Reduced Basis Methods for Elliptic Equations in sub-domains with A-Posteriori Error Bounds and Adaptivity.

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Abstract

We present an application in multi-parametrized subdomains based on a technique for the rapid and reliable prediction of linear-functional outputs of elliptic coercive partial differential equations with affine parameter dependence (reduced basis methods). The essential components are (*i*) (provably) rapidly convergent global reduced-basis approximations — Galerkin projection onto a space W_N spanned by solutions of the governing partial differential equation at N selected points in parameter space; (*ii*) *a posteriori* error estimation — relaxations of the error-residual equation that provide inexpensive bounds for the error in the outputs of interest; and (*iii*) off-line/on-line computational procedures — methods which decouple the generation and projection stages of the approximation process. The operation count for the on-line stage — in which, given a new parameter value, we calculate the output of interest and associated error bound — depends only on N (typically very small) and the parametric complexity of the problem; the method is thus ideally suited for the repeated and rapid evaluations required in the context of parameter estimation, design, optimization, and real-time control.

In [11] a rigorous *a posteriori* error bound framework for reduced-basis approximations of elliptic coercive equations is developed. The resulting error estimates are, in some cases, quite sharp: the ratio of the estimated error in the output to the true error in the output, or *effectivity*, is close to (but always greater than) unity. We use *a posteriori* bound error estimator applied also to an adaptive procedure in choosing the approximation space and its dimension, minimizing the estimated error or the *effectivity* [23].

The application is based on a parametrized geometry, divided in subdomains, each of them depending by geometrical quantities that can be useful for future haemodynamics applications [15], such as the bypass configuration problem (stenosis length, graft angle, artery diameter, incoming

bypass diameter and outflow length). For future development guidelines we suggest to see [19] and [16].

Key words: *elliptic parametrized partial differential equations; reduced-basis methods; output bounds; Galerkin approximation;* a posteriori *error estimation; adaptive approximation procedure.*

Introduction

The optimization, control, design and characterization of an engineering component or system requires the prediction of certain "quantities of interest," or performance metrics, which we shall denote *outputs* — for example velocity field, maximum stresses, maximum temperatures, heat transfer rates, flow rates, vorticity, or lifts and drags. These outputs are typically expressed as functionals of field variables associated with a parametrized partial differential equation which describes the physical behavior of the component or system. The parameters, which we shall denote *inputs*, serve to identify a particular "configuration" of the components: these inputs may represent design or decision variables, such as geometry, or characterization variables, such as physical properties — for example in inverse design problems. We thus get an implicit *input–output* relationship, evaluation of which demands solution of the underlying partial differential equations. See [22] for a detailed presentation of the state of the art of the design problem and some examples.

The development of computational methods is permitting *rapid* and *reliable* evaluation of this partial-differential-equation-induced input-output relationship in the design, optimization and control contexts. See recent developments in [18]. The approach used is based on the reduced-basis method, first introduced in the late 1970s for nonlinear structural analysis, and later developed more broadly in the 1980s and 1990s [1, 2, 3, 9, 10, 17]. The reduced-basis method recognizes that the field variable is not, in fact, some arbitrary member of the infinite-dimensional solution space associated with the partial differential equation; rather, the field variable resides, or evolves, on a much lower-dimensional manifold induced by the parametric dependence.

In the application we use *global* approximation spaces; second, we make rigorous *a posteriori error estimations*; and third, we exploit *off-line/ on-line* computational decompositions (see [1] for application of this strategy within the reduced–basis context). These three steps allow us — for the restricted but important class of "parameter-affine" problems — to reliably decouple the generation and projection stages of reduced-basis approximation, thereby effecting computational economies of several orders of magnitude.
In Section 1 we present the problem statement. In Section 2 we describe the *a posteriori* error estimation framework. In Section 3 we present the *a priori* convergence theory applied also to our output bounds and not only to approximate solution. In Sec 4 we study our procedure to control N more tightly and to apply our error bound adaptively in the choice of μ parameters family. Finally, in Section 5, we present the application and in the Section 6 the numerical results for our "model-problem" examples, in Section 7, we present our interest in future and further developments for flow viscous control and shape optimization [8] applied to haemodynamics, for example, within the reduced-basis framework.

1 Problem Formulation

1.1 Exact Statement

We first introduce a Hilbert space Y, and an associated inner product and a norm, $(\cdot,\cdot)$ and $\|\cdot\| \equiv (\cdot,\cdot)^{1/2}$, respectively. We next introduce the dual space of Y, Y', and the associated duality pairing between Y and Y', ${}_{Y'}\langle\cdot,\cdot\rangle_Y \equiv \langle\cdot,\cdot\rangle$. We then define, for any $\mu \in \mathcal{D}^\mu \subset \mathbb{R}^P$, the parametrized (distributional) operator $\mathcal{A}(\mu)\colon X \to X'$. We assume that $\mathcal{A}(\mu) = A(\Theta(\mu))$, where, for any $\Theta \in \mathbb{R}^Q_+$, $A(\Theta)\colon X \to X'$ is given by

$$A(\Theta) = A_0 + \sum_{q=1}^{Q} \Theta_q \, A_q \ ,$$

and the $\Theta_q\colon \mathcal{D}^\mu \to \mathbb{R}_+$, $q = 0,\ldots,Q$, are non-negative functions. Here $\mathbb{R}_+$ refers to the non-negative real numbers. The range of Θ is denoted $\mathcal{D}^\theta$; and we define $\theta^{\min}$ (≥ 0), $\theta^{\max}$ (assumed finite), and $\mathcal{D}^\theta_{\rm box} \subset \mathbb{R}^Q_+$ as

$$\begin{aligned} \theta_q^{\min} &\equiv \sup t_{\{t\in\mathbb{R}_+ \,|\, \Theta_q(\mu)\geq t,\ \forall \mu\in\mathcal{D}^\mu\}}, \qquad q = 1,\ldots,Q \ , \\ \theta_q^{\max} &\equiv \inf t_{\{t\in\mathbb{R}_+ \,|\, \Theta_q(\mu)\leq t,\ \forall \mu\in\mathcal{D}^\mu\}}, \qquad q = 1,\ldots,Q \ , \end{aligned}$$

and $\mathcal{D}^\theta_{\rm box} \equiv \Pi_{q=1}^Q [\theta_q^{\min}, \theta_q^{\max}]$, respectively.
Finally, we require that A_0 is continuous, symmetric, and coercive, and that the A_q, $q = 1,\ldots,Q$, are continuous, symmetric, and positive-semidefinite ($\langle A_q v, v\rangle \geq 0$, $\forall\, v \in Y$); it follows that $A(\Theta)$ (respectively, $\mathcal{A}(\mu)$) is continuous, symmetric, and coercive for all θ in $\mathcal{D}^\theta_{\rm box}$ (respectively, for all μ in $\mathcal{D}^\mu$).
The general formulation for the problem can then be stated as: given a $\mu \in \mathcal{D}^\mu$, and linear functional $F \in Y'$, evaluate the output

$$s(\mu) = \langle F, u(\mu)\rangle \ ,$$

where $u(\mu) \in Y$ is the unique solution of $A(\Theta(\mu))\, u(\mu) = F$; we shall interpret the latter as

$$\langle A(\Theta(\mu))\, u(\mu), v\rangle = \langle F, v\rangle, \quad \forall\, v \in Y \ . \tag{1.1}$$

Output $s(\mu)$ may also be interpreted as the energy of the solution; $s(\mu) = \langle F, u(\mu)\rangle = \langle A(\Theta(\mu))\, u(\mu), u(\mu)\rangle$ — and is hence strictly positive. The output $s(\mu)$ is "compliant," and the operator $A(\Theta)$ is symmetric; however, the formulation is readily extended [11] to treat both noncompliant outputs, $s(\mu) = \langle L, u(\mu)\rangle$ for given $L \in Y'$, and non-symmetric, but still coercive, operators. We may also express our output as

$$s(\mu) = \langle F, A^{-1}(\Theta(\mu))F\rangle \ . \tag{1.2}$$

Here, for any $\theta \in \mathcal{D}^\theta_{\rm box}$, $A^{-1}(\Theta)\colon Y' \to Y$ is the (continuous, symmetric, coercive) inverse of $A(\Theta)$; further, $\forall\, G \in Y'$, $\langle A(\Theta)\, A^{-1}(\Theta)\, G, v\rangle = \langle G, v\rangle$, $\forall\, v \in Y$.

1.2 Galerkin Approximation

The $u(\mu)$ of (1.1) are, in general, not calculable. In order to construct our reduced-basis space we will therefore require a *finite*-dimensional "truth" approximation to Y, which we shall denote $\tilde{Y}$; $\tilde{Y}$ is an $\mathcal{N}$-dimensional subspace

of Y. For example, for $\Omega \subset \mathbb{R}^{d=1,\,2,\,\text{or}\,3}$, and $Y \subset H^1(\Omega) \equiv \{v \in L^2(\Omega),\ \nabla v \in (L^2(\Omega))^d\}$ (here $L^2(\Omega)$ is the Lebesgue space of square-integrable functions over Ω), $\tilde{Y}$ will typically be a finite element approximation space associated with a very fine triangulation of Ω. In general, we expect that $\mathcal{N}$ will be very large. See [14] for more detailed studies on FEM approximation.
Our Galerkin approximation can be stated as: given a $\mu \in \mathcal{D}^\mu$, evaluate the output

$$\tilde{s}(\mu) = \langle F, \tilde{u}(\mu) \rangle \ , \tag{1.3}$$

where $\tilde{u}(\mu) \in \tilde{Y}$ is the unique solution of

$$\langle A(\Theta(\mu))\, \tilde{u}(\mu), v \rangle = \langle F, v \rangle, \quad \forall\, v \in \tilde{Y} \ . \tag{1.4}$$

As before, the output can be expressed as a (strictly positive) energy: $\tilde{s}(\mu) = \langle F, \tilde{u}(\mu) \rangle = \langle A(\Theta(\mu))\, \tilde{u}(\mu), \tilde{u}(\mu) \rangle$.
It shall prove convenient to express (1.3)–(1.4) in terms of a (in reality any) basis for $\tilde{Y}$, $\{\phi_i,\ i = 1, \ldots, \mathcal{N}\}$. We first introduce the matrices $\underline{\tilde{A}}_q \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$, $q = 0, \ldots, Q$, as $\tilde{A}_{q\,i\,j} = \langle A\phi_j, \phi_i \rangle$, $1 \le i, j \le \mathcal{N}$; it is readily shown that $\underline{\tilde{A}}_0$ (respectively, $\underline{\tilde{A}}_q$, $q = 1, \ldots, Q$) is symmetric positive-definite (respectively, symmetric positive-semidefinite). For any $\Theta \in \mathcal{D}^\theta_{\text{box}}$, we then define $\underline{\tilde{A}}(\Theta) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ as

$$\underline{\tilde{A}}(\Theta) = \underline{\tilde{A}}_0 + \sum_{q=1}^{Q} \Theta_q\, \underline{\tilde{A}}_q \ ;$$

