PRESSURE-CORRECTION ALGEBRAIC SPLITTING METHODS FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In this paper we present a new family of methods for the effective numerical solution of the incompressible unsteady Navier-Stokes equations. These methods resort to an algebraic splitting of the discretized problem based on inexact LU block-factorizations of the corresponding matrix (following [21]). In particular, we will start from inexact algebraic factorizations of *algebraic Chorin-Temam* and *Yosida* type and introduce a *pressure correction step* aiming at improving the time accuracy. One of the schemes obtained in this way (the Algebraic Chorin-Temam Pressure Correction method) resembles a method previously introduced in the framework of differential projection schemes (see [24], [19]). The stability and the dependence of splitting error on the time step of the new methods is investigated and tested on several numerical cases.

Key words. Incompressible Navier-Stokes equations, Fractional Steps Schemes.

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1. Introduction. The numerical computation of the unsteady Navier-Stokes equations for incompressible flows in real applications requires the solution of linear systems of large dimensions. These systems are typically not definite nor well conditioned, and therefore the set up of efficient methods is mandatory. Perhaps one of the most successful approaches is provided by the class of the projection methods at the differential (see e.g. [2] and, more recently, [12]) and the algebraic level (see [18] and [20], [21]). These method typically compute the velocity and pressure fields separately, (i) by computing an *auxiliary* (or *intermediate*) velocity; (ii) by solving a suitable problem for the pressure; (iii) by correcting the velocity (*end-of-step velocity*), by forcing the incompressibility constraint. In [21] we have introduced a general class of algebraic splitting methods that can be regarded in this framework. We recall, in particular, the *Algebraic Chorin-Temam* scheme (see [18]) and the *Yosida* scheme (see [20]). These methods are based on a splitting that reduces the computational effort, without affecting the time accuracy of the solution driven by the time discretization. This is true for first order time discretizations, while for higher order accuracy the set up of suitable splittings is still an open problem (see e.g. in the framework of differential schemes, [8]).

In the present paper, we aim at investigating splitting methods that arise whenever, besides the velocity, also the pressure is obtained after a suitable *correction step*. This step is set up in order to reduce the error associated to the splitting and obtain definitively a higher order of accuracy in time.

After a brief introduction to algebraic splitting methods (Sect. 2), we will therefore provide a general approach for setting up such *pressure correction* schemes (Sect. 3). Then, we will in particular analyze the schemes arising from the pressure correction of both the Algebraic Chorin-Temam and the Yosida methods. The former is investigated in Sect. 4. We analyze the splitting error reduction induced by the pressure correction step and prove that in the Stokes (linear) case the scheme feature

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unconditional stability when starting from a Backward Difference (implicit) time discretization. Then (Sect. 4.4), we establish some formal analogies between our scheme and other pressure correction schemes proposed at a differential level (see [24]).

The pressure correction formulation of the Yosida method is investigated in Sect. 5. In particular, it is possible to prove that the consistency error induced by the splitting on the matrix to be solved at each time step depends on the cube of the time step, which is an improvement of the original Yosida scheme introduced by the pressure correction. On the other hand, we prove that the stability of this scheme is in general conditional, even if applied to the Stokes problem discretized with a Backward Difference (implicit) scheme.

In Sect. 6 we provide several numerical results, testing the properties of pressure-correction schemes. We analyze in particular the improvements induced by the pressure correction when applied to high order time discretization schemes. Some conclusions are drawn in Sect. 7.

2. Inexact algebraic factorizations for the Navier-Stokes Equations. Consider an open and bounded domain $\Omega \subset \mathbb{R}^d$ for $d = 2, 3$ with boundary $\partial\Omega$ for a time $t \geq 0$. The Navier-Stokes equations for an incompressible flows in terms of the velocity, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, and the pressure, $p = p(\mathbf{x}, t)$, read as

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \triangle \mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{2.1}$$

for any $(\mathbf{x}, t) \in \Omega \times (0, T]$, with $T > 0$. This system must be completed with the initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x})$ (where $\mathbf{u}^0(\mathbf{x})$ is a given function) and suitable boundary conditions on $\partial\Omega$. Since in the framework of algebraic splitting methods there is a complete independence of the numerical methods of the boundary conditions, we do not specify a specific boundary set. It is however understood that some (reasonable) boundary conditions are prescribed on $\partial\Omega$.

In order to have a quantitative evaluation of the flow field in real applications, a numerical approximation has to be carried out. To this aim, the problem has to be discretized with respect to time and space variables. Concerning the *space discretization* issue, we will basically refer to the *Galerkin method* and, in particular, to the *finite element method* (FEM). The most part of what follows can be however applied to other space discretization methods as well. For any details concerning the FEM discretization of the Navier-Stokes problem, we refer, e.g., to [22]. In fact, we choose functional spaces for the approximate velocity and pressure fields which satisfy the *inf-sup* or *LBB condition* (see, e.g., [1]). We will denote by $N_{\mathbf{u}}$ the number of velocity degrees of freedom and by N_p the number of pressure degrees of freedom.

For what concerns the *time discretization*, we will refer to classic backward differences methods. Namely, we consider a decomposition of the time interval into N sub-intervals (t^n, t^{n+1}) with $t^n = n\Delta t$, where $\Delta t = T/N$ is the uniform positive time step and collocate the equation in the instants $t^n = n\Delta t$. For the treatment of the nonlinear convective term, we resort to the usual (semi-implicit) linearization $(\mathbf{u}(t^{n+1}) \cdot \nabla)\mathbf{u}(t^{n+1}) \simeq (\mathbf{u}(t^n) \cdot \nabla)\mathbf{u}(t^{n+1})$ or similar featuring higher order of time accuracy (see, e.g., [8]).

The fully discretized and linearized incompressible Navier-Stokes equations at the time t^{n+1} read therefore:

$$\mathcal{A}\mathbf{w}^{n+1} = \mathbf{b}^{n+1}, \tag{2.2}$$

where the vector $\mathbf{b}^k = \left[b_1^k, b_2^k\right]^T$ contains forcing terms and contributions of the boundary conditions, $\mathbf{w}^{n+1} = (\mathbf{u}^{n+1}, \mathbf{p}^{n+1})^T$ denotes the vector of the nodal values of the discrete velocity and pressure and

$$\mathcal{A} = \begin{bmatrix} \mathrm{C} & \mathrm{D}^T \\ \mathrm{D} & 0 \end{bmatrix}. \tag{2.3}$$

Here D the discrete divergence operator (i.e. $d_{ij} = -\int_\Omega \nabla\cdot\mathbf{v}_j q_i$, where $\{\mathbf{v}_k\}$ band $\{q_k\}$ are the basis functions for the velocity and the pressure respectively). D^T denotes the discrete gradient operator and C collects contributions from the time derivative and advection and diffusion operators. More specifically, we denote:

$$\mathrm{C} = \frac{\alpha}{\Delta t}\mathrm{M} + \mathrm{K}$$

where M is the velocity mass matrix, α is the coefficient of the Backward Difference scheme (briefly, BDF) at hand for the velocity field at time t^{n+1} and K corresponds to the discretization of the diffusive and of the convective terms. In the case of the Stokes problem, K corresponds just to the Laplacian of the velocity and it is therefore symmetric and positive definite (s.p.d.).

System (2.2) typically features large dimensions and bad conditioning properties. The splitting between the computation of the velocity field from that of the pressure is almost mandatory when large three-dimensional problems are faced. This can be obtained through inexact block LU decompositions. These strategies stem from the following "exact" *LU*-block factorization of $\mathcal{A}$

$$\mathcal{A} = \begin{bmatrix} \mathrm{C} & 0 \\ \mathrm{D} & -\mathrm{D}\mathrm{C}^{-1}\mathrm{D}^T \end{bmatrix} \begin{bmatrix} \mathrm{I} & \mathrm{C}^{-1}\mathrm{D}^T \\ 0 & \mathrm{I} \end{bmatrix}.$$

Since the inverse C^{-1} is seldom available, we can set up different schemes, achieving a reduction in the computational cost by suitably approximating C^{-1} with a matrix H_1 in the L-block and H_2 in the U-block. This leads to the following *inexact block LU factorization* (see [21] and also [4] and [5]):

$$\widehat{\mathcal{A}} = \begin{bmatrix} \mathrm{C} & 0 \\ \mathrm{D} & -\mathrm{D}\mathrm{H}_1\mathrm{D}^T \end{bmatrix} \begin{bmatrix} \mathrm{I} & \mathrm{H}_2\mathrm{D}^T \\ 0 & \mathrm{I} \end{bmatrix} = \begin{bmatrix} \mathrm{C} & \mathrm{C}\mathrm{H}_2\mathrm{D}^T \\ \mathrm{D} & \mathrm{D}(\mathrm{H}_2 - \mathrm{H}_1)\mathrm{D}^T \end{bmatrix}. \tag{2.4}$$

