OPTIMAL FLOW CONTROL FOR NAVIER–STOKES EQUATIONS: DRAG MINIMIZATION

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Abstract

Optimal control and shape optimization techniques have an increasing role in Fluid Dynamics problems governed by Partial Differential Equations (PDEs). In this paper we consider the problem of drag minimization for a body in relative motion in a fluid by controlling the velocity through the body boundary. With this aim we handle with an optimal control approach applied to the steady incompressible Navier–Stokes equations. We use the Lagrangian functional approach and we adopt the Lagrangian multiplier method for the treatment of the Dirichlet boundary conditions, which include the control function itself. Moreover we express the drag coefficient, which is the functional to be minimized, through the variational form of the Navier–Stokes equations. In this way we can derive, in a straightforward manner, the adjoint and sensitivity equations associated with the optimal control problem, even in presence of Dirichlet control functions. The problem is solved numerically by an iterative optimization procedure applied to state and adjoint PDEs which we approximate by the finite element method.

Keywords: optimal control problems; Navier–Stokes equations; drag minimization; Dirichlet boundary control; Lagrangian multiplier method; finite element approximation.

Introduction

A popular problem in Fluid Dynamics consists in minimizing the drag coefficient of a body in relative motion with a fluid [4, 9, 10, 12, 16, 19, 23]; in particular, one case commonly studied is that of the steady incompressible Navier–Stokes equations with constant density and viscosity. This problem can be recasted in the theory of optimal control for Partial Differential Equations [1, 21], or as a problem of shape optimization [12, 23]. In this paper we minimize the drag coefficient of the body by acting on

the velocity at the boundaries of the body itself. This problem, which corresponds to regulate the aspiration or the blowing of the boundary layer, can be formulated as an optimal control problem, for which the control function is the Dirichlet boundary condition [3, 9, 10, 15, 16, 18]. With this aim we adopt the Lagrangian functional approach [4, 5, 6, 8] instead of calculating directly the gradient of the functional subject to minimization [9, 10, 15, 16]. In this contest the main difficulty consists in the treatment of the Dirichlet boundary control function, which does not match with the variational setting provided by the Lagrangian functional approach, unless that suitable lifting terms are introduced.
Different approaches have been considered in literature to overcome this difficulty in the field of Navier–Stokes equations. In [3] the Nitsche's method ([26]) is adopted in order to suppress the recirculation in a backward facing step in a channel by regulating the inflow velocity field. In [17, 18] both linear and non–linear penalized Neumann control approaches for the solution of optimal control problems with Dirichlet control functions are adopted. Moreover in [18] a Lagrangian multiplier method ([2]) is proposed for the treatment of the Dirichlet control function (this strategy is also considered in [22] for optimal control problems governed by elliptic PDEs). The approaches presented in [17, 18] are applied for the minimization of the vorticity and the "distance" between the velocity field and a desired one; sequential quadratic programming (SQP) methods (see e.g. [13]) have been adopted for the solution of these optimal control problems.
In this paper we adopt, for our drag minimization problem, the Lagrangian multiplier method for the treatment of the Dirichlet velocity control function on the body, in analogy with the approaches outlined in [18] and in [22] (for elliptic PDEs). However for the evaluation of the drag, or more in general the force acting on the body, we exploit the variational form of the Navier–Stokes equations [11], instead of using the definition, the latter being directly related to the integral of the stress acting on the body boundaries. The two approaches, that are equivalent at the "continuous level", are in fact different at the "discrete level". The one that we follow in this paper yields more accurate computation of the drag coefficient [11] and, in the contest of optimization, reduces the propagation of the discretization errors in the course of the iterative optimization procedure. Moreover, this approach allows us to identify the Lagrangian multiplier as the inward directed normal stress, to express the drag coefficient in terms of this multiplier and, consequently, to obtain a simple expression for the adjoint equations.
For the numerical solution of the optimal control problem we adopt the steepest–descent method [1], while the PDE system is discretized by means of the Finite Element (FE) method. Finally, we report some numerical results concerning drag minimization (for low Reynolds number), which prove the effectiveness of the outlined procedures.

An outline of this work is as follows. In Sec.1, we recall the Lagrangian functional approach for optimal control problems in an abstract setting. Then we consider the drag coefficient minimization problem for the incompressible Navier–Stokes equations, introducing the evaluation of the drag coefficient by means of the variational form. Finally, we write the Lagrangian functional, adopting the Lagrangian multiplier method for the treatment of the Dirichlet boundary conditions, and we formulate the adjoint Navier–Stokes equations and the sensitivity equation. In Sec.2 we discuss the aspects related to the numerical resolution of the optimal control problem: in particular we recall the steepest–descent method and the techniques adopted for the numerical solution of the Stokes and Navier–Stokes equations. Finally, in Sec.3 we present some numerical results and compare them for different values of the Reynolds number.

1 Mathematical Model: the Optimal Control Problem

In this section we consider the optimal control problem in an abstract setting; then we treat a flow control problem governed by the Navier–Stokes equations for the minimization of the drag acting on a solid object.

1.1 The general setting for optimal control

Let us start with an abstract control problem, which, for the sake of simplicity, we consider depending on scalar variables; however this formalism can be extended in straightforward manner to vectorial problems, like the Navier–Stokes equations which we will consider in Sec.1.3.

The optimal control problem reads:

$$\text{find } \ u \in \mathcal{U}, \ \ u = \ \text{argmin } J(v,u), \ \ \text{with } \ A(v) = f + B(u) \ \text{ and } \ v \in \mathcal{V}, \tag{1}$$

where $J(v,u)$ is the *cost functional*, A a differential operator on $\mathcal{V}$ with values in $\mathcal{V}'$, B a differential operator on $\mathcal{U}$ into $\mathcal{V}'$, f a source term, $\mathcal{V}$ and $\mathcal{U}$ two Hilbert spaces. The equation $A(v) = f + B(u)$, which incorporates appropriate boundary conditions, is called the *state equation*, v the state variable and u the *control* variable. Let us notice that, in general, $A(\cdot)$ and $B(\cdot)$ are non linear differential operators in v and u respectively.

For the analysis of the optimal control problem (1) we adopt the *Lagrangian functional* framework ([4, 5, 6, 8]). This approach sets the control problem as a constrained minimization problem, for which a Lagrangian functional is defined; the minimum, if exists, is a stationary "point" of the Lagrangian functional. Conversely, the classical approach developed by J.L. Lions [21] allows a full analysis of the problem, providing existence and uniqueness results; incidentally, this technique is not always straightforward as the Lagrangian functional one, which, however, does not always provide a suitable analysis of the optimal control problem.

