On the distribution of the limit proportion for a two-color, randomly reinforced urn with equal reinforcement distributions

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Abstract

We consider a two-color randomly reinforced urn with equal reinforcement distributions and we characterize the distribution of the urn's limit proportion as the unique continuous solution of a functional equation involving unknown probability distributions on $[0, 1]$.

1 Introduction

An urn contains initially $b > 0$ black balls and $w > 0$ white balls. The urn is sequentially sampled. At time $n = 1, 2, \dots$ a ball is drawn from the urn and its color is observed: if the sampled ball is black, it is replaced in the urn together with a random number M_n of balls of color black, if the sampled ball is white it is replaced in the urn together with a random number N_n of balls of color white. The processes $\{M_n\}$ and $\{N_n\}$ are two independent sequences of i.i.d., non-negative and bounded random variables with distributions μ and ν respectively. This urn scheme is called a two-color, randomly reinforced urn.

A randomly reinforced urn generates some interesting stochastic processes: the sequences $\{B_n\}$ and $\{W_n\}$ of the number of balls, black and white respectively, present in the urn at time $n = 1, 2, ...$; the sequence $\{Z_n\}$ of the urn proportions where, for $n = 1, 2, ...$,

$$Z_n = \frac{B_n}{B_n + W_n};$$

and finally, the sequence $\{X_n\}$ of the colors generated by the urn, where X_n is 1 or 0 according to the color black or white of the ball drawn from the urn at time $n = 1, 2, ...$ In this paper we mostly focus on the process of urn proportions $\{Z_n\}$. In Muliere.et.al. (2005) it is proved that the sequence $\{Z_n\}$ is eventually a bounded super- or sub-martingale and it thus converges almost surely to a random limit $Z_\infty \in [0, 1]$. When the first moment of μ is strictly greater than the first moment of ν, the random variable Z_∞ is equal to 1 with probability one. On the other extreme, May.et.al. (2005) show that if the moments of μ and ν are all equal, i.e. μ and ν coincide, then Z_∞ has no point masses; however, the exact distribution of Z_∞ is unknown but in a few particular cases, the only non trivial one being the Polya's urn where μ and ν are point masses at a non-negative real number m and Z_∞ has distribution Beta$(b/m, w/m)$. The main result of this paper states that, when $\mu = \nu$, the distribution of the limit proportion Z_∞, regarded as function of the initial urn composition (b, w), is the unique continuous solution, satisfying some boundary conditions, of a specific functional equation in which the unknowns are distribution functions on $[0, 1]$.

The study of the distribution of Z_∞ has its origins in the seminal paper of Athreya (1969) and stems from theoretical and applicative motives. In May.et.al (2005) and in Muliere.et.al. (2005) it is proved that, conditionally on Z_∞, the random variables of the sequence of colors $\{X_n\}$ generated by a randomly reinforced urn are asymptotically i.i.d. Bernoulli(Z_∞); hence the distribution of Z_∞ may represent the prior distribution for a Bayesian adopting the randomly reinforced urn scheme as a metaphor for the construction of the statistical model. Moreover, in Muliere.et.al.(2005) and in Paganoni and Secchi (2005) it is stressed that a two-color, randomly reinforced urn implements a sequential, randomized, response-driven design for clinical trials where the experimenter is willing to bias, along the experiment, the allocation probability toward the better treatment; for a well informed review on urns and response-adaptive, randomized designs see Rosenberger (2002). To be specific, suppose that μ and ν represent the distributions of responses after treatment, say A and B respectively: if the mean response after A is greater than the mean response after B, allocating the n-th patient in the clinical trial to A or B according to the color of n-th ball drawn from a two-color randomly reinforced urn with reinforcements equal to responses after treatments guarantees that patients will be assigned to treatment A with higher and higher probability along the experiment, since $Z_\infty = 1$ almost surely. Hence, for testing hypothesis about treatment effects, it is important to know the distribution of Z_∞ when $\mu = \nu$, i.e. there is no difference between treatment effects. Indeed, a randomly reinforced urn where both reinforcement distributions are equal to the same μ, is the scheme considered in this paper.

After setting the notation and specifying the probabilistic model for a randomly reinforced urn with equal reinforcement distributions in the next section, in Section 3 we introduce the function $\mathcal{F}$ that, given a probability distribution μ, maps any admissible couple (b, w) in the distribution of the limit proportion of a two-color, randomly reinforced urn with reinforcement distributions equal to μ and initial composition (b, w). After discussing some interesting properties regarding the class of distributions in the range of $\mathcal{F}$, we end the section with the introduction of a functional equation solved by $\mathcal{F}$ and with the statement of the main result of the paper: namely that $\mathcal{F}$ is the unique continuous solution, satisfying some boundary conditions, of this functional equation. In Section 4 we prove that $\mathcal{F}$ is indeed continuous, while in Section 5 we conclude the proof of the main result by showing that a suitable transformation of $\mathcal{F}$ is the unique fixed point of a certain operator. An example will conclude the paper.

2 Model specification

On a rich enough probability space, define two independent sequences $\{M_n\}$ and $\{U_n\}$ of real valued random variables. The variables of the sequence $\{M_n\}$ are i.i.d with probability distribution μ : the support of μ is contained in $[0, \beta]$, with $\beta > 0$. Moreover it is assumed that $\mu(\{0\}) < 1$ for avoiding a trivial case. The random variables of the sequence $\{U_n\}$ are i.i.d. with uniform distribution on $[0, 1]$. Finally, let b and w be two non-negative real numbers such that $b + w > 0$.

Set $B_0 = b$, $W_0 = w$, and, for $n \geq 0$, let

$$\begin{cases} B_{n+1} &= B_n + M_{n+1}X_{n+1} \\ W_{n+1} &= W_n + M_{n+1}(1 - X_{n+1}). \end{cases} \tag{1}$$

where, for $n = 1, 2, ...$, the variable X_n is the indicator of the event $\{U_n \leq B_{n-1}(B_{n-1} + W_{n-1})^{-1}\}$. Then the law of the sequence $\{(B_n, W_n)\}$ is that of the stochastic process counting, along time, the number of black and white balls present in a randomly reinforced urn with initial composition (b, w) and reinforcement distributions both equal to μ whereas the law of the sequence $\{X_n\}$ is that of the process of colors generated by the same urn.

For $n = 0, 1, 2, ...$ let

$$Z_n = \frac{B_n}{B_n + W_n};$$

Z_n indicates the proportions of black balls in the urn before the $(n+1)$-th ball is sampled. The process $\{Z_n\}$ is a bounded martingale with respect to the filtration $\{\sigma(X_1, M_1, ..., X_{n-1}, M_{n-1})\}$ and it converges almost surely to a random variable $Z_\infty \in [0, 1]$ (May.et.al. 2005): moreover, the distribution of Z_∞ has no atoms and is completely determined once the parameters (b, w, μ) are specified, even though its analytical expression is unknown. We will say that the distribution of Z_∞ is that of the limit proportion of a randomly reinforced urn with initial composition (b, w) and reinforcement distributions equal to μ; this paper is focused on this distribution and its properties. Our approach will be to fix the reinforcement distribution μ and to explore how the distribution of Z_∞ varies according to changes in the urn's initial composition (b, w). When the argument requires it, we will draw the reader's attention on the dependency of the law of the stochastic elements generated by the urn on its initial composition (b, w), by using the obvious notations $B_n(b, w), W_n(b, w), X_n(b, w), Z_n(b, w), Z_\infty(b, w)$.