The corresponding algebraic fractional-step methods require at the generic time-level t^{n+1} the solution of the following systems:

$$\text{L-step} \quad \begin{cases} \mathrm{C}\widetilde{\mathbf{u}}^{n+1} = \mathbf{b}_1^{n+1}, \\ \mathrm{D}\widetilde{\mathbf{u}}^{n+1} - \mathrm{D}\mathrm{H}_1\mathrm{D}^T\tilde{\mathbf{p}}^{n+1} = \mathbf{b}_2^{n+1}, \end{cases} \tag{2.5}$$

$$\text{U-step} \quad \begin{cases} \mathbf{p}^{n+1} = \widetilde{\mathbf{p}}^{n+1}, \\ \mathbf{u}^{n+1} + \mathrm{H}_2\mathrm{D}^T\mathbf{p}^{n+1} = \widetilde{\mathbf{u}}^{n+1}. \end{cases} \tag{2.6}$$

Different choices can be pursued for the approximant matrices H_1 and H_2. In particular, we could take

$$\mathrm{H}_1 = \mathrm{H}_2 = \frac{\Delta t}{\alpha}\mathrm{M}^{-1}.$$

or else

$$\mathrm{H}_1 = \frac{\Delta t}{\alpha}\mathrm{M}^{-1}, \mathrm{H}_2 = \mathrm{C}^{-1},$$

The former choice yields a scheme (see [18]) that can be considered the *algebraic* counterpart of the *Chorin-Temam scheme* (ACT briefly), because of the formal analogy with the original differential-splitting method. The latter choice yields the so-called *Yosida method*, introduced in [21] and analyzed in [20]. See also [] and [5].

The main difference between the two possibilities is that in the Chorin-Temam scheme only the discretized momentum equation is perturbed, while in the Yosida scheme only the mass conservation equation is perturbed (see [25]). In the sequel, we will say that an algebraic fractional-step method is of *Yosida type* if $\mathrm{H}_1 = \mathrm{H}$ and $\mathrm{H}_2 = \mathrm{C}^{-1}$, of *Chorin-Temam* type if $\mathrm{H}_1 = \mathrm{H}_2 = \mathrm{H}$, being H any convenient approximation of C^{-1}. Note that the approximation of C^{-1} with $\Delta t/\alpha \mathrm{M}^{-1}$ stems from a truncation to the first term of the well known Neumann expansion:

$$\mathrm{C}^{-1} = \frac{\Delta t}{\alpha}\left(\mathrm{I}_{N_\mathbf{u}} + \frac{\Delta t}{\alpha}\mathrm{M}^{-1}K\right)^{-1}\mathrm{M}^{-1} = \frac{\Delta t}{\alpha}\sum_{i=0}^{\infty}\left(-\Delta t \mathrm{M}^{-1}K\right)^i \mathrm{M}^{-1}.$$

Here $\mathrm{I}_{N_\mathbf{u}}$ (I_{N_p}) denotes the identity matrix of dimension $N_\mathbf{u}$ (N_p). α is the coefficient of the term evaluated at t^{n+1} in the time discretization scheme adopted. For the Implicit Euler scheme, $\alpha = 1$, for a BDF scheme of order two, $\alpha = 3/2$, for a BDF scheme of order three, $\alpha = 11/6$. In order to improve the accuracy of the inexact factorization, one could choose H by taking more terms in the Neumann expansion. This strategy has been analyzed in [25] and it can lead to some relevant instabilities even for the Stokes problem. In the sequel, we will set:

$$\mathrm{H} = \frac{\Delta t}{\alpha}\mathrm{M}^{-1}, \tag{2.7}$$

so that

$$\mathrm{C}^{-1} = \mathrm{H} + \mathcal{O}(\Delta t).$$

As previously pointed out, note that for both approaches the pressure is only predicted (in the L-step) while the velocity is first predicted in the L-step (with the so-called *intermediate velocity*, $\widetilde{\mathbf{u}}^{n+1}$) and next corrected in the U-step, computing the *end-step velocity*. One could expect some improvements in the accuracy of computation by resorting to a correction also for the pressure field, leading to an *end-step pressure*. In the next Section we investigate such schemes.

REMARK A popular modification of the schemes presented above is the so called *incremental approach*. This approach can be applied to differential projection methods and to algebraic splittings as well. It basically consists of a reformulation of the time-discrete Navier-Stokes problem in such a way that the pressure field p^{n+1} is computed as the sum of an *extrapolation* $\sigma_{n+1}p$ (which is a linear combination of the pressure at the previous time steps) and an *increment* $\delta_{n+1}p$. This reformulation, for the algebraic (fully discretized) problem reads:

$$\mathcal{A}\begin{bmatrix}\mathbf{U}^{n+1}\\ \mathbf{P}^{n+1}\end{bmatrix} = \begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix} \Rightarrow \mathcal{A}\begin{bmatrix}\mathbf{U}^{n+1}\\ \delta_{n+1}\mathbf{P}\end{bmatrix} = \begin{bmatrix}\mathbf{b}_1 - \sigma_{n+1}\mathbf{P}\\ \mathbf{b}_2\end{bmatrix}, \tag{2.8}$$

having set $\mathbf{P}^{n+1} = \sigma_{n+1}\mathbf{P} + \delta_{n+1}\mathbf{P}$. In this way, possible errors introduced by the splitting affect the pressure increment rather than the pressure itself. This modification improves the accuracy of the solution. For instance, in the Van Kan scheme (see [14], [23]), we have:

$$\sigma_{n+1}p = p^n, \quad \delta_{n+1}p = p^{n+1} - \sigma_{n+1} = p^{n+1} - p^n, \tag{2.9}$$

which coupled with a Crank-Nicolson time discretization yields a second order scheme (see also [8]). Recent results (see [13]) show that the incremental approach can improve the accuracy of the solution also for higher order time discretizations, e.g. coupling a BDF of order three with an incremental approach by setting:

$$\sigma_{n+1}p = 2p^n - p^{n-1}, \quad \delta_{n+1}p = p^{n+1} - \sigma_{n+1} = p^{n+1} - 2p^n + p^{n-1}. \tag{2.10}$$

In our analysis we do not consider the incremental approach, in order to put in evidence the role of the pressure corrections proposed in the next sections. However, in the numerical results section, we will present and comment the impact of the incremental modification on pressure corrected schemes.

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REMARK In literature, there is an open discussion about pros and cons of algebraic or differential approaches in the splitting (see [9]). Here, we investigate essentially an algebraic approach, even if we do not claim that this necessarily leads to more accurate results. In fact, as previously pointed out, the algebraic approach has for sure the advantage of including all the possible boundary conditions (not only Dirichlet conditions) without taking care of the set up of special (approximate) pressure conditions which is conversely needed in differential splitting schemes. This makes the algebraic approach appealing in many real problems.

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3. Pressure Correction Algebraic Schemes. Let us consider the following *modified* inexact LU factorization of the matrix $\mathcal{A}$, defined in (2.3),

$$\widehat{\mathcal{A}} = \begin{bmatrix} \mathrm{C} & 0 \\ \mathrm{D} & -\mathrm{D}\mathrm{H}_1\mathrm{D}^T \end{bmatrix} \begin{bmatrix} \mathrm{I} & \mathrm{H}_2\mathrm{D}^T\mathrm{R} \\ 0 & \mathrm{Q} \end{bmatrix} = \begin{bmatrix} \mathrm{C} & \mathrm{C}\mathrm{H}_2\mathrm{D}^T\mathrm{R} \\ \mathrm{D} & \mathrm{D}(\mathrm{H}_2\mathrm{D}^T\mathrm{R} - \mathrm{H}_1\mathrm{D}^T\mathrm{Q}) \end{bmatrix}, \tag{3.1}$$

where R and Q are square $N_p \times N_p$ matrices that we choose in order to minimize the difference $\mathcal{A} - \widehat{\mathcal{A}}$ in some sense. This (generic) factorization leads to new algebraic fractional step methods, where the L-step is still given by (2.5), while the U-step becomes:

$$\text{U-step} \quad \begin{cases} \mathrm{Q}\mathbf{p}^{n+1} = \widetilde{\mathbf{p}}^{n+1}, \\ \mathbf{u}^{n+1} + \mathrm{H}_2\mathrm{D}^T\mathrm{R}\mathbf{p}^{n+1} = \widetilde{\mathbf{u}}^{n+1}. \end{cases} \tag{3.2}$$

Since we have now a pressure correction $(3.2)_1$, we give to schemes in the form (2.5), (3.2) the name of *Pressure Correction methods.*

We still distinguish two approaches, the Chorin-Temam, and the Yosida one.