We define the Lagrangian functional as ([5]):

$$\mathcal{L}(v,z,u) := J(v,u) +_{\mathcal{V}} \langle z, f + B(u) - A(v) \rangle_{\mathcal{V}'}, \tag{2}$$

where $z \in \mathcal{V}$ is the Lagrangian multiplier. The optimum of the control problem $\mathbf{x}^* := (v^*, z^*, u^*) \in \mathcal{X}$ ($\mathcal{X} := \mathcal{V} \times \mathcal{V} \times \mathcal{U}$), if it exists, is a stationary point of the Lagrangian functional $\mathcal{L}(\mathbf{x})$, with $\mathbf{x} := (v,z,u) \in \mathcal{X}$, i.e.:

$$\nabla \mathcal{L}(\mathbf{x}^*)[\mathbf{y}] = \mathbf{0}, \qquad \mathbf{y} \in \mathcal{X}, \tag{3}$$

where differentiation is in Fréchet sense [20]. The system (3) is regarded as the Euler–Lagrange system, for which we have the following three equations:

$$\begin{cases} \mathcal{L}_{,v}\,(v,z,u)[\vartheta] = 0, & \text{adjoint equation,} \\ \mathcal{L}_{,z}\,(v,z,u)[\phi] = 0, & \text{state equation,} \\ \mathcal{L}_{,u}\,(v,z,u)[\psi] = 0, & \text{"sensitivity" equation,} \end{cases} \tag{4}$$

where we have assumed $\mathbf{y} = (\vartheta, \phi, \psi)$. The first equation of the system (4) is regarded as the *adjoint equation*, for which $z \in \mathcal{V}$ is defined as the adjoint variable. The third equation is related to the sensitivity of the Lagrangian functional $\mathcal{L}(v,z,u)$ w.r.t. the control variable $u \in \mathcal{U}$.

Let us apply the general expression for the Lagrangian functional (3) to the abstract control problem stated in Eq.(1); to this aim we consider the weak form of the state equation, given in Eq.(1) in distributional sense, which reads:

$$\text{find } v \in \mathcal{V} \quad : \quad a(v)(\phi) = (f, \phi) + b(u)(\phi), \quad \forall \phi \in \mathcal{V}, \tag{5}$$

where $a(\cdot)(\cdot)$ and $b(\cdot)(\cdot)$ are the semi–linear forms (linear in the second argument) associated respectively to the operators $A(\cdot)$ and $B(\cdot)$. The Lagrangian functional (2) reads:

$$\mathcal{L}(v, z, u) = J(v, u) + (f, z) + b(u)(z) - a(v)(z). \tag{6}$$

By differentiating $\mathcal{L}(v, p, u)$ w.r.t. $v \in \mathcal{V}$, we obtain the adjoint equation in weak form:

$$\text{find } z \in \mathcal{V} \quad : \quad a'(v)(z, \vartheta) = J,_v (v, u)(\vartheta), \quad \forall \vartheta \in \mathcal{V}, \tag{7}$$

where, for the sake of simplicity, we indicate $a,_v (v)(z, \vartheta)$ as $a'(v)(z, \vartheta)$; let us notice that the adjoint equation is linear in $z \in \mathcal{V}$ even if the state equation is non–linear in the state variable. Similarly, by differentiating w.r.t. $u \in \mathcal{U}$ the third equation of the Euler–Lagrange system (4), we obtain:

$$J,_u (v, u)(\psi) + b'(u)(z, \psi) = 0, \quad \psi \in \mathcal{U}, \tag{8}$$

where we indicate $b,_u (u)(z, \psi)$ with $b'(u)(z, \psi)$. From the previous equation it is possible to extract the sensitivity function $\delta u \in \mathcal{U}$, which we will define in Sec.1.3 for the problem under investigation.

1.2 Drag minimization for Navier–Stokes equations

We consider a body embedded in a $2D$ flow governed by the steady incompressible Navier–Stokes equations for a Newtonian fluid with constant density and viscosity.

The goal consists in minimizing the drag coefficient by regulating the flow $\mathbf{u}$ across the boundary Γ_{CTRL} of the body in relative steady motion with the fluid; see Fig.s 1 and 2. By defining $\mathbf{v}$ as the velocity field, p as the pressure, μ as dynamic viscosity coefficient, ρ as the density, the Navier–Stokes system reads [14, 26]:

$$\begin{cases} -\nabla \cdot \mathbb{T}(\mathbf{v}, p) + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{0}, & \text{in } \Omega, \\ \nabla \cdot \mathbf{v} = 0, & \text{in } \Omega, \\ \mathbf{v} = \mathbf{v}_\infty, & \text{on } \Gamma_{IN}, \\ \mathbf{v} \cdot \hat{\mathbf{n}} = 0, \quad (\mathbb{T}(\mathbf{v}, p)\hat{\mathbf{n}}) \cdot \hat{\mathbf{t}} = 0, & \text{on } \Gamma_{SYM}, \\ \mathbb{T}(\mathbf{v}, p)\hat{\mathbf{n}} = \mathbf{0}, & \text{on } \Gamma_{OUT}, \\ \mathbf{v} = \mathbf{0}, & \text{on } \Gamma_{NS}, \\ \mathbf{v} = \mathbf{u}, & \text{on } \Gamma_{CTRL}, \end{cases} \tag{9}$$

where $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ are respectively the outward directed normal and tangential unit vectors on the boundary Γ_i; the *stress tensor* $\mathbb{T}(\mathbf{v}, p)$ reads:

$$\mathbb{T}(\mathbf{v}, p) = \mu(\nabla \mathbf{v} + \nabla^T \mathbf{v}) - p\mathbb{I}, \tag{10}$$

being $\mathbb{I}$ the identity tensor. We impose inflow boundary conditions on Γ_{IN}, symmetry conditions on Γ_{SYM}, no stress conditions on Γ_{OUT}, no slip conditions on Γ_{NS} and Dirichlet conditions ($\mathbf{u}$ is the control variable) on the control boundary Γ_{CTRL}, which corresponds to impose the velocity on Γ_{CTRL}. Let us notice that $\cup_i \Gamma_i = \partial\Omega$ and $\Gamma_i \cap \Gamma_j = \emptyset$, $\forall i, j$.

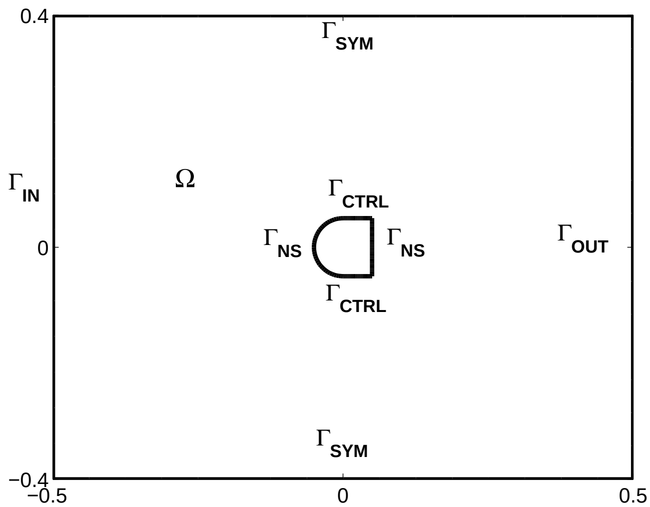

Figure 1: Domain for the control problem and boundary conditions.

