbol{n}\, d\gamma = (\boldsymbol{f}, \boldsymbol{v}_h), \\ d(q_h, \boldsymbol{u}_h^l) = 0. \end{cases}$$

2. `Compute` $k_h^{l+1} = ((\boldsymbol{f}, \boldsymbol{u}_h^l) - \|\boldsymbol{u}_h^l\|_a^2 - ((\boldsymbol{u}_h^{l-1} \cdot \nabla)\boldsymbol{u}_h^l, \boldsymbol{u}_h^l))/Q.$
3. `Convergence test: if` $|k_h^{l+1} - k_h^l| < \varepsilon$ `then exit`
4. `else` $l = l + 1$

`end`

Remark 3 *In the case of unsteady problem, Algorithm 2 fails if there exists a* $\widetilde{t}$ *such that* $Q(\widetilde{t})=0$*.*

In the particular case of a linear problem with $\boldsymbol{f} = \boldsymbol{0}$, we do not even need to resort to a fixed point iteration. Indeed, in this case the velocity $\boldsymbol{u}$ depends linearly on k. Therefore, denoting with $\widetilde{\boldsymbol{u}}$ the solution for $k = 1$, $\boldsymbol{u}$ is obtained by $\boldsymbol{u} = k\widetilde{\boldsymbol{u}}$. Therefore, from (46), we obtain $k = -k^2\|\widetilde{\boldsymbol{u}}\|_a^2/Q$, yielding the following explicit expression for k:

$$k = -\frac{Q}{\|\widetilde{\boldsymbol{u}}\|_a^2}.$$

4.2 Mean pressure problem

We can extend Algorithm 1 to the problems shown in Section 3. Let us notice that if the control variable is a set of scalars (i.e. for Problem 5 and 9), the adjoint problem could be still solved out of the iterative cicle. In particular, referring for instance to Problem 5, if $(\widetilde{\boldsymbol{\lambda}}_u, \widetilde{\lambda}_p)$ is the solution of

$$\begin{cases} a(\widetilde{\boldsymbol{\lambda}}_u, \boldsymbol{v}) + d(\widetilde{\lambda}_p, \boldsymbol{v}) = 0, & \forall \boldsymbol{v} \in \boldsymbol{V}_{div}, \\ d(q, \widetilde{\boldsymbol{\lambda}}_u) = \displaystyle\int_\Omega q \, d\omega, & \forall q \in H^1(\Omega), \end{cases} \tag{47}$$

we can build the solution of the adjoint problem (A) in the iterative loop by setting

$$\boldsymbol{\lambda}_u = \Big[\sum_{i=0}^m \Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p(\boldsymbol{k})\, d\gamma - P_i\Big)\Big]\widetilde{\boldsymbol{\lambda}}_u, \qquad \lambda_p = \Big[\sum_{i=0}^m \Big(\frac{1}{|\Gamma_i|}\int_{\Gamma_i} p(\boldsymbol{k})\, d\gamma - P_i\Big)\Big]\widetilde{\lambda}_p$$

Therefore, introducing two inf-sup compatible subspaces $\boldsymbol{V}_{div,h}$ and Q_h^1 of $\boldsymbol{V}_{div}$ and $H^1(\Omega)$, respectively, we obtain the following algorithm for solving Problem 5:

Algorithm 3

1. `Solve problem (47) discretized in space, giving the solution` $\widetilde{\boldsymbol{\lambda}}_{u,h}$ `and` $\widetilde{\lambda}_{p,h}$.
2. `Loop: given` $k_{j,h}^1$, $j = 0, \ldots, m$, `and` ε, `set` $l = 1$ `and do until convergence`

   - `Solve` $\forall \boldsymbol{v}_h \in \boldsymbol{V}_{div,h}$ `and` $\forall q_h \in Q_h^1$

$$\begin{cases} a(\boldsymbol{u}_h^l, \boldsymbol{v}_h) + d(p_h^l, \boldsymbol{v}_h) + \displaystyle\sum_{i=0}^m \int_{\Gamma_i} k_{i,h}^l \boldsymbol{v}_h \cdot \boldsymbol{n}\, d\gamma - (\boldsymbol{f}, \boldsymbol{v}_h) = 0, \\ d(q_h, \boldsymbol{u}_h^l) = 0. \end{cases}$$