Pemantle (1990) introduces the following time-dependent version of the Polya's urn. Let $F : \{1, 2, ...\} \to [0, \infty)$ be any function and consider an urn containing initially $b \geq 0$ black balls and $w \geq 0$ white balls, with $b + w > 0$. The urn is sequentially sampled: at time $n = 1, 2, ...$ a ball is drawn from the urn and reintroduced in it together with $F(n)$ balls of the same color. Pemantle studies the behavior of the sequence of the successive proportions V_n of black balls in the urn. The link between Pemantle's urn and our two-color, randomly reinforced urn with equal reinforcement distributions is evident. In fact, let $\mathcal{M}$ be the sigma-field generated by the random variables $M_1, M_2, \ldots$. Given $\mathcal{M}$, the conditional law of the process $\{(B_n, W_n)\}$ (and therefore also the conditional laws of the processes $\{X_n\}$ and $\{Z_n\}$) is the same as that of the process counting the number of black and white balls in Pemantle's urn, once we define his reinforcement function F by setting $F(n) = M_n$ for $n = 1, 2, \ldots$. In the next sections we will make use of the following two results proved in Pemantle's (1990).

Theorem 2.1 *For any function F, the sequence of successive proportions $\{V_n\}$ of black balls in a Pemantle's urn directed by F is a martingale converging almost surely to a random variable $V \in [0,1]$. If F is not the null function and is bounded by some constant $\beta > 0$, then V has no atoms on $[0,1]$.*

Theorem 2.2 *Consider two Pemantle's urns; both urns have initial composition equal to (b, w). The first urn is directed by the function $F^{(1)}$, the second by the function $F^{(2)}$. Let $\{V_n^{(1)}\}$ and $\{V_n^{(2)}\}$ represent the sequences of successive proportions of black balls in the two urns. If, for all $n = 1, 2, \ldots$,*

$$\frac{F^{(1)}(n)}{b + w + \sum_{i=1}^{n-1} F^{(1)}(i)} \geq \frac{F^{(2)}(n)}{b + w + \sum_{i=1}^{n-1} F^{(2)}(i)}$$

then $\mathrm{E}(h(V_{n+1}^{(1)})) \geq \mathrm{E}(h(V_{n+1}^{(2)}))$, *for all continuous convex $h : [0,1] \to [0,2]$.*

3 The function $\mathcal{F}$ and the characteristic equation

Fix a probability distribution μ on the interval $[0, \beta]$, with $\beta > 0$. Let $S = [0, \infty) \times [0, \infty) \setminus (0, 0)$ and indicate with $\mathcal{P}$([0,1]) the space of distribution functions with support in $[0, 1]$; define

$$\mathcal{F} : S \to \mathcal{P}([0,1])$$

to be the function that maps any couple (b, w) of non-negative real numbers with positive sum in the distribution $\mathcal{F}(b, w)$ of the limit proportion $Z_\infty(b, w)$ of a randomly reinforced urn with initial composition (b, w) and reinforcement distributions equal to μ.

For $x \in R$, let δ_x be the distribution of the point mass at x.

Theorem 3.1

(i) $\mathcal{F}(b,0)=\delta_1$ *for all* $b>0$;

(ii) $\mathcal{F}(0,w)=\delta_0$ *for all* $w>0$;

(iii) *For all* $(b,w)\in S$,

$$\mathcal{F}(b,w)(x)+\mathcal{F}(w,b)(1-x)=1$$

for $x\in R$.

(iv) *for all* $c\geq 0$ *and all* $(b,w)\in S$ *such that* $b/(b+w)=c$,

$$\int x\mathcal{F}(b,w)(dx)=c,$$

i.e. the distribution $\mathcal{F}(b,w)$ *has constant mean along the line*

$$\frac{b}{b+w}=c.$$

Proof. Properties (i) and (ii) are trivial; to prove (iii) note that the distribution of the limit proportion of a randomly reinforced urn with reinforcement distributions equal to μ and initial composition equal to (b,w) must be the same as the distribution of 1 minus the limit proportion of the same urn when the initial composition is (w,b), i.e. the distribution of $Z_\infty(b,w)$ is the same as the distribution of $1-Z_\infty(w,b)$; now (iii) follows, since the distribution of Z_∞ has no point masses (May.et.al 2005). Finally, property (iv) is true because, for all (b,w), the sequence $\{Z_n(b,w)\}$ is a bounded martingale converging to $Z_\infty(b,w)$ (May.et.al 2005) and thus

$$E(Z_\infty(b,w))=Z_0(b,w)=\frac{b}{b+w}.$$

□

Endow the space S with the lower-right-quadrant order relationship such that $(b,w)\preceq(\bar{b},\bar{w})$ if and only if $b\leq\bar{b}$ and $w\geq\bar{w}$; give $\mathcal{P}([0,1])$ the stochastic order such that $G\leq_{st}H$ if and only if $1-G(x)\leq 1-H(x)$ for all $x\in[0,1]$. The proof of the next result uses for the first time a coupling argument that will frequently appear in the rest of the paper.

Theorem 3.2 $\mathcal{F}$ *is monotonic.*

Proof. We need to prove that if (b,w) and $(\bar{b},\bar{w})$ are states of S such that $(b,w) \preceq (\bar{b},\bar{w})$, then

$$\mathcal{F}(b,w) \leq_{st} \mathcal{F}(\bar{b},\bar{w}).$$

Consider two different randomly reinforced urns. The first urn has initial composition (b,w) and generates the counting process $\{(B_n, W_n)\}$ according to the dynamics described in (1). Analogously, the second urn has initial composition $(\bar{b},\bar{w})$ and generates the counting process $\{(\bar{B}_n, \bar{W}_n)\}$. The urns are coupled in the sense that the processes $\{M_n\}$, $\{U_n\}$ and $\{\bar{M}_n\}$, $\{\bar{U}_n\}$ appearing in (1) for the definition of $\{(B_n, W_n)\}$ and $\{(\bar{B}_n, \bar{W}_n)\}$ respectively are identical; i.e. we assume that

$$P[M_n = \bar{M}_n,\ U_n = \bar{U}_n \text{ for all } n] = 1.$$

Notice that $\dfrac{b}{b+w} \leq \dfrac{\bar{b}}{\bar{b}+\bar{w}}$; hence $U_1 \leq \dfrac{\bar{b}}{\bar{b}+\bar{w}}$ if $U_1 \leq \dfrac{b}{b+w}$ and

$$P[(B_1, W_1) \preceq (\bar{B}_1, \bar{W}_1)] = 1.$$

By induction on n,

$$P[(B_n, W_n) \preceq (\bar{B}_n, \bar{W}_n) \ \text{ for all } n] = 1$$

and thus

$$P[Z_\infty(b,w) \leq Z_\infty(\bar{b},\bar{w})] = 1.$$

□

Since the support of μ is bounded above by β, it is natural to conjecture that when the initial number of balls in the urn is large, the limit proportion Z_∞ will be close to its mean value; we will in fact prove the conjecture by means of the next theorem. For $(b,w) \in S$, let $B(b/\beta, w/\beta)$ indicate a random variable with distribution $\mathrm{Beta}(b/\beta, w/\beta)$ on $[0,1]$.