$\underline{\tilde{A}}(\Theta)$ is symmetric positive-definite for all $\Theta \in \mathcal{D}^\theta_{\text{box}}$. In the same way we introduce $\underline{\tilde{F}} \in \mathbb{R}^{\mathcal{N}}$ as $\tilde{F}_i = \langle F, \phi_i \rangle$, $1 \le i \le \mathcal{N}$.
Our approximation can then be restated as: given a $\mu \in \mathcal{D}^\mu$, evaluate the output

$$\tilde{s}(\mu) = \underline{\tilde{F}}^T\, \underline{\tilde{u}}(\mu) \ ,$$

where $\underline{\tilde{u}}(\mu) \in \mathbb{R}^{\mathcal{N}}$ is the unique solution of

$$\underline{\tilde{A}}(\Theta(\mu))\, \underline{\tilde{u}}(\mu) = \underline{\tilde{F}} \ ; \tag{1.5}$$

here ^{T} refers to the algebraic transpose. Note that $\tilde{u}(\mu)$ and $\underline{\tilde{u}}(\mu) = (\tilde{u}_1, \ldots, \tilde{u}_{\mathcal{N}})$ are related via

$$\tilde{u}(\mu) = \sum_{j=1}^{\mathcal{N}} \tilde{u}_j(\mu)\, \phi_j \ .$$

As always, our compliance output can be expressed as an energy:

$$\tilde{s}(\mu) = \underline{\tilde{u}}^T(\mu)\, \underline{\tilde{A}}(\Theta(\mu))\, \underline{\tilde{u}}(\mu) \ ,$$

or, equivalently,

$$\tilde{s}(\mu) = \underline{\tilde{F}}^T\, \underline{\tilde{A}}^{-1}(\Theta(\mu))\, \underline{\tilde{F}} \ , \tag{1.6}$$

where $\underline{\tilde{A}}^{-1}(\Theta)$ is the (symmetric, positive-definite) inverse of $\underline{\tilde{A}}(\Theta)$. Note that since $\mathcal{N}$ is large, solution of (1.5), and hence evaluation of $\tilde{s}(\mu)$, will be computationally expensive.

1.3 Reduced-Basis Method

We introduce some "μ" samples $S_N^\mu = \{\mu^1, \ldots, \mu^N\}$, where $\mu^n \in \mathcal{D}^\mu$, $n = 1, \ldots, N$. We then define our reduced-basis space $W_N = \text{span}\{\tilde{\zeta}_n, n = 1, \ldots, N\}$, where $\tilde{\zeta}_n = \tilde{u}(\mu^n)$, $n = 1, \ldots, N$. Recall that $\tilde{u}(\mu^n)$ is the solution of (1.4) for $\mu = \mu^n$. We denote $\underline{\tilde{\zeta}}_n = \underline{\tilde{u}}(\mu^n)$, $n = 1, \ldots, N$. In the first step, given a $\mu \in \mathcal{D}^\mu$, we find $s_N(\mu) = \langle F, u_N(\mu)\rangle$, where $u_N(\mu) \in W_N$ satisfies

$$\langle A(\Theta(\mu))\, u_N(\mu), v\rangle = \langle F, v\rangle, \quad \forall\, v \in W_N \ .$$

We may also express the output as an energy, $s_N(\mu) = \langle A(\Theta(\mu))\, u_N(\mu), u_N(\mu)\rangle$. In terms of our basis functions, we define the symmetric positive-definite matrix $\underline{A}_N(\mu) \in \mathbb{R}^{N\times N}$ as $A_{N\,i\,j}(\mu) = \langle A(\Theta(\mu))\, \tilde{\zeta}_j, \tilde{\zeta}_i\rangle$, $1 \le i, j \le N$, and the vector $\underline{F}_N \in \mathbb{R}^N$ as $F_{N\,i} = \langle F, \tilde{\zeta}_i\rangle$, $1 \le i \le N$. It is a simple matter to observe that

$$\underline{A}_N(\Theta) = \underline{A}_{N\,0} + \sum_{q=1}^{Q} \Theta_q\, \underline{A}_{N\,q} \ , \tag{1.7}$$

where $(A_{Nq})_{i\,j} = \langle A_q\, \tilde{\zeta}_j, \tilde{\zeta}_i\rangle$, $1 \le i, j \le N$, $0 \le q \le Q$; note that the $\underline{A}_{Nq} \in \mathbb{R}^{N\times N}$, $0 \le q \le Q$, are *independent* of θ.
We can then restate the formulation as: given a $\mu \in \mathcal{D}^\mu$, find $s_N(\mu) = \underline{F}_N^T \underline{u}_N(\mu)$, where $\underline{u}_N(\mu) \in \mathbb{R}^N$ is the unique solution to

$$\underline{A}_N(\Theta(\mu))\, \underline{u}_N(\mu) = \underline{F}_N \ .$$

Note that $u_N(\mu) = \sum_{j=1}^N u_{N\,j}(\mu)\, \tilde{\zeta}_j$. The output may also be expressed as $s_N(\mu) = \underline{u}_N^T(\mu)\, \underline{A}_N(\Theta(\mu))\, \underline{u}_N(\mu) = \underline{F}_N^T\, \underline{A}_N^{-1}(\Theta(\mu))\, \underline{F}_N$.

2 Reduced-Basis Output Bounds

2.1 Introduction

In the previous step we have seen how to compute our *predictor*, $s_N(\mu)$; in the next step we are going to compute our *bounds*, $s_N^-(\mu) \le \tilde{s}(\mu) \le s_N^+(\mu)$. The latter may be interpreted as *a posteriori* estimators.

2.2 Bound Conditioner

We first define the error $\tilde{e}(\mu) \in Y$ as $\tilde{e}(\mu) = \tilde{u}(\mu) - u_N(\mu)$ and residual $R \in Y'$ as $\langle R(\mu), v\rangle \equiv \langle F - A(\Theta(\mu))\, u_N(\mu), v\rangle$, $\forall\, v \in Y$; and then $\underline{\tilde{R}}(\mu) \in \mathbb{R}^{\mathcal{N}}$ as $\tilde{R}_i(\mu) = \langle R(\mu), \phi_i\rangle$ $i = 1, \ldots, \mathcal{N}$. We note that

$$\underline{\tilde{R}}(\mu) = \underline{\tilde{F}} - \underline{\tilde{A}}(\Theta(\mu))\, \underline{u}_N(\mu) \ , \tag{2.1}$$

where $\underline{u}_N(\mu) \in \mathbb{R}^{\mathcal{N}}$ is given by

$$u_N(\mu) = \sum_{n=1}^{N} u_{Nn}(\mu)\, \underline{\tilde{\zeta}}_n \ ; \tag{2.2}$$

by construction: $\underline{\tilde{\zeta}}_n = \sum_{i=1}^{\mathcal{N}} \tilde{u}_i(\mu^n)\,\phi_i$. We then introduce a symmetric, continuous, and coercive *bound conditioner* [22] $\mathcal{C}(\mu) : Y \to Y'$ such that the minimum and maximum eigenvalues

$$\rho_{min}(\mu) \equiv \min_{v\in Y} \frac{\langle \tilde{A}(\mu)v, v\rangle}{\langle \mathcal{C}(\mu)v, v\rangle}, \tag{2.3}$$

$$\rho_{max}(\mu) \equiv \max_{v\in Y} \frac{\langle \tilde{A}(\mu)v, v\rangle}{\langle \mathcal{C}(\mu)v, v\rangle}, \tag{2.4}$$

satisfy the following spectral condition:

$$1 \le \rho_{min}(\mu), \quad \rho_{max} \le \rho, \tag{2.5}$$

for some (preferably small) constant $\rho \in \mathbb{R}$.
We note that:

$$\rho_{min}(\mu) \le \frac{\langle \tilde{A}(\mu)v, v\rangle}{\langle \mathcal{C}(\mu)v, v\rangle} \le \rho_{max}(\mu) \forall v \in Y, \tag{2.6}$$

and for any $\hat{e} \in Y$ and $\tilde{e} \in Y$ such that:

$$\langle \tilde{A}(\mu)\tilde{e}, v\rangle = \langle \underline{\tilde{R}}(\mu), v\rangle,\ \forall v \in Y, \tag{2.7}$$

and

$$\langle \mathcal{C}(\mu)\hat{e}, v\rangle = \langle \underline{\tilde{R}}(\mu), v\rangle,\ \forall v \in Y, \tag{2.8}$$

we can show that:

$$\rho_{min}(\mu) \le \frac{\langle \tilde{A}(\mu)\tilde{e}, v\rangle}{\langle \mathcal{C}(\mu)\hat{e}, v\rangle} \le \rho_{max}(\mu). \tag{2.9}$$

In addition to the spectral condition (2.5), we also require a "computational invertibility" hypothesis, in particular that $\mathcal{C}^{-1}(\mu)$ be of the form:

$$\mathcal{C}^{-1}(\mu) = \sum_{i\in\mathcal{I}(\mu)} \alpha_i(\mu)\,\mathcal{C}_i^{-1}$$

where (i) $\mathcal{I}(\mu) \subset \{1, \ldots, I\}$ is a parameter-dependent set of indices, I is a finite (preferably small) integer, and each $\mu \in \mathcal{D}^\mu$. (ii) The $\mathcal{C}_i : Y \to Y'$ are parameter-*indipendent* symmetric, coercive operators.

2.3 Error and Output Bounds

We now find $\underline{\hat{e}}(\mu) \in \mathbb{R}^{\mathcal{N}}$ such that

$$\underline{\mathcal{C}}(\mu)\,\underline{\hat{e}}(\mu) = \underline{\tilde{R}}(\mu)\ ; \tag{2.10}$$

this equation will of course have a unique solution since $\mathcal{C}(\mu)$ is symmetric positive-definite.

We can now define our lower and upper bounds as

$$s_N^-(\mu) = s_N(\mu)\ ,$$

and

$$s_N^+(\mu) = s_N(\mu) + \Delta_N(\mu)\ ,$$

where $\Delta_N(\mu)$, the bound gap, is given by

$$\begin{aligned}\Delta_N(\mu) &\equiv \underline{\hat{e}}^T(\mu)\,\underline{\mathcal{C}}(\mu)\,\underline{\hat{e}}(\mu)\\ &= \underline{\tilde{R}}^T(\mu)\,\underline{\mathcal{C}}^{-1}(\mu)\,\underline{\tilde{R}}(\mu)\\ &= \underline{\tilde{R}}^T(\mu)\,\underline{\hat{e}}(\mu)\ .\end{aligned}$$

The previous expressions for the bound gap will be useful in the theoretical and computational context.