In the first case, we obtain:

$$\widehat{\mathcal{A}} = \begin{bmatrix} \mathrm{C} & \mathrm{C}\mathrm{H}\mathrm{D}^T\mathrm{R} \\ \mathrm{D} & \mathrm{D}(\mathrm{H}\mathrm{D}^T\mathrm{R} - \mathrm{H}\mathrm{D}^T\mathrm{Q}) \end{bmatrix}, \tag{3.3}$$

If we select $\mathrm{R} = \mathrm{Q}$, the (discrete) mass conservation equation is satisfied without any approximation. This choice will be investigated in the next Section.

In the second case, we have:

$$\widehat{\mathcal{A}} = \begin{bmatrix} \mathrm{C} & \mathrm{D}^T\mathrm{R} \\ \mathrm{D} & \Sigma\mathrm{R} - \mathrm{SQ} \end{bmatrix}, \tag{3.4}$$

where:

$$\Sigma = \mathrm{DC}^{-1}\mathrm{D}^T, \quad \mathrm{S} = \mathrm{DHD}^T. \tag{3.5}$$

Matrix $-\Sigma$ is the so-called pressure *Schur complement* of $\mathcal{A}$. Observe that thanks to (2.7), S is $\mathcal{O}(\Delta t)$. In this case, if $\mathrm{R} = \mathrm{I}_{N_p}$, the (discrete) momentum equation is fulfilled exactly. We investigate this choice in Section 5.

4. The Algebraic Chorin-Temam Pressure Correction Scheme. Let us investigate the choice $\mathrm{H}_1 = \mathrm{H}_2$, with $\mathrm{R} = \mathrm{Q}$. Since we are introducing a pressure correction in the ACT method, we will call this choice *Algebraic Chorin-Temam - Pressure Correction* (CTPC) scheme.

The splitting error matrix is:

$$\widehat{\mathcal{E}} = \mathcal{A} - \widehat{\mathcal{A}} = \begin{bmatrix} 0 & \mathrm{D}^T - \widehat{\mathrm{D}}^T \\ 0 & 0 \end{bmatrix}, \qquad \widehat{\mathrm{D}}^T = \mathrm{CHD}^T\mathrm{Q} \tag{4.1}$$

thus the splitting error vanishes if

$$\mathrm{CHD}^T\mathrm{Q} = \mathrm{D}^T. \tag{4.2}$$

Matrix equation (4.2) is an overdetermined problem. In order to obtain a solution, multiply both the sides of (4.2) by the matrix DH, yielding:

$$\mathrm{BQ} = \mathrm{S}$$

where

$$\mathrm{B} = \mathrm{DHCHD}^T. \tag{4.3}$$

This implies that the matrix equation (4.2) is solved up to a non-zero matrix Z such that $\mathrm{DHZ} = 0$. Observe that if the inf-sup condition is fulfilled, matrix B is non singular, then we can compute:

$$\mathrm{Q} = \mathrm{B}^{-1}\mathrm{S}. \tag{4.4}$$

REMARK

In the case of the Stokes problem, C is s.p.d. and the matrix Q corresponds to solving (4.2) in the least square sense, where the solution yields the minimal error in the norm $\|\cdot\|_{\mathrm{C}}$.

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At each time step, the CTPC scheme reads (we neglect the time index $n+1$ for the sake of simplicity):

1. *Intermediate velocity computation*: $\mathrm{C}\widetilde{\mathbf{U}} = \mathbf{b}_1$;

2. *Intermediate pressure computation*: $\mathrm{S}\widetilde{\mathbf{P}} = \mathrm{D}\widetilde{\mathbf{U}} - \mathbf{b}_2$;
3. *End-of-step pressure computation*: $\mathrm{S}\mathbf{P} = \mathrm{B}\widetilde{\mathbf{P}}$;
4. *End-of-step velocity computation*: $\mathbf{U} = \widetilde{\mathbf{U}} - \mathrm{H}\mathrm{D}^T\widetilde{\mathbf{P}}$.

Observe that the two systems for the pressure computation share the same matrix S. This is useful if S can be factorized once at all, allowing an effective direct strategy for the solution of the related systems (see Sect. 6).

REMARK

In order to solve equation (4.2), another possibility resorts to manipulate it in the form: $\mathrm{H}\mathrm{D}^T\mathrm{Q} = \mathrm{C}^{-1}\mathrm{D}^T$ and then solve it in the least square sense with respect to the matrix norm $\|\cdot\|_{\mathrm{H}^{1/2}}$. This strategy yields:

$$\mathrm{Q} = \mathrm{S}^{-1}\Sigma.$$

This means that the pressure correction step reads:

$$\mathrm{Q}\mathbf{P} = \widetilde{\mathbf{P}} \Rightarrow \Sigma\mathbf{P} = \mathrm{S}\widetilde{\mathbf{P}}.$$

This step involves the pressure Schur complement Σ, and it is, in fact, an (ineffective) reformulation of the pressure matrix method (see [22]), corresponding to exploiting the exact factorization (2.3). Therefore it is not feasible.

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4.1. Stability and splitting error analysis of CTPC.

4.1.1. Preliminary results. We start with some preliminary notation and lemmas.

Starting from the identity

$$\mathrm{C}\mathrm{H} = \mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1, \qquad \mathrm{E}_1 = \mathrm{K}\mathrm{H} \tag{4.5}$$

we have

$$\mathrm{B} = \mathrm{D}\mathrm{H}\left(\mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1\right)\mathrm{D}^T = \mathrm{S} + \mathrm{W} = \mathrm{S}\left(\mathrm{I}_{N_p} + \mathrm{E}_2\right), \tag{4.6}$$

where

$$\mathrm{W} = \mathrm{D}\mathrm{H}\mathrm{K}\mathrm{H}\mathrm{D}^T, \qquad \mathrm{E}_2 = \mathrm{S}^{-1}\mathrm{W}. \tag{4.7}$$

Observe that W is $\mathcal{O}(\Delta t^2)$ thanks to (2.7). Consequently, E_2 is $\mathcal{O}(\Delta t)$. From (4.6) we have:

$$\mathrm{B}^{-1} = \left(\mathrm{I}_{N_p} + \mathrm{E}_2\right)^{-1}\mathrm{S}^{-1} \Rightarrow \mathrm{B}^{-1}\mathrm{S} = \left(\mathrm{I}_{N_p} + \mathrm{E}_2\right)^{-1}. \tag{4.8}$$

Therefore, matrix $\widehat{\mathrm{D}}^T$ introduced in (4.1) admits the following factorization:

$$\hat{\mathrm{D}}^T = \mathrm{C}\mathrm{H}\mathrm{D}^T\mathrm{B}^{-1}\mathrm{S} = \left(\mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1\right)\mathrm{D}^T\left(\mathrm{I}_{N_p} + \mathrm{E}_2\right)^{-1}. \tag{4.9}$$

Observe that if H is proportional to $\mathrm{I}_{N_\mathbf{u}}$, as it happens in a Finite Difference discretization, it is possible to verify that the matrix DH is the Moore-Penrose pseudo-inverse of $\mathrm{D}^T\mathrm{S}^{-1}$. In general (for a Finite Element or a Spectral discretization) this is not true. In fact, it is possible to verify that $\mathrm{H}^{1/2}\mathrm{D}^T\mathrm{S}^{-1}$ is the Moore-Penrose pseudo-inverse of $\mathrm{D}\mathrm{H}^{1/2}$. However, for the purpose of the present work, it is useful the following Lemma.