Lemma 3.1 *For every $j \geq 1$ and $(b,w) \in S$,*

$$\mathrm{E}[Z_\infty^j(b,w)] \leq \mathrm{E}[B^j(b/\beta, w/\beta)]. \tag{2}$$

Proof. Let $N \geq 1$. Given $M_1 = m_1, M_2 = m_2, ..., M_N = m_n, ...$, note that

$$\mathrm{E}[Z_N^j(b,w) | M_1 = m_1, M_2 = m_2, ...] = \mathrm{E}\left[{V_N}^j\right] \tag{3}$$

where $V_1, V_2, ...$ is the sequence of proportion of black balls in a Pemantle's urn with initial composition (b,w) and reinforcement directed by F, with $F(n) = m_n$ for $n \leq N$ and $F(n) = 0$ for $n > N$. We now use a trick learned in Pemantle (1990). If it is not the case that $F(1) \geq F(2) \geq F(3) \geq ...$, let $F^{(1)}(n) = F(n)$ for every n except for two indices $k \geq 1$ and $k+1$, where $F^{(1)}(k) = F(k+1) >$

$F(k) = F^{(1)}(k+1)$, and consider a Pemantle's urn with intial composition (b, w) and reinforcements directed by $F^{(1)}$; then Proposition 2 of Pemantle (1990) proves that

$$\mathrm{E}[h(V_n)] \leq \mathrm{E}[h(V_n^{(1)})],$$

for all $n \geq 1$ and for all continuous, convex $h : [0,1] \to [0,2]$, where $V_n^{(1)}$ indicates the successive proportions of black balls in the urn directed by $F^{(1)}$. By repeatedly applying this result and by setting $h(x) = x^m$, for $x \in [0,1]$, we obtain

$$\mathrm{E}[V_N^j] \leq \mathrm{E}[(V_N^{(2)})^j] \tag{4}$$

where, for $n = 1, 2, ...$, $V_n^{(2)}$ are the successive proportions of black balls in a Pemantle's urn with reinforcements directed by a function $F^{(2)}$ with the same values as F but rearranged in descending order. Finally, let $\bar{V}_1, \bar{V}_2, ...$ be the sequence of proportions of black balls in a Polya's urn with initial composition (b, w) and constant reinforcement equal to β; Theorem 2.2 implies that

$$\mathrm{E}\left[(V_N^{(2)})^j\right] \leq \mathrm{E}\left[\bar{V}_N^j\right]. \tag{5}$$

Let $\mathcal{M}$ be the sigma-field generated by $M_1, M_2, ...$; equations (3),(4) and (5) prove that, for $N \geq 1$ and $(b, w) \in S$,

$$\mathrm{E}[Z_N^j(b,w)|\mathcal{M}] \leq \mathrm{E}[\bar{V}_N^j].$$

Equation (2) now follows by computing the expected values on both sides of the previous inequality and applying the Dominated Convergence Theorem. □

Theorem 3.3 *For every $\eta > 0$ and $\varepsilon > 0$ there exists $K = K(\eta, \varepsilon)$ such that*

$$P[|Z_\infty(b,w) - \frac{b}{b+w}| > \eta] < \varepsilon$$

if $b + w > K$.

Proof. Let $(b, w) \in S$ and $\eta > 0$. Then

$$\begin{aligned} P[|Z_\infty(b,w) - \frac{b}{b+w}| > \eta] &\leq \frac{1}{\eta^2}\mathrm{Var}(Z_\infty(b,w)) \\ &\leq \frac{1}{\eta^2}\frac{bw}{(b+w)^2}\,\frac{\beta}{b+w+\beta} \\ &\leq \frac{1}{\eta^2}\,\frac{1}{4}\,\frac{\beta}{b+w}; \end{aligned}$$

the first inequality is Chebichev's while the second one follows from (2) with $j = 2$ and the fact that the expected value of $Z_\infty(b,w)$ is $b/(b+w)$. The theorem is thus proved by setting $K = \beta(4\eta^2\varepsilon)^{-1}$. $\square$

Theorem 3.3 has interesting consequences in applications, for instance when one wants to approximate the distribution of Z_∞ by means of a Monte Carlo simulation. In fact, given $\epsilon, \eta > 0$, let

$$\tau(\epsilon,\eta) = \inf\{n \geq 0 : B_n + W_n > K(\eta,\epsilon)\}.$$

Since μ is not concentrated on 0, $\tau(\epsilon,\eta)$ is finite almost surely; the next corollary states that by approximating the distribution of Z_∞ with that of $Z_{\tau(\epsilon,\eta)}$ we don't loose much. Its proof follows from Theorem 3.3 and the observation that the process $\{(B_n, W_n)\}$ is Markov. Hence the strong Markov property holds and implies that, given the sigma-field generated by the stopping time $\tau(\epsilon,\eta)$, the conditional law of the process

$$((B_{\tau(\epsilon,\eta)}, W_{\tau(\epsilon,\eta)}), (B_{\tau(\epsilon,\eta)+1}, W_{\tau(\epsilon,\eta)+1}), ...)$$

is that of the process counting the successive number of balls of color black and white respectively in a randomly reinforced urn with reinforcement distributions equal to μ and initial composition equal to $(B_{\tau(\epsilon,\eta)}, W_{\tau(\epsilon,\eta)})$.

Corollary 3.1 *For all $(b,w) \in S$ and $\epsilon, \eta > 0$,*

$$P[|Z_\infty(b,w) - Z_{\tau(\epsilon,\eta)}(b,w)| > \eta] < \epsilon.$$

Remark 3.1 *If μ has support contained in $[\alpha,\beta]$ with $\alpha > 0$, then*

$$P[\tau(\epsilon,\eta) \leq \frac{\beta}{4\alpha\eta^2\epsilon} + 1] = 1.$$

Finally, we consider the variance of $Z_\infty(b,w)$. The next result states that for $b + w =$ constant, the variance of $Z_\infty(b,w)$ reaches its maximum value when $b = w$, and that for $b =$ constant $\cdot\, w$, i.e. along the lines from the origin, the variance of $Z_\infty(b,w)$ decreases when w increases: see Figure 1 for illustration.

Theorem 3.4

(i) For every $(b,w) \in S$,

$$\mathrm{Var}(Z_\infty(b,w)) \leq \mathrm{Var}(Z_\infty(\frac{b+w}{2}, \frac{b+w}{2})).$$

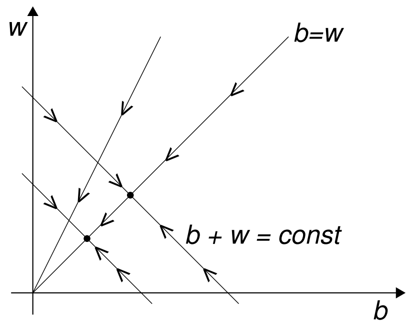

Figure 1: Variance of $Z_\infty(b,w)$.

(ii) For every $(b,w) \in S$ *and* $t > 1$,

$$\mathrm{Var}(Z_\infty(b,w)) \geq \mathrm{Var}(Z_\infty(tb,tw)).$$

Proof. For proving (i), set $\Delta^2 Z_{n+1} = Z_{n+1}^2 - Z_n^2$ for $n = 0, 1, ...$; we claim that, for $(b,w) \in S$,

$$\mathrm{E}(\Delta^2 Z_{n+1}(b,w)) \leq \mathrm{E}(\Delta^2 Z_{n+1}(\frac{b+w}{2}, \frac{b+w}{2})). \tag{6}$$

Since $\mathrm{Var}(Z_{n+1}) = \mathrm{E}[\sum_{i=0}^n \Delta^2 Z_{i+1}]$, from (6) it follows that

$$\mathrm{Var}(Z_{n+1}(b,w)) \leq \mathrm{Var}(Z_{n+1}(\frac{b+w}{2}, \frac{b+w}{2}))$$

for all n; letting $n \to \infty$ we get (i) from the Dominated Convergence Theorem.