- Compute the adjoint solutions

$$\begin{cases} \boldsymbol{\lambda}^l_{u,h} = \Big[\sum_{i=0}^{m} \Big(\frac{1}{|\Gamma_i|} \int_{\Gamma_i} p^l_h \, d\gamma - P_i \Big) \Big] \widetilde{\boldsymbol{\lambda}}_{u,h}, \\ \lambda^l_{p,h} = \Big[\sum_{i=0}^{m} \Big(\frac{1}{|\Gamma_i|} \int_{\Gamma_i} p^l_h \, d\gamma - P_i \Big) \Big] \widetilde{\lambda}_{p,h} \end{cases}$$

- Convergence test: if $\Big| \int_{\Gamma_j} \boldsymbol{\lambda}^l_{u,h} \cdot \boldsymbol{n} \, d\gamma \Big| < \varepsilon, \, \forall j = 0, \ldots, m$

  then exit

- else $k^{l+1}_{j,h} = k^l_{j,h} + \tau^l \int_{\Gamma_j} \boldsymbol{\lambda}^l_{u,h} \cdot \boldsymbol{n} \, d\gamma \quad \forall j = 0, \ldots, m$

  and set $l = l + 1$

3. end

Also in this case we can consider an expression analogous to (45) to determine τ dinamically.

For what concernes Problem 9, a similar algorithm can be derived. In this case both the adjoint problem out of the iterative cicle and the problem in the iterative cicle are solved with the augmented formulation, in particular resorting to the *GMRes+Schur complement* scheme introduced in [14].

5 Numerical results

In this Section we present a few numerical results meant to validate the algorithms introduced in Section 4. In particular, in Section 5.1 we focus on the flow rate problem, while in Section 5.2 on the mean pressure problem.

5.1 Flow rate problems

All simulations of this Section and of Section 5.2 have been implemented using the 2D finite element library *Freefem++* (see [7]), with a discretization time step $\Delta t = 0.01\,s$ and using conforming $\mathbb{P}_2 - \mathbb{P}_1$ elements. In the first simulations we present, the computational domain is a rectangle R of $6 \times 1\,cm$ (see Figure 3, left), while the fluid kinematic viscosity is $\mu = 0.035\,cm^2/s$. This value is typical from problems in computational haemodynamics. We imposed both a

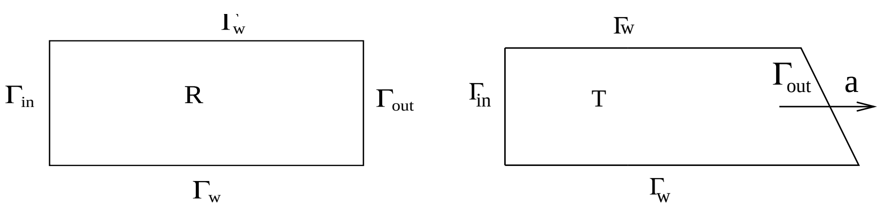

Figure 3: Computational domains R and T.

steady ($Q = 0.1\,cm^2/s$) and a pulsatile ($Q = 0.15 + 0.1\cos(2\pi t)\,cm^2/s$) flow rate at the inlet Γ_{in} of R. We have used an almost uniform triangular grid with spacing $h = 0.05\, cm$ and we have discretized the Stokes system.

In order to verify the accuracy of the solutions obtained with the two algorithms, we compare them with the analytical (Poiseuille) solution in the steady case, whereas in the unsteady case with the solution obtained using the *GM-Res+Schur complement* (GS) algorithm for the resolution of the same problem with the augmented formulation (see [14, 16], where it is extensively tested). In particular, Table 1 shows the quantities $D_I = \|\boldsymbol{u}_{GS} - \boldsymbol{u}_I\|_{L^2(\Gamma_{in})}/\|\boldsymbol{u}_{GS}\|_{L^2(\Gamma_{in})}$ and $D_{II} = \|\boldsymbol{u}_{GS} - \boldsymbol{u}_{II}\|_{L^2(\Gamma_{in})}/\|\boldsymbol{u}_{GS}\|_{L^2(\Gamma_{in})}$, where $\boldsymbol{u}_{GS}, \boldsymbol{u}_I$ and $\boldsymbol{u}_{II}$ are the numerical solutions obtained with the GS algorithm, with Algorithm 1 and with Algorithm 2, respectively. In Figure 4, the axial velocity at the inlet Γ_{in}, com-