LEMMA 4.1. *Matrix* $\mathrm{I}_{N_\mathbf{u}} - \mathrm{D}^T\mathrm{S}^{-1}\mathrm{DH}$ *is similar to the matrix*

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I}_{N_\mathbf{u}-N_p} \end{bmatrix}.$$

Proof.
The singular value decomposition of the matrix $\mathrm{H}^{1/2}\mathrm{D}^T$ reads:

$$\mathrm{H}^{1/2}\mathrm{D}^T = \mathrm{U}^T\Pi\mathrm{V}, \tag{4.10}$$

where U is an orthogonal $N_\mathbf{u} \times N_\mathbf{u}$ matrix, V is an orthogonal $N_p \times N_p$ matrix and Π is the $N_\mathbf{u} \times N_p$ matrix such that

$$\Pi_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ \sigma_i & \text{if} \quad i = j \end{cases}$$

where $\{\sigma_i\}$ are the singular values. Observe that, thanks to the inf-sup condition, $\sigma_i \neq 0$ for any $i = 1, \ldots, N_p$. Now, we have:

$$\mathrm{I}_{N_\mathbf{u}} - \mathrm{D}^T\mathrm{S}^{-1}\mathrm{DH} = \mathrm{H}^{-1/2}\left(\mathrm{I}_{N_\mathbf{u}} - \mathrm{H}^{1/2}\mathrm{D}^T\left(\mathrm{DH}^{1/2}\mathrm{H}^{1/2}\mathrm{D}^T\right)^{-1}\mathrm{DH}^{1/2}\right)\mathrm{H}^{1/2}$$

and, thanks to (4.10),

$$\mathrm{I}_{N_\mathbf{u}} - \mathrm{H}^{1/2}\mathrm{D}^T\left(\mathrm{DH}^{1/2}\mathrm{H}^{1/2}\mathrm{D}^T\right)^{-1}\mathrm{DH}^{1/2} = \mathrm{I}_{N_\mathbf{u}} - \mathrm{U}^T\Pi\mathrm{V}\left(\mathrm{V}^T\Pi^T\mathrm{U}^T\mathrm{U}\Pi\mathrm{V}\right)^{-1}\mathrm{V}^T\Pi^T\mathrm{U}.$$

Observe that $\Pi^T\Pi$ is the $N_p \times N_p$ diagonal matrix with the square of the singular values on the diagonal. In the sequel we will set $\Pi_0^2 = \Pi^T\Pi$. The thesis is a consequence of the fact that U and V are orthogonal and that

$$\Pi\Pi_0^{-2}\Pi^T = \begin{bmatrix} \mathrm{I}_{N_p} & 0 \\ 0 & 0 \end{bmatrix}.$$

□

4.1.2. Splitting error analysis. We are now in position of investigating the splitting error matrix $\mathcal{A} - \hat{\mathcal{A}}$ associated to the CTPC scheme, that is:

$$\widehat{\mathcal{E}}_{CTPC} = \begin{bmatrix} 0 & \mathrm{D}^T - \widehat{\mathrm{D}}^T \\ 0 & 0 \end{bmatrix}.$$

Setting:

$$\mathrm{E} = \mathrm{D}^T - \widehat{\mathrm{D}}^T \tag{4.11}$$

from the definition of $\widehat{\mathrm{D}}^T$ we can straightforwardly verify that

$$\mathrm{DHE} = 0. \tag{4.12}$$

This was to be expected, since we have solved the overdetermined problem (4.2), by projecting it into the subspace image of DH.

From (4.9), it follows that

$$\mathrm{E} = \mathrm{D} - (\mathrm{I}_{N_\mathrm{u}} + \mathrm{E}_1)\,\mathrm{D}^T \left(\mathrm{I}_{N_p} + \mathrm{E}_2\right)^{-1}. \tag{4.13}$$

Assuming that Δt is small enough to exploit the Neumann expansion, which makes sense since E_2 is $\mathcal{O}(\Delta t)$:

$$\left(\mathrm{I}_{N_p} + \mathrm{E}_2\right)^{-1} = \sum_{k=0}^{\infty} (-\mathrm{E}_2)^k,$$

from (4.6), (4.7), (4.8), (4.5) we have:

$$\begin{aligned} &\mathrm{D}^T - (\mathrm{I}_{N_\mathrm{u}} + \mathrm{E}_1)\,\mathrm{D}^T \left(\mathrm{I}_{N_p} - \mathrm{E}_2 + \mathrm{E}_2^2 - \ldots\right) = \\ &\mathrm{D}^T - (\mathrm{I}_{N_\mathrm{u}} + \mathrm{E}_1)\,\mathrm{D}^T \left(\mathrm{I}_{N_p} - \mathrm{E}_2 + \mathcal{O}(\Delta t^2)\right), \end{aligned} \tag{4.14}$$

yielding:

$$\mathrm{E} = -\mathrm{E}_1\mathrm{D}^T \boxed{+\mathrm{D}^T\mathrm{E}_2} + \mathcal{O}(\Delta t^2).$$

In the boxed term we put in evidence the specific contribution on the error given by the pressure correction. Since both the matrices KHD^T and $\mathrm{D}^T\mathrm{S}^{-1}\mathrm{W}$ are $\mathcal{O}(\Delta t)$, we conclude that the splitting error is at least first order in time. Unfortunately, this conclusion does not give significant improvements in terms of order of accuracy with respect to the original ACT scheme. The main difference is that in the latter scheme the splitting error is dominated by the term KHD^T, while in the CTPC scheme the matrix error is dominated by:

$$-\left(\mathrm{I}_{N_\mathrm{u}} - \mathrm{D}^T\mathrm{S}^{-1}\mathrm{DH}\right)\mathrm{E}_1\mathrm{D}^T. \tag{4.15}$$

Thanks to Lemma 4.1, it is to be expected that the matrix into brackets, having N_p null eigenvalues, will reduce the error associated to the scheme, in comparison with the ACT scheme, even if it is not possible to prove that the order of accuracy of the scheme is improved. Numerical results confirm that the scheme is in general only first order accurate in time (see Sect. 6).

However, for some special space discretization, it is interesting to point out the following circumstance.

PROPOSITION 4.2. *If* $\mathrm{KHD}^T = \nu\mathrm{D}^T\mathrm{M}_p^{-1}\mathrm{DHD}^T$, *the splitting error matrix* E *vanishes.*

Proof.

From (4.13) it follows that

$$\mathrm{E}\left(\mathrm{I}_{N_p} + \mathrm{E}_2\right) = \mathrm{D}^T + \mathrm{D}^T\mathrm{E}_2 - \mathrm{D}^T - \mathrm{E}_1\mathrm{D}^T = \mathrm{D}^T\mathrm{E}_2 - \mathrm{E}_2\mathrm{D}^T \tag{4.16}$$

If $\mathrm{E}_1\mathrm{D}^T = \mathrm{KHD}^T = \nu\mathrm{D}^T\mathrm{M}_p^{-1}\mathrm{DHD}^T$, then we have:

$$\mathrm{E}_2 = \nu\mathrm{S}^{-1}\mathrm{DHD}^T\mathrm{M}_p^{-1}\mathrm{DHD}^T = \nu\mathrm{M}_p^{-1}\mathrm{S}.$$

Recalling that $\mathrm{S} = \mathrm{DHD}^T$, from (4.16), it follows:

$$\mathrm{E}\left(\mathrm{I}_{N_p} + \mathrm{E}_2\right) = \nu\mathrm{D}^T\mathrm{M}_p^{-1}\mathrm{S} - \nu\mathrm{D}^T\mathrm{M}_p^{-1}\mathrm{DHD}^T = 0.$$

Thus, since $\mathrm{I}_{N_p} + \mathrm{E}_2$ is invertible, the thesis follows.

□

Observe that the hypothesis in Proposition 4.2 reinterpreted at the continuous level yields the identity:

$$-\nu\triangle\nabla = -\nu\nabla\cdot(\nabla)\nabla.$$

Moreover, in the context of Finite Difference space discretization the same hypothesis has been advocated in [16] as a *compatibility condition* holding for special set of velocity boundary conditions in rectangular domains.

4.1.3. Stability analysis. The main result is the following Proposition.

PROPOSITION 4.3. *Consider the Stokes problem discretized with an Implicit Euler time discretization scheme ($\alpha = 1$). Then, the CTPC scheme is* unconditionally stable.

Proof. We consider a problem where the forcing term and the boundary conditions are null as well, since they do not influence the stability properties of the scheme.

The CTPC scheme actually resorts to solve the problem:

$$\begin{bmatrix} \mathrm{C} & \hat{\mathrm{D}}^T \\ \mathrm{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{P}^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t}\mathrm{M}\mathbf{U}^n \\ \mathbf{0} \end{bmatrix}. \tag{4.17}$$

where $\mathrm{C} = \frac{1}{\Delta t}\mathrm{M} + \mathrm{K}$ (with K s.p.d.).

