Control and Inverse Problems Theory Approaches for Solving Some Hydrodynamics Equations

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Abstract – In this study, we apply optimal control approaches and adjoint equations to construct iterative processes for some problems of hydrodynamics. First, the iterative processes are proposed for an abstract problem given by the system of operator equations. Second, the general methodology is applied to the generalized Stokes equations perturbed by a skew-symmetric bounded operator, to the stationary linearized Navier–Stokes equations (the Oseen problem) and to the non-stationary Stokes problem. The convergence rates of the iterative algorithms are studied.

In some physical processes of fluid mechanics, the internal and pressure forces are the main forces that affect the velocity of particle motion. In some of these mathematical problems like ones for the Stokes or Navier–Stokes equations, these forces are expressed in equations as right-hand-side functions of a pressure gradient. We usually consider the solution both as an unknown function of the velocity vector $\phi(x)$ and as an unknown function of pressure $p(x)$. However, in the theory of boundary value problems the statements are very often encountered in which, besides a 'regular' solution (for the Stokes equations we understand it as a velocity vector function), all of the source functions or some of them are sought. These problems belong to the class of inverse problems. Thus, in view of the above interpretation, it is possible to consider (to some extent!) the Stokes problem, the Navier–Stokes problem, etc. as inverse problems since, along with a fluid velocity vector, we also seek a pressure function that defines one of the source functions. But then it may be assumed that in investigating and solving these problems many special algorithms and approaches developed in the classical theory of inverse problems [1, 10] may prove useful. Moreover, it may happen that if these special approaches are ignored while examining and especially numerically solving these problems, many difficulties can emerge (this is the case in some problems of fluid mechanics).

The approaches of the optimal control theory [1–5] in turn happened

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to be fruitful in examining and solving the inverse problems. It should be emphasized that most problems in the control theory are essentially inverse problems.

In view of the above assumptions, the author examines in this study the following approach to the solution of problems of the given class exemplified by a boundary value problem for the Stokes and Oseen systems: the pressure function is regarded as an 'additional' unknown to the 'main' component of a problem's solution, and the continuity condition from the system of equations is regarded as one of the 'observation conditions' ('observational data', etc.), which is prescribed to close the system. Then the problem, which is considered as an inverse one, is included in the family of optimal control problems depended on a regularization parameter. In the following, the optimal control problems are examined and solved by classical methods. At the final stage, we must show that we obtain the original problem and corresponding results for the trivial value of the regularization parameter (possibly, with additional restrictions on the initial data of the problem).

The above approach will be applied here to some of the boundary value problems for the Stokes equations perturbed by a skew-symmetric bounded operator and for the stationary linearized Navier–Stokes equations. Having formulated the original problem, the author reduces it to an optimal control problem, in which the pressure function $p(x)$ is a 'control', while the condition $\operatorname{div} \phi = 0$ belongs to the 'observation conditions' and is included in a cost functional. Later, having investigated the solvability of the optimal control problem, the author arrives at the well-known results on the solvability of the problem as a consequence of the statements obtained. Then the author considers some examples of iterative processes for the solution of the problems in question, justify them, and estimate their convergence rates. The realization of these processes consists of the solution of a sequence of classical elliptic (or parabolic) problems by the well-studied numerical methods (including the simplest ones) such as the finite element method and the finite difference method. It should be emphasized that in solving these problems the condition $\operatorname{div} \phi = 0$ is absent; as shown in [13], this condition considerably complicates the algorithms, especially in multidimensional problems and in the case of a domain of complex form (in addition, it imposes certain restrictions on the order of the employed finite elements in the finite element method).

In conclusion, we present some results of numerical experiments.

1. GENERAL APPROACH TO THE SOLUTION OF A CLASS OF PROBLEMS

Let $X_{\mathrm{c}}, H_0, H_{\mathrm{c}}, H_{\mathrm{ob}}$ be Hilbert spaces, let W, Y be Banach spaces (where 'ob' stands for 'observation' and 'c' for 'control'), and let $W^*, \ldots, Y^*$ be adjoint spaces, where $H_0 \equiv H_0^*$, $H_{\mathrm{ob}} \equiv H_{\mathrm{ob}}^*$, and $H_{\mathrm{c}} \equiv H_{\mathrm{c}}^*$ are basic. The inclusions $W \subseteq Y \subset H_0 \subset Y^* \subseteq W^*$ and $X_{\rm c} \subseteq H_{\rm c} \subseteq X_{\rm c}^*$ are assumed to be continuous and dense. It is also assumed that $W = W^{**}$.

Consider a class of problems given by the system of equations

$$L\phi = f + Bu, \qquad C\phi = h + Du \tag{1}$$

where f, h are the vectors of the given data, u is the data to be determined concurrently with ϕ [i.e. the pair ϕ, u is the solution of (1)]; L, B, C, D are linear operators; L and B are closed; the operator L acts from W into Y^*, B acts from $H_{\rm c}$ into Y^*, C acts from W into $H_{\rm ob}$, and D acts from $H_{\rm c}$ into $H_{\rm ob}$. We assume that the domain of L coincides with the domain of C, the domain of L is dense in W, while its range is dense in Y^*, i.e. $D(L) = D(C)$, $\overline{D(L)} = W$, $\overline{R(L)} = Y^*$, and L is a continuously invertible operator. Here the operators B, D are bounded and $D(B) = D(D) = X_{\rm c}$. If $f \in Y^*$ and $u \in H_{\rm c}$ are given, we assume that the first equation of (1) has the solution (which is generally weak) $\phi \in W$, and the estimate $\|\phi\|_W \leq C(\|f\|_{Y^*} + \|u\|_{H_{\rm c}})$ is valid.

Along with (1), we consider the optimal control problem: find $\phi = \phi(u)$ and the 'control' $u \in X_{\rm c}$ such that

$$L\phi = f + Bu, \qquad J_\alpha(u, \phi(u)) = \inf_{v \in D(B)} J_\alpha(v, \phi(v)), \tag{2}$$

where

$$\begin{gathered} J_\alpha(v, \phi(v)) = \alpha\|v\|^2_{\Lambda_{\rm c}} + \|C\phi - h - Dv\|^2_{H_{\rm ob}} \\ L\phi(v) = f + Bv, \qquad \alpha \geq 0 \\ \|v\|^2_{\Lambda_{\rm c}} = (\Lambda_{\rm c} v, v)_{H_{\rm c}} = \|\Lambda_{\rm c}^{1/2} v\|^2_{H_{\rm c}} \end{gathered} \tag{3}$$

and $\Lambda_{\rm c} : X_{\rm c} \to X_{\rm c}^*$ is a symmetric positive definite operator with domain $D(\Lambda_{\rm c})$, $\overline{D(\Lambda_{\rm c})} = X_{\rm c}$, the norm $\|v\|_{\Lambda_{\rm c}}$ being equivalent to the norm $\|v\|_{X_{\rm c}}$ in $X_{\rm c}$; $\alpha \geq 0$ is a parameter. If ϕ, u is the solution of (1), it is also a solution of (2) with $\alpha = 0$. Thus, for trivial value of α the problem (2) can be regarded as a weak statement of the problem (1) with the second equation considered in the 'least squares' sense.