2.4 Minimum Coefficient Bound Conditioner

We now consider a method for choosing the $\mathcal{C}_i$ and the associated α_i called *Minimum Coefficient Bound Conditioner*.
To begin we recall our separability assumption on $\underline{\tilde{A}}(\Theta)$:

$$\underline{\tilde{A}}(\Theta(\mu)) = \underline{\tilde{A}}_0 + \sum_{q=1}^{Q} \Theta_q(\mu)\,\underline{\tilde{A}}_q \ \forall\ \mu \in \mathcal{D}^{\mu}_{box},$$

where the $\Theta(\mu) : \mathcal{D}^{\mu} \to \mathbb{R}$ and the $\underline{\tilde{A}}_q : Y \to Y'$. We now define

$$\underline{\tilde{\mathcal{A}}}(\theta) = \underline{\tilde{\mathcal{A}}}_0 + \sum_{q=1}^{Q} \theta_q\,\underline{\tilde{\mathcal{A}}}_q$$

where the $\theta \in \mathbb{R}^Q$ and the $\underline{\tilde{\mathcal{A}}}(\theta) : Y \to Y'$ and $\underline{\tilde{A}}_q \equiv \underline{\tilde{\mathcal{A}}}_q$. If $\Theta_q(\mu) = \theta_q$, we may then write

$$\underline{\tilde{A}}(\Theta(\mu)) = \underline{\tilde{\mathcal{A}}}(\theta)$$

where $\Theta : \mathcal{D}^{\mu} \to \mathcal{D}^{\theta}$ and $\mathcal{D}^{\theta} \equiv Range(\Theta) \in \mathbb{R}^Q$. Then we introduce I points θ_i as samples $S^{\theta}_I = \{\theta_1, \ldots, \theta_I\}$, where $\theta_i \in \mathcal{D}^{\theta}_{\text{box}}$, $i = 1, \ldots, I$. We choose

$$\alpha_i(\mu) = \Big(\min_{1 \le q \le Q} \Big(\frac{\Theta^q(\mu)}{\theta^q_i} \Big) \Big)^{-1}, \tag{2.11}$$

and

$$\mathcal{C}_i = \sum_{q=1}^{Q} \theta^q_i\,\underline{\tilde{\mathcal{A}}}_q \tag{2.12}$$

Our bound conditioner $\mathcal{C}(\mu) \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ is

$$\mathcal{C}(\mu) = \left(\sum_{i \in \mathcal{I}(\mu)} \alpha_i(\mu) {\mathcal{C}_i}^{-1}(\theta^i) \right)^{-1} . \tag{2.13}$$

Clearly, $\mathcal{C}^{-1}$ and hence $\mathcal{C}$ are symmetric positive-definite. In words, $\mathcal{C}^{-1}$ is an approximation to ${\underline{\tilde{A}}}^{-1}(\Theta(\mu))$ In our application we used a single-point conditioner (labelled SP). We set $I = 1$, $S^{\theta}_I = \{\overline{\theta}\}$, $|\mathcal{I}(\mu)| = 1$, $\mathcal{I}(\mu) = \{1\}$. For further information and a detailed understanding of bound conditioner families (and special cases) we suggest to see [5, 11, 12].

2.5 Bounding Properties

It remains to demonstrate our claim that $s_N^-(\mu) \leq \tilde{s}(\mu) \leq s_N^+(\mu)$ for all $N \geq 1$. First for all $\mu \in \mathcal{D}^\mu$, and all $N \geq 1$, $s_N^-(\mu) \leq \tilde{s}(\mu)$. We have that

$$\begin{aligned}
\tilde{s}(\mu) - s_N(\mu) &= \langle F, \tilde{u}(\mu) - u_N(\mu) \rangle \\
&= \langle A(\Theta(\mu))\, \tilde{u}(\mu), \tilde{u}(\mu) - u_N(\mu) \rangle \\
&= \langle A(\Theta(\mu))\, (\tilde{u}(\mu) - u_N(\mu)), \tilde{u}(\mu) \rangle \\
&= \langle A(\Theta(\mu))\, (\tilde{u}(\mu) - u_N(\mu)), \tilde{u}(\mu) - u_N(\mu) \rangle \qquad (2.14) \\
&\geq 0
\end{aligned}$$

from the definition of $s(\mu)$, (1.4), symmetry of A, Galerkin orthogonality, and coercivity, respectively.

This lower bound proof is a standard result in variational approximation theory. We now turn to the upper bound to demonstrate for all $\mu \in \mathcal{D}^\mu$, and all $N \geq 1$, $s_N^+(\mu) \geq \tilde{s}(\mu)$. We first define $\underline{\tilde{e}} \in \mathbb{R}^{\mathcal{N}}$ as $\underline{\tilde{e}} = \underline{\tilde{u}} - \underline{u}_N$; it then follows from (1.5) and (2.1) that

$$\underline{\tilde{A}}(\Theta(\mu))\, \underline{\tilde{e}}(\mu) = \underline{\tilde{R}}(\mu)\ , \qquad (2.15)$$

which is the usual error-residual relationship. It then follows from (2.14) that

$$\begin{aligned}
\tilde{s}(\mu) - s_N(\mu) &= \underline{\tilde{e}}^T(\mu)\, \underline{\tilde{A}}(\Theta(\mu))\, \underline{\tilde{e}}(\mu) \\
&= \underline{\tilde{R}}^T(\mu)\, \underline{\tilde{A}}^{-1}(\Theta(\mu))\, \underline{\tilde{R}}(\mu)\ .
\end{aligned} \qquad (2.16)$$

It thus only remains to write that

$$\eta_N(\mu) \equiv \frac{s_N^+(\mu) - s_N(\mu)}{\tilde{s}(\mu) - s_N(\mu)} = \frac{\Delta_N(\mu)}{\tilde{s}(\mu) - s_N(\mu)} = \frac{\underline{\tilde{R}}^T(\mu) \underline{\mathcal{C}}^{-1}(\mu) \underline{\tilde{R}}(\mu)}{\underline{\tilde{R}}^T(\mu) \underline{\tilde{A}}^{-1}(\Theta(\mu)) \underline{\tilde{R}}(\mu)}, \qquad (2.17)$$

using the boundaries properties (2.6) and (2.9) and equations (2.10-(2.15) we have:

$$\eta_N(\mu) \equiv \frac{\underline{\hat{e}}^T(\mu)\, \underline{\mathcal{C}}(\Theta(\mu))\, \underline{\hat{e}}(\mu)}{\underline{\tilde{e}}^T(\mu)\, \underline{\tilde{A}}(\Theta(\mu))\, \underline{\tilde{e}}(\mu)} = \frac{\langle \tilde{C}(\mu)\hat{e}, \hat{e} \rangle}{\langle \underline{\tilde{A}}(\Theta(\mu))\tilde{e}, \tilde{e} \rangle} \qquad (2.18)$$

therefore

$$\rho_{min}(\mu) \leq \eta_N(\mu) \leq \rho_{max}(\mu); \qquad (2.19)$$

by construction, $\rho_{min}(\mu) \geq 1$ for all $\mu \in \mathcal{D}^\mu$ and therefore:

$$\eta_N(\mu) \geq 1, \qquad (2.20)$$

and $s_N^+(\mu) \geq s(\mu)$ as required. Note that the result (2.19) also indicates the sharpness of our bounds: it follows from (2.5) that:

$$\eta_N(\mu) \leq \rho. \qquad (2.21)$$

This result provides insight as to the properties of a good bound conditioner. Clearly, we wish $\rho_{max}(\mu)$ to be as close to unity, and hence as close to ρ_{min}, as possible.

2.6 Computational Procedure: Off-line/On-line Decomposition

We review here arguments given in detail in [11]; early applications of this approach may be found in [1]. The theoretical and empirical results let us to apply separate *off-line* and *on-line* computational procedures that exploit the dimension reduction.

In an *off-line* stage, we find the $\tilde{\underline{\zeta}}_n$, $n = 1, \ldots, N$ ($N \times \tilde{\underline{A}}$-solves), and form the $\underline{A}_{N\,q}$, $0 \le q \le Q$ ($((Q+1)N^2 \times \tilde{\underline{A}}$-inner products), and $\underline{F}_N$ ($N \times \mathcal{N}$ operations). In the *on-line* stage — given any new μ — we need only form $\underline{A}_N(\mu)$ from the $\underline{A}_{Nq}$ ($(Q+1)N^2$ operations), find $\underline{u}_N(\mu)$ ($O(N^3)$ operations), and evaluate $s_N(\mu)$ (N operations). The most important point is that the on-line complexity (and storage — $O(QN^2)$) is *independent* of the very large dimension of the truth space $\tilde{Y}, \mathcal{N}$; in particular, since N is typically very small (as suggested in the previous Sections), "real-time" response is obtained.

2.6.1 Computational Procedures for the Upper Bound $s_N^+(\mu)$

A computational procedure for the Upper Bound is very important and very useful in view of adaptivity procedures applied to basis $\tilde{\underline{\zeta}}^n$ storage and to test results with a cheap and fast procedure. See [11] for detailed bound conditioners presentation. We first note from (2.13)–(2.10) that

$$\underline{\hat{e}}(\mu) = \sum_{i \in \mathcal{I}(\mu)} \alpha_i(\mu)\, \tilde{\underline{A}}^{-1}(\theta^j) \left[\underline{\tilde{F}} - \sum_{q=0}^{Q} \sum_{n=1}^{N} \Theta_q(\mu)\, u_{Nn}(\mu)\, \tilde{\underline{A}}_q\, \tilde{\underline{\zeta}}_n \right] ;$$

recall that $\Theta_0 = 1$. It follows that we may express $\underline{\hat{e}}(\mu)$ as

$$\underline{\hat{e}}(\mu) = \sum_{i \in \mathcal{I}(\mu)} \alpha_j(\mu) \left[\underline{\tilde{z}}^i_{00} + \sum_{q=0}^{Q} \sum_{n=1}^{N} \Theta_q(\mu)\, u_{Nn}(\mu)\, \underline{\tilde{z}}^i_{qn} \right] ,$$

where for all $i \in \{1, \ldots, M\}$, $\tilde{\underline{A}}(\theta^i)\underline{\tilde{z}}^i_{00} = \underline{\tilde{F}}$, and $\tilde{\underline{A}}(\theta^i)\underline{\tilde{z}}^i_{qn} = -\tilde{\underline{A}}_q \tilde{\underline{\zeta}}_n$, $0 \le q \le Q$, $1 \le n \le N$. We may thus express our bound gap $\Delta_N(\mu)$ as the following (2.22):

$$\begin{aligned}
\Delta_N(\mu) &= \tilde{\underline{R}}^T(\mu)\, \underline{\hat{e}}(\mu) \\
&= \sum_{i \in \mathcal{I}(\mu)} \alpha_i(\mu) \left[\underline{\tilde{F}} - \sum_{q=0}^{Q} \sum_{n=1}^{N} \Theta_q(\mu)\, u_{Nn}(\mu)\, \tilde{\underline{A}}_q\, \tilde{\underline{\zeta}}_n \right]^T \\
&\qquad \left[\underline{\tilde{z}}^i_{00} + \sum_{q'=0}^{Q} \sum_{n'=1}^{N} \Theta_{q'}(\mu)\, u_{Nn'}(\mu)\, \underline{\tilde{z}}^i_{q'n'} \right] \\
&= \sum_{i \in \mathcal{I}(\mu)} \alpha_i(\mu) \left[c^i + \sum_{q=0}^{Q} \sum_{n=1}^{N} \Theta_q(\mu)\, u_{Nn}(\mu)\, \Lambda^i_{qn} \right. \\
&\qquad \left. + \sum_{q=0}^{Q} \sum_{n=1}^{N} \sum_{q'=0}^{Q} \sum_{n'=1}^{N} \Theta_q(\mu)\, \Theta_{q'}(\mu)\, u_{Nn}(\mu)\, u_{Nn'}(\mu)\, \Gamma^i_{qq'nn'} \right]
\end{aligned}$$

where for all $i \in \{1, \ldots, M\}$, $c^i = \underline{\tilde{F}}^T \underline{\tilde{z}}^i_{00}$, $\Lambda^i_{qn} = \underline{\tilde{F}}^T \underline{\tilde{z}}^i_{qn} - \underline{\tilde{\zeta}}^T_n \tilde{A}_q \underline{\tilde{z}}^i_{00}$ for $0 \le q \le Q$, $1 \le n \le N$, and $\Gamma^i_{qq'nn'} = -\underline{\tilde{\zeta}}^T_n \tilde{A}_q \underline{\tilde{z}}^i_{q'n'}$ for $0 \le q, q' \le Q$, $1 \le n, n' \le N$.