Eliminating the pressure unknowns in (4.17), we obtain:

$$\mathbf{U}^{n+1} = \frac{1}{\Delta t}\mathrm{C}^{-1}\mathrm{M}\mathbf{U}^n - \frac{1}{\Delta t}\mathrm{C}^{-1}\hat{\mathrm{D}}^T\left(\mathrm{D}\mathrm{C}^{-1}\hat{\mathrm{D}}^T\right)^{-1}\mathrm{D}\mathrm{C}^{-1}\mathrm{M}\mathbf{U}^n. \tag{4.18}$$

Observe that:

$$\begin{aligned}\hat{\mathrm{D}}^T\left(\mathrm{D}\mathrm{C}^{-1}\hat{\mathrm{D}}^T\right)^{-1} = \\ (\mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1)\,\mathrm{D}^T\left(\mathrm{I}_{N_p} + \mathrm{E}_2\right)^{-1}\left(\mathrm{D}\mathrm{C}^{-1}\left(\mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1\right)\mathrm{D}^T\left(\mathrm{I}_{N_p} + \mathrm{E}_2\right)^{-1}\right)^{-1} \\ = (\mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1)\,\mathrm{D}^T\left(\mathrm{D}\mathrm{C}^{-1}\left(\mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1\right)\mathrm{D}^T\right)^{-1}.\end{aligned}$$

Moreover, from the definition of E_1 and C, we have:

$$\mathrm{C}^{-1}\left(\mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1\right) = \Delta t\mathrm{M}^{-1}\left(\mathrm{I}_{N_\mathbf{u}} + \Delta t\mathrm{K}\mathrm{M}^{-1}\right)^{-1}\left(\mathrm{I}_{N_\mathbf{u}} + \mathrm{E}_1\right) = \Delta t\mathrm{M}^{-1},$$

so that $\hat{\mathrm{D}}^T\left(\mathrm{D}\mathrm{C}^{-1}\hat{\mathrm{D}}^T\right)^{-1}$ reduces to $\left(\mathrm{I}_{N_\mathbf{u}} + \Delta t\mathrm{K}\mathrm{M}^{-1}\right)\mathrm{D}^T\mathrm{S}^{-1}$. We have therefore that the second matrix on the right hand side of (4.18) becomes:

$$\frac{1}{\Delta t}\mathrm{C}^{-1}\hat{\mathrm{D}}^T\left(\mathrm{D}\mathrm{C}^{-1}\hat{\mathrm{D}}^T\right)^{-1}\mathrm{D}\mathrm{C}^{-1}\mathrm{M} = \mathrm{M}^{-1}\mathrm{D}^T\mathrm{S}^{-1}\mathrm{D}\mathrm{C}^{-1}\mathrm{M},$$

yielding:

$$\mathbf{U}^{n+1} = \frac{1}{\Delta t}\left(\mathrm{M}^{-1} - \mathrm{M}^{-1}\mathrm{D}^T\left(\mathrm{D}\mathrm{M}^{-1}\mathrm{D}^T\right)^{-1}\mathrm{D}\mathrm{M}^{-1}\right)\mathrm{M}\mathrm{C}^{-1}\mathrm{M}\mathbf{U}^n, \tag{4.19}$$

or equivalently:

$$\mathbf{U}^{n+1} = \mathrm{M}^{-1}\left(\mathrm{I}_{N_\mathbf{u}} - \mathrm{D}^T\left(\mathrm{D}\mathrm{M}^{-1}\mathrm{D}^T\right)^{-1}\mathrm{D}\mathrm{M}^{-1}\right)\mathrm{M}\left(\mathrm{I}_{N_\mathbf{u}} + \Delta t\mathrm{M}^{-1}\mathrm{K}\right)\mathbf{U}^n. \tag{4.20}$$

We prove that for any $\Delta t > 0$

$$||\mathrm{M}^{-1}\left(\mathrm{I}_{N_\mathrm{u}} - \mathrm{D}^T\left(\mathrm{DM}^{-1}\mathrm{D}^T\right)^{-1}\mathrm{DM}^{-1}\right)\mathrm{M}\left(\mathrm{I}_{N_\mathrm{u}} + \Delta t\mathrm{M}^{-1}\mathrm{K}\right)^{-1}||_2 < 1 \tag{4.21}$$

For a generic matrix X, we will denote by ρ_X its spectral radius.

First of all, observe that

$$||\left(\mathrm{I}_{N_\mathrm{u}} - \mathrm{D}^T\left(\mathrm{DM}^{-1}\mathrm{D}^T\right)^{-1}\mathrm{DM}^{-1}\right)||_2 = 1. \tag{4.22}$$

This is, in fact, a consequence of Lemma 4.1, since the matrix in (4.22) corresponds exactly to the matrix $\mathrm{I}_{N_\mathrm{u}} - \mathrm{D}^T\mathrm{S}^{-1}\mathrm{DH}$ considered in the Lemma. If follows that the matrix $\mathrm{M}^{-1}\left(\mathrm{I}_{N_\mathrm{u}} - \mathrm{D}^T\left(\mathrm{DM}^{-1}\mathrm{D}^T\right)^{-1}\mathrm{DM}^{-1}\right)\mathrm{M}$ has still unit spectral radius, being similar to the one in (4.22). Now, due to the positiveness of $\Delta t\mathrm{M}^{-1}\mathrm{K}$ with respect to the scalar product weighted by the s.p.d mass matrix M, we have that:

$$\rho_{(\mathrm{I}_{N_\mathrm{u}} + \Delta t\mathrm{M}^{-1}\mathrm{K})^{-1}} < 1.$$

Therefore, we have:

$$\begin{aligned} &||\mathrm{M}^{-1}\left(\mathrm{I}_{N_\mathrm{u}} - \mathrm{D}^T\left(\mathrm{DM}^{-1}\mathrm{D}^T\right)^{-1}\mathrm{DM}^{-1}\right)\mathrm{M}\left(\mathrm{I}_{N_\mathrm{u}} + \Delta t\mathrm{M}^{-1}\mathrm{K}\right)^{-1}||_2 \leq \\ &\quad ||\mathrm{M}^{-1}\left(\mathrm{I}_{N_\mathrm{u}} - \mathrm{D}^T\left(\mathrm{DM}^{-1}\mathrm{D}^T\right)^{-1}\mathrm{DM}^{-1}\right)||_2||\left(\mathrm{I}_{N_\mathrm{u}} + \Delta t\mathrm{M}^{-1}\mathrm{K}\right)^{-1}||_2 < 1 \end{aligned} \tag{4.23}$$

The first inequality is due to (4.22) and the fact that each matrix of the product is s.p.d., so its norm corresponds to its spectral radius. (4.18) and (4.23) yield

$$||\mathbf{U}^{n+1}||_2 < ||\mathbf{U}^n||_2$$

for any $\Delta t > 0$, giving the unconditional stability result. □

REMARK With a similar approach, it can be proved that the unconditional stability holds also for any implicit (unconditionally stable) BDF time discretization.

◇

REMARK It is worth wile pointing out the fact that on the stability of the scheme the error matrix E_2 does not play any role. Actually, since the error matrix E_1 of the present scheme is the same of the original ACT scheme, the unconditional stability of CTPC could be directly inferred from the unconditional stability of ACT. E_2 is therefore set up in such a way that it should reduce the splitting error without affecting the stability of the scheme. Other schemes featuring a high order splitting error do not share the same stability properties. As we will see, also the pressure correction of the Yosida scheme is affected by instabilities.

◇

4.2. CTPC and differential pressure correction schemes. The introduction of a pressure correction for improving the numerical solution is not new in the field of projection methods for the Navier-Stokes equations. In Timmermans *et al.* [24] a variant of the second order Van Kan scheme (see [14]) is proposed for improving the pressure computation in the framework of splitting (differential) schemes. Following Prohl ([19]), who has extensively analyzed this scheme, we present the corresponding formulation starting from the Chorin-Temam method. Suppose to have homogeneous Dirichlet conditions on the whole boundary of the computational domain Ω. First,

compute the intermediate velocity $\widetilde{\mathbf{u}}^{n+1}$ as the solution of the advection diffusion semi-discrete problem:

$$\frac{1}{\Delta t}\left(\widetilde{\mathbf{u}}^{n+1}-\mathbf{u}^{n}\right)-\nu\triangle\widetilde{\mathbf{u}}^{n+1}+\left(\mathbf{u}^{n}\cdot\nabla\right)\widetilde{\mathbf{u}}^{n+1}=\mathbf{f}. \tag{4.24}$$

with $\widetilde{\mathbf{u}}^{n+1}=\mathbf{0}$ on $\partial\Omega$. Then, compute an "intermediate" pressure as the solution of the Poisson problem:

$$\Delta t\triangle\widetilde{p}^{n+1}=\nabla\cdot\widetilde{\mathbf{u}}^{n+1} \tag{4.25}$$

with $\partial_{\mathbf{n}}\widetilde{p}^{n+1}=0$ on $\partial\Omega$, where $\mathbf{n}$ is the outward normal unit vector to $\partial\Omega$. The end-of-step velocity is now given by

$$\mathbf{u}^{n+1}=\widetilde{\mathbf{u}}^{n+1}-\Delta t\nabla\widetilde{p}^{n+1} \tag{4.26}$$

If we take $\widetilde{p}^{n+1}$ as the end-of-step pressure we have actually the classical Chorin-Temam scheme. In the Timmermans proposal, we take:

$$p^{n+1}=\widetilde{p}^{n+1}-\nu\nabla\cdot\widetilde{\mathbf{u}}^{n+1}. \tag{4.27}$$

At the semi-discrete level (time-discrete and space-continuous) it is possible to verify that this scheme is *strongly consistent* with the Stokes problem, that means that solving the Chorin-Temam method with the correction (4.27) amounts exactly to solve the Stokes equations, without any splitting error. This strong consistency however fails to be verified in the case of the Navier-Stokes problem. Actually, in this case, a Lagrangian treatment of the time derivative needs to be pursued ([24]). Prohl has moreover proved that this pressure correction introduces a "smoothing effect" only on the pressure error in the interior domain, and however, the pressure correction step does not improve the order of accuracy of the method.