If we exclude $\phi = L^{-1}(f + Bu)$ from (2), the functional $J_\alpha(u, \phi(u)) \equiv J_\alpha(u)$ has the form

$$J_\alpha(u) = \alpha\|u\|^2_{X_{\rm c}} + \|Au - g\|^2_{H_{\rm ob}} \tag{4}$$

where

$$A = CL^{-1}B - D, \qquad g = h - CL^{-1}f.$$

The variational equation corresponding to (4) reads:

$$\mathcal{A}_\alpha u \equiv \alpha\Lambda_{\rm c} u + A^* A u = A^* g \tag{5}$$

while the complete system of variational equations, which is equivalent to the problem (2), is

$$\begin{gathered} L\phi(u) = f + Bu, \qquad L^*q = C^*(C\phi - h - Du) \\ \alpha\Lambda_c u + B^*q - D^*(C\phi - h - Du). \end{gathered} \tag{6}$$

If $\alpha > 0$, the operator of equation (5) is symmetric and positive definite. Using these properties, it is possible to use various iterative procedures for solving (5). The simplest of them can be written as

$$\Lambda_c u^{k+1} = \Lambda_c^k u^k - \tau_k(\alpha\Lambda_c u^k - A^*Au^k - A^*g), \quad k = 0, 1, \ldots \tag{7}$$

with some parameters $\{\tau_k\}$. If

$$\mathrm{Sp}(\Lambda_c^{-1}A^*A) \in [\mathbf{C}_0, \mathbf{C}_1], \quad \mathbf{C}_0, \mathbf{C}_1 = \mathrm{const}, \quad \mathbf{C}_0 \geq 0,\ \mathbf{C}_1 > 0 \tag{8}$$

then we can put

$$\tau_k = \tau = \frac{2}{2\alpha + \mathbf{C}_0 + \mathbf{C}_1}, \quad k = 0, 1, \ldots. \tag{9}$$

With these values of parameters $\{\tau_k\}$, the estimate of the convergence rate is valid [1, 12]:

$$\|u - u^k\|_{X_c} \leq C\left(\frac{\mathbf{C}_1 - \mathbf{C}_0}{2\alpha + \mathbf{C}_0 + \mathbf{C}_1}\right)^k \to 0, \quad k \to \infty, \tag{10}$$

where the constant C does not depend on k and u [with ϕ, q, the element u is the solution of (6)].

In terms of the operators of the problem (6) the algorithm (7) reads (for given u^k):

$$\begin{gathered} L\phi^k = f + Bu^k, \quad L^*q^k = C^*(C\phi^k - h - Du^k) \\ \Lambda_c w^k = B^*q - D^*(C\phi^k - h - Du^k) \\ u^{k+1} = u^k - \tau(\alpha u^k + w^k), \quad k = 0, 1, \ldots. \end{gathered} \tag{11}$$

From the estimate $\|\phi\|_W \leq C(\|f\|_{Y^*} + \|u\|_{H_c})$ and (10) we have also the estimate

$$\|\phi - \phi^k\|_W \leq C\left(\frac{\mathbf{C}_1 - \mathbf{C}_0}{2\alpha + \mathbf{C}_0 + \mathbf{C}_1}\right)^k \to 0, \quad k \to \infty, \tag{12}$$

where $C = \mathrm{const} > 0$ and $\phi = \phi(\alpha)$ is the component of the solution of the system (6).

If the system (1) is well-posed, we can put $\alpha \equiv 0$ in all the above iterative algorithms. Generally, if the problem (1) is ill-posed, we need to consider $\alpha \to +0$. The number of iterations in (11) should be adjusted to the value of α and to the error estimates of numerical methods to be used for solving all the equations from (11) at each step of iterations [10, 14].

Assume that the problem (1) is well-posed. Then we can put $\alpha \equiv 0$, $\Lambda_c = I$ (i.e. $X_c = H_c$) and the algorithm (11) has the 'geometric convergence rate' in this case (independently of the 'symmetry property' of $L, \ldots, D$). To implement the algorithm (11) one needs only to invert the operators L and L^*.

The methodology described earlier has been applied in [3] to solve the 'generalized Stokes problem'. In the following sections, we consider applications of this general methodology to the Stokes equations perturbed by a skew-symmetric bounded operator and to the stationary linearized Navier–Stokes problem (the Oseen problem).

2. COMPUTATIONAL PROCESS FOR SOLVING THE PERTURBED STOKES SYSTEM

Let us use the optimal control methods to study and construct algorithms for solving the 'generalized' Stokes problem perturbed by the skew-symmetric boundary operator:

$$\begin{gathered} -\alpha\Delta\phi + b\phi + K\phi = f - \nabla p \quad \text{in} \ \ \Omega \\ \operatorname{div}\phi = 0 \quad \text{in} \ \ \Omega, \qquad \int_\Omega p\,\mathrm{d}x = 0, \quad \phi = 0 \quad \text{on} \ \ \Gamma, \end{gathered} \tag{13}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with Lipschitz boundary $\Gamma \equiv \partial\Omega$, $x = (x_1, x_2, \ldots, x_n)$, $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$, $p(x)$ is a scalar function ('pressure'), $a = \text{const} > 0$, $b = \text{const} \geq 0$, $f = (f_1, f_2, \ldots, f_n)$ is a given vector function from $(L_2(\Omega))^n$, $K\phi \equiv (-l\phi_2, l\phi_1, 0, \ldots, 0)$, $l = l_0 + l_1 x_2$, $l_0, l_1 = \text{const}$. If $l \equiv 0$, the problem (13) is called 'the generalized Stokes problem'; for $l \equiv 0$ and $b \equiv 0$, the problem (13) is the classical Dirichlet boundary value problem for the stationary Stokes equations. Instead of the Dirichlet boundary condition, other boundary conditions can be considered (for example, 'periodicity conditions', see Section 5).

We introduce the spaces: $W \equiv (\overset{\circ}{W}{}^1_2(\Omega))^n$, $H_0 \equiv (L_2(\Omega))^n$, $H_c \equiv L_2^{(0)}(\Omega) = \{v : v \in L_2(\Omega), \int_\Omega v\,\mathrm{d}x = 0\}$. The problem can be written in the operator form

$$L\phi = f + Bp, \qquad C\phi = 0, \tag{14}$$

i.e. in the form (1) with $h \equiv 0$, $D \equiv 0$, $p \equiv u$, where

$$L = A_0 + K, \quad (A_0\phi, \psi) \equiv (a\nabla\phi, \nabla\psi) + (b\phi, \psi) \quad \forall\, \phi, \psi \in W$$

$$(Bp, \psi) \equiv (p, \operatorname{div}\psi)_{L_2(\Omega)} \quad \forall \psi \in W, \quad C\psi \equiv W, \quad C\psi \equiv \operatorname{div}\psi$$

$$L : W \to W^*, \quad B : H_c \to W^*, \quad C : W \to H_c$$

$$D(L) = W, \quad D(B) = H_c, \quad D(C) = W, \quad (\cdot,\cdot) \equiv (\cdot,\cdot)_{H_0}, \ \|\cdot\| \equiv \|\cdot\|_{H_0}.$$

Let us introduce the functional J_α:

$$J_\alpha(p,\phi) \equiv \alpha \|p\|^2_{W_2^m(\Omega)} + \|C\phi\|^2_{H_c},$$

where $m = 0$ or $m = 1$, $\alpha = \text{const} \geq 0$, and consider the set of problems given by (2). Then in (5) we have (with $p \equiv u$ and $\Lambda_c \equiv \Lambda_m$):

$$\Lambda_0 p = p, \qquad m = 0$$

$$(\Lambda_1 p, \widetilde{p})_{L_2(\Omega)} \equiv (\nabla p, \nabla \widetilde{p})_{L_2(\Omega)} + (p, \widetilde{p})_{L_2(\Omega)}$$

$$\forall\, p, \widetilde{p} \in W_2^1(\Omega), \qquad m = 1$$

$$L \equiv A_0 + K, \quad A = -\text{div}(A_0 + K)^{-1}\nabla$$

and equations (6) read:

$$\begin{gathered} A_0\phi + K\phi = f + Bp, \qquad A_0 q - Kq = C^*C\phi \\ \alpha\Lambda_m p + B^* q = 0, \quad m = 0, 1. \end{gathered} \tag{15}$$

In the sequel, only the case '$m = 0$' is considered (some specific properties of the problems arising for $m = 1$ are discussed in [3]).

Let us study the properties of the operator $\mathcal{A}_0 \equiv A^*A$. To prove the positive definiteness of $\mathcal{A}_0$ we consider the form $(Ap,p)_{L_2(\Omega)} \equiv \|p\|^2_A \,\forall\, p \in H_c$. If $\phi \equiv -(A_0 + K)^{-1}\nabla p$, then

$$\begin{aligned} \|p\|^2_A = (\text{div}\phi, p) = -(\phi, \nabla p) = (\phi, A_0\phi + K\phi) = (\phi, A_0\phi) \\ \geq (a\lambda_1 + b)\|\phi\|^2 > 0, \qquad p \not\equiv 0, \end{aligned} \tag{16}$$

where $\lambda_1 = \min_i\ \lambda_i > 0$ and $\{\lambda_1\}$ are eigenvalues of the eigenvalue problem:

$$-\Delta\varphi_j = \lambda_j\varphi_j \quad \text{in}\ \ \Omega, \qquad \varphi_j = 0 \quad \text{on}\ \ \Gamma, \qquad \|\varphi_j\|_{L_2(\Omega)} = 1. \tag{17}$$

From (16) we conclude that the operators $\mathcal{A}_0,\ A,\ A^*$ are positive definite.