The off-line/on-line decomposition is now clear. In the *off-line* stage we compute the $\underline{\tilde{z}}^i_{00}$ and $\underline{\tilde{z}}^i_{qn}$ ($M((Q+1)N+1)$ $\underline{\tilde{A}}$-solves) and the c^i, Λ^i_{qn}, and $\Gamma^i_{qq'nn'}$ (predominated by M $((Q+1)^2N^2+(Q+1)N)$ $\underline{\tilde{A}}$-inner products). In the on-line stage we need "only" perform the sum (2.22), which requires $|\mathcal{I}(\mu)|((Q+1)^2N^2+(Q+1)N+1)$ operations. The essential point is that the on-line complexity (and storage — $O(M(Q+1)^2N^2)$) is independent of $\mathcal{N}$.

We note that the off-line/on-line decomposition depends critically on the "separability" of $\underline{\tilde{C}}^{-1}$ as a sum of products of parameter-*dependent* functions (the $\alpha_i(\mu)$) and parameter-*independent* operators (the $\underline{\tilde{A}}^{-1}(\theta^i)$). In turn, it is the direct approximation of $\underline{\tilde{A}}^{-1}(\Theta(\mu))$ (i.e., by a convex combination of $\underline{\tilde{A}}^{-1}(\theta^i)$) rather than of $\underline{\tilde{A}}(\Theta(\mu))$ (e.g., by a convex combination of $\underline{\tilde{A}}(\theta^i)$) that permits us to achieve this separability while simultaneously pursuing a "high-order" bound conditioner achieving some fixed (known, certain) accuracy — as measured by $\Delta_N(\mu)$ — at a lower computational effort. See [13].

3 A-Priori Convergence Theory:

3.1 Framework

We recall *a priori* framework for a general A-Priori convergence theory. Depending on the context and application, we will either invoke the lower bound ($s^-_N(\mu)$) or upper bound ($s^+_N(\mu)$) as our estimator for $\tilde{s}(\mu)$. For example, in an optimization exercise in which $\tilde{s}(\mu)$ enters as a constraint $\tilde{s}(\mu) \le s^{\max}$ (respectively, $\tilde{s}(\mu) \ge s^{\min}$), we will replace this condition with $s^+_N(\mu) \le s^{\max}$ (respectively, $s^-_N(\mu) \ge s^{\min}$) so as to ensure satisfaction/feasibility even in the presence of approximation errors. The rigorous bounding properties proven in Section 2.5 provide the requisite certainty.

But we of course also require accuracy: if, in the optimization context cited above, $s^+_N(\mu)$ or $s^-_N(\mu)$ is not close to $\tilde{s}(\mu)$, then our design may be *seriously suboptimal*. Since $|s^+_N(\mu) - \tilde{s}(\mu)| \le |s^+_N(\mu) - s^-_N(\mu)| = \Delta_N(\mu)$ and $|\tilde{s}(\mu) - s^-_N(\mu)| \le |s^+_N(\mu) - s^-_N(\mu)| = \Delta_N(\mu)$, it is the convergence of $\Delta_N(\mu)$ as a function of N that we must understand. In particular, from (2.18) and (2.14) we may write

$$\begin{aligned} \Delta_N(\mu) = s^+_N(\mu) - s^-_N(\mu) &= (\tilde{s}(\mu) - s_N(\mu)) \left(\frac{s^+_N(\mu) - s^-_N(\mu)}{\tilde{s}(\mu) - s_N(\mu)} \right) \\ &= \langle A(\Theta(\mu))\, \tilde{e}(\mu), \tilde{e}(\mu) \rangle\, \eta_N(\mu) \ , \end{aligned}$$

where $\tilde{e}(\mu) = \tilde{u}(\mu) - u_N(\mu)$. In some sense, the first factor, $\langle A(\Theta(\mu))\tilde{e}(\mu), \tilde{e}(\mu)\rangle$, measures the error in the solution $\tilde{u}(\mu) - u_N(\mu)$, while the second factor, the effectivity $\eta_N(\mu)$, measures the ratio of the actual and estimated errors; the former should be small, while the latter should be close to unity. As we shall see, this two-step factorization is important not only as a theoretical construct: it is this factorization which permits us to achieve high accuracy while simultaneously honoring our bound requirements. We would thus like to understand the convergence of $\Delta_N(\mu)$ to zero as a function of N. In particular, we consider the

case in which $A(\mu) = A_0 + \mu A_1$ (and hence $\Theta_1(\mu) = \mu$), and $\mu \in \mathcal{D}^\mu \equiv [0, \mu^{\max}]$. From our continuity and coercivity assumptions, there exists a positive real constant γ_1 such that

$$\langle A_1 v, v\rangle \leq \gamma_1 \langle A_0 v, v\rangle \ ; \tag{3.1}$$

it thus follows that $\langle A(\mu)v, v\rangle \leq (1 + \mu^{\max}\gamma_1)\,\langle A_0 v, v\rangle$. Defining $\|\cdot\|^2 \equiv \langle A_0 \cdot, \cdot\rangle$, we may thus write

$$\Delta_N(\mu) \leq (1 + \mu^{\max}\gamma_1)\,\|\tilde{u}(\mu) - u_N(\mu)\|^2\,\eta_N(\mu)\ . \tag{3.2}$$

3.2 Best Approximation

It remains to bound $\|\tilde{u}(\mu) - u_N(\mu)\|$ and $\eta_N(\mu)$; and, in particular, to understand the convergence rate of $\|\tilde{u}(\mu) - u_N(\mu)\| \to 0$ and $\eta_N(\mu) \to 1$ (or at least a constant) as N increases.
The proofs for both $\|\tilde{u}(\mu) - u_N(\mu)\|$ [6] and $\eta_N(\mu)$ implicate a particular "optimal" logarithmic point distribution which we thus impose *a priori*. In particular, we introduce an upper bound for γ_1, γ, and a "log increment" $\delta_N = (\ln(\gamma\mu^{\max} + 1))/(N-1)$; we then define

$$\mu^n = \exp\{-\ln\gamma + (n-1)\delta_N\} - \gamma^{-1}, \quad 1 \leq n \leq N\ ,$$

and take $S_N^\mu = \{\mu^1, \ldots, \mu^N\}$. Clearly, $\ln(\mu^n + \gamma^{-1})$ is uniformly distributed. Note that for $N \geq N_{\rm crit} \equiv 1 + e\,\ln(\gamma\mu^{\max} + 1)$, $\delta_N \leq e^{-1} < 1$.

We remind to the main result in [6, 7]:

For $N \geq N_{\rm crit} \equiv 1 + e\,\ln(\gamma\mu^{\max} + 1)$ and all $\mu \in \mathcal{D}^\mu$ we find:

$$\|\tilde{u}(\mu) - u_N(\mu)\| \leq (1 + \mu^{\max}\gamma_1)^{1/2}\,\|\tilde{u}(0)\|\,e^{-(\frac{N}{N_{\rm crit}})}\ ,$$

See Theorem 3 of [6] (for $c^* = 1$).
We see that we obtain *exponential* convergence, *uniformly* for all μ in $\mathcal{D}^\mu$. Furthermore, our convergence threshold parameter $N_{\rm crit} = 1 + e\,\ln(\gamma\mu^{\max} + 1)$, and exponential convergence rate $1/N_{\rm crit}$, depend only weakly — *logarithmically* — on γ_1 and $\mu^{\max}$ (which together comprise the continuity-coercivity ratio). In short, we expect extremely rapid convergence even for large parameter ranges. To obtain a bound for $\eta_N(\mu)$ (limited and close to unity) see [21].

4 Adaptation Procedure for Basis Construction

Given the higher powers of N that appear in our complexity estimates, it is crucial (both as regards online and offline effort) to control N more tightly. To this end, we may gainfully apply our *a posteriori* error bounds adaptively. We first construct, offline, an approximation that, over most of the domain, exhibits an error (in the H^1-norm) less than ϵ_d^{prior}: we begin with a first point $\mu^1 (S_{N'=1} = \{\mu^1\})$; we next (inexpensively) evaluate $\Delta_{N'=1}(\mu)$ over a large test sample of parameter points in $\mathcal{D}^\mu, \Sigma^{prior}$; we then choose for μ^2 (and hence $S_{N'=2} = \{\mu^1, \mu^2\}$) the maximizer of $\Delta_{N'=1}(\mu)$ over Σ^{prior}. We repeat this process until the maximum of $\Delta_{N'=N^{prior}}(\mu)$ over Σ^{prior} is less than ϵ_d^{prior}. Then, online, given a new value of the parameter, μ, and an error tolerance $\epsilon_d^{post}(\mu)$, we essentially repeat this adaptive process - but now our sample points are drawn from $S_{N^{prior}}$, and the test sample is a singleton - μ. Typically we

choose $\epsilon_d^{prior} \ll \epsilon_d^{post}(\mu)$ since our test is not exhaustive; and therefore, typically, $N^{post}(\mu) \ll N^{prior}$. With the adaptive process we get higher accuracy at lower N: modest reductions in N can translate into measurable performance improvements. This procedures is very important not only to get a computationally cheaper and faster procedure but also to avoid ill-conditioning in matrix assembling procedures.

5 Multi-parameters Application

We proceed to specify the model problem we have studied. The physical domain $\Omega \subset \mathbb{R}^2$ is divided in four subdomains $\Omega^i, i = 1, ..., 4$, a "T"-shaped region with Γ boundary, divided in Γ_d and Γ_n and associated, respectively, Dirichlet or Neumann condition. See Figure (1) and (2).

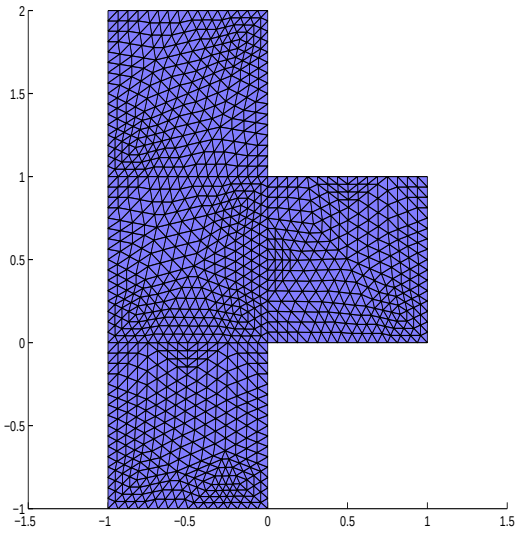


Figure 1: Mesh for the reduced-basis problem (normalized domain).