Here, we want to establish some relations between the Timmermans method and our CTPC scheme. Indeed, exploiting (4.25) we can eliminate $\widetilde{\mathbf{u}}^{n+1}$ in (4.27), yielding:

$$p^{n+1}=\widetilde{p}^{n+1}-\nu\Delta t\Delta\widetilde{p}^{n+1}=\left(\mathcal{I}-\nu\Delta t\Delta\right)\widetilde{p}^{n+1} \tag{4.28}$$

where $\mathcal{I}$ denotes the identity operator.

On the other hand, concerning the CTPC scheme applied to the Stokes problem, we assume that matrix $\Delta t\mathrm{K}\mathrm{M}^{-1}\mathrm{D}^{T}$ can be factorized as $\nu\Delta t\mathrm{D}^{T}\mathrm{M}_{p}^{-1}\mathrm{D}\mathrm{M}^{-1}\mathrm{D}^{T}$ (as we have assumed in Proposition 4.2). The end-of-step pressure of the CTPC schemes becomes:

$$\begin{aligned}\mathbf{P}^{n+1}&=\mathrm{S}^{-1}\mathrm{B}\widetilde{\mathbf{P}}^{n+1}=\left(\mathrm{I}_{N_p}+\mathrm{S}^{-1}\mathrm{W}\right)\widetilde{\mathbf{P}}^{n+1}=\\&\left(\mathrm{I}_{N_p}+\Delta t^{2}\mathrm{S}^{-1}\mathrm{D}\mathrm{M}^{-1}\mathrm{K}\mathrm{M}^{-1}\mathrm{D}^{T}\right)\widetilde{\mathbf{P}}^{n+1}=\left(\mathrm{I}_{N_p}+\nu\Delta t\mathrm{M}_{p}^{-1}\mathrm{D}\mathrm{M}^{-1}\mathrm{D}^{T}\right)\widetilde{\mathbf{P}}^{n+1}.\end{aligned}$$

A formal analogy with (4.28) can be drawn if we read matrix $\nu\mathrm{D}\mathrm{M}^{-1}\mathrm{D}^{T}$ as a discrete counterpart of the pressure operator $-\nu\Delta$.

The algebraic reformulation, actually, has some advantages. As already pointed out (also in [24]), in the differential framework there is the problem of determining boundary conditions for the pressure problem, which in the algebraic approach are not required. Moreover, we point out that the CTPC scheme naturally embodies the presence of a convective term, and it does not necessarily need a Lagrangian treatment of the time derivative or to an explicit treatment of the convective term, as required in the Timmermans work.

We finally point out that the CTPC scheme provides *discrete-divergence* solutions, which is not true for the Timmermans method (whose divergence is null at the continuous level).

5. The Algebraic Yosida Pressure Correction scheme. Let us consider now the pressure correction approach applied following the Yosida strategy, i.e. $\mathrm{H}_2 = \mathrm{C}^{-1}$ and $\mathrm{R} = \mathrm{I}_{N_p}$. The splitting error matrix in this case is:

$$\widehat{\mathcal{E}}_{YPC} = \mathcal{A} - \widehat{\mathcal{A}}_{YPC} = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{SQ} - \Sigma \end{bmatrix}. \tag{5.1}$$

The problem of finding out a matrix such that the splitting error vanishes this time is clearly well posed and the solution is:

$$\Sigma - \mathrm{SQ} = \mathbf{0} \Rightarrow \mathrm{Q} = \mathrm{S}^{-1}\Sigma, \tag{5.2}$$

corresponding to the pressure correction step:

$$\Sigma \mathbf{P}^{n+1} = \mathrm{S}\widetilde{\mathbf{P}}^{n+1}.$$

This is another formulation of the pressure matrix method (see [22]) and therefore this is not interesting in the perspective of the present work.

However, we would like to introduce an approximate computation of Q which is computationally affordable. It corresponds to solve again an overdetermined problem related to (5.2). Multiply the two sides of the first equation in (5.2) by $\mathrm{D}^T\mathrm{M}_p^{-1}$, yielding:

$$\mathrm{D}^T\mathrm{M}_p^{-1}\mathrm{DC}^{-1}\mathrm{D}^T = \mathrm{D}^T\mathrm{M}_p^{-1}\mathrm{DHD}^T\mathrm{Q}.$$

Since $\mathrm{D}^T\mathrm{M}_p^{-1}\mathrm{D}$ is non singular if the inf-sup condition is fulfilled, we can write:

$$\mathrm{C}^{-1}\mathrm{D}^T = \mathrm{HD}^T\mathrm{Q} \Rightarrow \mathrm{DHD}^T = \mathrm{DHCHD}^T\mathrm{Q}$$

corresponding to the choice:

$$\mathrm{Q} = \mathrm{B}^{-1}\mathrm{S}$$

which is exactly the same matrix set up for the CTPC scheme (see (4.4), (4.3)). Dropping again the time index $n+1$, the Yosida-Pressure Correction scheme reads therefore:

1. *Intermediate velocity computation*: $\mathrm{C}\widetilde{\mathbf{U}} = \mathbf{b}_1$;
2. *Intermediate pressure computation*: $\mathrm{S}\widetilde{\mathbf{P}} = \mathrm{D}\widetilde{\mathbf{U}} - \mathbf{b}_2$;
3. *End-of-step pressure computation*: $\mathrm{SP} = \mathrm{B}\widetilde{\mathbf{P}}$;
4. *End-of-step velocity computation*: $\mathrm{CU} = \mathbf{b}_1 - \mathrm{D}^T\mathbf{P}$.

Observe that still the two problems for both the intermediate and end-of-step pressures are solved by solving the same matrix S.

5.1. Stability and splitting error analysis.

5.1.1. Splitting error analysis. Starting from (5.1), the splitting error is given by the block (2,2) of $\mathcal{E}_{YPC}$ where:

$$\mathrm{E}_{YPC} = \mathrm{SB}^{-1}\mathrm{S} - \mathrm{DC}^{-1}\mathrm{D}^T.$$

Exploting (4.6), (4.7), (4.8) and (4.11) the previous matrix can be reduced to:

$$\begin{aligned}
\mathrm{E}_{YPC} &= \mathrm{S}\left(\mathrm{I}_{N_p}+\mathrm{E}_2\right)^{-1} - \mathrm{DH}\left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)^{-1}\mathrm{D}^T = \\
&\mathrm{DHD}^T\left(\mathrm{I}_{N_p}+\mathrm{E}_2\right)^{-1} - \mathrm{DH}\left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)^{-1}\mathrm{D}^T = \\
&\mathrm{DH}\left(\mathrm{D}^T\left(\mathrm{I}_{N_p}+\mathrm{E}_2\right)^{-1} - \left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)^{-1}\mathrm{D}^T\right) = \\
&\mathrm{DH}\left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)^{-1}\left(\left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)\mathrm{D}^T\left(\mathrm{I}_{N_p}+\mathrm{S}^{-1}\mathrm{W}\right)^{-1} - \mathrm{D}^T\right) \\
&= -\mathrm{DH}\left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)^{-1}\mathrm{E}
\end{aligned} \tag{5.3}$$

The splitting error associated to the YPC scheme can now be estimated, by assuming that Δt is small enough to exploit the Neumann expansion:

$$\left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)^{-1} = \sum_{i=0}^{\infty}(-\mathrm{E}_1)^i \tag{5.4}$$

This makes sense, since as we have pointed out $\mathrm{E}_1 = \mathcal{O}(\Delta t)$.

PROPOSITION 5.1. *The splitting error matrix associated to the YPC scheme is* $\mathcal{O}(\Delta t^3)$

Proof.

By exploiting the Neumann expansion (5.4), we have from (5.3) and (4.12):

$$\mathrm{DH}\left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)^{-1}\mathrm{E} = \mathrm{DHE} - \mathrm{DHE}_1\mathrm{E} + \mathrm{DHE}_1^2\mathrm{E} - \ldots = -\mathrm{DHE}_1\mathrm{E} + \text{high order terms.} \tag{5.5}$$

On the other hand, we have that $\mathrm{H} = \mathcal{O}(\Delta t)$, $\mathrm{E}_1 = \mathcal{O}(\Delta t)$ and (from the analysis carried out in Sect. 4.1) $\mathrm{E} = \mathcal{O}(\Delta t)$, so that the thesis is proven.