In order to prove (6), let $\mathcal{M}$ be the sigma-field generated by $M_1, M_2, ...$, and compute

$$\begin{aligned} \mathrm{E}({Z_{n+1}}^2 | \mathcal{M}) &= \mathrm{E}(\mathrm{E}({Z_{n+1}}^2 | Z_n, \mathcal{M}) | \mathcal{M}) \\ &= \mathrm{E}(Z_n^2 + H_{n+1}^2 Z_n - H_{n+1}^2 Z_n^2 | \mathcal{M}), \end{aligned} \tag{7}$$

where

$$H_n = H_n(b,w) = \frac{M_n}{b + w + \sum_{i=1}^n M_i}$$

for $n = 1, 2, ...$ Hence

$$\mathrm{E}(Z_{n+1}^2 - {Z_n}^2 | \mathcal{M}) = H_{n+1}^2 \, \mathrm{E}(Z_n - {Z_n}^2 | \mathcal{M}). \tag{8}$$

For $n = 0, 1, ...,$ set $W_n = \mathrm{E}(Z_n - {Z_n}^2 | \mathcal{M})$. Theorem 2.1 states that the proportion of black balls in a Pemantle's time-dependent version of the Polya's urn is a martingale; thus $\mathrm{E}(Z_{n+1}|\mathcal{M}) = \mathrm{E}(Z_n|\mathcal{M})$ for $n = 0, 1, 2, ...$ Hence, (7) implies that, for all n,

$$W_{n+1} = W_n(1 - {H_{n+1}}^2),$$

and then

$$W_n = W_0 \prod_{i=1}^{n} (1 - {H_i}^2).$$

Hence (8) can be rewritten as

$$\mathrm{E}({Z_{n+1}}^2 - {Z_n}^2 | \mathcal{M}) = {H_{n+1}}^2 \, W_0 \prod_{i=1}^{n} (1 - {H_i}^2);$$

taking expectations, we see that

$$\mathrm{E}(\Delta^2 Z_{n+1}) = W_0 \, \mathrm{E}[{H_{n+1}}^2 \prod_{i=1}^{n} (1 - {H_i}^2)]. \tag{9}$$

Finally, to obtain (6) from (9), it is sufficient to notice that

$$H_n(b, w) = H_n(\frac{b+w}{2}, \frac{b+w}{2})$$

and that

$$W_0(b, w) = \frac{b}{b+w} \left(1 - \frac{b}{b+w}\right)$$

reaches its maximum value when $b = w$.

For proving (ii), consider two urns, the first with initial composition $(b, w) \in S$ and the second with initial composition (tb, tw), with $t > 1$. The urns are coupled in the sense that, as in the proof of Theorem 3.2, the same sequences $\{M_n\}$ and $\{U_n\}$ generate the urns successive compositions by means of (1). Notice that, for $n = 1, 2, ...,$

$$\frac{M_n}{b + w + \sum_{i=1}^{n-1} M_i} \geq \frac{M_n}{t(b+w) + \sum_{i=1}^{n-1} M_i}.$$

By conditioning to $\mathcal{M}$ and setting $h(x) = (x - \frac{b}{b+w})^2$, for $x \in [0, 1]$, it follows from Theorem 2.2 that

$$\mathrm{E}(h(Z_n(b, w))|\mathcal{M}) \geq \mathrm{E}(h(Z_n(tb, tw))|\mathcal{M}).$$

Now (ii) follows by taking expectations. □

By conditioning on X_1 and M_1 and computing the expected values, we see that $\mathcal{F}$ must satisfy the following condition: for all $(b, w) \in S$,

$$\mathcal{F}(b,w) = \frac{b}{b+w}\int_{[0,\beta]} \mathcal{F}(b+k,w)\mu(dk) + \frac{w}{b+w}\int_{[0,\beta]} \mathcal{F}(b,w+k)\mu(dk). \tag{10}$$

We call (10) the *characteristic equation* of a randomly reinforced urn with reinforcement distributions equal to μ. The following question arises naturally: does equation (10) characterize the class of distributions for limit proportions of randomly reinforced urns with reinforcement distributions equal to μ and initial compositions varying in S? In other words: is the function $\mathcal{F}$ introduced in this section the unique map from S to $\mathcal{P}([0,1])$ that solves the functional equation (10)? Without further conditions, the answer to this question is no; for instance, any constant function mapping S to a fixed element of $\mathcal{P}([0,1])$ is a solution of (10), albeit what we already know about $\mathcal{F}$ implies that $\mathcal{F}$ is not a constant function. The main result of this paper states that $\mathcal{F}$ is the unique continuous solution of (10) satisfying some boundary conditions. In order to have a precise statement of the theorem, consider S as a subset of the space R^2 with the euclidean metric and endow $\mathcal{P}([0,1])$ with the Wasserstein metric defined, for all $F, G \in \mathcal{P}([0,1])$, as

$$d_W(F,G) = \int_0^1 |F(x) - G(x)|dx. \tag{11}$$

Theorem 3.5 *The function $\mathcal{F}$ is the unique solution of (10) among the continuous functions $G : S \to \mathcal{P}[0,1]$ satisfying the following three conditions:*

(a) $G(0,w) = \delta_0$ for $w > 0$;

(b) $G(b,0) = \delta_1$ for $b > 0$;

(c) for every $\varepsilon > 0$, there is a $K = K(\varepsilon)$ such that

$$d_W(G(b,w), \delta_{\frac{b}{b+w}}) < \varepsilon$$

if $b + w > K$.

We already know that $\mathcal{F}$ satisfies conditions (a) and (b) of the theorem: in the next section we will prove that $\mathcal{F}$ is indeed continuous on S and satisfies condition (c). Then, in Section 5, through a suitable transformation of the space S and the function $\mathcal{F}$, we will prove that $\mathcal{F}$ is the unique continuous solution of (10) satisfying (a)-(c).

Remark 3.2 *When modelling our two-color randomly reinforced urn with reinforcement distributions equal to* μ*, we assumed* $\mu(\{0\}) < 1$*. For all Borel subsets* B *of* R *define*

$$\nu(B) = \frac{\mu(B \bigcap (0,\beta])}{1-\mu(\{0\})};$$

hence ν *is the conditional distribution of* M_1*, given that* $M_1 > 0$*. Note that,* $\mathcal{F} : S \to \mathcal{P}([0,1])$ *solves* (10) *if and only if*

$$\begin{aligned}&[1-\mu(\{0\})]\mathcal{F}(b,w)\\ &\quad = \frac{b}{b+w}\int_{(0,\beta]} \mathcal{F}(b+k,w)\mu(dk) + \frac{w}{b+w}\int_{(0,\beta]} \mathcal{F}(b,w+k)\mu(dk),\end{aligned}$$

i.e. if and only if

$$\mathcal{F}(b,w) = \frac{b}{b+w}\int_{(0,\beta]} \mathcal{F}(b+k,w)\nu(dk) + \frac{w}{b+w}\int_{(0,\beta]} \mathcal{F}(b,w+k)\nu(dk) \quad (12)$$

for all $(b,w) \in S$*. In light of Theorem 3.5, we can thus say that* $\mathcal{F}$ *is also the unique solution of* (12) *among the continuous functions* $G : S \to \mathcal{P}([0,1])$ *satisfying conditions (a)-(c) of the theorem.*

4 $\mathcal{F}$ is continuous

Before proceeding to prove that $\mathcal{F}$ is continuous, we recall that the distance d_W, introduced in (11), metrizes the weak convergence in $\mathcal{P}([0,1])$. Moreover, by the Kantorovich-Rubinstein theorem,

$$d_W(F,G) = \inf\{\mathrm{E}(|X-Y|) : X \sim F, Y \sim G\} \quad (13)$$

where the infimum is taken over all joint distributions for (X,Y) with marginals equal to F and G respectively (see Gibbs and Su (2002) for a review on this and other distances for probability distribution functions).