5.1 Geometrical Model

The geometrical model is based on a simple and standard configuration made up of four square subdomains parametrized in their dimensions (lengths and angles). The application is a very preliminary study of a possible "macro" configuration of interest for a bypass anastomosis for example. We have chosen five parameters:

- the angle θ for the incoming branch of the bypass.
- the diameters t and D, respectively, of the bypass and the artery.
- the lengths L and S, respectively, the outflow length and the distance between the incoming new branch and the occlusion caused by a stenosis. See Figure (2).

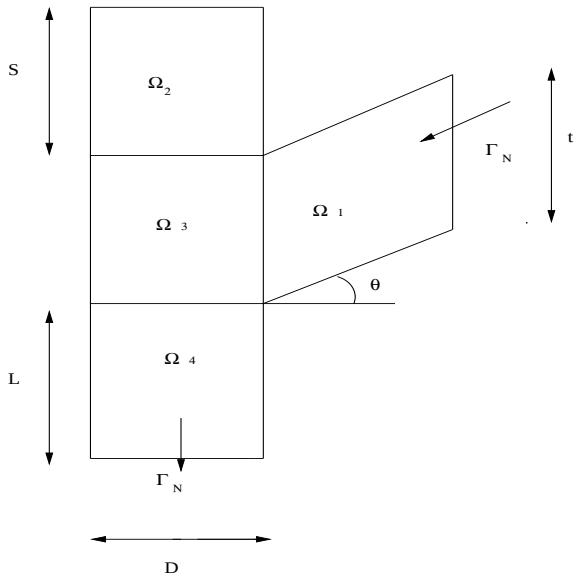


Figure 2: Scheme for the reduced-basis multi-parameter problem.

We underline that this is a macro geometrical structure, useful to study with low computational cost and sharp error bounds a possible configuration to be later optimized by optimal shape design and control tools. It's a case studied to validate a methodological approach and to get a deep understanding of reduced-basis methodologies. The method is based on the affine mapping procedures from a sub-domain of reference (square) to the true one ($\Omega \rightarrow \overline{\Omega}$). Preliminar tests were made studying a simpler case based only on a square domain.

5.2 Formulation

We present the strong form of the equations governing our preliminary problem (in $\overline{\Omega}$), from which we derive the weak statement; we then reformulate the problem in terms of a reference (parameter-independent) domain (Ω), thus recovering the abstract formulation of Section 1. We consider an elliptic equation, for example the equation of steady heat transfer with symmetric thermal diffusivity tensor $\overline{k}_{ij}$ (Poisson equation when $\overline{k}_{ij} = 1$) in $\overline{\Omega}^i \subset \mathbb{R}^d$ domain, for $i = 1, \ldots, 4$ with boundary $\overline{\Gamma}$. The field $\overline{u}$ (i.e for example temperature) satisfies the partial differential equation

$$-\frac{\partial}{\partial \overline{x}_i}\Big(\overline{k}_{ij}\frac{\partial \overline{u}}{\partial \overline{x}_j}\Big) = \overline{f} \quad in\ \overline{\Omega}, \tag{5.3}$$

with boundary conditions (see Figure(2))

$$\overline{u} = 0\ on\ \overline{\Gamma}_D, \quad \overline{k}_{ij}\frac{\partial \overline{u}}{\partial \overline{x}_j}\overline{e}_i^n = b\ on\ \overline{\Gamma}_N, \tag{5.4}$$

where $\overline{f}$ can be seen as the rate of heat generated per unit volume, b is the prescribed heat flux input on the surface $\overline{\Gamma}_N$, and $\overline{e}_i^n$ is the unit outward normal.

We now derive the weak form of the governing equations. We introduce the function space:

$$\overline{Y} = \{\overline{v} \in (H^1(\overline{\Omega}))^{p=1} | \overline{v} = 0 \ on \overline{\Gamma}_D\}, \tag{5.5}$$

and the associated norm

$$||\overline{v}||_{\overline{Y}} = \Big(\int_{\overline{\Omega}} \sum_{i=1}^{d} \Big(\frac{\partial \overline{v}}{\partial \overline{x}_i} \Big)^2 d\overline{\Omega} \Big)^{1/2}. \tag{5.6}$$

Multiplying (5.3) by a test function $\overline{v} \in \overline{Y}$, integrating over $\overline{\Omega}$ by parts and applying the divergence theorem we have

$$-\int_{\overline{\Omega}} \overline{v} \frac{\partial}{\partial \overline{x}_i} \Big(\overline{k}_{ij} \frac{\partial \overline{u}}{\partial \overline{x}_j} \Big) d\overline{\Omega} = -\int_{\overline{\Gamma}} \overline{v} \overline{k}_{ij} \frac{\partial \overline{u}}{\partial \overline{x}_j} \overline{e}_i^n d\overline{\Gamma} + \int_{\overline{\Omega}} \frac{\partial \overline{v}}{\partial \overline{x}_i} \overline{k}_{ij} \frac{\partial \overline{u}}{\partial \overline{x}_j} d\overline{\Omega} = \int_{\overline{\Omega}} \overline{f} \overline{v} d\overline{\Omega} \ \ \forall \overline{v} \in \overline{Y}. \tag{5.7}$$

Substituting boundary condition and by the fact that $\overline{v} = 0$ on $\overline{\Gamma}_D$, we obtain as our weak statement

$$\langle \overline{\mathcal{A}} \overline{u}, \overline{v} \rangle = \langle \overline{F}, \overline{v} \rangle, \ \forall \overline{v} \in \overline{Y}, \tag{5.8}$$

where

$$\langle \overline{\mathcal{A}} \overline{w}, \overline{v} \rangle = \int_{\overline{\Omega}} \frac{\partial \overline{v}}{\partial \overline{x}_i} \overline{k}_{ij} \frac{\partial \overline{w}}{\partial \overline{x}_j} d\overline{\Omega}, \tag{5.9}$$

$$\langle \overline{F}, \overline{v} \rangle = \langle \overline{F}_f, \overline{v} \rangle + \langle \overline{F}_b, \overline{v} \rangle, \tag{5.10}$$

here,

$$\langle \overline{F}_f, \overline{v} \rangle = \int_{\overline{\Omega}} \overline{f} \overline{v} d\overline{\Omega}, \quad \langle \overline{F}_b, \overline{v} \rangle = \int_{\overline{\Gamma}_N} \overline{b} \overline{v} d\overline{\Gamma}. \tag{5.11}$$

In our case $\overline{\Omega} = \bigcup_{r=1}^{R} \overline{\Omega}^r$, $R = 4$, so that the weak statement takes the form 5.8 where

$$\langle \overline{\mathcal{A}} \overline{w}, \overline{v} \rangle = \sum_{r=1}^{R} \int_{\overline{\Omega}^r} \frac{\partial \overline{v}}{\partial \overline{x}_i} \overline{k}_{ij}^r \frac{\partial \overline{w}}{\partial \overline{x}_j} d\overline{\Omega}, \tag{5.12}$$

$$\langle \overline{F}, \overline{v} \rangle = \langle \overline{F}_f, \overline{v} \rangle + \langle \overline{F}_b, \overline{v} \rangle, \tag{5.13}$$

and,

$$\langle \overline{F}_f, \overline{v} \rangle = \sum_{r=1}^{R} \int_{\overline{\Omega}^r} \overline{f}^r \overline{v} d\overline{\Omega}, \quad \langle \overline{F}_b, \overline{v} \rangle = \sum_{r=1}^{R} \int_{\overline{\Gamma}_N{}^r} \overline{b}^r \overline{v} d\overline{\Gamma}. \tag{5.14}$$

5.3 Affine mapping

The partition in subdomains $\overline{\Omega}^r$ is done such that there exists a reference domain Ω where, for any $\overline{x} \in \overline{\Omega}^r$, $r = 1, \ldots, R$, its image $x \in \Omega^r$ is given by

$$x = \mathcal{G}^r(\mu; \overline{x}) = G^r(\mu)\overline{x} + g^r, \quad 1 \le r \le R; \tag{5.15}$$

we thus write

$$\frac{\partial}{\partial \overline{x}_i} = \frac{\partial x_j}{\partial \overline{x}_i} \frac{\partial}{\partial x_j} = G_{ji}(\mu) \frac{\partial}{\partial x_j} \tag{5.16}$$

where $x \in \Omega$, $\overline{x} \in \overline{\Omega}$, $G^r(\mu) \in \mathbb{R}^{d \times d}$ is a picewise-constant matrix, $g^r(\mu) \in \mathbb{R}^d$ is a piecewise-constant vector, and $\mathcal{G}(\mu) : \overline{\Omega} \to \Omega$ is a piecewise-affine geometric

mapping. We then denote the boundary of Ω as Γ, where $\Gamma(\mu, \overline{\Gamma})$. We now define the function space Y as $Y(\Omega) = \overline{Y}(\mathcal{G}^{-1}(\mu;\Omega)) = \overline{Y}(\overline{\Omega})$ such that

$$Y = \{v \in (H^1(\Omega))^{p=1} | v = 0 \ on \Gamma_D\}, \tag{5.17}$$

and for any function $\overline{w} \in \overline{Y}$, we define $w \in Y$ such that $w(x) = \overline{w}(\mathcal{G}^{-1}(\mu;x))$. Futhermore, we have

$$d\overline{\Omega} = detG^{-1}(\mu)d\Omega, \ d\overline{\Gamma} = |G^{-1}(\mu)e^t|d\Gamma, \tag{5.18}$$

where e^t is a unit vector tangent to the boundary Γ, and

$$|G^{-1}e^t| = \Big(\sum_{i=1}^{d}(G_{ij}e_j^t)^2\Big)^{1/2}. \tag{5.19}$$

It then follows that $\langle \mathcal{A}(\mu)w, v\rangle = \langle \overline{\mathcal{A}}\overline{w}, \overline{v}\rangle$ and $\mathcal{A}(\mu)$ given by

$$\langle \mathcal{A}w, v\rangle = \sum_{r=1}^{R}\int_{\Omega^r}\Big(G_{ii'}^r(\mu)\frac{\partial w}{\partial x_i}\Big)\overline{k}_{i'j'}^r\Big(G_{jj'}^r(\mu)\frac{\partial v}{\partial x_j}\Big)det(G^r(\mu))^{-1}d\Omega, \tag{5.20}$$

or

$$\langle \mathcal{A}w, v\rangle = \sum_{r=1}^{R}\int_{\Omega^r}\frac{\partial w}{\partial x_i}\Big(G_{ii'}^r(\mu)\overline{k}_{i'j'}^r G_{jj'}^r(\mu)det(G^r(\mu))^{-1}\Big)\frac{\partial v}{\partial x_j}d\Omega \ \forall w, v \in Y, \tag{5.21}$$

and $\langle F(\mu)w, v\rangle = \langle \overline{F}\overline{w}, \overline{v}\rangle$ and $F(\mu)$ given by

$$\langle F(\mu), v\rangle = \langle F_f, v\rangle + \langle F_b, v\rangle, \tag{5.22}$$

where

$$\langle F_f, v\rangle = \sum_{r=1}^{R}\int_{\Omega^r}\Big(\overline{f}^r det(G^r(\mu))^{-1}\Big)vd\Omega, \ \ \langle F_b, v\rangle = \sum_{r=1}^{R}\int_{\Gamma_N{}^r}\Big(\overline{b}^r|(G^r(\mu))^{-1}e^t|\Big)vd\Gamma. \tag{5.23}$$