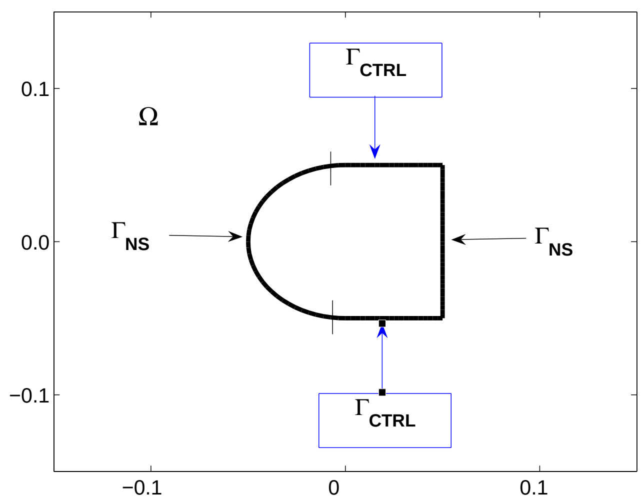

Figure 2: Boundary conditions on the body.

The functional to minimize is the drag coefficient $c_D(\mathbf{v}, p)$, for which the problem that we consider is:

$$\text{find } \mathbf{u} = \ \operatorname{argmin} c_D(\mathbf{v}, p), \quad \text{with} \ \ c_D(\mathbf{v}, p) := -\frac{1}{q_\infty d} \oint_{\Gamma_{BODY}} (\mathbb{T}(\mathbf{v}, p)\hat{\mathbf{n}}) \cdot \hat{\mathbf{v}}_\infty \, d\Gamma, \tag{11}$$

and $(\mathbf{v}, p)$ depends on $\mathbf{u}$ through the state equations (9). The minus sign takes into account that, by convention, the force is positive if acting on the fluid. In Eq.(11) $\Gamma_{BODY} := \Gamma_{NS} \cup \Gamma_{CTRL}$, $q_\infty := \frac{1}{2}\rho V_\infty^2$, with $\mathbf{v}_\infty = V_\infty \hat{\mathbf{v}}_\infty$, $\hat{\mathbf{v}}_\infty$ is the unit vector

directed as the incoming flow and d is the characteristic dimension of the body. Let us notice that the drag $q_\infty d c_D$ assumes the dimensions of a force per unit length.
We assume a parabolic profile for the control flow, written in the form:

$$\mathbf{u} = U\mathbf{g}(\mathrm{x}), \quad \text{with} \quad \mathbf{g}(\mathrm{x}) := -4\frac{(\mathrm{x}-\mathrm{x}1_{CTRL})(\mathrm{x}2_{CTRL}-\mathrm{x})}{(\mathrm{x}2_{CTRL}-\mathrm{x}1_{CTRL})^2}\hat{\mathbf{n}}_{CTRL}, \tag{12}$$

where $\mathrm{x}1_{CTRL}$ and $\mathrm{x}2_{CTRL}$ are the abscissae of the endpoints of the boundary Γ_{CTRL}, while $\hat{\mathbf{n}}_{CTRL}$ is the outward directed unit vector normal to Γ_{CTRL}. The effective control variable is the parameter $U \in \mathbb{R}$, which is the maximum value (in modulus) of the parabolic flow of Eq.(12).

Weak formulation

For the analysis of the optimal control problem we introduce, similarly to what done in Sec.1.1 for the general case, the weak form of the Navier–Stokes equations. This can be done by introducing suitable functional spaces for $\mathbf{v}$ and p [14, 26], which account properly for Dirichlet boundary conditions; an alternative approach consists in introducing the boundary conditions by means of a Lagrange multiplier approach [2, 18, 22]. This last approach, which we will follow, allows a straightforward treatment of the boundary conditions in the analysis of the optimal control problem, without introducing lifting terms.

Let us introduce the following forms:

$$a(\mathbf{v},\mathbf{\Phi}) := \int_\Omega \mu(\nabla\mathbf{v}+\nabla^T\mathbf{v})\cdot\nabla\mathbf{\Phi}\; d\Omega, \tag{13}$$

$$b(p,\mathbf{\Phi}) := -\int_\Omega p\nabla\cdot\mathbf{\Phi}\; d\Omega, \tag{14}$$

$$c(\mathbf{v},\mathbf{v},\mathbf{\Phi}) := \int_\Omega \rho(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{\Phi}\; d\Omega. \tag{15}$$

The Navier–Stokes equations in weak form, with boundary conditions introduced through a Lagrange multiplier, read:

$$\begin{gathered}
\text{find} \quad \mathbf{v}\in[H^1(\Omega)]^2, \quad p\in L^2(\Omega), \quad \mathbf{w}\in[H^{-\frac{1}{2}}(\Gamma_{DS})]^2 \quad : \\
a(\mathbf{v},\mathbf{\Phi})+b(p,\mathbf{\Phi})+c(\mathbf{v},\mathbf{v},\mathbf{\Phi})+\int_{\Gamma_D}\mathbf{w}\cdot\mathbf{\Phi}\; d\Gamma+\int_{\Gamma_{SYM}}\mathbf{w}\cdot\hat{\mathbf{n}}\;\mathbf{\Phi}\cdot\hat{\mathbf{n}}\; d\Gamma=0, \\
b(\varphi,\mathbf{v})=0, \\
\int_{\Gamma_D}(\mathbf{v}-\mathbf{v}_D)\cdot\mathbf{\Lambda}\; d\Gamma+\int_{\Gamma_{SYM}}\mathbf{v}\cdot\hat{\mathbf{n}}\;\mathbf{\Lambda}\cdot\hat{\mathbf{n}}\; d\Gamma=0, \\
\forall\mathbf{\Phi}\in[H^1(\Omega)]^2, \quad \forall\varphi\in L^2(\Omega), \quad \forall\mathbf{\Lambda}\in[H^{-\frac{1}{2}}(\Gamma_{DS})]^2,
\end{gathered} \tag{16}$$

where $\Gamma_D := \Gamma_{IN}\cup\Gamma_{NS}\cup\Gamma_{CTRL}$, $\Gamma_{DS} := \Gamma_D\cup\Gamma_{SYM}$, $[H^1(\Omega)]^2$ and $[H^{-\frac{1}{2}}(\Gamma_{DS})]^2$ are the usual Sobolev spaces, while $\mathbf{v}_D$, which we consider in the Sobolev space $[H^{\frac{1}{2}}(\Gamma_D)]^2$, reads (Eq.(9)):

$$\mathbf{v}_D := \begin{cases} \mathbf{v}_\infty, & \text{on } \Gamma_{IN}, \\ \mathbf{0}, & \text{on } \Gamma_{NS}, \\ \mathbf{u}, & \text{on } \Gamma_{CTRL}. \end{cases} \tag{17}$$

In particular, the control variable $\mathbf{u}\in[H^{\frac{1}{2}}(\Gamma_{CTRL})]^2$. The Lagrange multiplier $\mathbf{w}\in[H^{-\frac{1}{2}}(\Gamma_{DS})]^2$ is introduced to allow the variable $\mathbf{v}\in[H^1(\Omega)]^2$ to fit the Dirichlet boundary conditions, which includes the control variable $\mathbf{u}$. Let us observe that we assume