Now, using simple calculations, we can write down the equalities (with $\phi \equiv -(A_0 + K)^{-1}\nabla p$):

$$\begin{aligned} (Ap,p)_{L_2(\Omega)} \equiv \|p\|^2_A = (\phi, A_0\phi) = (-\nabla p - K\phi,\ A_0^{-1}(-\nabla p - K\phi)) \\ = (\nabla p, A_0^{-1}\nabla p) + (K\phi, A_0^{-1}K\phi) + 2(K\phi, A_0^{-1}\nabla p). \end{aligned}$$

Since

$$\begin{aligned} (K\phi,\ A_0^{-1}\nabla p) = (K\phi, -\phi - A_0^{-1}K\phi) = (K\phi, \phi) - (K\phi, A_0^{-1}K\phi) \\ = -(K\phi, A_0^{-1}K\phi) \end{aligned}$$

we have

$$(Ap,p)_{L_2(\Omega)} \equiv \|p\|^2_A = (A_0^{-1}\nabla p, \nabla p) - (A_0^{-1}K\phi, K\phi). \tag{18}$$

The relations also hold true [1, 3]:

$$\frac{\mu_1\lambda_1}{(a\lambda_1+b)}\,\|p\|^2_{L_2(\Omega)} \le (A_0^{-1}\nabla p, \nabla p) \le \frac{1}{a}\,\|p\|^2_{L_2(\Omega)} \quad \forall\, p \in H_{\rm c}, \tag{19}$$

where $1 > \mu_1 = \mu_1(\Omega) = \mathrm{const} > 0$ (estimates for the constant μ are studied in [7]). From (18), (19) we get (with $\|l\|_\infty \equiv \|l\|_{L_\infty(\Omega)}$)

$$\|p\|^2_A \le \|p\|^2_{L_2(\Omega)}/a$$

$$(\nabla p,\, A_0^{-1}\nabla p) = \|p\|^2_A + (A_0^{-1}K\phi, K\phi)$$

$$\le \|p\|^2_A + \frac{\|l\|^2_\infty}{a\lambda_1+b}\|\phi\|^2 \le \left(1+\frac{\|l\|^2_\infty}{(a\lambda_1+b)^2}\right)\|p\|^2_A \tag{20}$$

$$\frac{\mu_1\lambda_1(a\lambda_1+b)}{(a\lambda_1+b)^2+\|l\|^2_\infty}\,\|p\|^2_{L_2(\Omega)} \le \|p\|^2_A \le \frac{1}{a}\|p\|^2_{L_2(\Omega)}.$$

The estimates and relations like (20) are valid for the operator A^* too. To prove them it is sufficient to replace K by $-K$ in the above considerations.

We obtain the spectrum bounds for the operator $\mathcal{A}_0 = A^*A$. If $p, \widetilde{p} \in H_{\rm c}$, $\phi \equiv -(A_0+K)^{-1}\nabla p$, $\widetilde{\phi} \equiv -(A_0+K)^{-1}\nabla\widetilde{p}$, we have

$$(Ap, \widetilde{p}\,)_{L_2(\Omega)} = (\phi, -\nabla\widetilde{p}\,) = (a\nabla\phi, \nabla\widetilde{\phi}) + (b\phi, \widetilde{\phi}) + (K\phi, \widetilde{\phi})$$

$$\le \|p\|_A\|\widetilde{p}\,\|_A + \frac{\|l\|_\infty}{a\lambda_1+b}\|p\|_A\|\widetilde{p}\|_A$$

$$\le \frac{1}{a}\left(1+\frac{\|l\|_\infty}{a\lambda_1+b}\right)\|p\|_{L_2(\Omega)}\|\widetilde{p}\,\|_{L_2(\Omega)}.$$

Hence,

$$\|A\| \le \frac{1}{a}\left(1+\frac{\|l\|_\infty}{a\lambda_1+b}\right). \tag{21}$$

Since $\|p\|^2_A = (Ap,p)_{L_2(\Omega)} \le \|Ap\|_{L_2(\Omega)}\|p\|_{L_2(\Omega)}$, from (20) we have

$$\frac{\mu_1\lambda_1(a\lambda_1+b)}{(a\lambda_1+b)^2+\|l\|^2_\infty} \le \frac{\|Ap\|_{L_2(\Omega)}}{\|p\|_{L_2(\Omega)}} \quad \forall\, p \ne 0. \tag{22}$$

Now, from (21), (22) it follows that

$$\mathrm{Sp}(A^*A) \in [\mathbf{C}_0, \mathbf{C}_1], \tag{23}$$

where

$$
\begin{gathered}
\mathbf{C}_0 = \left(\frac{\mu_1 \lambda_1 (a\lambda_1 + b)}{(a\lambda_1 + b)^2 + \|l\|_\infty^2} \right)^2, \qquad \mathbf{C}_1 = \frac{1}{a^2} \left(1 + \frac{\|l\|_\infty}{a\lambda_1 + b} \right)^2 \\
\xi \equiv \frac{\mathbf{C}_0}{\mathbf{C}_1} = \frac{\mu_1^2 (1 + b_0)^4}{((1 + b_0)^2 + a_0^2)^2 (1 + b_0 + a_0)^2} \\
a_0 \equiv \frac{\|l\|_\infty}{a\lambda_1}, \qquad b_0 \equiv \frac{b}{a\lambda_1}.
\end{gathered} \tag{24}
$$

Let us formulate the above statements as the following assertions.

Proposition 2.1 The assertions hold true:

(1) The operators $A,\ A^*,\ \mathcal{A}_0 = A^* A$ are positive definite.

(2) $\mathrm{Ker(A)} = \mathrm{Ker(A^*)} = \mathrm{Ker}(\mathcal{A}_0) = \{0\}$.

(3) The problem (15) is well-posed for $\alpha \geq 0$.

(4) $\mathrm{Sp}(A^* A) \in [\mathbf{C}_0, \mathbf{C}_1]$, where $\mathbf{C}_0, \mathbf{C}_1$ are given by (24).

To solve (15) the iterative algorithm (11) can be chosen. Assume that $\alpha = 0,\ m = 0$. Then this algorithm reads (in the 'classical' form of equations):

$$
\begin{gathered}
-a\Delta \phi^k + b\phi^k + K\phi^k = f - \nabla p^k \quad \text{in } \Omega \\
\phi^k = 0 \quad \text{on } \Gamma, \qquad \int_\Omega p^k \, \mathrm{d}x = 0 \\
-a\Delta q^k + b q^k - K q^k = -\nabla \mathrm{div}\, \phi^k \quad \text{in } \Omega \\
q^k = 0 \quad \text{on } \Gamma \\
p^{k+1} = p^k - \tau \,\mathrm{div}\, q^k, \quad k = 0, 1, \ldots,
\end{gathered} \tag{25}
$$

where $\tau = 2/(\mathbf{C}_0 + \mathbf{C}_1)$, and constants $\mathbf{C}_0, \mathbf{C}_1$ are given by (24).

Let ϕ, p be the solution of (13). Then it is easy to see that $\phi, p, q \equiv 0$ is the solution of the problem (15) as $\alpha = 0$. Using this remark and the estimate (10), we have the following assertion.

Proposition 2.2 The functions $\{\phi^k\}, \{p^k\}$ constructed by the iterative process (25) can be regarded as the approximate solutions of the problem (13) and the estimate

$$
\|\phi^k - \phi\|_W + \|p - p^k\|_{H_c} \leq C \left(\frac{1 - \xi}{1 + \xi} \right)^k \to 0, \quad k \to \infty, \tag{26}
$$

where $C = \mathrm{const}$ and $\xi = \mathbf{C}_0 / \mathbf{C}_1$ holds true.