The abstract problem is then recovered for

$$\langle \mathcal{A}w, v\rangle = \sum_{r=1}^{R}\int_{\Omega^r}\frac{\partial w}{\partial x_i}k_{ij}^r\frac{\partial v}{\partial x_j}d\Omega \ \forall w, v \in Y, \tag{5.24}$$

$$\langle F(\mu), v\rangle = \langle F_f(\mu), v\rangle + \langle F_b(\mu), v\rangle, \tag{5.25}$$

$$\langle F_f, v\rangle = \sum_{r=1}^{R}\int_{\Omega^r} f^r v d\Omega, \ \ \langle F_b, v\rangle = \sum_{r=1}^{R}\int_{\Gamma_N{}^r} b^r v d\Gamma. \tag{5.26}$$

Here $k_{ij}^r(\mu)$ is given by

$$k_{ij}^r = G_{ii'}^r(\mu)\overline{k}_{i'j'}^r G_{jj'}^r(\mu)det(G^r(\mu))^{-1}, \tag{5.27}$$

and $b^r(\mu)$, $f^r(\mu)$ are given by

$$f^r(\mu) = \overline{f}^r det(G^r(\mu))^{-1}, \ b^r(\mu) = \overline{b}^r|(G^r(\mu))^{-1}e^t|. \tag{5.28}$$

Furthermore, clearly we may define

$$\Theta^{q(i,j,r)}(\mu) = k_{ij}^r(\mu), \ \langle \mathcal{A}^{q(i,j,r)}w, v\rangle = \int_{\Omega^r}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j}d\Omega, \tag{5.29}$$

for $1 \le r \le R$, $1 \le i, j \le d$.

5.4 Model Problem

We now consider our model problem in detail. As already said the problem can be seen as a problem involving the flow of heat in a T-shaped region containing an internal heat source as shown in Figure (1). Our output of interest is

$$s(\mu) = \sum_{r=1}^{R} \frac{1}{\xi_r} \int_{\overline{\Omega}^r} \overline{u} d\overline{\Omega}, \tag{5.30}$$

for $\mu = \{t, D, L, S, \theta\} \in \mathcal{D}^\mu \subset \mathbb{R}^P$, where ξ_r is a normalizing factor (related to geometrical quantities in the subdomains $\overline{\Omega}^r$). Our problem can then be formulated as: given a $\mu \in \mathcal{D}^\mu \subset \mathbb{R}^P$, find $s(\mu) = \langle \overline{L}, \overline{u} \rangle$ where $\overline{u} \in \overline{Y}$ is the solution to:

$$\langle \overline{\mathcal{A}} \overline{w}, \overline{v} \rangle = \langle \overline{F}, \overline{v} \rangle, \forall \overline{v} \in \overline{Y}; \tag{5.31}$$

here, $\langle \overline{L}, \overline{v} \rangle = \langle \overline{F}, \overline{v} \rangle$, $\forall \overline{v} \in \overline{Y}$. In our case $\mathcal{D}^\mu$ is given by $[t_{min}, t_{max}] \times [D_{min}, D_{max}] \times [L_{min}, L_{max}] \times [S_{min}, S_{max}] \times [\theta_{min}, \theta_{max}]$, i.e: $[0.1, 1.5] \times [0.1, 1.5] \times [0.1, 5.0] \times [0.1, 5.0] \times [0^o, 60^o]$. We have $\overline{b} = 0$, $\overline{f} = \{\frac{1}{\xi_r}\}$ defined on $\overline{\Omega}^r$, so that:

$$\xi_1 = t,\ \xi_2 = S \cdot D,\ \xi_3 = t \cdot D,\ \xi_4 = L \cdot D. \tag{5.32}$$

The affine mapping $\mathcal{G}(\overline{x})(\mu) : \overline{\Omega} \to \Omega$ is given by (5.15) and we have

$$G^1(\mu) = \begin{pmatrix} 1 & -\tan(\theta) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{t} \end{pmatrix},\ G^2(\mu) = \begin{pmatrix} \frac{1}{D} & 0 \\ 0 & \frac{1}{S} \end{pmatrix} \tag{5.33}$$

$$G^3(\mu) = \begin{pmatrix} \frac{1}{D} & 0 \\ 0 & \frac{1}{t} \end{pmatrix},\ G^4(\mu) = \begin{pmatrix} \frac{1}{D} & 0 \\ 0 & \frac{1}{L} \end{pmatrix}. \tag{5.34}$$

Each mapping function is defined in $\overline{\Omega}^r$, furthermore $g^r = 0\ \forall r$. We have

$$d\overline{\Omega}^1 = det(G^1)^{-1}(\mu) d\Omega^1 = t d\Omega,\ d\overline{\Gamma}^1 = |det(G^1)^{-1}(\mu) e^t| d\Gamma^1 = t d\Gamma, \tag{5.35}$$

$$d\overline{\Omega}^2 = det(G^2)^{-1}(\mu) d\Omega^2 = LD d\Omega,\ \ d\overline{\Gamma}^2 = |det(G^2)^{-1}(\mu) e^t| d\Gamma^2 = LD d\Gamma, \tag{5.36}$$

$$d\overline{\Omega}^3 = det(G^3)^{-1}(\mu) d\Omega^3 = tD d\Omega,\ \ d\overline{\Gamma}^3 = |det(G^3)^{-1}(\mu) e^t| d\Gamma^3 = tD d\Gamma, \tag{5.37}$$

$$d\overline{\Omega}^4 = det(G^4)^{-1}(\mu) d\Omega^4 = SD d\Omega,\ d\overline{\Gamma}^4 = |det(G^4)^{-1}(\mu) e^t| d\Gamma^4 = SD d\Gamma, \tag{5.38}$$

Figure (3) and (4) show the affine mapping procedures on two subdomains (Ω^1 and Ω^3 for example)
We may now re-formulate our problem in terms of our reference domain: find $s(\mu) = \langle L, u \rangle$ where $u \in Y$ is the solution to:

$$\langle \mathcal{A} w, v \rangle = \langle F, v \rangle, \forall v \in Y; \tag{5.39}$$

where $\langle L, v \rangle = \langle F, v \rangle \forall v \in Y$.

$$\langle \mathcal{A} w, v \rangle = \sum_{r=1}^{R} \langle \mathcal{A}^r w, v \rangle = \langle F, v \rangle, \forall v \in Y, \tag{5.40}$$

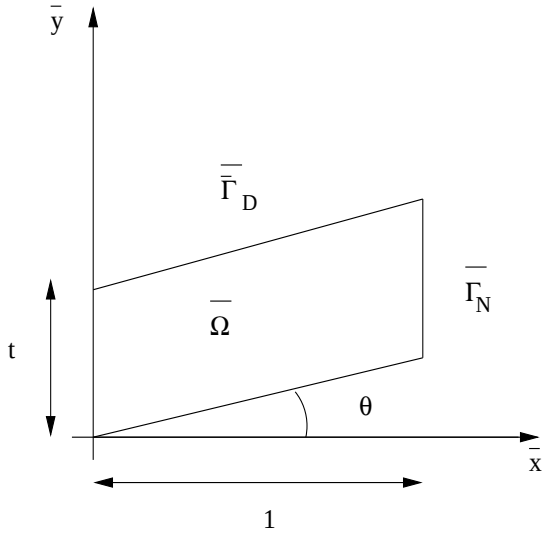


Figure 3: Scheme of parameter-dependent domain Ω^1 undergoing vertical shear and rotation.

$$\langle F, v\rangle = \sum_{r=1}^{R} \int_{\Omega_r} v d\Omega, \ \forall v \in Y \tag{5.41}$$

Here:

$$\langle \mathcal{A}^r w, v\rangle = \int_{\Omega_r} \frac{\partial v}{\partial x_i} k_{ij}^r \frac{\partial w}{\partial x_j} d\Omega, \ \forall w, v \in Y \tag{5.42}$$

and the effective diffusivity tensors $k_{ij}^r(\mu) = G_{ii'}(\mu)\overline{k}_{i'j'}^r G_{jj'}(\mu) det G^{-1}(\mu)$ are given by:

$$k^1 = \begin{bmatrix} t & -\tan\theta \\ -\tan\theta & \frac{1+\tan^2\theta}{t} \end{bmatrix} \tag{5.43}$$

$$k^2 = \begin{bmatrix} \frac{S}{D} & 0 \\ 0 & \frac{D}{S} \end{bmatrix} \tag{5.44}$$

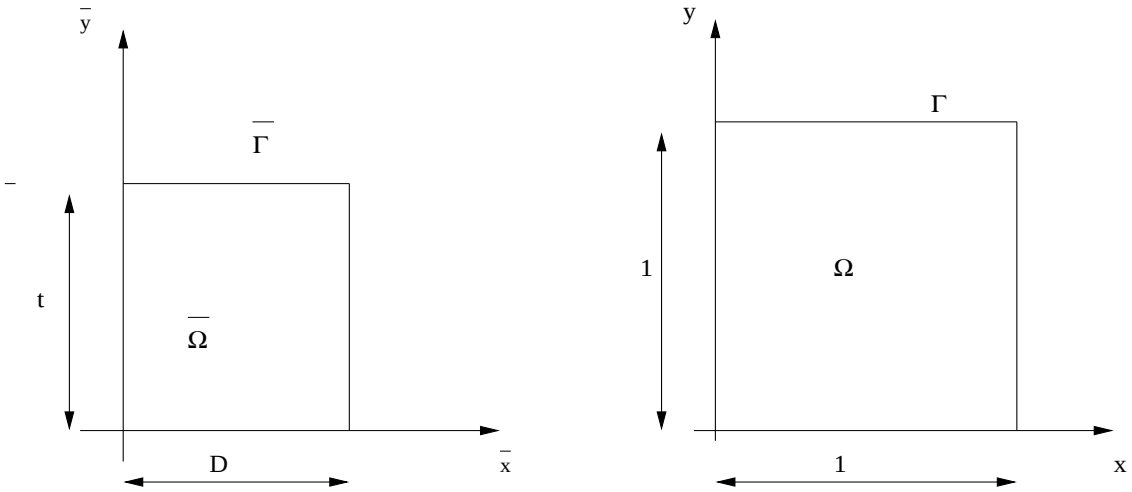
$$k^3 = \begin{bmatrix} \frac{t}{D} & 0 \\ 0 & \frac{D}{t} \end{bmatrix} \tag{5.45}$$

$$k^4 = \begin{bmatrix} \frac{L}{D} & 0 \\ 0 & \frac{D}{L} \end{bmatrix}. \tag{5.46}$$

The abstract problem formulation is then given for $P = 5$ (number of parameters), $R = 4$ (subdomains) and $Q = 9$ (different bilinear forms contributes to $\mathcal{A}$ over different portions of domain).