$\hat{\mathbf{n}}$ (i.e. Γ_{SYM}) to be "sufficiently regular" so that in Eq.(16) $\mathbf{w} \in [H^{-\frac{1}{2}}(\Gamma_{SYM})]^2$ implies $\mathbf{w} \cdot \hat{\mathbf{n}} \in H^{-\frac{1}{2}}(\Gamma_{SYM})$, $\mathbf{v} \in [H^{\frac{1}{2}}(\Gamma_{SYM})]^2$ implies $\mathbf{v} \cdot \hat{\mathbf{n}} \in H^{\frac{1}{2}}(\Gamma_{SYM})$, and so on. This is the case of the problem under investigation (see Fig.1), for which Γ_{SYM} is composed by two distinct segments. Moreover the boundary integrals of Eq.(16) can be seen as duality pairs: ${}_{H^{-\frac{1}{2}}(\Gamma_i)}\langle \cdot, \cdot \rangle_{H^{\frac{1}{2}}(\Gamma_i)}$ or ${}_{H^{\frac{1}{2}}(\Gamma_i)}\langle \cdot, \cdot \rangle_{H^{-\frac{1}{2}}(\Gamma_i)}$. Finally, in view of the finite element approximation, problem (16) will be reformulated in weak form [26]; the solution $\tilde{\mathbf{v}}$ is sought for in $[H^1_{|\Gamma_{DS}}(\Omega)]^2$, the test function $\mathbf{\Phi} \in [H^1_{|\Gamma_{DS}}(\Omega)]^2$, where $[H^1_{|\Gamma_{DS}}(\Omega)]^2 := \{\mathbf{s} \in [H^1(\Omega)]^2 : \ \mathbf{s}_{|\Gamma_D} = \mathbf{0} \text{ and } (\mathbf{s} \cdot \hat{\mathbf{n}})_{|\Gamma_{SYM}} = 0\}$, by modifying the source term in order to account for the non–homogeneous Dirichlet data. Thereby the direct evaluation of the variable $\mathbf{w} \in [H^{-\frac{1}{2}}(\Gamma_{DS})]^2$ is not necessary.

Drag evaluation

The drag coefficient c_D, defined in Eq.(11), can be evaluated once the Navier–Stokes equations are resolved and the velocity field $\mathbf{v}$ and the pressure p are known. However, the computation of c_D by means of the definition (11) can lead to inaccurate results even if the computational grid is quite fine, as we better specify later. Better results can be achieved by adopting an alternative expression for c_D, which we indicate with $\tilde{c}_D$, dependent on the variational forms used for the Navier–Stokes equations, as reported in [6, 11, 16].

In order to provide the expression of $\tilde{c}_D$, we express c_D (11) in the following form:

$$c_D(\mathbf{v}, p) = \frac{1}{q_\infty d} \oint_{\partial\Omega} (\mathbb{T}(\mathbf{v}, p)\hat{\mathbf{n}}) \cdot \mathbf{\Phi}_\infty \, d\Gamma, \tag{18}$$

where:

$$\mathbf{\Phi}_\infty \in [H^1(\Omega)]^2 \quad \text{with} \quad \mathbf{\Phi}_{\infty|\Gamma_{BODY}} = -\hat{\mathbf{v}}_\infty, \quad \mathbf{\Phi}_{\infty|\partial\Omega\setminus\Gamma_{BODY}} = \mathbf{0}. \tag{19}$$

By adopting the Gauss theorem and the Green's identity ([11, 14]), Eq.(18) becomes:

$$\begin{aligned} c_D(\mathbf{v}, p) &= \frac{1}{q_\infty d} \int_\Omega \nabla \cdot (\mathbb{T}(\mathbf{v}, p)^T \mathbf{\Phi}_\infty) \, d\Omega \\ &= \frac{1}{q_\infty d} \int_\Omega \left(\nabla \cdot \mathbb{T}(\mathbf{v}, p) \cdot \mathbf{\Phi}_\infty + \mathbb{T}(\mathbf{v}, p) \cdot \nabla \mathbf{\Phi}_\infty\right) \, d\Omega. \end{aligned} \tag{20}$$

Owing to Eq.s (9), (13), (14) and (15), we have the following representations:

$$\int_\Omega \nabla \cdot \mathbb{T}(\mathbf{v}, p) \cdot \mathbf{\Phi}_\infty \, d\Omega = c(\mathbf{v}, \mathbf{v}, \mathbf{\Phi}_\infty), \tag{21}$$

$$\int_\Omega \mathbb{T}(\mathbf{v}, p) \cdot \nabla \mathbf{\Phi}_\infty \, d\Omega = a(\mathbf{v}, \mathbf{\Phi}_\infty) + b(p, \mathbf{\Phi}_\infty), \tag{22}$$

for which, being $b(\varphi, \mathbf{v}) = 0$, $\forall \varphi \in L^2(\Omega)$, $\tilde{c}_D$ reads:

$$\tilde{c}_D(\mathbf{v}, p) = \frac{1}{q_\infty d} \mathcal{A}(\mathbf{v}, p, \mathbf{\Phi}_\infty, \varphi), \qquad \forall \varphi \in L^2(\Omega), \tag{23}$$

where:

$$\mathcal{A}(\mathbf{v}, p, \mathbf{\Phi}, \varphi) := a(\mathbf{v}, \mathbf{\Phi}) + b(p, \mathbf{\Phi}) + b(\varphi, \mathbf{v}) + c(\mathbf{v}, \mathbf{v}, \mathbf{\Phi}). \tag{24}$$

The form for the drag coefficient provided in Eq.(23) allows accurate computations and will be adopted in this work. Let us notice that c_D and $\tilde{c}_D$ are equivalent at the continuous level, however the representations (20) and (23) lead to different approximations at the discrete level. For instance, by adopting the FE method with continuous, piecewise polynomials of degree k for $\mathbf{v}$ and $k-1$ for p (Taylor–Hood elements, see [7, 26, 27]), the order of convergence of $\tilde{c}_{Dh}$ to the exact value c_D is $2k$, while for c_{Dh} is only k [11], being c_{Dh} and $\tilde{c}_{Dh}$ the computed quatities corresponding to c_D and $\tilde{c}_D$. This fact assumes great importance in the optimization contest, where the adjoint variables depend on the sensitivity of the cost functional (in this case c_D) with respect to the state variables. The choice of the representation, as verified by numerical tests, not only influences the computation of the adjoint variables, but it also affects the accuracy of the result obtained by the optimization procedure.
Let us observe that, according to Eq.(16), the variational form $\mathcal{A}(\mathbf{v}, p, \mathbf{\Phi}, \varphi)$ can be expressed as:

$$\mathcal{A}(\mathbf{v}, p, \mathbf{\Phi}, \varphi) = -\int_{\Gamma_D} \mathbf{w} \cdot \mathbf{\Phi} \, d\Gamma - \int_{\Gamma_{SYM}} \mathbf{w} \cdot \hat{\mathbf{n}} \, \mathbf{\Phi} \cdot \hat{\mathbf{n}} \, d\Gamma, \tag{25}$$

for which, upon the definition of Φ_∞ given in Eq.(19), the drag coefficient $\tilde{c}_D$ assumes the alternative expression:

$$\hat{c}_D(\mathbf{w}) = \frac{1}{q_\infty d} \int_{\Gamma_{BODY}} \mathbf{w} \cdot \hat{\mathbf{v}}_\infty \, d\Gamma. \tag{26}$$

Let us notice that Eq.(26) allows us to identify the Lagrange multiplier $\mathbf{w} \in [H^{-\frac{1}{2}}(\Gamma_{DS})]^2$ as the inward normal stress, more precisely:

$$\mathbf{w} = -\mathbb{T}(\mathbf{v}, p)\hat{\mathbf{n}}, \qquad \text{on} \quad \Gamma_{DS}. \tag{27}$$

This last result allows us to derive in a straightforward manner the adjoint equations associated to the control problem, as we will show in the following Section.