Theorem 4.1 $\mathcal{F}$ *is continuous on* S*.*

Proof.
First case. Continuity on the axes
Consider a point $(0,\bar{w})$ with $\bar{w} > 0$ and note that $\mathcal{F}(0,\bar{w}) = \delta_0$ because of (ii) of Theorem 3.1. For $(b,w) \in S$,

$$d_W(\mathcal{F}(b,w),\mathcal{F}(0,\bar{w})) = \int_0^1 [1-\mathcal{F}(b,w)(x)]\,dx = \mathrm{E}[Z_\infty(b,w)] = \frac{b}{b+w}.$$

Hence $\lim_{(b,w)\to(0,\bar{w})} d_W(\mathcal{F}(b,w),\mathcal{F}(0,\bar{w}))=0$. This proves that $\mathcal{F}$ is continuous on the axis $b=0$; by symmetry, $\mathcal{F}$ is continuous also on the axis $w=0$.

Second case. Continuity at the inner points of S.
Consider a point $(\bar{b},\bar{w})\in S$ with $\bar{b}>0$ and $\bar{w}>0$. For $(b,w)\in S$, compare two randomly reinforced urns with reinforcement distributions equal to μ and initial compositions $(\bar{b},\bar{w})$ and (b,w) respectively. As in the proof of Theorem 3.2, the two urns are coupled in the sense that the same processes $\{M_n\}$ and $\{U_n\}$ generate both $\{(B_n(b,w),W_n(b,w))\}$ and $\{(B_n(\bar{b},\bar{w}),W_n(\bar{b},\bar{w}))$ according to the dynamics described in (1); let $Z_\infty(\bar{b},\bar{w})$ and $Z_\infty(b,w)$ be the limit proportions for the two urns.

From (13) and the triangular inequality, it follows that

$$\begin{aligned} &d_W(\mathcal{F}(b,w),\mathcal{F}(\bar{b},\bar{w}))\\ &\quad\le \mathrm{E}(|Z_\infty(b,w)-Z_\infty(\bar{b},\bar{w})|)\\ &\quad\le \mathrm{E}(|Z_\infty(b,w)-Z_N(b,w)|)+\\ &\qquad+\mathrm{E}(|Z_\infty(\bar{b},\bar{w})-Z_N(\bar{b},\bar{w})|)+\mathrm{E}(|Z_N(b,w)-Z_N(\bar{b},\bar{w})|) \end{aligned}\tag{14}$$

for all $N\ge 1$.

Next we prove that the first two terms of the last right member can be taken arbitrarily small for N large enough. In fact, for all $(b,w)\in S$ and $m\ge 1$,

$$\begin{aligned} &\mathrm{E}(|Z_\infty(b,w)-Z_m(b,w)|)\\ &\quad\le \mathrm{E}^{1/2}(|Z_\infty(b,w)-Z_m(b,w)|^2)\\ &\quad= \mathrm{E}^{1/2}\left(\mathrm{E}(|Z_\infty(b,w)-Z_m(b,w)|^2|(B_1,W_1),...,(B_m,W_m))\right)\\ &\quad= \mathrm{E}^{1/2}\left(\mathrm{E}(Z_\infty^2(B_m,W_n)|(B_m,W_m))-(\frac{B_m}{B_m+W_m})^2\right)\\ &\quad\le \mathrm{E}^{1/2}\left(\frac{B_mW_m}{(B_m+W_m)^2}\frac{\beta}{B_m+W_m+\beta}\right); \end{aligned}$$

the last equality is true because the process $\{(B_n,W_n)\}$ is Markov and, moreover, $\mathrm{E}(Z_\infty(b,w)|(B_1,W_1),...,(B_m,W_m))=B_m(B_m+W_m)^{-1}$; the last inequality follows from Lemma 3.1 as in the proof of Theorem 3.3. Since $\mu(\{0\})<1$,

$$P[\lim_{n\to+\infty}\sum_{i=1}^n M_i=+\infty]=1;$$

therefore, for every $\epsilon>0$ and $K>0$ there is an $m=m(K,\epsilon)$ such that

$$P[\sum_{i=1}^m M_i>K]\ge 1-\epsilon^2.$$

For $(b,w)\in S$, let F be the event that is true when

$$B_m(b,w)+W_m(b,w)+\beta=\beta+b+w+\sum_{i=1}^m M_i>K;$$

then $P[F] \geq 1-\epsilon^2$. Set $K = \dfrac{\beta}{4\,\varepsilon^2}$ and $N = m(K,\epsilon)$. Then, for all $(b,w) \in S$,

$$\begin{aligned}
&\mathrm{E}(|Z_\infty(b,w) - Z_N(b,w)|) \\
&\quad \leq \mathrm{E}^{1/2}\left(\frac{B_N W_N}{(B_N+W_N)^2}\frac{\beta}{B_N+W_N+\beta}\right) \\
&\quad = \left(\mathrm{E}(\frac{B_N W_N}{(B_N+W_N)^2}\frac{\beta}{B_N+W_N+\beta}; F) + \mathrm{E}(\frac{B_N W_N}{(B_N+W_N)^2}\frac{\beta}{B_N+W_N+\beta}; F^c)\right)^{1/2} \\
&\quad \leq \left(\frac{1}{4}\,\mathrm{E}(\frac{\beta}{B_N+W_N+\beta}; F) + P[F^c]\right)^{1/2} \\
&\quad \leq (\epsilon^2 P[F] + P[F^c])^{1/2} \\
&\quad \leq \epsilon\sqrt{2}. \qquad (15)
\end{aligned}$$

Now consider the term $\mathrm{E}(|Z_N(b,w) - Z_N(\bar{b},\bar{w})|)$ in (14). For $i = 1, 2, ..., N$, let

$$A_i = \{X_i(b,w) \neq X_i(\bar{b},\bar{w})\}$$

and set $A = \bigcup_{i=1}^N A_i$. Then

$$P(A_1) = \left|\frac{b}{b+w} - \frac{\bar{b}}{\bar{b}+\bar{w}}\right|$$

and, for $i = 1, ..., N-1$,

$$P(A_{i+1}|(\bigcup_{j=1}^{i} A_i)^c) \leq \left|\frac{b}{b+w} - \frac{\bar{b}}{\bar{b}+\bar{w}}\right|.$$

Hence,

$$\begin{aligned}
P(A) &= P(A_1 \cup A_2 \cup \cdots \cup A_N) \\
&= P(A_1 \cup (A_1 \cap {A_2}^C) \cup \cdots \cup (A_N \cap (A_1 \cup \cdots \cup A_{N-1})^C)) \\
&\leq P(A_1) + P(A_1|{A_2}^C) + \cdots + P(A_N|(A_1 \cup \cdots \cup A_{N-1})^C) \\
&\leq N\left|\frac{b}{b+w} - \frac{\bar{b}}{\bar{b}+\bar{w}}\right|.
\end{aligned}$$