6 Numerical Results

In this section we present some numerical results obtained with the configuration previously described. We used A-posteriori error bounds and adaptivity



(a)

Figure 4: Scheme for parameter-dependent domain Ω^3 and reference domain undergoing both stretch and shear.

procedures in basis-building process. Table 1 shows a preliminary test based on two-parameters configuration and adaptivity procedures. We can see that the dimension of N is very small in reduced-basis model. With adaptivity procedures we get a 60% saving in computational cost related to Off-line procedures. The information about the error bound is provided by Δ_N. Table 2 gives us

N	Δ_N
7	0.1
9	0.01
13	0.001
20	0.0001

Table 1: Δ_N and N using two-parameters configuration (L and D) and adaptivity procedure (Section 4). Without adaptivity procedure we would need $N = 50$ to get a $\Delta_N < 10^{-3}$, we reach a consistent computational load reduction.

information about convergence (Δ_N), varying N, and about the effectivity η_N, studying a complete five-parameters configuration. With $N = 45$, for example, we get $\Delta_N \leq 10^{-4} = \varepsilon_d^{prior}$. Effectivity is near unity (always ≥ 1). The results are obtained testing at least 1000 different configurations Σ^{prior} and different parameters combinations (see Section 4), the ones in the table are the worst gotten testing all the random configurations considered. It's important to underline that the adaptive procedure tested in the reduced-basis off-line building phase has permitted us to keep under control the conditioning number of the reduced-basis matrix, avoiding ill-conditioning problems caused by the random choice of parameters.
Figure (5) and (6) show the solution of the problem given by Finite-Elements

N	η_N	Δ_{Nmax}	N	η_N	Δ_{Nmax}
1	3.1438	0.064589558	31	1.2802	0.000234237
2	4.4864	0.051445346	32	1.4427	0.000220116
3	4.6687	0.052944041	33	1.4321	0.000192997
4	4.5023	0.050876316	34	1.3848	0.000171898
5	4.0258	0.043368854	35	1.6566	0.000189594
6	2.9801	0.025041318	36	1.651	0.000176161
7	2.521	0.019830625	37	1.814	0.000185761
8	2.3531	0.018128973	38	1.8877	0.000192532
9	1.8848	0.012419358	39	1.9003	0.000187474
10	1.7454	0.010821934	40	1.8567	0.000179674
11	1.5322	0.008960269	41	1.6172	0.000140013
12	1.5339	0.008963254	42	1.6207	0.000139235
13	1.5327	0.008874298	43	1.6361	0.000140171
14	1.5052	0.008636955	44	1.7266	0.000146752
15	2.2945	0.008465466	45	1.6297	0.000101771
16	2.1233	0.008245741	46	1.6366	$9.74091E-05$
17	2.0121	0.007101543	47	1.6187	$9.63095E-05$
18	1.9571	0.005058438	48	1.6283	$9.64255E-05$
19	2.2285	0.003577518	49	1.481	$8.87554E-05$
20	2.3294	0.003595009	50	1.4757	$8.78055E-05$
21	1.9196	0.001392253	51	1.4802	$8.46522E-05$
22	1.882	0.00132827	52	2.8702	$8.46041E-05$
23	1.673	0.000950676	53	2.9196	$8.46041E-05$
24	1.6219	0.000783639	54	2.8235	$8.46041E-05$
25	1.655	0.000795108	55	2.6366	$8.47554E-05$
26	1.2313	0.000360924	56	2.7151	$7.8055E-05$
27	1.1243	0.000301648	57	2.6354	$7.65224E-05$
28	1.1096	0.000269211	58	2.6772	$7.46041E-05$
29	1.108	0.000268716	59	2.3726	$4.26101E-05$
30	1.1698	0.000275998	60	2.4882	$4.45111E-05$

Table 2: Δ_{Nmax} and N using five-parameters configuration and adaptivity procedure to get a $\Delta_{Nmax} < 10^{-4} = \varepsilon_d^{prior}$ in the worst case. Note values of the effectivity η_N.

method and by Reduced-Basis for a certain combination of parameters (i.e $L = 1.0,\ D = 1.5, S = 1.0,\ t = 1.31,\ \theta = 16^o$). Figure (7) shows the distribution of the error $\tilde{e}$ in Ω (difference between u_N and $\tilde{u}$) using $N = 60$ basis. This is not the error over the output $s(\mu)$ calculated by A-posteriori error bounds, but it can be considered as a good indication how reduced-basis method is able to provide results close to the Finite-Elements solution. Figure (8) shows the convergence of the method used in logarithmic scale increasing N, while Figure (9) shows the upper and lower bound error estimator and their convergence to the true value of the output quantity of interest. Figure (10) and (11) represent effectivity η_N and the upper (η_{max}) and lower limit (η_{min}, i.e. unity). Figure (12) and (13) show parameters distribution in the parameters spaces during the off-line reduced-basis matrix construction. The former shows a two-parameters distribution region ($\mathcal{D}^{\mu} \subset \mathbb{R}^2$) around the first couple of parameters chosen in the centre of the region. Note that the parameters are distributed around the original starting point. The latter is a possible five-parameters distribution in the case studied applying the adaptive procedure.

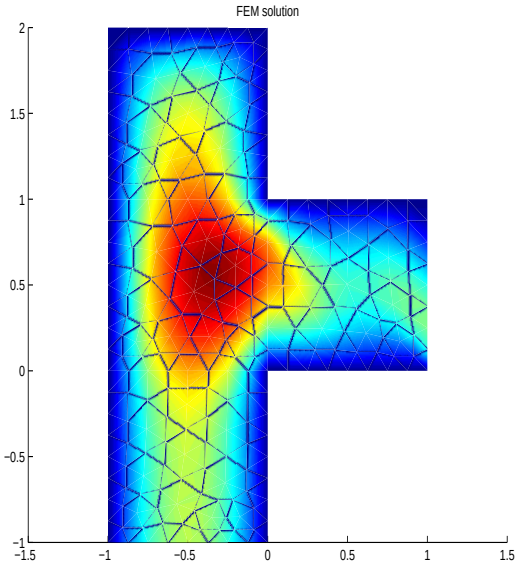


Figure 5: Solution using FEM-Galerkin method for a 5-paramters configuration.

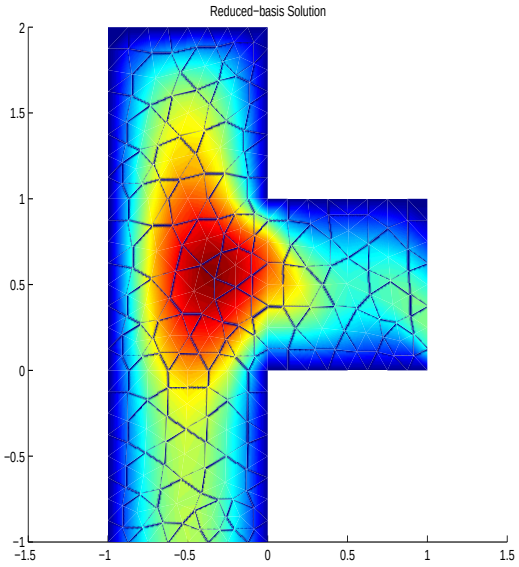


Figure 6: Solution using RB method with $N = 60$.

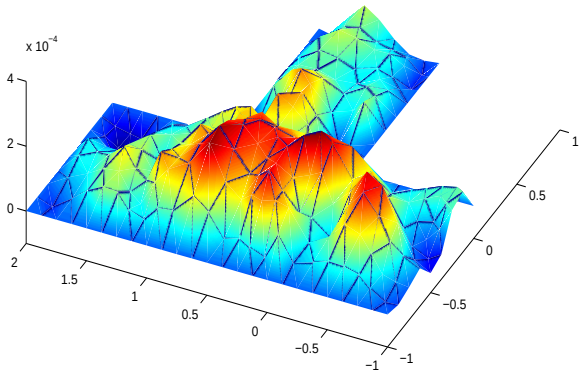


Figure 7: Distributed error $\tilde{e}$ over the domain Ω between $\tilde{u}$ and u_N.

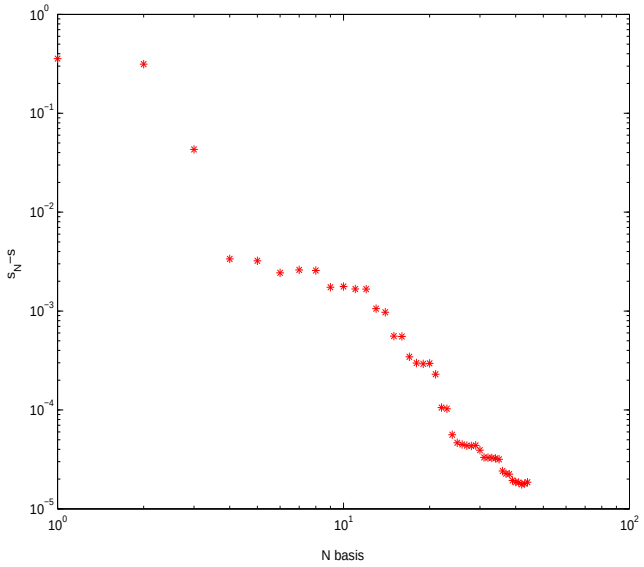


Figure 8: Convergence of the difference $s_N - s$ increasing N.

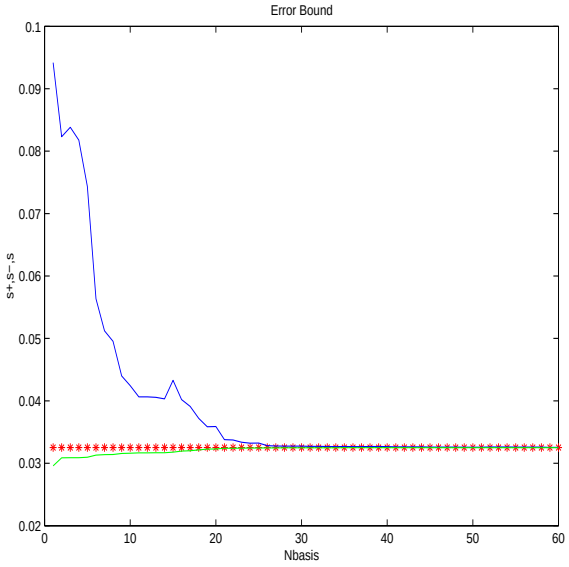


Figure 9: Error Bounds s^+, s^- and their convergence to $s(*)$.