1.3 The optimal control system for the Navier–Stokes equations

We apply the optimal control analysis based on the Lagrangian functional given in Sec.1.1 to the drag minimization problem defined in Eq.(11).

According to Eq.s (16), (24) and (26), the associated Lagrangian functional is defined as:

$$\begin{aligned} \mathcal{L}(\mathbf{v}, p, \mathbf{w}, \mathbf{z}, q, \mathbf{r}, \mathbf{u}) \quad := \quad & \hat{c}_D(\mathbf{w}) - \mathcal{A}(\mathbf{v}, p, \mathbf{z}, q) \\ & - \int_{\Gamma_D} \mathbf{w} \cdot \mathbf{z} \, d\Gamma - \int_{\Gamma_{SYM}} \mathbf{w} \cdot \hat{\mathbf{n}} \, \mathbf{z} \cdot \hat{\mathbf{n}} \, d\Gamma \\ & - \int_{\Gamma_D \setminus \Gamma_{CTRL}} (\mathbf{v} - \mathbf{v}_D) \cdot \mathbf{r} \, d\Gamma - \int_{\Gamma_{CTRL}} (\mathbf{v} - \mathbf{u}) \cdot \mathbf{r} \, d\Gamma \\ & - \int_{\Gamma_{SYM}} \mathbf{v} \cdot \hat{\mathbf{n}} \, \mathbf{r} \cdot \hat{\mathbf{n}} \, d\Gamma. \end{aligned} \tag{28}$$

By differentiating $\mathcal{L}(\cdot)$ with respect to $(\mathbf{z}, q, \mathbf{r})$ we obtain the state equation in weak form reported in Eq.(16). Similarly, by differentiating $\mathcal{L}(\cdot)$ with respect to the state variables

$(\mathbf{v}, p, \mathbf{w})$, we obtain the adjoint equations. Their weak form reads:

$$
\begin{gathered}
\text{find} \quad \mathbf{z} \in [H^1(\Omega)]^2, \quad q \in L^2(\Omega) \quad , \mathbf{r} \in [H^{-\frac{1}{2}}(\Gamma_{DS})]^2 \quad : \\
a(\boldsymbol{\Theta}, \mathbf{z}) + b(q, \boldsymbol{\Theta}) + c'(\mathbf{v}, \boldsymbol{\Theta}, \mathbf{z}) + \int_{\Gamma_D} \boldsymbol{\Theta} \cdot \mathbf{r} \; d\Gamma + \int_{\Gamma_{SYM}} \boldsymbol{\Theta} \cdot \hat{\mathbf{n}} \; \mathbf{r} \cdot \hat{\mathbf{n}} \; d\Gamma = 0, \\
b(\vartheta, \mathbf{z}) = 0, \\
\int_{\Gamma_D} \boldsymbol{\Upsilon} \cdot \mathbf{z} \; d\Gamma + \int_{\Gamma_{SYM}} \boldsymbol{\Upsilon} \cdot \hat{\mathbf{n}} \; \mathbf{z} \cdot \hat{\mathbf{n}} \; d\Gamma = \hat{c}_D(\boldsymbol{\Upsilon}), \\
\forall \boldsymbol{\Theta} \in [H^1(\Omega)]^2, \quad \forall \vartheta \in L^2(\Omega), \quad \forall \boldsymbol{\Upsilon} \in [H^{-\frac{1}{2}}(\Gamma_{DS})]^2,
\end{gathered}
\tag{29}
$$

where, according to Eq.(15):

$$
c'(\mathbf{v}, \boldsymbol{\Theta}, \mathbf{z}) := c(\boldsymbol{\Theta}, \mathbf{v}, \mathbf{z}) + c(\mathbf{v}, \boldsymbol{\Theta}, \mathbf{z}). \tag{30}
$$

This adjoint problem corresponds to the linearized Navier–Stokes equations around $\mathbf{v}$ with the following boundary conditions:

$$
\begin{cases}
\mathbf{z} = \mathbf{0}, & \text{on } \Gamma_{IN}, \\
\mathbf{z} = \hat{\mathbf{v}}_\infty / (q_\infty d), & \text{on } \Gamma_{BODY}, \\
\mathbf{z} \cdot \hat{\mathbf{n}} = 0, \quad (\mathbb{T}(\mathbf{z}, q)\hat{\mathbf{n}}) \cdot \hat{\mathbf{t}} = 0, & \text{on } \Gamma_{SYM}, \\
\mathbb{T}(\mathbf{z}, q)\hat{\mathbf{n}} = \mathbf{0}, & \text{on } \Gamma_{OUT}.
\end{cases}
\tag{31}
$$

Finally, by differentiating $\mathcal{L}(\cdot)$ with respect to the control variable $\mathbf{u}$, we have:

$$
\int_{\Gamma_{CTRL}} \boldsymbol{\Psi} \cdot \mathbf{r} \; d\Gamma = 0, \qquad \forall \boldsymbol{\Psi} \in [H^{\frac{1}{2}}(\Gamma_{CTRL})]^2. \tag{32}
$$

After considering the form (12) for $\mathbf{u}$, the previous equation reads:

$$
\int_{\Gamma_{CTRL}} \psi \mathbf{g}(\mathrm{x}) \cdot \mathbf{r} \; d\Gamma = 0, \qquad \forall \psi \in \mathbb{R}. \tag{33}
$$

The sensitivity δU assumes the following form:

$$
\delta U = \int_{\Gamma_{CTRL}} \mathbf{g}(\mathrm{x}) \cdot \mathbf{r} \; d\Gamma, \tag{34}
$$

which is identically equal to zero at the optimum. Let us notice that the sensitivity δU depends on the adjoint stress $\mathbf{r}$; by similar arguments to those considered for the computation of the drag coefficient and by manipulating Eq.(29), Eq.(34) equivalently reads:

$$
\delta U = -\mathcal{A}'(\mathbf{G}, \vartheta, \mathbf{z}, q), \quad \forall \vartheta \in L^2(\Omega), \tag{35}
$$

where:

$$
\mathcal{A}'(\boldsymbol{\Theta}, \vartheta, \mathbf{z}, q, \mathbf{v}) := a(\boldsymbol{\Theta}, \mathbf{z}) + b(q, \boldsymbol{\Theta}) + b(\vartheta, \mathbf{z}) + c'(\boldsymbol{\Theta}, \mathbf{v}, \mathbf{z}), \tag{36}
$$

being:

$$
\mathbf{G} \in [H^1(\Omega)]^2, \quad \text{with} \quad \mathbf{G}_{|\Gamma_{CTRL}} = \mathbf{g}(\mathrm{x}), \quad \mathbf{G}_{|\partial\Omega \setminus \Gamma_{CTRL}} = \mathbf{0}. \tag{37}
$$

2 Numerical Discretization

To solve the optimal control problem, we consider the Galerkin–Finite Element method for the numerical solution of the state and adjoint Navier–Stokes equations and an iterative approach for functional minimization.