Moreover, when A^c is true $|Z_N(b,w) - Z_N(\bar{b},\bar{w})| \leq \left|\dfrac{b}{b+w} - \dfrac{\bar{b}}{\bar{b}+\bar{w}}\right|$. Therefore,

$$\begin{aligned}
&\mathrm{E}(|Z_N(b,w) - Z_N(\bar{b},\bar{w})|) \\
&\quad = \mathrm{E}(|Z_N(b,w) - Z_N(\bar{b},\bar{w})|; A^C) + \mathrm{E}(|Z_N(b,w) - Z_N(\bar{b},\bar{w})|; A) \\
&\quad \leq (2N+1)\left|\frac{b}{b+w} - \frac{\bar{b}}{\bar{b}+\bar{w}}\right|. \qquad (16)
\end{aligned}$$

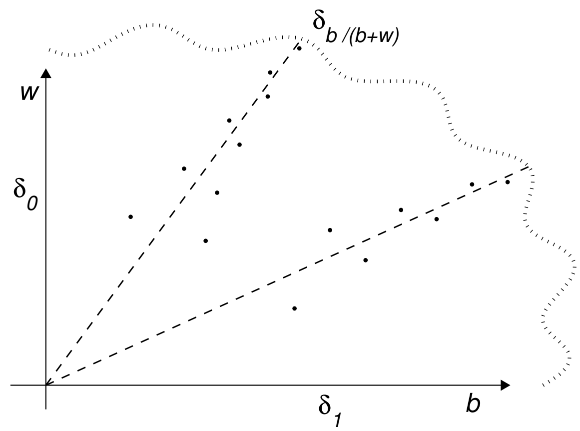

Figure 2: Continuity of $\mathcal{F}$ on the points of the projective line.

To finish the proof, let $\epsilon > 0$. From (14), (15) and (16) it follows that there is an $N = N(\epsilon)$ such that

$$d_W(\mathcal{F}(b,w), \mathcal{F}(\bar{b},\bar{w})) \leq \varepsilon 2\sqrt{2} + (2N+1)\left|\frac{b}{b+w} - \frac{\bar{b}}{\bar{b}+\bar{w}}\right|.$$

Hence

$$\lim_{(b,w)\to(\bar{b},\bar{w})} d_W(\mathcal{F}(b,w), \mathcal{F}(\bar{b},\bar{w})) \leq \varepsilon 2\sqrt{2}.$$

Since ε is arbitrary, this proves that

$$\lim_{(b,w)\to(\bar{b},\bar{w})} d_W(\mathcal{F}(b,w), \mathcal{F}(\bar{b},\bar{w})) = 0.$$

□

An immediate corollary of Theorem 3.3 is that $\mathcal{F}$ is continuous at the points of the projective line corresponding to the directions $b(b+w)^{-1}$, if we set $\mathcal{F}$ equal to $\delta_{\frac{b}{b+w}}$ in these points. Figure 2 illustrates the next result.

Corollary 4.1 *For every $\varepsilon > 0$ there exists $K = K(\epsilon)$ such that if $b+w > K$, then*

$$d_W(\mathcal{F}(b,w),\ \delta_{\frac{b}{b+w}}) < \varepsilon.$$

We conclude the section by describing the behavior of $\mathcal{F}$ near the origin of R^2. Note that $\mathcal{F}$ cannot be extended with continuity in $(0,0)$. In fact $\mathcal{F}(b,w)$ has constant mean c along the points $(b,w) \in S$ such that $b(b+w)^{-1} = c$: hence, the weak limit of $\mathcal{F}(b,w)$ for $(b,w) \to (0,0)$ does not exist.

Theorem 4.2 *For every $\varepsilon > 0$ there is a neighborhood U_ε of $(0,0)$ such that*

$$d_W(\mathcal{F}(b,w), \frac{b}{b+w}\delta_1 + \frac{w}{b+w}\delta_0) < \varepsilon,$$

for $(b,w) \in U_\varepsilon$.

Proof. As stated in Remark 3.2, $\mathcal{F}$ solves (10) if and only if, for all $(b,w) \in S$,

$$\mathcal{F}(b,w) = \frac{b}{b+w}\int_{(0,\beta]} \mathcal{F}(b+k,w)\nu(dk) + \frac{w}{b+w}\int_{(0,\beta]} \mathcal{F}(b,w+k)\nu(dk)$$

where ν is the conditional probability distribution of M_1, given that $M_1 > 0$. Since the support of ν is contained in $(0,\beta]$, for every $\varepsilon > 0$ there is an $\alpha > 0$ such that $\nu((0,\alpha]) < \varepsilon/2$. Let $b < \alpha\cdot\varepsilon/2$ and $w < \alpha\cdot\varepsilon/2$ and set $y = b(b+w)^{-1}$. Then

$$\begin{aligned}
&d_W(\mathcal{F}(b,w),\ y\,\delta_1 + (1-y)\,\delta_0) \\
&\quad= \int_0^1 |\,\mathcal{F}(b,w)(x) - (1-y)|\ dx \\
&\quad= \int_0^1 \left| y\int \mathcal{F}(b+k,w)(x)\nu(dk) + (1-y)\int \mathcal{F}(b,w+k)(x)\nu(dk) - (1-y)\right| dx \\
&\quad\le \int_0^1 y\int \mathcal{F}(b+k,w)(x)\nu(dk)dx + \int_0^1 (1-y)\int [1-\mathcal{F}(b,w+k)(x)]\nu(dk)dx \\
&\quad= y\int \frac{w}{b+w+k}\ \nu(dk) + (1-y)\int \frac{b}{b+w+k}\ \nu(dk) \\
&\quad\le y\left(\frac{\varepsilon}{2} + \frac{w}{b+w+\alpha}\left(1-\frac{\varepsilon}{2}\right)\right) + (1-y)\left(\frac{\varepsilon}{2} + \frac{b}{b+w+\alpha}\left(1-\frac{\varepsilon}{2}\right)\right) \\
&\quad\le y\left(\frac{\varepsilon}{2} + \frac{\frac{\varepsilon}{2}\,\alpha}{\alpha}\right) + (1-y)\left(\frac{\varepsilon}{2} + \frac{\frac{\varepsilon}{2}\,\alpha}{\alpha}\right) \\
&\quad= \varepsilon.
\end{aligned}$$

□

Remark 4.1 *Theorem 4.2 implies that $\mathcal{F}$ can be extended with continuity in the origin $(0,0)$ along the lines $b(b+w)^{-1}$ = constant.*

5 $\mathcal{F}$ is the unique solution of the characteristic equation

Given the probability distribution μ, in order to describe the distribution of the next state of a randomly reinforced urn with reinforcement distributions equal to μ one needs to know either the current number of black and white balls contained in the urn or, equivalently, the current total number of balls in the urn and the proportion of black balls. This fact and the behavior of the function $\mathcal{F}$ near the origin and at infinity, suggest to transform the state space S according to the map

$$\tau : \begin{cases} x = \dfrac{1}{b+w}, \\ y = \dfrac{b}{b+w}. \end{cases}$$

Let $S^* = [0, \infty) \times [0, 1]$. For $(x, y) \in (0, \infty) \times [0, 1]$ define

$$\mathcal{F}^*(x, y) = \mathcal{F}(\tau^{-1}(x, y));$$

set

$$\mathcal{F}^*(0, y) = \delta_y$$

for $y \in [0, 1]$.