Note that $\xi = \mu_1^2$ for $b \equiv 0$ and $l \equiv 0$. Since μ_1 depends only on the geometry of Ω [7, 9], we conclude: if $b \equiv 0$ and $l \equiv 0$, the convergence rate of the iterative algorithm (25) depends only on the geometry of Ω.

Thus, if we have some fast procedures to solve elliptic problems, applying the iterative procedure like (25), we can effectively solve the generalized Stokes problem perturbed by the bounded skew-symmetric operator K.

3. COMPUTATIONAL PROCESS FOR THE OSEEN PROBLEM

Let us consider the problem (13):

$$\begin{gathered} -a\Delta\phi + b\phi + K\phi = f - \nabla p \quad \text{in} \;\; \Omega, \\ \operatorname{div}\phi = 0 \quad \text{in} \;\; \Omega, \quad \int\limits_{\Omega} p\, d\Omega = 0 \quad \text{on} \quad \Gamma, \end{gathered} \tag{27}$$

where K is determined as follows:

$$K\phi \equiv (U,\nabla) \equiv \sum_{i=1}^{n} U_i \frac{\partial}{\partial x_i}\,\phi \tag{28}$$

with given vector function $U = (U_1, U_2, \ldots, U_n)$. For simplicity we assume that

$$\begin{gathered} U \in (C^{(1)}(\Omega))^n, \\ \operatorname{div} U = 0 \quad \text{in} \;\; \Omega, \qquad U = 0 \quad \text{on} \; \Gamma, \\ U_\infty \equiv \|U\|_{C(\Omega)} = \left\| \left(\sum_{i=1}^{n} U_i^2 \right)^{1/2} \right\|_{C(\Omega)} < \infty. \end{gathered} \tag{29}$$

Other coefficients from (27) are described in the previous section.

The problem (27) is the linearized stationary Navier–Stokes problem (the 'Oseen problem'). The operator of the form (27) can be written as (14). The system of variational equations corresponding to the set of the optimal control problems with cost functional $J_\alpha(p,\phi)$ is given by (15).

Let us study some properties of the operators $A \equiv -\operatorname{div}(A_0 + K)^{-1}\nabla$, A^*, $\mathcal{A}_0 \equiv A^*A$, which correspond to the problem (27).

Relations (16), (18), (19) and the estimate $\|p\|_A^2 \le \|p\|_{L_2(\Omega)}^2/a$ also hold true. Then from

$$\begin{aligned} (\nabla p, A_0^{-1}\nabla p)_{L_2(\Omega)} &= \|p\|_A^2 + (A_0^{-1}K\phi, K\phi) \le \|p\|_A^2 + \frac{U_\infty^2}{a(\lambda_1 a + b)}(a\nabla\phi, \nabla\phi) \\ &= \|p\|_A^2 + \frac{U_\infty^2}{a(\lambda_1 a + b)}\Big((a\nabla\phi,\nabla\phi) + (b\phi,\phi)\Big) - \frac{U_\infty^2 (b\phi,\phi)}{a(\lambda_1 a + b)} \end{aligned}$$

it follows that

$$\frac{U_\infty^2}{a(a\lambda_1 + b)}\,(b\phi,\phi) + (\nabla p, A_0^{-1}\nabla p)_{L_2(\Omega)} \le \Big(1 + \frac{U_\infty^2}{a(a\lambda_1 + b)}\Big)\,\|p\|_A^2$$

and

$$\frac{\mu_1\lambda_1}{a\lambda_1 + b}\,\|p\|_{L_2(\Omega)}^2 + \frac{U_\infty^2}{a(a\lambda_1 + b)}\,(b\phi,\phi) \le \Big(2 + \frac{U_\infty^2}{a(\lambda_1 a + b)}\Big)\,\|p\|_A^2. \tag{30}$$

Hence the estimates

$$\frac{a\mu_1\lambda_1}{a(\lambda_1 a+b)+U_\infty^2}\,\|p\|^2_{L_2(\Omega)} \le \|p\|^2_A \le \frac{1}{a}\,\|p\|^2_{L_2(\Omega)} \tag{31}$$

hold true.

The estimates and relations (16), (18), (19), (31) for the operator A^* are also valid for this problem.

By virtue of the same reasoning as was used for proving (21), (28), we have

$$\frac{\mu_1\lambda_1}{(a\lambda_1+b)+U_\infty^2/a} \le \frac{\|Ap\|_{L_2(\Omega)}}{\|p\|_{L_2(\Omega)}} \quad \forall\, p\neq 0 \tag{32}$$

and

$$\begin{aligned}
|(Ap,\widetilde{p})_{L_2(\Omega)}| &= |(a\nabla\phi,\nabla\widetilde{\phi}) + (b\phi,\widetilde{\phi}) + (\phi,(U,\nabla)\widetilde{\phi})| \\
&\le \Big((a\nabla\phi,\nabla\phi)+(b\phi,\phi)+(A_0\phi,\phi)\Big)^{1/2} \\
&\times\Big((a\nabla\widetilde{\phi},\nabla\widetilde{\phi})+(b\widetilde{\phi},\widetilde{\phi})+(A_0^{-1}K\widetilde{\phi},K\widetilde{\phi})\Big)^{1/2} \\
&\le \frac{\sqrt{2}}{a}\,\|p\|_{L_2(\Omega)}\|\widetilde{p}\|_{L_2(\Omega)}
\end{aligned}$$

$$\|A\| \le \frac{\sqrt{2}}{a}. \tag{33}$$

From (32), (33) it follows that

$$\mathrm{Sp}\,(A^*A) \in [\mathbf{C}_0,\mathbf{C}_1], \tag{34}$$

where

$$\mathbf{C}_0 = \left(\frac{\mu_1\lambda_1}{(a\lambda_1+b)+U_\infty^2/a}\right)^2, \qquad \mathbf{C}_1 = \frac{2}{a^2} \tag{35}$$

$$\xi = \frac{\mathbf{C}_0}{\mathbf{C}_1} = \frac{\mu_1^2}{2(1+b/(a\lambda_1)+U_\infty^2/(a^2\lambda_1))^2}. \tag{36}$$

Thus, the following assertion is valid.

Proposition 3.1 If $U_\infty < \infty$, we have:
(1) The operators A, A^*, $\mathcal{A}_0 \equiv A^*A$ are positive definite.
(2) The problem (27) and the problems (15) with $K\phi \equiv (U,\nabla)\phi$ are well-posed.
(3) $\mathrm{Sp}\,(A^*A) \in [\mathbf{C}_0,\mathbf{C}_1]$, where $\mathbf{C}_0,\mathbf{C}_1$ are given by (35).
(4) The iterative process (25) (with $K\phi \equiv (U,\nabla)\phi$) to be applied to the solution of (27) converges and for its convergence rate the estimate (26) is valid, where ξ is defined in (36).

Now let us consider the system (15) with small positive α and $m=0$. In the 'classical form' this system reads:

$$-a\Delta\phi + b\phi + K\phi = f - \nabla p \quad \text{in } \Omega, \qquad \int_\Omega p\,\mathrm{d}x = 0, \qquad \phi = 0 \quad \text{on } \Gamma,$$

$$-a\Delta q + bq - Kq = -\nabla \mathrm{div}\,\phi \quad \text{in} \ \ \Omega, \qquad q = 0 \quad \text{on} \ \ \Gamma,$$
$$\alpha p + \mathrm{div}\,q = 0 \quad \text{in} \ \ \Omega, \tag{37}$$

where $K\phi \equiv (U, \nabla)\phi$, $\phi = \phi(\alpha)$, $p = p(\alpha)$. We denote by ϕ_0, p_0 the solution of (27). Since p, p_0 satisfy the equations

$$\alpha p + A^*Ap = A^*g, \qquad A^*Ap_0 = A^*g$$

$$A^*A(p - p_0) = -\alpha p, \quad \|A(p - p_0)\|^2_{H_c} = -\alpha(p, p - p_0)_{H_c},$$

where $g \equiv \mathrm{div}(A_0 + K)^{-1} f$, using the properties of A^*A, A, A^*, we derive the estimates:

$$\alpha\,\|p\|^2_{H_c} + C\|p\|^2_{H_c} \le \widetilde{C}\|f\|_{W^*}\|p\|_{H_c}$$
$$\|p\|_{H_c} \le C\|f\|_{W^*} \quad \forall\,\alpha \ge 0$$
$$C\|p - p_0\|^2_{H_c} \le \alpha\,\|p\|_{H_c}\|p - p_0\|_{H_c} \le C\alpha\|f\|_{W^*}\|p - p_0\|_{H_c}$$
$$\|p - p_0\|_{H_c} \le \alpha\,C\|f\|_{W^*}$$
$$\|\phi - \phi_0\|_W \le C\|p - p_0\|_{H_c} \le \alpha C\,\|f\|_{W^*},$$

where the constants $C, \widetilde{C}$ do not depend on $\alpha, p, \ldots, f$. Hence,

$$\|\phi - \phi_0\|_W + \|p - p_0\|_{H_c} \le \alpha C\,\|f\|_{W^*} \to 0, \quad \alpha \to +0 \tag{38}$$

and $\phi \cong \phi_0$, $p \cong p_0$ as $0 < \alpha \ll 1$.