2.1 The optimization iterative method

We consider, in an abstract framework, the steepest–descent method [1, 28]. Referring to Sec.1.1, the approach reads:

1. assume an initial control variable u^0;
2. solve the state equation/s;
3. solve the adjoint equation/s, given the control variable and once the state variables are computed;
4. extract the sensitivity δu^j from the third equation of the Euler–Lagrange system (Eq.(8));
5. compute the norm of the sensitivity $\|\delta u^j\|_{\mathcal{U}}$;
6. compare the norm of the sensitivity with a stopping criterium tolerance Tol_{IT}:
   - if $\|\delta u^j\|_{\mathcal{U}} \geq Tol_{IT}$, perform an iterative step on the control variable:
   $$u^{j+1} = u^j - \tau^j \delta u^j, \tag{38}$$
   and return to point 2;
   - if $\|\delta u^j\|_{\mathcal{U}} < Tol_{IT}$, stop.

The relaxation parameter τ^j can be chosen on the basis of the optimal control properties ([1, 28]).
Let us notice that for the optimal control problem under investigation, according to Eq.(35), the norm of sensitivity reads:

$$\|\delta u\|_{\mathcal{U}} = |\delta U| = |\mathcal{A}'(\mathbf{G}, \vartheta, \mathbf{z}, q)|, \quad \forall \vartheta \in L^2(\Omega), \tag{39}$$

where $\mathbf{G}$ is chosen according with Eq.(37).

2.2 Numerical solution of state and adjoint equations

Both the state and adjoint equations enforce the Dirichlet boundary conditions in weak form, by means of Lagrangian multipliers, through the state and adjoint stresses $\mathbf{w}$ and $\mathbf{r}$. However, for the numerical solution we can re–write these equations in a conventional manner introducing appropriate functional spaces and lifting terms. In this way, we omit to compute explicitly the state and adjoint stresses $\mathbf{w}$ and $\mathbf{r}$.
The state equations consist of the steady Navier–Stokes equations, while the adjoint ones correspond to a generalized steady Stokes problem. In fact the adjoint equations are linear in both the adjoint variables (velocity and pressure), being the term $c'(\mathbf{v}, \mathbf{\Theta}, \mathbf{z})$ (30) linear w.r.t. the adjoint velocity $\mathbf{z}$. Let us notice that the adjoint equations depend on the state variables uniquely through the term $c'(\mathbf{v}, \mathbf{\Theta}, \mathbf{z})$; for this reason the FE matrix corresponding to this term needs to be recomputed at each step of the optimization iterative procedure.
First, we recall briefly a method for the solution of the Stokes equations, which is employed to solve the adjoint equations and to initialize the state Navier–Stokes equations. Then we consider the Newton method, with Uzawa preconditioning for the solution of the Navier–Stokes equations. For the numerical solution of Stokes and Navier–Stokes equations, both steady and time dependent, see e.g. [14, 24, 26].

Solution of steady Stokes equations

In order to satisfy the *inf–sup* (LBB) condition [26] we consider the $\mathbb{P}^2$–$\mathbb{P}^1$ continuous FE pair for the velocity and the pressure respectively (for both state and adjoint ones). By considering a generic Stokes problem (see e.g. Eq.(9) without the non linear term) the corresponding discretized problem reads:

$$\begin{cases} A\mathbf{V} + B^T\mathbf{P} = \mathbf{F}, \\ B\mathbf{V} = \mathbf{0}, \end{cases} \tag{40}$$

where the matrices A and B corresponds to the variational forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ respectively (Eq.s (13) and (14)), $\mathbf{F}$ is the source term vector (including the lifting terms), while $\mathbf{V}$ and $\mathbf{P}$ are respectively the vectors of the unknown FE coefficients for the velocity and pressure. The problem is resolved by computing firstly the pressure vector $\mathbf{P}$ and then the velocity $\mathbf{V}$ vector:

$$BA^{-1}B^T\mathbf{P} = BA^{-1}\mathbf{F}, \tag{41}$$

$$A\mathbf{V} = \mathbf{F} - B^T\mathbf{P}. \tag{42}$$

Both systems admit unique solution, being the matrix A symmetric and positive definite and the rank of B maximum (due to the FE pair considered), that can be obtained by means of the Conjugate Gradient method (CG) [25], being the matrix $BA^{-1}B^T$ symmetric and positive definite.

Remark 2.1 *In the case of the generalized Stokes problem for the adjoint equation, the matrix A arises from the variational forms $a(\cdot,\cdot)$ and $c'(\cdot,\cdot,\cdot)$ (Eq.s (13) and (30)), for which A is no longer symmetric; in this case for the solution of the linear systems (41) and (42) the GMRES method [25] has been adopted.*

Solution of steady Navier–Stokes equations

The Navier–Stokes equations for an incompressible fluid with constant properties can be seen as a compact perturbation of the Stokes equations, with the addition of a non linear term in the velocity variable. The system can be solved by adopting the Newton method, solving at each step of the iterative procedure a generalized Stokes problem, which arises by linearizing the Navier–Stokes equations. These linearized equations at the generic step $k+1$, obtained from Eq.(9), read:

$$\begin{cases} -\nabla\cdot\mathbb{T}(\mathbf{v}^{(k+1)}, p^{(k+1)}) + \mathbf{c}'(\mathbf{v}^{(k)}, \mathbf{v}^{(k+1)}) + \mathbf{c}'(\mathbf{v}^{(k+1)}, \mathbf{v}^{(k)}) = \mathbf{c}'(\mathbf{v}^{(k)}, \mathbf{v}^{(k)}), & \text{in } \Omega, \\ \nabla\cdot\mathbf{v}^{(k+1)} = 0, & \text{in } \Omega, \\ B.C.s, & \text{on } \partial\Omega, \end{cases} \tag{43}$$

where $\mathbf{v}^{(k)}$ is the velocity computed at the previous iterative step and $\mathbf{c}'(\cdot,\cdot)$ is defined as:

$$\mathbf{c}'(\mathbf{r},\mathbf{q}) := \rho(\mathbf{r}\cdot\nabla)\mathbf{q}. \tag{44}$$

The Newton algorithm reads:

1. solve the Stokes problem, computing the initial pressure and velocity field;
2. by considering the velocity computed at the previous step $\mathbf{v}^{(k)}$, solve the generalized Stokes problem, corresponding to Eq.(43), obtaining $\mathbf{v}^{(k+1)}$ and $p^{(k+1)}$.