Theorem 5.1 *$\mathcal{F}^*$ is continuous on S^*.*

Proof. Because of Theorem 4.1, we need to prove the continuity of $\mathcal{F}^*$ only in the states $(0, y) \in S^*$, with $y \in [0, 1]$. Let $\bar{y} \in [0, 1]$ and consider the state $(0, \bar{y})$; the triangular inequality implies that, for every $(x, y) \in S^*$,

$$d_W(\mathcal{F}^*(x, y), \mathcal{F}^*(0, \bar{y})) = d_W(\mathcal{F}^*(x, y), \delta_{\bar{y}}) \leq d_W(\mathcal{F}^*(x, y), \delta_y) + d_W(\delta_{\bar{y}}, \delta_y).$$

However,

$$d_W(\mathcal{F}^*(x, y), \delta_y) = d_W(\mathcal{F}(\frac{y}{x}, \frac{1-y}{x}), \delta_y),$$

and Corollary 4.1 implies that this quantity converges to 0 when $x \to 0$, whereas

$$d_W(\delta_{\bar{y}}, \delta_y) = \int_0^1 |\bar{F}_{\delta_{\bar{y}}}(x) - \bar{F}_{\delta_y}(x)| \, dx = |y - \bar{y}|$$

that goes to 0 when $y \to \bar{y}$. □

The function $\mathcal{F}^*$ maps any $(x, y) \in (0, \infty) \times [0, 1]$ in the distribution $\mathcal{F}(yx^{-1}, (1-y)x^{-1})$ of the limit proportion Z_∞ of a randomly reinforced urn with reinforcement distributions equal to μ and initial composition equal to $(yx^{-1}, (1-y)x^{-1})$. The results of Section 3 allow some instructive descriptions of the distribution $\mathcal{F}^*(x, y)$. For instance: along the horizontal lines y = constant, the mean of $\mathcal{F}^*(x, y)$ is constant while the variance increases with x. Along the vertical lines x = constant, the variance of

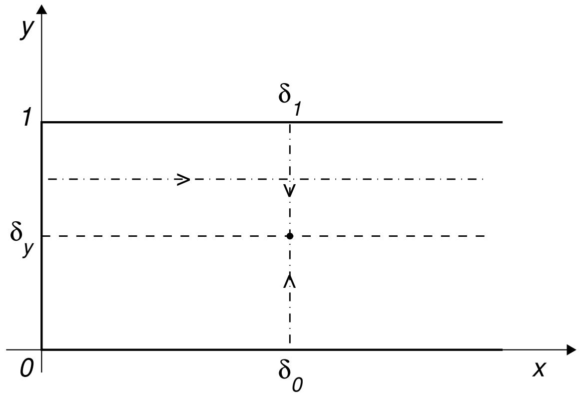

Figure 3: Boundary values of $\mathcal{F}^*$ and the variance of $\mathcal{F}^*(x, y)$.

$\mathcal{F}^*(x, y)$ increases as y moves from 0 to 1/2 and decreases for y moving from 1/2 to 1, reaching its maximum value at $y = 1/2$. Moreover, for $(x, y) \in S^*$,

$$\mathcal{F}^*(x, 0) = \delta_0,\ \mathcal{F}^*(x, 1) = \delta_1 \text{ and } \mathcal{F}^*(0, y) = \delta_y.$$

See Figure 3 for illustration.

Finally, note that $\mathcal{F}^*$ satisfies the characteristic equation

$$\begin{aligned} &\mathcal{F}^*(x, y) \\ &= y \int \mathcal{F}^* \left(\frac{x}{1 + kx}, \frac{y + kx}{1 + kx} \right) \mu(dk) + (1 - y) \int \mathcal{F}^* \left(\frac{x}{1 + kx}, \frac{y}{1 + kx} \right) \mu(dk). \end{aligned} \tag{17}$$

Let $\mathbb{C}(S^*)$ be the space of continuous function $G : S^* \to \mathcal{P}([0, 1])$ such that, for every $(x, y) \in S^*$,

$$G(x, 0) = \delta_0,\ G(x, 1) = \delta_1 \text{ and } G(0, y) = \delta_y.$$

For $(x, y) \in S^*$ and $G \in \mathbb{C}(S^*)$, define

$$A^*(G)(x, y) = y \int G \left(\frac{x}{1 + kx}, \frac{y + kx}{1 + kx} \right) \mu(dk) + (1 - y) \int G \left(\frac{x}{1 + kx}, \frac{y}{1 + kx} \right) \mu(dk);$$

then $A^*(G)(x, y) \in \mathcal{P}([0, 1])$. Note that, for every $G \in \mathbb{C}(S^*)$, the function $A^*(G) \in \mathbb{C}(S^*)$; hence we may regard A^* as an operator mapping $\mathbb{C}(S^*)$ into $\mathbb{C}(S^*)$.

Theorem 5.2 *$\mathcal{F}^*$ is the unique fixed point of A^*.*

Proof. For $c \in (0, +\infty)$, let $S_c^* = [0, c] \times [0, 1]$ and consider the space $\mathbb{C}(S_c^*)$ of the continuous functions $G : S_c^* \to \mathcal{P}([0, 1])$ such that, for every $(x, y) \in S^*$,

$$G(x, 0) = \delta_0,\ G(x, 1) = \delta_1 \text{ and } G(0, y) = \delta_y.$$

Set A_c^* to be the restriction of A^* to $\mathbb{C}(S_c^*)$; note that A_c^* maps $\mathbb{C}(S_c^*)$ into $\mathbb{C}(S_c^*)$. Let $\mathcal{F}_c^*$ be the restriction of $\mathcal{F}^*$ to S_c^*. Note that, for every $c > 0$, $\mathcal{F}_c^*$ is a fixed point of A_c^* since $\mathcal{F}^*$ satisfies (17): if $\mathcal{F}_c^*$ is the unique fixed point of A_c^*, then the theorem is proved because $S^* = \bigcup_{c>0} S_c^*$.

By way of contradiction, assume that, for $c > 0$, there is a $\mathcal{G}^* \in \mathbb{C}(S_c^*)$ that is a fixed point of A_c^* different from $\mathcal{F}_c^*$. Then we claim there is a $(\bar{x}, \bar{y}) \in S_c^*$ such that

$$d_W(\mathcal{F}_c^*(\bar{x}, \bar{y}), \mathcal{G}^*(\bar{x}, \bar{y})) > d_W(A_c^*(\mathcal{F}_c^*)(\bar{x}, \bar{y}), A_c^*(\mathcal{G}^*)(\bar{x}, \bar{y})) \tag{18}$$

which contradicts the assumption that both $\mathcal{F}_c^*$ and $\mathcal{G}^*$ are fixed points of A_c^*.