	Time	D_I	D_{II}
steady simulation		0.0000003	0.0000050
unsteady simulation	$t = 1.25\, s$	0.000230	0.000160
	$t = 1.30\, s$	0.000154	0.000156
	$t = 1.60\, s$	0.000348	0.000348

Table 1: Relative differences of the numerical solutions obtained with Algorithm 1 and Algorithm 2, compared with the reference solution.

puted with the two algorithms in the steady (top, left) and in the unsteady case, is compared with the reference solution. We point out that the differences are hardly noticeable.

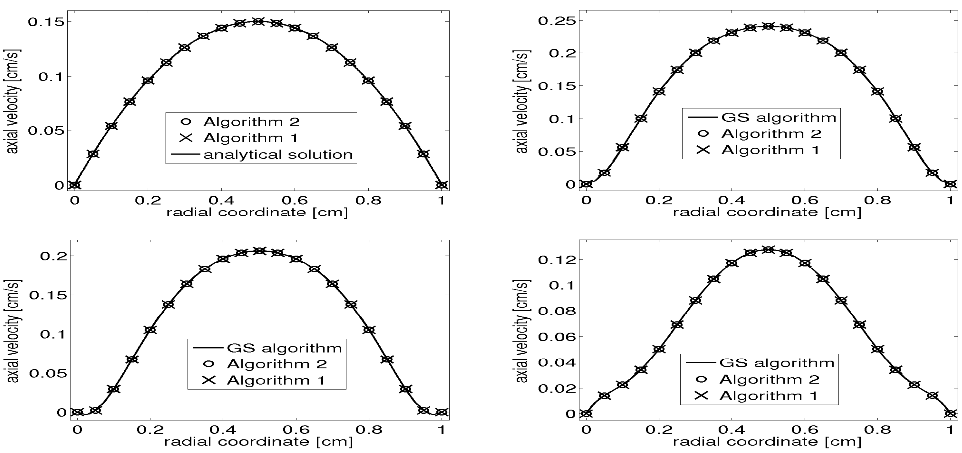

Figure 4: Axial velocity computed with Algorithm 1 and Algorithm 2 and reference solution - steady simulation (top, left) and unsteady simulation, $t = 1.25\, s$ (top, right), $t = 1.30\, s$ (bottom, left) and $t = 1.60\, s$ (bottom, right) - toll = 10^{-7}.

The number of iterations for the three algorithms are shown in Table 2. In the unsteady case, we refer to mean values. For Algorithm 1 we compare the performances obtained with different values of the relaxation parameter τ. An Aitken acceleration procedure has been implemented to speed up convergence, proving to be effective. For what concerns Algorithm 2 we have considered only the Aitken accelerated case. We observe that, since the forcing term is zero, in the steady case Algorithm 2 converge in just 1 iteration, as pointed out in Section 4.1.1. We observe that Algorithm 1 seems to converge faster than Algorithm 2. Therefore, it is faster also from the point of view of the computational time, since the differential problem to be solved at each iteration is the same in both algorithms (i.e. a generalized Stokes problem with Neumann boundary conditions at the artificial sections). Moreover, we recall that the GS algorithm converge exactly in $m + 1$ iterations per time step, where m is the number of prescribed flow rates (see [14]). Therefore, Algorithm 1 with the choice $\tau = 2\tau_N^l$ seems to run faster, since also in the GS algorithm the differential problem to be solved at each iteration is a generalized Stokes problem with Neumann boundary conditions.

	Alg. 1 $\tau = -1$	Alg. 1 $\tau = \tau_N^l$	Alg. 1 $\tau = \tau_N^l$+Aitken	Alg. 1 $\tau = 2\tau_N^l$	Alg.2 +Aitken	GS
Steady case	87	22	4	2	1	2
Unsteady case	-	5.08	3.81	1.98	4.50	2.00

Table 2: Number of iterations for the convergence of Algorithm 1, Algorithm 2 and GS algorithm (in average for the unsteady case).