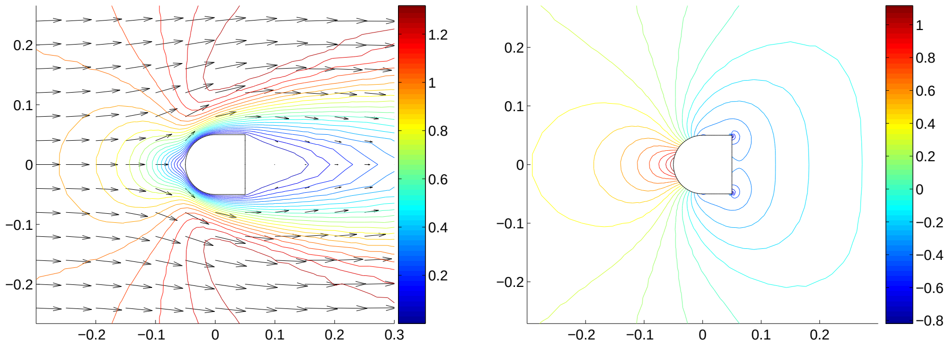

Figure 3: Zoom of velocity field (left) and pressure (right) for the initial state flow ($\mathbb{R}e = 10$).

3. compute the appropriate norms of the incremental differences $\mathbf{v}^{(k+1)} - \mathbf{v}^{(k)}$ and $p^{(k+1)} - p^{(k)}$ and compare them with a prescribed tolerance;
4. if the stopping criterium is fulfilled assume $\mathbf{v}^{(k+1)}$ and $p^{(k+1)}$ as solutions of the Navier–Stokes equations; otherwise, set $\mathbf{v}^{(k)} = \mathbf{v}^{(k+1)}$, $p^{(k)} = p^{(k+1)}$ and return to point 2.

A preconditioning matrix can be used for the solution of the linear system (41) corresponding to the Stokes (or generalized Stokes) equations, which are recursively solved in the Newton method. In particular, we consider the Uzawa preconditioning matrix D, which corresponds to the bilinear form $d(p, q) := \int_\Omega pq \; d\Omega$.
No stabilization terms are used in the discrete Navier–Stokes equations, as we are interested in flows at low Reynolds numbers (inferior to 100).

Remark 2.2 *At each step of the optimization iterative procedure we solve the Navier–Stokes state equations, for which an initialization by means of a Stokes problem occurs. In order to save computational costs, we can assume as initial guess the state velocity computed at the previous step of the optimization procedure, instead of computing it by means of a Stokes problem solver. The validity of this procedure is confirmed by numerical tests (see Sec.3).*

3 Numerical Results

We present some numerical results concerning the drag minimization control problem outlined in Sec.1, carried out by means of the FE library *FreeFEM++* [29]. In particular, referring to Eq.s (9) and (11) we chose $\rho = 1$, $V_\infty = 1$ and $d = 0.1$, for which, defining the Reynolds number:

$$\mathbb{R}e := \frac{\rho V_\infty d}{\mu}, \tag{45}$$

the dynamic viscosity coefficient can be expressed as:

$$\mu = \frac{\rho V_\infty d}{\mathbb{R}e}. \tag{46}$$

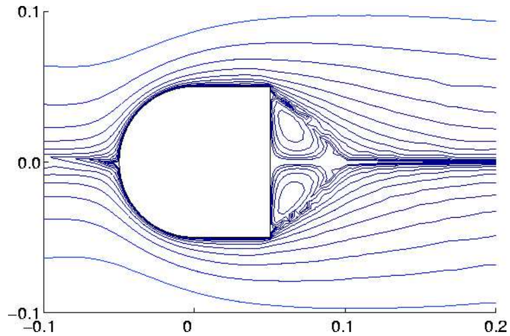

Figure 4: Streamlines for the initial state flow (zoom) ($\mathbb{R}e = 10$).

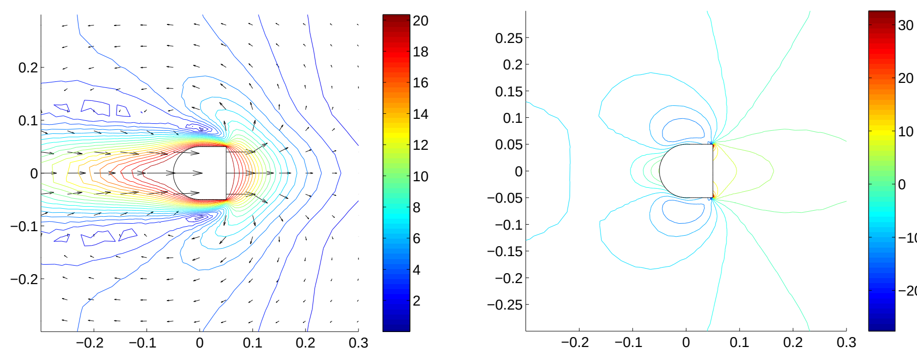

Figure 5: Zoom of adjoint velocity field (left) and adjoint pressure (right) for the initial adjoint flow ($\mathbb{R}e = 10$).

At the initial step of the optimization procedure, we assume $\mathbf{u} = \mathbf{0}$, for which, according to Eq.(12) $U = 0$ and $U_{av} = \frac{2}{3}U = 0$, being U_{av} the average value of $\mathbf{u}$ on Γ_{CTRL}.
Firstly we consider the case for which $\mathbb{R}e = 10$. In Fig.3 we report the velocity field and pressure, solution of the state Navier–Stokes equations, obtained for $U_{av} = 0$; in Fig.4, the streamlines for the state velocity field are reported. In Fig.5, we show the adjoint velocity and pressure, computed at the initial step of the optimization iterative procedure (corresponding to $U_{av} = 0$). We assume the relaxation parameter $\tau^j = \tau = 0.5$ (Eq.(38)) and we consider as tolerance for the stopping criterium of the optimization method outlined in Sec.2.1 $Tol_{IT} = \frac{1}{1000}|\delta U^0|$, where the apex 0 means at the initial step, while δU is reported in Eq.(39). At the end of the optimization procedure, we obtain the velocity field and the pressure reported in Fig.6; in Fig.7 the streamlines of the optimal velocity field show the effects of the optimization. In particular, the initial drag coefficient $c_D = 3.83$, corresponding to $U_{av} = 0$, is reduced, by means of this procedure, to the optimal one, which is $c_D = 3.31$, obtained for $U_{av} = -0.967$. In Fig.8 we report the behavior of the drag coefficient c_D and the sensitivity $|\delta U|$ (normalized w.r.t. $|\delta U^0|$) versus the number of iterations of the optimization procedure, for which the convergence occurs in 24 iterations. In Fig.9 we show the values assumed from

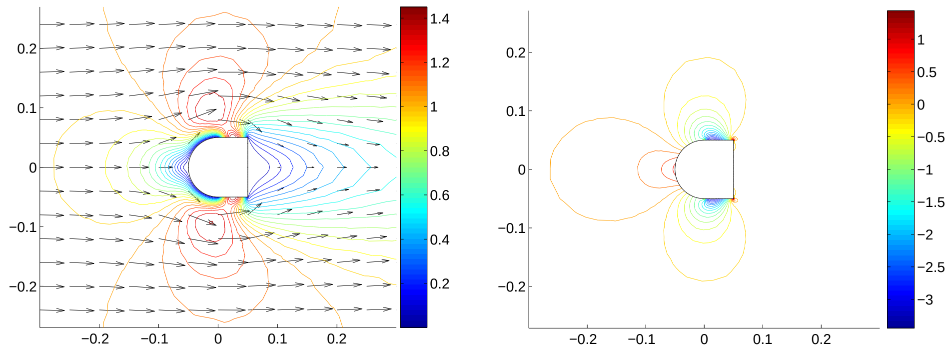

Figure 6: Zoom of velocity field (left) and pressure (right) for the optimal state flow ($\mathbb{R}e = 10$).