To prove (18), for $(x, y) \in S_c^*$, define

$$\Phi_{\mathcal{F}_c^*,\ \mathcal{G}^*}(x, y) = d_W(\mathcal{F}_c^*(x, y), \mathcal{G}^*(x, y)).$$

Observe that $\Phi_{\mathcal{F}_c^*,\ \mathcal{G}^*}$ is a continuous function defined on the compact set S_c^*. Moreover $\Phi_{\mathcal{F}_c^*,\ \mathcal{G}^*} \geq 0$ and, for $(x, y) \in S_c^*$,

$$\Phi_{\mathcal{F}_c^*,\ \mathcal{G}^*}(x, 0) = \Phi_{\mathcal{F}_c^*,\ \mathcal{G}^*}(x, 1) = \Phi_{\mathcal{F}_c^*,\ \mathcal{G}^*}(0, y) = 0.$$

However,$\Phi_{\mathcal{F}_c^*,\ \mathcal{G}^*} \not\equiv 0$ since, by assumption, $G^* \neq \mathcal{F}_c^*$. Let

$$M = \max\{\Phi_{\mathcal{F}_c^*,\ \mathcal{G}^*}(x, y) : (x, y) \in S_c^*\}.$$

Then $\Phi^{-1}_{\mathcal{F}_c^*,\ \mathcal{G}^*}(M)$ is a closed subset of $(0, c] \times (0, 1)$. Indicate with $\Pi_1 : R^2 \to R$ the projection on the first coordinate defined by setting $\Pi_1(z, w) = z$, for $(z, w) \in R^2$. Since Π_1 is an open map, $\Pi_1(\Phi^{-1}_{\mathcal{F}_c^*,\ \mathcal{G}^*}(M))$ is a closed subset of $(0, c]$. Let

$$\bar{x} = \min\left\{x : x \in \Pi_1(\Phi^{-1}_{\mathcal{F}_c^*,\ \mathcal{G}^*}(M))\right\}$$

and $\bar{y}$ such that $\Phi_{\mathcal{F}^*,\ \mathcal{G}^*}(\bar{x}, \bar{y}) = M$: the point $(\bar{x}, \bar{y})$ makes (18) true.

In fact, firstly note that, for $(x, y) \in S_c^*$ with $x < \bar{x}$, $\Phi_{\mathcal{F}^*,\ \mathcal{G}^*}(x, y) < M$. Then compute

$$
\begin{aligned}
&d_W(A_c^*(\mathcal{F}_c^*)(\bar{x},\bar{y}), A_c^*(\mathcal{G}^*)(\bar{x},\bar{y})) \\
&\quad = \int_0^1 |A_c^*(\mathcal{F}^*)(\bar{x},\bar{y})(t) - A_c^*(\mathcal{G}^*)(\bar{x},\bar{y})(t)|\ dt \\
&\quad = \int_0^1 \Bigg| \bar{y} \int \mathcal{F}^*(\frac{\bar{x}}{1+k\bar{x}}, \frac{\bar{y}+k\bar{x}}{1+k\bar{x}})(t)\mu(dk) + (1-\bar{y}) \int \mathcal{F}^*(\frac{\bar{x}}{1+k\bar{x}}, \frac{\bar{y}}{1+k\bar{x}})(t)\mu(dk) \\
&\qquad -\bar{y} \int \mathcal{G}^*(\frac{\bar{x}}{1+k\bar{x}}, \frac{\bar{y}+k\bar{x}}{1+k\bar{x}})(t)\mu(dk) - (1-\bar{y}) \int \mathcal{G}^*(\frac{\bar{x}}{1+k\bar{x}}, \frac{\bar{y}}{1+k\bar{x}})(t)\mu(dk) \Bigg|\ dt \\
&\quad \leq \bar{y} \int \Phi_{\mathcal{F}^*,\ \mathcal{G}^*}\left(\frac{\bar{x}}{1+k\bar{x}}, \frac{\bar{y}+k\bar{x}}{1+k\bar{x}}\right)\mu(dk) \\
&\qquad\qquad +(1-\bar{y}) \int \Phi_{\mathcal{F}^*,\ \mathcal{G}^*}\left(\frac{\bar{x}}{1+k\bar{x}}, \frac{\bar{y}}{1+k\bar{x}}\right)\mu(dk) \\
&\quad < \bar{y}\cdot M + (1-\bar{y})\cdot M \\
&\quad = \Phi_{\mathcal{F}^*,\ \mathcal{G}^*}(\bar{x},\bar{y}) \\
&\quad = d_W(\mathcal{F}^*(\bar{x},\bar{y}), \mathcal{G}^*(\bar{x},\bar{y})).
\end{aligned}
$$

This proves (18) and shows that $\mathcal{F}_c^*$ is the unique fixed point of A_c^*. □

Theorem 3.5 is now easy to obtain.

Proof. **[of Theorem 3.5]**. Theorem 4.1 proves that $\mathcal{F}$ is continuous, while $\mathcal{F}$ satisfies conditions (a)-(c) because of (i) and (ii) of Theorem 3.1 and because of Corollary 4.1. Assume that $G : S \to \mathcal{P}([0,1])$ is another continuous solution of (10) for which (a)-(c) are true; then, as in Theorem 5.1, show that $G^* : S^* \to \mathcal{P}([0,1])$ defined by setting $G^*(x,y) = G(\tau^{-1}(x,y))$, for $(x,y) \in (0,\infty)\times[0,1]$, and $G^*(0,y) = \delta_y$, for $y \in [0,1]$, is continuous on S^* and satisfies (a)-(c). Verify moreover that $A^*(G^*) = G^*$, i.e. that G^* solves (17). Hence Theorem 5.2 proves that $G^* = F^*$ and thus $G = F$. □

6 An example

When $\mu = \delta_\beta$, the point mass at a given $\beta > 0$, our two-color randomly reinforced urn with reinforcement distributions equal to μ becomes a Polya's urn and the unique continuous solution of (10) satisfying conditions (a)-(c) of Theorem 3.5 is

$$\mathcal{F}(b,w) = \mathrm{Beta}(b/\beta, w/\beta)$$

for $(b,w) \in S$, where, for $c,d > 0$, $\mathrm{Beta}(c,d)$ is the Beta distribution on $[0,1]$ whereas we define $\mathrm{Beta}(c,0) = \delta_1$ and $\mathrm{Beta}(0,d) = \delta_0$.

Now let $\mu = p\delta_0 + (1-p)\delta_\beta$ with $p \in (0,1)$ and $\beta > 0$: then, by Remark 3.2, Beta$(b/\beta, w/\beta)$ is again the unique continuous solution of (10) satisfying conditions (a)-(c) of Theorem 3.5.

The next step is to assume $\mu = p\delta_\alpha + (1-p)\delta_\beta$ with $0 < \alpha < \beta$ integers and $p \in (0,1)$. Let $M = p\alpha + (1-p)\beta$ be the mean of μ. A result by Athreya (1969) implies that the solution $\mathcal{F}(b,w)$ of (10) determined by Theorem 3.5 cannot be equal to Beta$(b/M, w/M)$, for $(b,w) \in S$. A tentative guess could be

$$\mathcal{F}(b,w) = p\text{Beta}(b/\alpha, w/\alpha) + (1-p)\text{Beta}(b/\beta, w/\beta) \tag{19}$$

for $(b,w) \in S$. This $\mathcal{F}$ does not solve (10): in fact, for $(b,w) \in S$, the second moment of $\mathcal{F}(b,w)$ defined as in (19) coincides with the second moment of the right member of (10) if and only if

$$\frac{bwp(1-p)(\beta-\alpha)^2}{(b+w)(b+w+\alpha)(b+w+\beta)(b+w+\alpha+\beta)} = 0$$

for all $(b,w) \in S$; but this can't happen since $0 < \alpha < \beta$ and $p \in (0,1)$. Hence in this case the analytical expression of $\mathcal{F}(b,w)$ is still unknown, although interesting approximations for it can be found by simulations, as in May.et.al. (2005).

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