To obtain the approximate solutions of (37) we can use the iterative procedure (11) with small positive α. For the system (37) this iterative procedure reads: for $k = 0, 1, 2, \ldots$ compute

$$-a\Delta\phi^k + b\phi^k + K\phi^k = f - \nabla p^k \quad \text{in} \ \ \Omega$$
$$\phi^k = 0 \quad \text{on} \ \ \Gamma, \qquad \int\limits_\Omega p^k\,\mathrm{d}\Omega = 0 \tag{39}$$
$$-a\Delta q^k + bq^k - Kq^k = -\nabla\mathrm{div}\,\phi^k \quad \text{in} \ \ \Omega, \qquad q^k = 0 \quad \text{on} \ \ \Gamma$$
$$p^{k+1} = p^k - \tau_k(\alpha p^k + \mathrm{div}\,q^k) \quad \text{in} \ \ \Omega.$$

If we set

$$\tau_k \equiv \tau = \frac{2}{2\alpha + \mathbf{C}_0 + \mathbf{C}_1} \tag{40}$$

then according to (10), (12) the estimate

$$\|\phi^k - \phi\|_W + \|p - p^k\|_{H_c} \le C\,\left(\frac{\mathbf{C}_1 + \mathbf{C}_0}{2\alpha + \mathbf{C}_0 + \mathbf{C}_1}\right)^k \to 0, \quad k \to \infty \tag{41}$$

is valid.

From (38), (41) it follows that

$$\|\phi_0 - \phi^k\|_W + \|p_0 - p^k\|_{H_c} \leq C \left(\alpha + \left(\frac{\mathbf{C}_1 - \mathbf{C}_0}{2\alpha + \mathbf{C}_1 + \mathbf{C}_0} \right)^k \right) \to 0$$
$$k \to \infty, \ \alpha \to +0. \tag{42}$$

Thus, if $\alpha > 0$ is sufficiently small, while $k(\alpha)$ is sufficiently large, then ϕ^k, p^k can be taken as the approximate solutions of (27).

Remark 3.1 The iterative process (39) with $\alpha > 0$ can be used to solve (27) when b or U_∞ is large.

Remark 3.2 Since $\mathbf{C}_0, \mathbf{C}_1$ in (40) are only the estimates of bounds of the spectrum of A^*A, the convergence rate of the algorithm (40), when (39) is used, can be slow. On the other hand, the following choice of $\{\tau_k\}$ can be proposed. By the theory of extremal problems [14] the parameters $\{\tau_k\}$ can be calculated as $\tau_k = (J_\alpha(u^k) - J_{\alpha,*})/\|J'_\alpha(u^k)\|^2$, where the value $J_{\alpha,*} \equiv \inf J_\alpha$ is assumed to be known. It often happens that $J_{\alpha,*}$ is unknown. However, if $\mathrm{Ker}\,(A^*) = \mathrm{Ker}\,(B^* L^{*-1} C^*) = \{0\}$, we have $\overline{R(A)} = H_c$ and $\inf J_\alpha(u) \to 0$ as $\alpha \to 0$ [1]. Hence, if α is sufficiently small, we can put $J_{\alpha,*} \cong 0$ and $\tau_k \cong J_\alpha(u^k)/\|J'_\alpha(u^k)\|^2$. Applying these considerations to the process (39), for τ_k we have the expression: $\tau_k \cong \|\mathrm{div}\,\phi^k\|^2_{L_2(\Omega)}/(4\|\mathrm{div}\,q^k\|^2_{L_2(\Omega)})$. The convergence rate of (39), when these values $\{\tau_k\}$ are used, can be much faster.

4. COMPUTATIONAL PROCESS FOR SOLVING NONSTATIONARY STOKES PROBLEM

Let us consider the nonstationary Stokes problem given as:

$$\begin{cases} u_t - a\Delta u - lv + bu = f_1 - p_x \text{ in } Q_T \equiv \Omega \times (0,T), \\ v_t - a\Delta v + lu + bv = f_2 - p_y \text{ in } Q_T, \\ u\Big|_{\partial\Omega} = v\Big|_{\partial\Omega} = 0, \forall t \in (0,T), \\ u\Big|_{t=0} = u_0(x,y), \quad v\Big|_{t=0} = v_0(x,y), \\ \mathrm{div}\phi \equiv u_x + v_y = 0 \text{ in } Q_T, \\ \int\limits_\Omega p \; dxdy = 0, \forall t, \end{cases} \tag{43}$$

where $\Omega \equiv (0,A) \times (0,B), \phi \equiv U \equiv (u,v), \ a, \ b, \ l - const, \ a > 0, \ b \geqslant 0, u_t \equiv \partial u/\partial t, \ldots, p_y \equiv \partial p/\partial y$.

Let us rewrite (43) in an operator form. Assume that:

$$H \equiv (L_2(Q_T))^2,$$

$$H^{(0)} = \{f \in L_2(Q_T) : \int_\Omega f\,dxdy = 0 \quad \forall t \in [0,T]\}$$

$$W = \{\phi \in H : ||\phi||_W = \left(||\phi_t||^2_{L_2(0,T;(\overset{\diamond}{W}{}^1_2(\Omega))^*)^2} + \right.$$

$$\left. + ||\phi||^2_{L_2(0,T;(\overset{\diamond}{W}{}^1_2(\Omega))^*)^2} \right)^{1/2} < \infty\}.$$

Then we can write (43) in a weak form: find $\phi \in W, p \in H^{(0)}$ such that

$$\xi(\phi, \widehat{\phi}) = f(\widehat{\phi}) + \eta(p, \widehat{\phi}) \quad \forall \widehat{\phi} \in W, \; \zeta(\phi, \widehat{p}) = 0 \quad \forall \widehat{p} \in H^{(0)}, \tag{44}$$

where

$$\phi \equiv (u, v), \quad \widehat{\phi} \equiv (\widehat{u}, \widehat{v}),$$

$$\xi(\phi, \widehat{\phi}) \equiv (\phi, -\widehat{\phi}_t) + (a\nabla\phi, \nabla\widehat{\phi}) + (b\phi, \widehat{\phi}) \quad +$$

$$+ \quad (-lv, \widehat{u}) + (lu, \widehat{v})_{(L_2(Q_T)} + (\phi(T), \widehat{\phi}(T))_{(L_2(\Omega))^2},$$

$$f(\widehat{\phi}) = (f, \widehat{\phi}) + (\phi_0, \widehat{\phi}(0))_{(L_2(\Omega))^2},$$

$$(\cdot,\cdot) \equiv (\cdot,\cdot)_H, \; \eta(p, \widehat{\phi}) \equiv (p, \mathrm{div}\widehat{\phi})_{L_2(\Omega)}, \; \zeta(\phi, \widehat{p}) \equiv (\mathrm{div}\phi, \widehat{p})_{L_2(\Omega)},$$

$$f \equiv (f_1, f_2), \quad \phi_0 = (u_0, v_0).$$

Let us define the operators L, B, C as follows: $(L\phi, \widehat{\phi}) \equiv \xi(\phi, \widehat{\phi}) \quad \forall \phi, \widehat{\phi} \in W$, $(Bp, \widehat{\phi}) \equiv \eta(p, \widehat{\phi}) \quad \forall p \in H^{(0)}, \widehat{\phi} \in W$, $(C\phi, \widehat{p}) \equiv \zeta(\phi, \widehat{p}) \quad \forall \phi \in W, \widehat{p} \in H^{(0)}$. Then the problem (43) can be rewritten as the system of two operator equations:

$$L\phi = f + Bp, \quad C\phi = 0, \tag{45}$$

i.e. in the form (1) with $p \equiv u, \; Du \equiv 0 \cdot u, \; h \equiv 0$. The system (43) is the "classical form" of (45).