□

In order to have a better insight to the benefits introduced by the pressure correction, let us suppose that, besides the Neumann expansion (5.4), it is possible to expand also $(\mathrm{I}_{N_p}+\mathrm{S}^{-1}\mathrm{W})^{-1}$:

$$\left(\mathrm{I}_{N_p}+\mathrm{E}_2\right)^{-1} = \sum_{i=0}^{\infty}(-\mathrm{E}_2)^i.$$

This still makes sense since also E_2 is $\mathcal{O}(\Delta t)$. Exploiting the Neumann expansions and recalling (4.8), we have:

$$\begin{aligned}
&\mathrm{E}_{YPC} = \mathrm{S}\left(\mathrm{I}_{N_p}+\mathrm{E}_2\right)^{-1} - \mathrm{DH}\left(\mathrm{I}_{N_\mathrm{u}}+\mathrm{E}_1\right)^{-1}\mathrm{D}^T = \\
&\mathrm{S}\left(\mathrm{I}_{N_p} - \boxed{\mathrm{E}_2 - \mathrm{E}_2^2 + \mathcal{O}(\Delta t^3)}\right) - \mathrm{DH}\left(\mathrm{I}_{N_\mathrm{u}} - \mathrm{E}_1 - E_1^2 + \mathcal{O}(\Delta t^3)\right)\mathrm{D}^T
\end{aligned}$$

The boxed terms are the contribution of the error due to the pressure correction, i.e. they were absent in the original Yosida scheme. This to outline how the pressure correction acts. Indeed, by recalling the definition of E_1 and E_2 in (3.5) and (4.7), the first two terms of the two Neumann expansions cancel themselves (in the original Yosida scheme only the first ones were canceled), yielding:

$$\begin{aligned}
\mathrm{E}_{YPC} &= \mathrm{WS}^{-1}\mathrm{W} - \mathrm{DH}\left(\mathrm{KH}\right)^2\mathrm{D}^T + \mathcal{O}(\Delta t^4) \\
&= -\mathrm{DHKH}\left(\mathrm{I}_{N_\mathrm{u}} - \mathrm{D}^T\mathrm{S}^{-1}\mathrm{DH}\right)\mathrm{KHD}^T + \mathcal{O}(\Delta t^4)
\end{aligned}$$

the first term being $\mathcal{O}(\Delta t^3)$ as already proven. Observe that the matrix into brackets in this term is the same investigated in Lemma 4.1 and its spectral radius is therefore 1.

REMARK From the expression of the splitting error (5.3) and Proposition 4.2 it follows that also for the YPC scheme *the splitting error vanishes* whenever $\mathrm{KHD}^T = \nu \mathrm{D}^T \mathrm{M}_p^{-1} \mathrm{DHD}^T$.

◇

5.1.2. Stability analysis. A different formulation of the matrix E_{YPC} is needed in order to carry out a stability analysis. Let us introduce the following QR factorization (remember that H is s.p.d.):

$$\mathrm{H}^{1/2}\mathrm{D}^T = \mathrm{UR}$$

where U is an orthogonal square ($N_{\mathbf{u}} \times N_{\mathbf{u}}$) matrix and, if the inf-sup condition holds, R is a triangular full-rank $N_{\mathbf{u}} \times N_p$ matrix such that:

$$\mathrm{R} = \begin{bmatrix} \mathrm{R}_0 \\ \mathbf{0} \end{bmatrix},$$

where R_0 is non-singular and square ($N_p \times N_p$). In this way, we have the Cholesky factorization:

$$\mathrm{S} = \mathrm{R}_0^T \mathrm{R}_0.$$

Matrix $\mathrm{SB}^{-1}\mathrm{S}$ therefore reads:

$$\begin{aligned} &\mathrm{SB}^{-1}\mathrm{S} = \mathrm{S}\left(\mathrm{DHCHD}^T\right)^{-1}\mathrm{S} = \\ &\mathrm{R}_0^T\mathrm{R}_0\left(\mathrm{R}^T\mathrm{U}^T\mathrm{H}^{1/2}\left(\mathrm{H}^{-1}+\mathrm{K}\right)\mathrm{H}^{1/2}\mathrm{UR}\right)^{-1}\mathrm{R}_0^T\mathrm{R}_0 = \\ &\mathrm{R}_0^T\left(\mathrm{R}_0^{-T}\mathrm{R}^T\mathrm{U}^T\mathrm{H}^{1/2}\left(\mathrm{H}^{-1}+\mathrm{K}\right)\mathrm{H}^{1/2}\mathrm{UR}\mathrm{R}_0^{-1}\right)^{-1}\mathrm{R}_0 = \\ &\mathrm{R}_0^T\left(\begin{bmatrix}\mathrm{I}_{N_p} & \mathbf{0}\end{bmatrix}\left(\mathrm{I}_{N_{\mathbf{u}}}+\mathrm{U}^T\mathrm{H}^{1/2}\mathrm{KH}^{1/2}\mathrm{U}\right)\begin{bmatrix}\mathrm{I}_{N_p} & \mathbf{0}\end{bmatrix}^T\right)^{-1}\mathrm{R}_0. \end{aligned} \tag{5.6}$$

Matrix Σ, on the other hand, can be resorted as follows:

$$\begin{aligned} &\Sigma = \mathrm{D}\left(\mathrm{H}^{-1}+\mathrm{K}\right)^{-1}\mathrm{D}^T = \mathrm{DH}^{1/2}\left(\mathrm{I}_{N_{\mathbf{u}}}+\mathrm{H}^{1/2}\mathrm{KH}^{1/2}\right)^{-1}\mathrm{H}^{1/2}\mathrm{D}^T = \\ &\mathrm{R}^T\mathrm{U}^T\left(\mathrm{I}_{N_{\mathbf{u}}}+\mathrm{H}^{1/2}\mathrm{KH}^{1/2}\right)^{-1}\mathrm{UR} = \begin{bmatrix}\mathrm{R}_0^T & \mathbf{0}\end{bmatrix}\left(\mathrm{I}_{N_{\mathbf{u}}}+\mathrm{U}^T\mathrm{H}^{1/2}\mathrm{KH}^{1/2}\mathrm{U}\right)^{-1}\begin{bmatrix}\mathrm{R}_0^T & \mathbf{0}\end{bmatrix}^T \end{aligned} \tag{5.7}$$

Now set:

$$\mathrm{N} = \mathrm{I}_{N_{\mathbf{u}}} + \mathrm{U}^T\mathrm{H}^{1/2}\mathrm{KH}^{1/2}\mathrm{U}$$

and since N is s.p.d., we denote:

$$\mathrm{N} = \begin{bmatrix} \mathrm{N}_{11} & \mathrm{N}_{12} \\ \mathrm{N}_{12}^T & \mathrm{N}_{22} \end{bmatrix}$$

where N_{11} is $N_p \times N_p$ and N_{22} is $(N_{\mathbf{u}} - N_p) \times (N_{\mathbf{u}} - N_p)$. Observe that, in the case of the Stokes problem (i.e. K s.p.d.), from the Sylvester criterion both the diagonal blocks are s.p.d. With these positions, we have that:

$$\mathrm{SQ} - \Sigma = \mathrm{R}_0^T\left(\mathrm{N}_{11}^{-1} - \left(\mathrm{N}_{11} - \mathrm{N}_{12}\mathrm{N}_{22}^{-1}\mathrm{N}_{12}^T\right)^{-1}\right)\mathrm{R}_0 \tag{5.8}$$

Observe that also matrix $\mathrm{N}_{11} - \mathrm{N}_{12}\mathrm{N}_{22}^{-1}\mathrm{N}_{12}^T$ is s.p.d., being the first $N_p \times N_p$ block-component of the inverse of N.