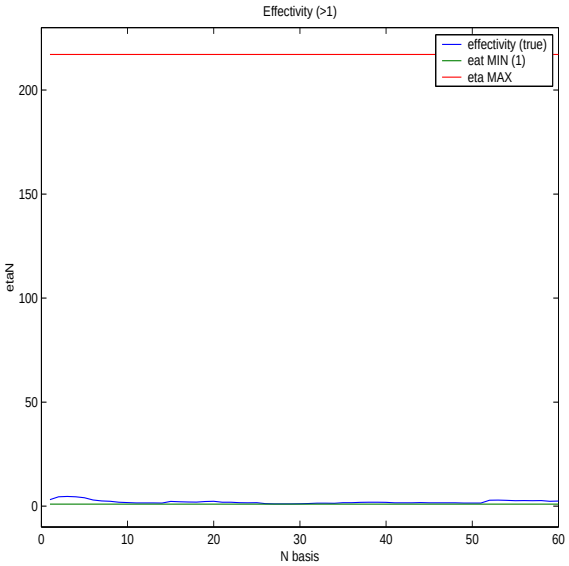


Figure 10: Effectivity η_N and its bounds $\eta_{min}(=1)$ and η_{max}

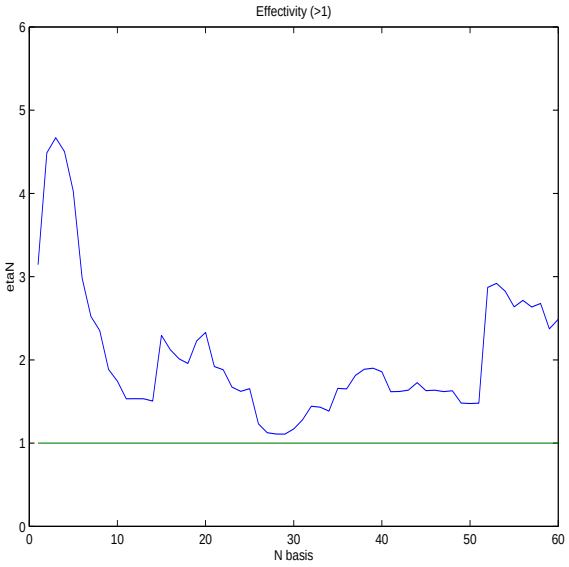


Figure 11: Effectivity η_N with lower Bound (unity). Zoom.

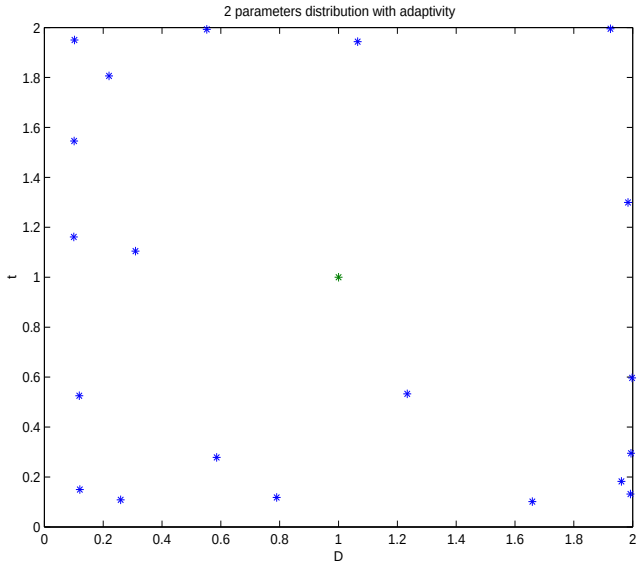


Figure 12: 2-Parameters (t, D) distribution using adaptivity in basis building off-line process. The starting configuration is in the centre of the 2-Parameters space.

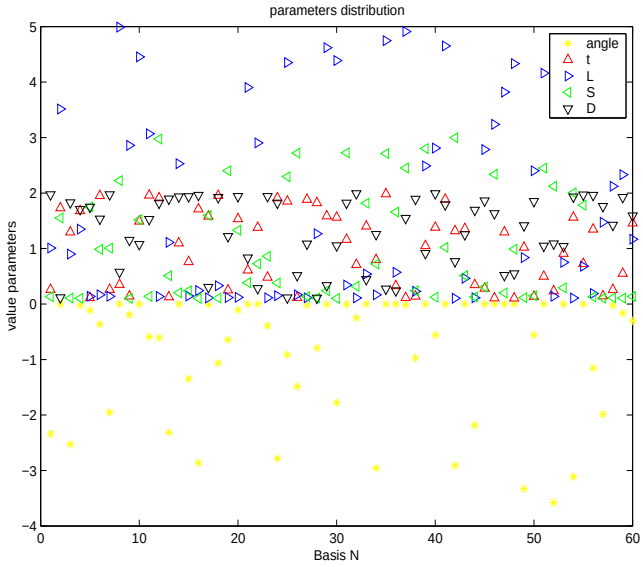


Figure 13: 5-Parameters distribution using adaptivity for reduced-basis matrix assembling.

7 Further and Future Developments

Future development guidelines could be guided by results and improvements provided in the field of reduced-basis by its extension to Navier-Stokes equation (also unsteady) and in problems involving non-affine mapping (i.e. shape design). See references [4] and [20] in progress. We refer to possible developments in the problem of the aorto-coronaric bypass anastomoses, for example, but the procedures can be generalized for other design problems in engineering.

7.1 First step

The first step will be the replacement of the elliptic state equation (laplacian) with Stokes equation (with Dirichlet in-flow condition and Neumann out-flow condition). See [18].

- Reformulation of the problem (new state equations, new bilinear forms) with the same geometry (4 subdomains).
- When this step is completed we can test the model with "macro" parameters in the haemodynamics background (5 macro-parameters).
- We get a possible starting configuration to apply the tools of shape optimization and flow control (see [16]) with an output of interest (i.e. vorticity).
- We can get a sensitivity analysis about macro-geometrical configuration changes (Diameter, Stenosis lentgh, Bypass angle,...).
- At the end of the study I get an optimized starting configuration with useful indication for bypass implanting procedures.

7.2 Second step

We study the problem at an intermediate level with an approach based on feedback procedures (a shell model).

- We used reduced-basis technique to get a preliminary configuration for the bypass problem (macro configuration).
- We apply "state of the art" tools for optimization based on flow control and optimal shape design technique, already available in [8], using the starting configuration gotten from reduced basis model application. We use steady Stokes equations. See [16].
- We take the new configuration (the micro configuration optimized by shape design tools) for a further feedback using unsteady Navier Stokes equations and other output of interest (such as unsteady quadratic functionals related to wall shear stress oscillations).

In conclusion we have a "shell" model with three inner feedback procedure (see Figure (14)):

- Reduced-Basis method on macro structure.

- Optimal Shape design and Flow Control on micro structure.
- Unsteady quantities and Navier Stokes equations test the final configuration.

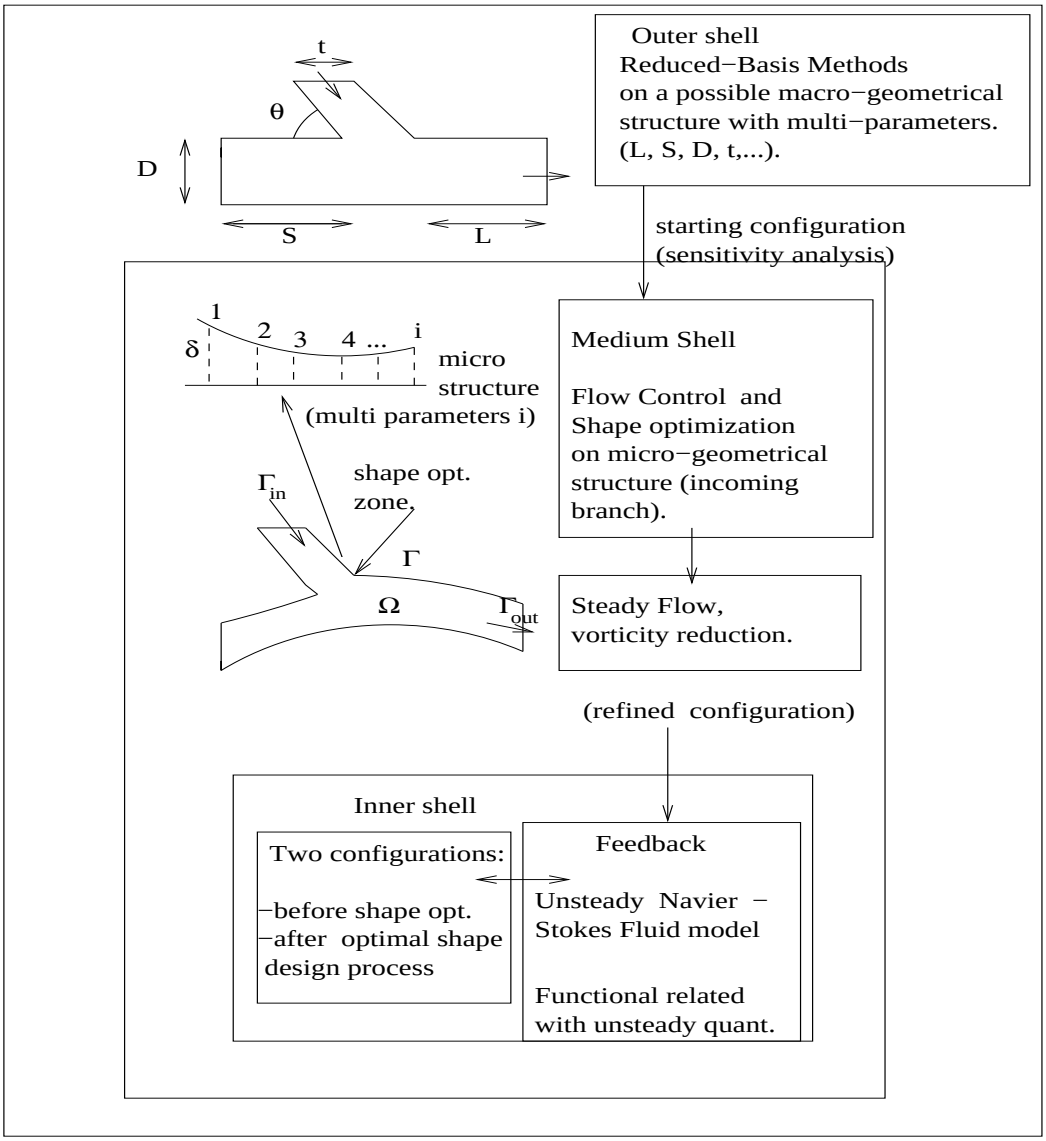


Figure 14: "Shell" model scheme made up by future steps.

7.3 Third step

The last step could be the full extension of reduced-basis model using more than five parameters. We can use parameters to model the wall shapes (pointwise) basing the procedures on non-affine mapping and on fictitious domain. The

problem could be solved by reduced-basis method first with a steady output, later with the unsteady output of our interest (reduced model all-based and sensitivity analysis). An other possible option could be the interface between optimal control (based on adjoint formulation) and reduced models used to solve both state and adjoint problem with an increasing number of geometrical shape parameters. In this last case optimization would be led by optimal control tools, while approximation by reduced-basis model. See Table 3.

Third Step	First Option	Second Option
Problem Approximation	Galerkin-Finite Elements and Reduced-basis for state equation	Galerkin-Finite Elements and Reduced-basis for state and adjoint equation
Optimization	Sensitivity Analysis	Optimal Control (adjoint formulation)

Table 3: Third step future developments options.

It's evident that to expand and apply reduced-basis theory on biomechanics problems (i.e. biomedical devices such as bypass) the two most important phases are the use of a great number of geometrical parameters (non-affine mapping) and the use of Navier-Stokes equations to model fluid flow.

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