In the second simulation, we consider the computational domain depicted in Figure 5 representing a 2D simplified model of a by-pass anastomosis. The space discretization step h is equal to $0.1\,cm$ and we have solved the Navier-Stokes equations with Algorithm 1 extended to the non-linear case, imposing the flow rates $Q_1 = 1 \cdot \cos(2\pi t)\,cm^2/s$ at the upper inlet and $Q_2 = 0.5 \cdot \cos(2\pi t)\,cm^2/s$ at the other inlet. In Figure 5, left, the axial velocity is shown, while on the right we show the difference with the solution obtained using the GS algorithm. We observe that the two numerical results are in excellent agreement. The mean number of iterations per time step required in this case by Algorithm 1 with the chioce $\tau = 2\tau_N^l$ is 19.92, against the $m+1 = 3$ needed by the GS algorithm. The slowness of convergence should be induced by the ill-conditioning of the problem in the 2 dimensional case (i.e. prescribing 2 flow rates). Indeed, in such a case it is well known that the convergence of the steepest descent method can be quite slow.

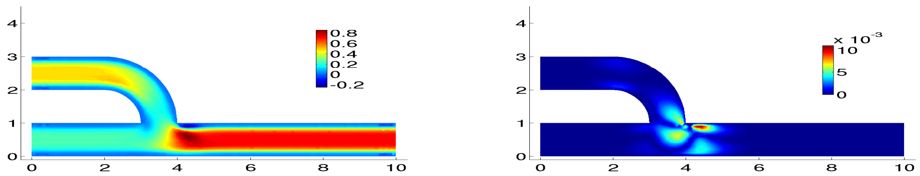

Figure 5: Anastomosis simulation - Algorithm 1 - $t = 1.2s$ - on the left the axial velocity, on the right the difference with the GS algorithm - toll = 10^{-9}.

5.2 Mean pressure problems

In this Section, we want to validate Algorithm 3 and its extensions to solve Problems 7, 8 and 9. In the first simulation, we impose a mean pressure $P = 1 \cdot \sin(2\pi t)\, g/(s^2\, cm)$ at the outlet Γ_{out} of the domain R (with $h = 0.1\, cm$) and we compare the performance of the strategy using the normal component of the normal stress as control variable (called in the sequel strategy PS, refer to Problem 5) with the one using the flow rates as control variables (strategy PF, refer to Problem 9). We consider the Stokes equations. Figure 6 and Table 3 show the good results obtained with both the strategies. We point out that strategy PF does not converge neither with the choice $\tau = \tau_N^l$ nor with $\tau = 2\tau_N^l$ and therefore we consider a static parameter τ in this case. The number of iterations (in average) per time step required by the two techniques is 2.00 for PS strategy with $\tau = 2\tau_N^l$, and 31.31 for PF strategy with $\tau = 10^{-6}$. In addiction we point out that for each iteration strategy PF needs to solve $m + 1 = 2$ generalized Stokes problems against just 1 required by strategy PS. Therefore, we conclude that strategy PS seems to be definitely the best one.

Time	PF	PS
$t = 1.10\, s$	$9.327 \cdot 10^{-5}$	$9.329 \cdot 10^{-5}$
$t = 1.40\, s$	$3.569 \cdot 10^{-5}$	$3.578 \cdot 10^{-5}$

Table 3: Relative errors in the $L^2(\Gamma)$ norm of the solutions obtained using the flow rate (PF) and the normal stress (PS), respectively, as control variable.

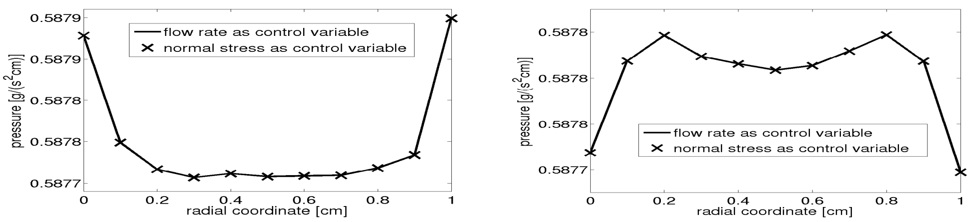

Figure 6: Mean pressure prescribed at the outlet at t=1.1 s, $P = 0.5878\, g/(s^2\, cm)$ (left) and at t=1.4 s, $P = 0.5878\, g/(s^2\, cm)$ (right) - toll $= 10^{-7}$.