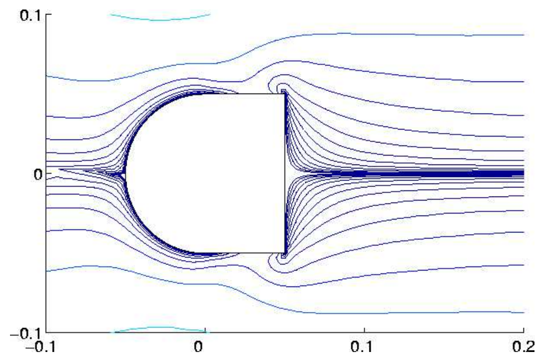

Figure 7: Streamlines for the optimal state flow ($\mathbb{R}e = 10$).

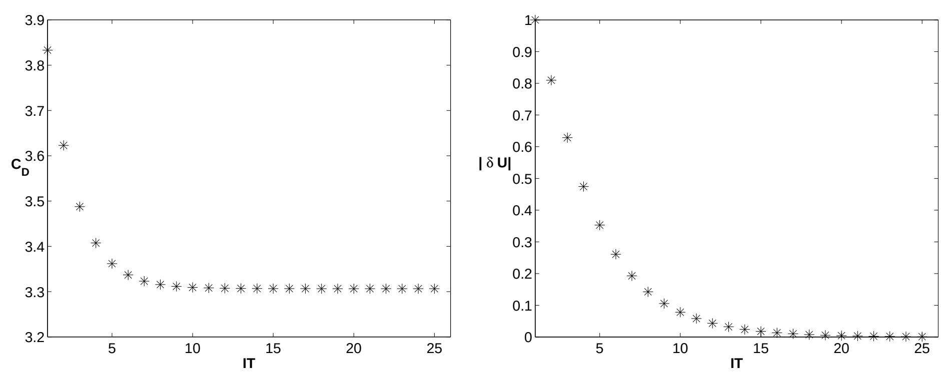

Figure 8: Drag coefficient (left) and sensitivity (right), normalized with δU^0, vs. number of iterations of the optimization procedure ($\mathbb{R}e = 10$).

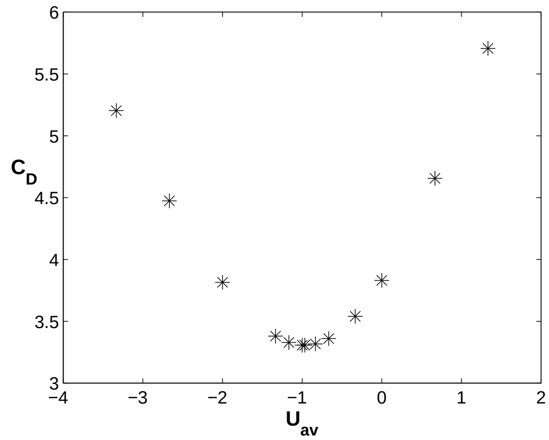

Figure 9: Drag coefficient vs. average value of control variable ($\mathbb{R}e = 10$).

$\mathbb{R}e$	c_D ($\mathbf{u} = \mathbf{0}$)	c_D (opt.)	U_{av} (opt.)
5	5.80	4.74	−1.77
10	3.83	3.31	−0.967
20	2.67	2.33	−0.580
50	1.77	1.52	−0.325

Table 1: Drag coefficients, minimum drag coefficients and average value (optimal) of control function for different $\mathbb{R}e$ values.

c_D for different values of U_{av}, for which we show that the local minimum value of c_D corresponds to the optimal one, assumed for $U_{av} = -0.967$.
In Table 1, we report the values of c_D (with $U_{av} = 0$) for different Reynolds numbers $\mathbb{R}e$, the corresponding optimal drag coefficients and optimal control functions U_{av}. In particular we find that, when $\mathbb{R}e$ increases, the modulus of the optimal U_{av} decreases.

In Remark 2.2 we have discussed a strategy for the reduction of the computational cost of the optimization procedure, for which we initialize the linearized Navier–Stokes state equations with the velocity field computed at previous step of the iterative procedure, instead of adopting a Stokes solver at each optimization step. Numerical tests, referring to the problem under investigation, outline that cost savings can be reached of about 37% and 25%, for $\mathbb{R}e = 5$ and $\mathbb{R}e = 10$, respectively.

The Stokes case

If we define the drag minimization control problem for the Stokes equations, the drag coefficient does not admit a positive minimum, but tends to $-\infty$; this is due to the linearity in the velocity of the Stokes equations and the drag coefficient. In Fig.10, we report the drag coefficient (for a prescribed μ) computed for different values of U_{av}, for which the linear dependence of c_D on U_{av} is outlined.

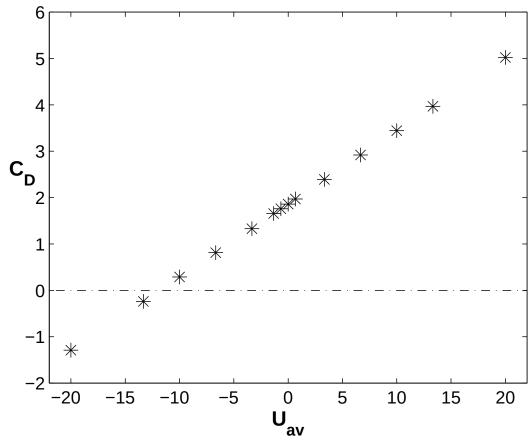

Figure 10: Drag coefficient vs. average value of control variable (the Stokes case).

Conclusions

In this work we have studied an optimization control problem for the drag reduction of a $2D$ body in relative motion with an incompressible fluid with constant properties. In particular, we have considered as control function the Dirichlet boundary conditions defined on a part of the boundary of the body itself. We have adopted the Lagrangian functional approach for the resolution of the optimal control problem, together with the Lagrangian multiplier method for the treatment of the Dirichlet boundary conditions. This has allowed a straightforward determination of the Euler–Lagrange system, without introducing lifting terms. We have proved the effectiveness of the procedure on numerical tests by computing the optimal flow for which the drag coefficient is minimum and comparing the results for different $\mathbb{R}e$ numbers.

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