If we set $X_C \equiv L_2(Q_T)$ and $\Lambda_C \equiv I$, $h \equiv 0$, $D\nu = 0, H_{ob} \equiv L_2(Q_T)$ in the functional J_α from (3), then the iterative process (11) in the "classical form" applying (45) (i.e. applying (43)) reads as follows: for given $p^{(k)}$ find the solution of the equations

$$\begin{cases} u^k_t - a\Delta u^k - lv^k + bu^k = f_1 - p^k_x \text{ in } Q_T \\ v^k_t - a\Delta v^k + lu^k + bv^k = f_2 - p^k_y \text{ in } Q_T \\ u^k\Big|_{t=0} \equiv u_0, \; v^k\Big|_{t=0} \equiv v_0; \; u^k\Big|_{\partial\Omega} = v^k\Big|_{\partial\Omega} = 0 \; \forall t \in (0,T) \end{cases} \tag{46}$$

$$
\begin{cases}
-q^k_{1\,t} - a\Delta q^k_1 + l q^k_2 + b q^k_1 = -(\mathrm{div}\phi^k)_x \text{ in } Q_T \\
-q^k_{2\,t} - a\Delta q^k_2 - l q^k_1 + b q^k_2 = -(\mathrm{div}\phi^k)_y \text{ in } Q_T \\
q^k_1\Big|_{t=T} \equiv 0,\ q^k_2\Big|_{t=T} \equiv 0;\ q^k_1\Big|_{\partial\Omega} = q^k_2\Big|_{\partial\Omega} = 0\ \forall t \in (0,T)
\end{cases} \tag{47}
$$

$$
p^{k+1} = p^k - \tau_k(\alpha p^k + \mathrm{div}\ \mathrm{Q}^k), \tag{48}
$$

where $\phi^k \equiv (u^k, v^k)$, $\mathrm{Q}^k \equiv (q^k_1, q^k_2)$, $\tau > 0$, $\alpha > 0$ are parameters.

Remark 4.1 The iterative parameters $\{\tau_k\}$ calculated according to Remark 3.2 have the following form:

$$
\tau_k \cong \frac{1}{4} \frac{\int\limits_0^T ||\mathrm{div}\phi^k||^2_{L_2(\Omega)} dt}{\int\limits_0^T ||\mathrm{div}\ \mathrm{Q}^k||^2_{L_2(\Omega)} dt}. \tag{49}
$$

5. NUMERICAL EXAMPLES

5.1 Let us consider the equations from (13) in R^2 under the 'periodicity condition':

$$
\begin{gathered}
-a\Delta\phi + b\phi + K\phi = -\nabla p + f(x_1, x_2) \quad \text{in } \ \Omega \\
\mathrm{div}\,\phi = 0 \quad \text{in } \ \Omega, \quad \int\limits_\Omega p \, \mathrm{d}x = 0 \\
\phi|_{x_i+1} = \phi|_{x_i}, \qquad p|_{x_i+1} = p|_{x_i} \quad \forall\, x \in \Omega, \quad i = 1, 2,
\end{gathered} \tag{50}
$$

where $\Omega = \{0 \le x_i < 1,\ i = 1, 2\}$, $K\phi = (-l\phi_2, l\phi_1)$, $l = \mathrm{const}$,

$$
f = f_N = f_{0,0} + \sum_{0<K_1,K_2<N} \sum_{m=1}^{2} f^{(m)}_{K_1,K_2} \varphi^{(m)}_{K_1,K_2}(x_1, x_2)
$$

$$
\varphi^{(1)}_{K_1,K_2} = \cos(2\pi(K_1x_1 + K_2x_2)), \quad \varphi^{(2)}_{K_1,K_2} = \sin(2\pi(K_1x_1 + K_2x_2))
$$

$$
f^{(m)}_{K_1,K_2} = 1/(1 + K_1^2 + K_2^2).
$$

If we take into account (50), the iterative process (39) reads:

$$
\begin{gathered}
-a\Delta\phi^k + b\phi^k + K\phi^k = -\nabla p^k + f \quad \text{in } \ \Omega, \qquad \int\limits_\Omega p^k \, \mathrm{d}x = 0 \\
-a\Delta q^k + bq^k - Kq^k = -\nabla \mathrm{div}\,\phi^k \quad \text{in } \ \Omega \\
p^{k+1} = p^k - \tau(\alpha p^k + \mathrm{div}\, q^k), \quad k = 0, 1, 2,
\end{gathered} \tag{51}
$$

where

$$a = 0.1, \quad b = 10, \quad N = 50, \quad \alpha = 0.001, \quad 0 \le l \le 30$$

$$\tau = 2/(2\alpha + \mathbf{C}_0 + \mathbf{C}_1)$$

$$\mathbf{C}_0 = 2\Big/\left(\left(a + \frac{b}{(2\pi)^2}\right)^2 + \frac{l^2}{(2\pi)^4}\right), \quad \mathbf{C}_1 = \frac{2}{a^2}.$$

Note that $\tau \cong 0.01$ for the above data.

For the stopping criterion of the iterative process (51) we used the relation

$$\max_m \max_{K_1,K_2} \frac{|(p^{k+1} - p^k, \varphi^{(m)}_{K_1,K_2})_{L_2(\Omega)}|}{|(p^{k+1}, \varphi^{(m)}_{K_1,K_2})_{L_2(\Omega)}|} \le 10^{-10}.$$

To solve the subproblems at each step of the iterative algorithm (51) the Fourier method was applied to calculate $\phi^k \equiv (\phi_1^k, \phi_2^k)$, $q^k \equiv (q_1^k, q_2^k)$ using the basis functions $\{\cos(2\pi(K_1x_1 + K_2x_2))\}$, $\{\sin(2\pi(K_1x_1 + K_2x_2))\}$.

Figures 1–4 show some computational results.

5.2 Let us solve the problem (43) by the proposed methods. Assume the test solution of (43) is the following:

$$u(t,x,y) = e^{-t^2}\frac{4\pi}{B}\sin^3\frac{\pi x}{A}\sin^3\frac{\pi y}{B}\cos\frac{\pi y}{B},$$

$$v(t,x,y) = -e^{-t^2}\frac{3\pi}{A}\sin^2\frac{\pi x}{A}\cos\frac{\pi x}{A}\sin^4\frac{\pi y}{B},$$

$$p(t,x,y) = 2\sqrt{t}\sin^2\frac{\pi x}{A}\sin^2\frac{\pi y}{B} - \frac{8\sqrt{t}}{AB}.$$

The functions f_1, f_2 corresponding to this test solution are equal zero on $\partial\Omega \times (0,T)$. We set $A = B = 4$, $T = 3$, $a = b = 1$, $l = 5$, $\alpha = 10^{-5}$. The figures 5-8 presents the results of calculating of the solution of (43).

At each step of the iterative process (46)- (48) all the subproblems have been solved by the eigenfunction method.