In the second set of simulations, we want to precribe a mean pressure $P = 1\, g/(s^2\, cm)$ at the outlet Γ_{out} of the domain T (see Figure 3, right). We indicate with $\boldsymbol{a}$ the axial direction. By minimizing functional (21), an undesiderable radial velocity and a non correct axial velocity at the outlet occur (Figures 7, top). To avoid these effects, we solve Problems 7 and 8, i.e. we minimize functional (40) with (41) and (42), respectively. We consider the same iterative scheme of Algorithm 3 (recall that in both cases we have to solve the adjoint problem in the iterative cicle, since the control variables is a set of vectors). Figures 7, middle and bottom, show that these strategies are able to reduce these effects of several order of magnitude.

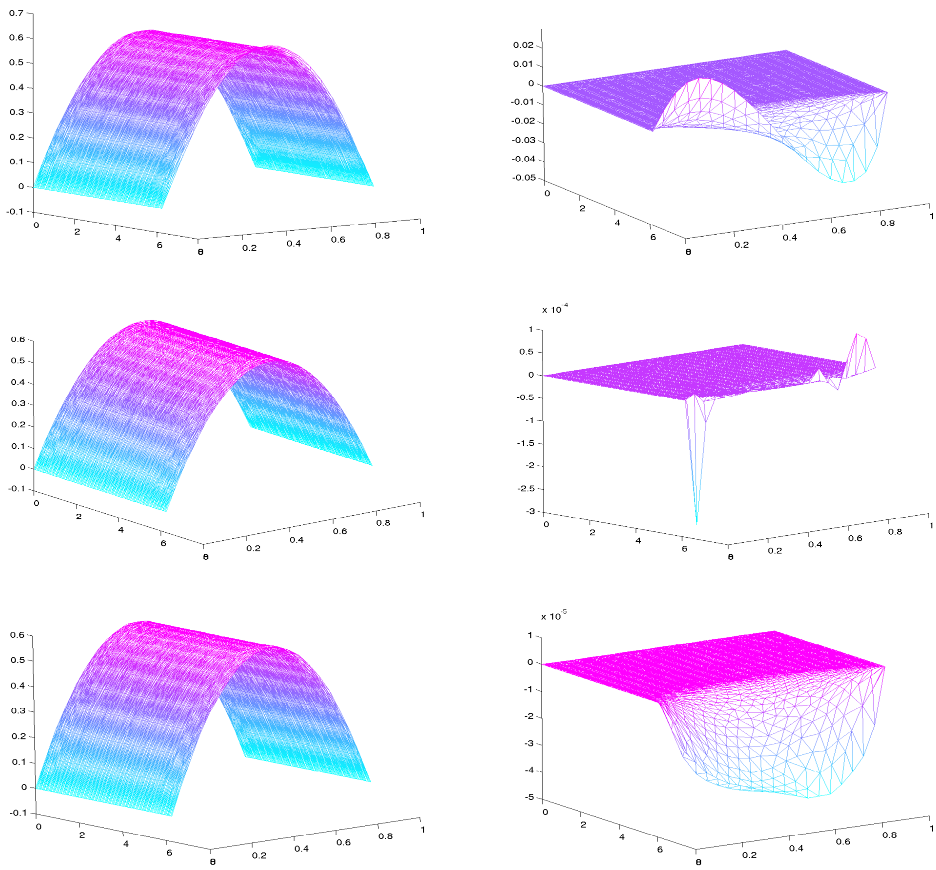

Figure 7: Axial (left) and tangential (right) velocity in cm^2/s obtained prescribing $P = 1\, g/(s^2\, cm)$ at the outlet of T minimizing (21) (top), (40) with (41) (middle) and (40) with (42) (bottom) - toll $= 10^{-7}$.

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