CONCLUSIONS

The results described in the previous sections show that the methodology of the solution of the above problems does not need explicit forms of the operators L^{-1} and $(L^*)^{-1}$. Therefore this methodology may be used to solve boundary value problems of general form. We can also suggest that the approaches discussed in this paper may be extended to the solution of the problems of the form

$$L(\varphi, u) = 0, \qquad C\varphi = \phi_{\text{ob}} \tag{52}$$

with a nonlinear state equation and to nonlinear inverse problems of general form. We also assume that this methodology can be applied to the solution of the Navier–Stokes problem:

$$
\begin{gathered}
-a\Delta\phi + b\phi + (\phi,\nabla)\phi = -\nabla p + f \quad \text{in} \ \ \Omega \subset R^n \\
\operatorname{div}\phi = 0 \quad \text{in} \ \ \Omega, \qquad \phi = 0 \quad \text{on} \ \ \Gamma, \quad \int\limits_{\Omega} p \, \mathrm{d}x = 0.
\end{gathered} \tag{53}
$$

The iterative process like (39) for the problem (53) is

$$
\begin{gathered}
-a\Delta\phi^k + b\phi^k + \sum_{i=1}^{n} \phi_i^k \frac{\partial \phi^k}{\partial x_i} = -\nabla p^k + f \quad \text{in} \ \ \Omega \\
\phi^k = 0 \quad \text{on} \ \ \Gamma, \quad \int\limits_{\Omega} p^k \, \mathrm{d}x = 0 \\
-a\Delta q^k + bq^k - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\phi_i^k q^k) + (\nabla \otimes \phi^k) q^k = -\nabla \operatorname{div}\phi^k \quad \text{in} \ \ \Omega \\
p^{k+1} = p^k - \tau_k(\alpha p^k + \operatorname{div} q^k), \quad k = 0, 1, 2, \ldots,
\end{gathered} \tag{54}
$$

where $(\nabla \otimes \phi^k)$ is the matrix with elements $(\nabla \otimes \phi^k)_{ij} \equiv (\partial\phi_j^k)/(\partial x_i)$, $i, j = 1, \ldots, n$. The problem of the convergence of the iterative algorithm (54) calls for special investigation. We hope that this study and the corresponding numerical algorithms may be investigated on the basis of nonlinear analysis by using the results of the optimal control theory and the theory of nonlinear inverse problems as well.

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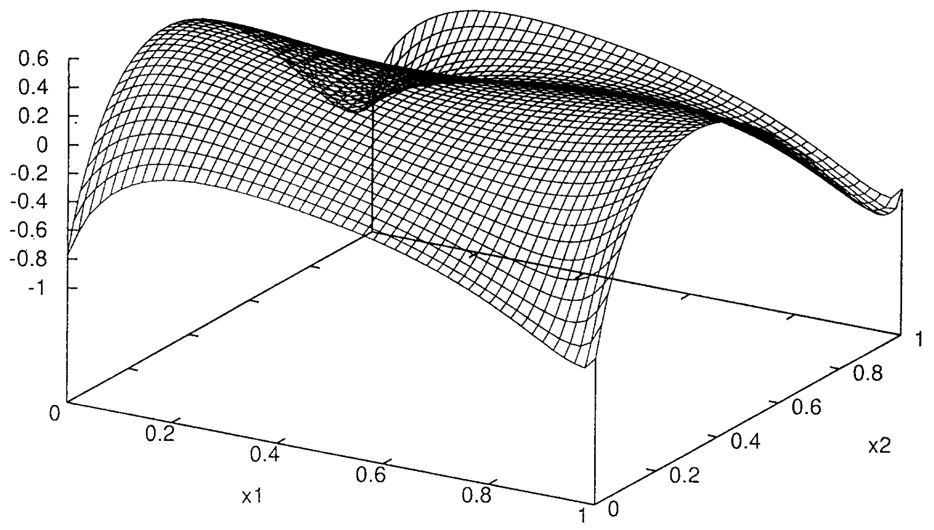

Figure 1: The function of calculated pressure $p(x_1, x_2)$.

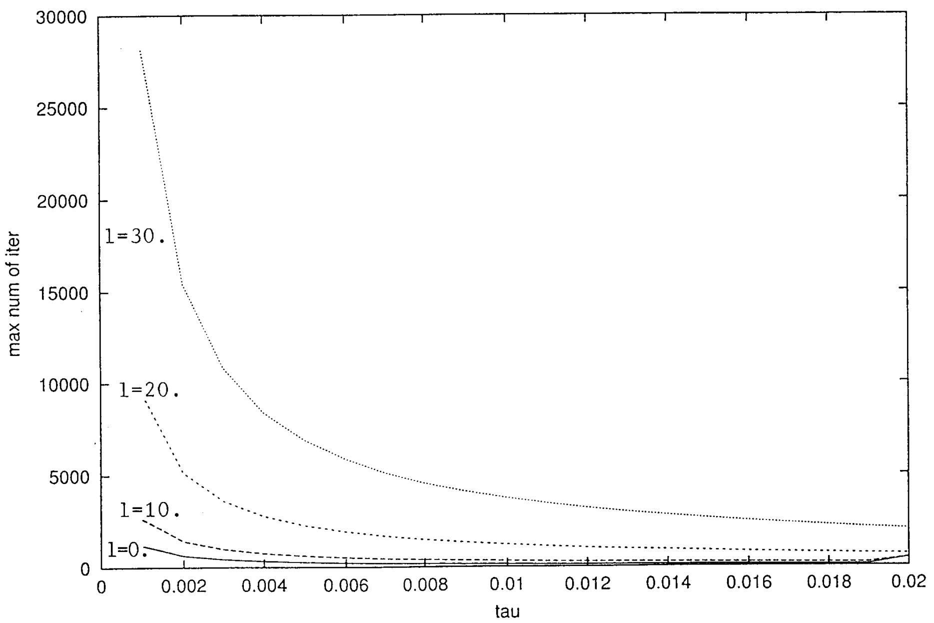

Figure 2: The dependence of the number of iterations on $\tau\,(0 \leq l \leq 30$, $\alpha = 0.001)$.

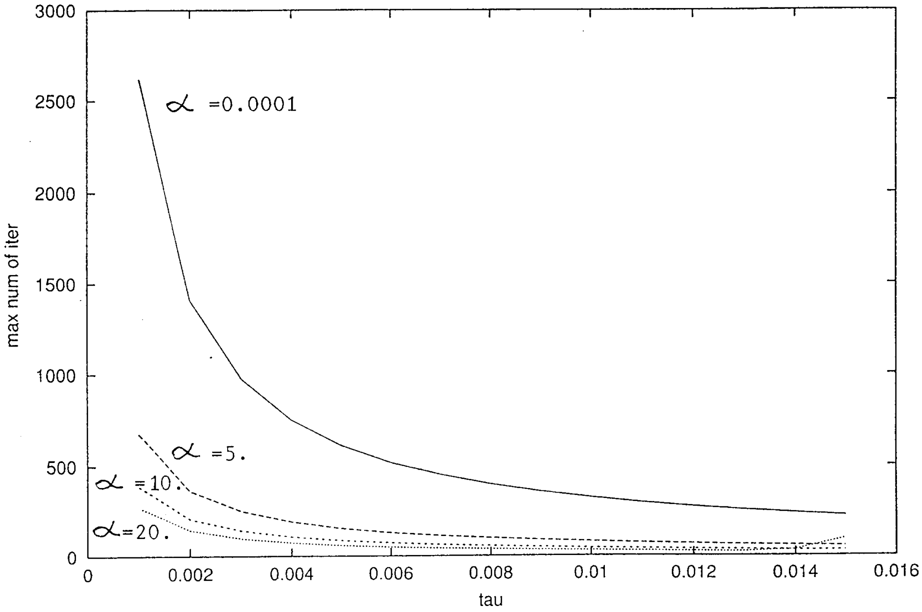

Figure 3: The dependence of the number of iterations on τ (for various values of α and for $l = 10$).

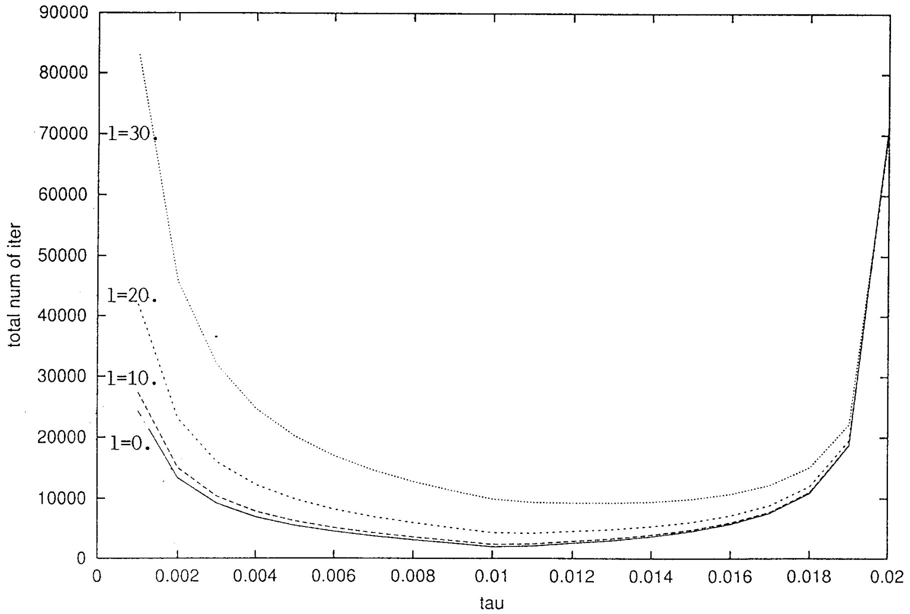

Figure 4: The dependence of the total number of iterations on τ for all harmonics (for various values of l and for $\alpha = 0.001$).

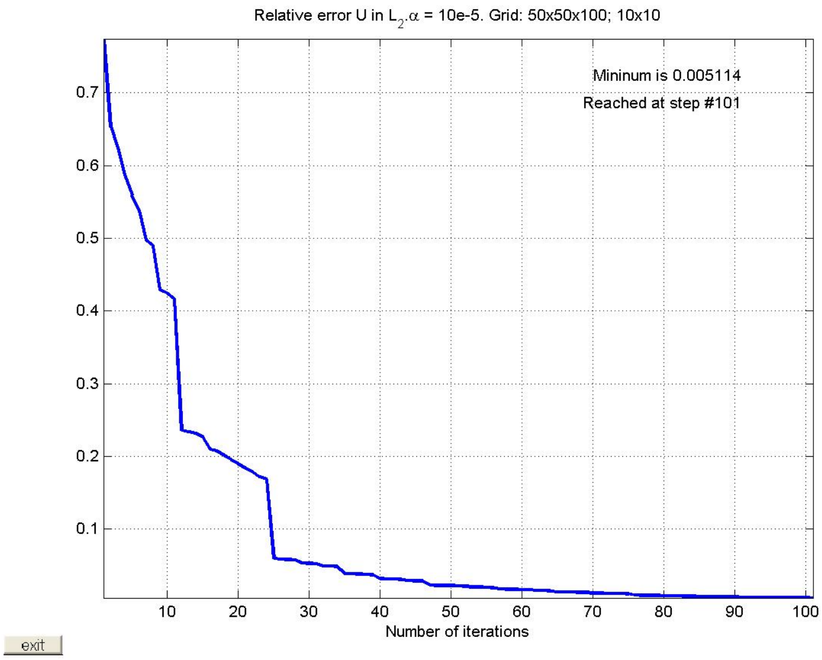

Figure 5: The relative error of the function u in $L_2(Q_T)$.

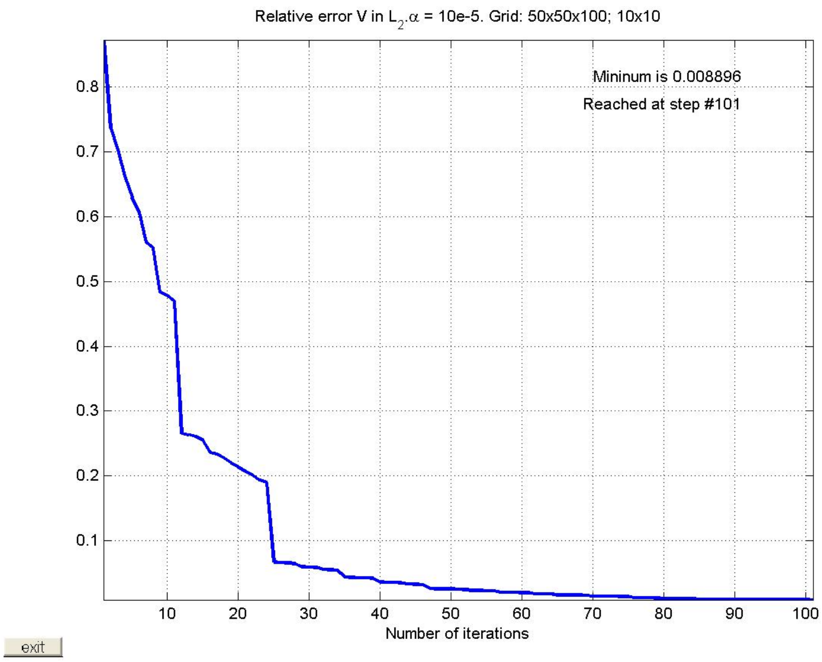

Figure 6: The relative error of the function v in $L_2(Q_T)$.

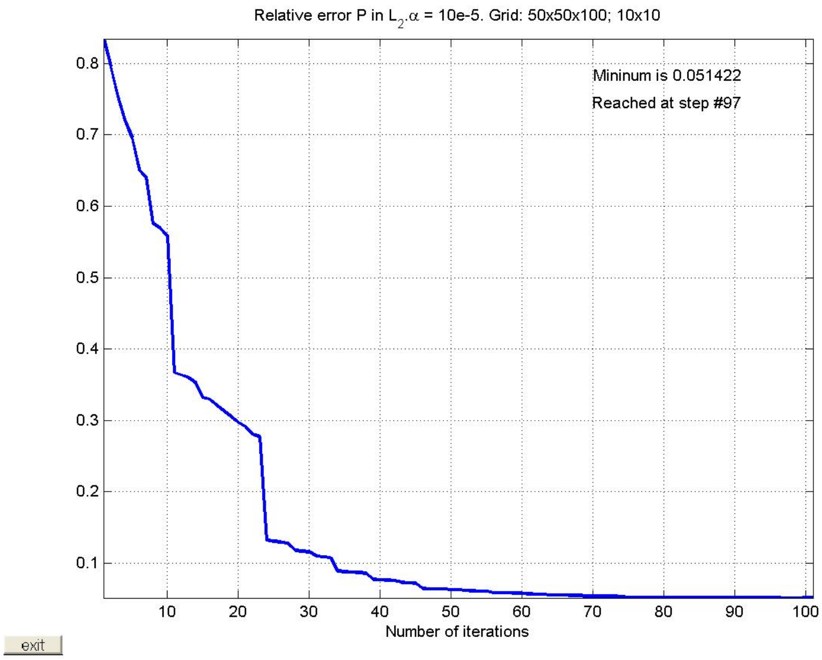

Figure 7: The relative error of the function p in $L_2(Q_T)$.

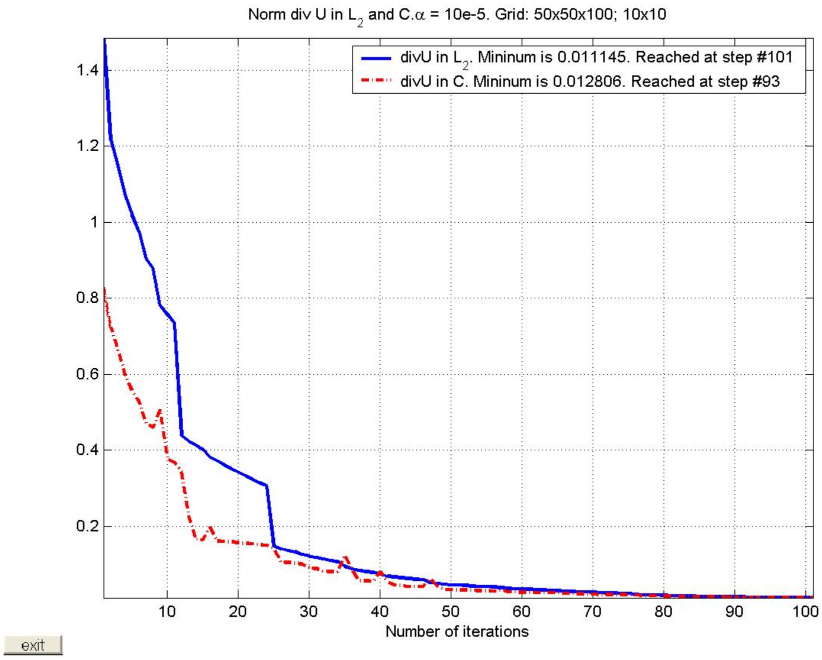

Figure 8: The norm of the divergence in $L_2(Q_T)$.