Since N_{11} is s.p.d., we can rearrange the previous matrix in the following way:

$$\mathrm{SQ} - \Sigma = \mathrm{R}_0^T \mathrm{N}_{11}^{-1/2} \left(\mathrm{I}_{N_p} - \left(\mathrm{I}_{N_p} - \mathrm{N}_{11}^{-1/2}\mathrm{N}_{12}\mathrm{N}_{22}^{-1}\mathrm{N}_{12}^T\mathrm{N}_{11}^{-1/2} \right)^{-1} \right) \mathrm{N}_{11}^{-1/2}\mathrm{R}_0. \quad (5.9)$$

Starting from this reformulation of the error matrix, unfortunately we are not able to prove an unconditional stability result even for the Stokes problem with an implicit Euler time discretization. Indeed, we have the following results:

PROPOSITION 5.2. *In the Stokes case, discretized with the implicit Euler scheme, error matrix* $\mathrm{SQ} - \Sigma$ *is symmetric and negative semi-definite.*

Proof. Resorting to (5.9), consider the scalar product:

$$s = \mathbf{x}^T \mathrm{R}_0^T \mathrm{N}_{11}^{-1/2} \left(\mathrm{I}_{N_p} - \left(\mathrm{I}_{N_p} - \mathrm{N}_{11}^{-1/2}\mathrm{N}_{12}\mathrm{N}_{22}^{-1}\mathrm{N}_{12}^T\mathrm{N}_{11}^{-1/2} \right)^{-1} \right) \mathrm{N}_{11}^{-1/2}\mathrm{R}_0\mathbf{x}$$

We prove that $s \leq 0$ for each $\mathbf{x} \in \mathbb{R}^{N_p}$.

Set:

$$\mathrm{V} = \mathrm{N}_{11}^{-1/2}\mathrm{N}_{12}\mathrm{N}_{22}^{-1}\mathrm{N}_{12}^T\mathrm{N}_{11}^{-1/2}.$$

and

$$\mathrm{J} = \left(\mathrm{I}_{N_p} - \mathrm{V} \right)^{-1}.$$

As previously pointed out, J is s.d.p. The thesis amounts therefore to prove that $\mathrm{I}_{N_p} - \mathrm{J}$ is negative semi-definite. First of all observe that if we set:

$$\mathbf{v} = \mathrm{J}\mathbf{y}$$

we obtain by definition:

$$\mathbf{v} - \mathrm{V}\mathbf{v} = \mathbf{y} \Rightarrow \mathrm{J}\mathbf{y} - \mathrm{V}\mathrm{J}\mathbf{y} = \mathbf{y} \Rightarrow \left(\mathrm{I}_{N_p} - \mathrm{J} \right) \mathbf{y} = -\mathrm{V}\mathrm{J}\mathbf{y}$$

Therefore, since J is s.d.p. we get, with obvious notation:

$$s = -\mathbf{y}^T \mathrm{V}\mathrm{J}\mathbf{y} = -\mathbf{y}^T \mathrm{J}^{1/2}\mathrm{J}^{-1/2}\mathrm{V}\mathrm{J}^{1/2}\mathrm{J}^{1/2}\mathbf{y} = -\mathbf{z}^T \mathrm{J}^{-1/2}\mathrm{V}\mathrm{J}^{1/2}\mathbf{z}$$

where $\mathbf{y} = \mathrm{N}_{11}^{-1/2}\mathrm{R}_0\mathbf{x}$ and $\mathbf{z} = \mathrm{J}^{1/2}\mathbf{y}$. Since V and J are both symmetric and, by construction, share the same set of orthogonal eigenvectors, it is possible to verify that $\mathbf{z}^T \mathrm{J}^{-1/2}\mathrm{V}\mathrm{J}^{1/2}\mathbf{z} \geq 0$, yielding the thesis. □

The previous result is negative in view of the stability analysis of the scheme. Actually, it implies that the scheme introduces a mass source in the fluid. This clearly reflects negatively on the stability of the scheme.

PROPOSITION 5.3. *In the case of the Stokes problem discretized in time with the implicit Euler method, the Y-PC scheme is conditionally stable.*

Proof. The Y-PC scheme associated to a Backward Euler time discretization, in a homogeneous Dirichlet case with null forcing terms, resorts to solve at each time step the system:

$$\begin{bmatrix} \mathrm{C} & \mathrm{D}^T \\ -\mathrm{D} & \mathrm{SQ} - \Sigma \end{bmatrix} \begin{bmatrix} \mathbf{U}^{n+1} \\ \mathbf{P}^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{\Delta t}\mathrm{M}\mathbf{U}^n \\ \mathbf{0} \end{bmatrix}$$

The conditional stability is proven by multiplying the two sides by $\left[\mathbf{U}^{n+1}\mathbf{P}^{n+1}\right]^T$ and applying the Young inequality. We obtain, indeed:

$$\frac{1}{2\Delta t}\mathbf{U}^{n+1}\mathrm{M}\mathbf{U}^{n+1} + \mathbf{U}^{n+1}\mathrm{K}\mathbf{U}^{n+1} - |\mathbf{P}^{n+1}\left(\mathrm{SQ}-\Sigma\right)\mathbf{P}^{n+1}| \leq \frac{1}{2\Delta t}\mathbf{U}^{n}\mathrm{M}\mathbf{U}^{n}.$$

Since $\mathrm{SQ}-\Sigma$ vanishes when Δt tends to zero, it is possible to select a time step Δt_{max} such that for each $\Delta t \leq \Delta t_{max}$ we have:

$$\mathbf{U}^{n+1}\mathrm{K}\mathbf{U}^{n+1} - |\mathbf{P}^{n+1}\left(\mathrm{SQ}-\Sigma\right)\mathbf{P}^{n+1}| \geq 0,$$

yielding the (conditional) stability of the scheme. □

The actual impact of this conditional stability on numerical results will be discussed in Sect. 6.

REMARK In [25] the use of inexact algebraic factorizations (Yosida and Algebraic Chorin-Temam without pressure corrections) as preconditioners for the Navier-Stokes problem has been extensively investigated. Following the same idea, the inexact factorizations with pressure corrections can be used in the same fashion. The outcome is a fast preconditioner which seems to be an effective generalization of the well known Cahouet-Chabard preconditioner for the Stokes problem. Preliminary numerical results about this preconditioner can be found in [7].

◇

6. Numerical results. In this section we present some numerical results* that confirm the analysis carried out above and give a deeper insight into the real accuracy and stability properties of the methods presented. In particular, we consider BDF schemes of order 1, 2 and 3, both in the *non incremental* and *incremental* formulations. In particular, in the incremental case, we refer to the approach proposed in [13], with a pressure increment like in (2.9) for BDF of order 2 and like in (2.10) for BDF or order 3 (see Remark in Sect. 2).

We refer to 2D Navier-Stokes problem on the unit domain $(0,1)^2$ in the time interval $(0,1)$ where (time-dependent) Dirichlet boundary conditions for the velocity, initial conditions and the forcing term are prescribed in such a way that the analytical solution is (see [9]):

$$\begin{aligned} u(x,y,t) &= \sin(x)\sin(y+t), \\ v(x,y,t) &= \cos(x)\cos(y+t), \\ p(x,y,t) &= \cos(x)\sin(y+t). \end{aligned} \tag{6.1}$$

Similar results have been obtained also for other test cases, such as the Kim and Moin (see [15]) and the Timmermans (see [24]) ones.

For what concerns the space-discretization, we have adopted an *inf-sup* compatible couple of Finite Elements spaces. In particular, for the numerical results of the present section, we resorted to a piecewise linear functions space $\mathbf{P^1}$ for the pressure fields and we have used $\tilde{\mathbf{P}}^2 = \mathbf{P}^2 \oplus b$ finite elements for each component of the velocity, where b is a cubic bubble function. Following [3], the role of the bubble function is to give non-singular (velocity) mass lumped matrices, which is useful in solving systems for matrix S.

*Numerical results of the present section have been obtained with a Matlab code developed by the authors.

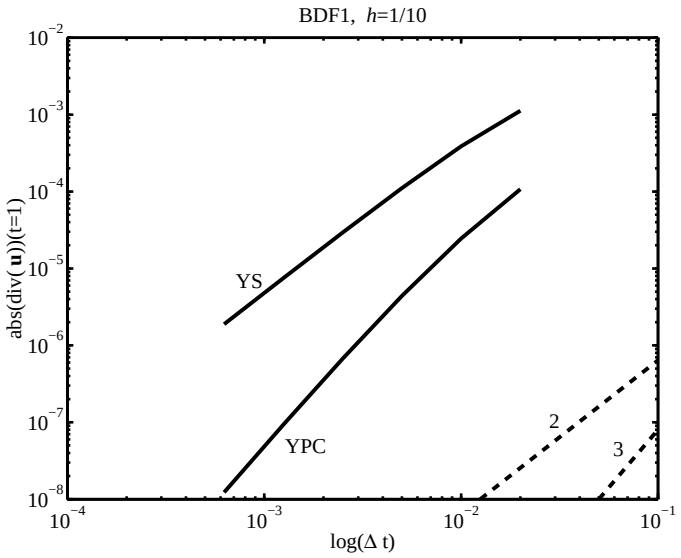

FIG. 6.1. *Divergence of the computed velocity field in $T_f = 1$ for different values of Δt. Comparison between the YS and YPC schemes, with a BDF1 (implicit Euler) time discretization scheme.*

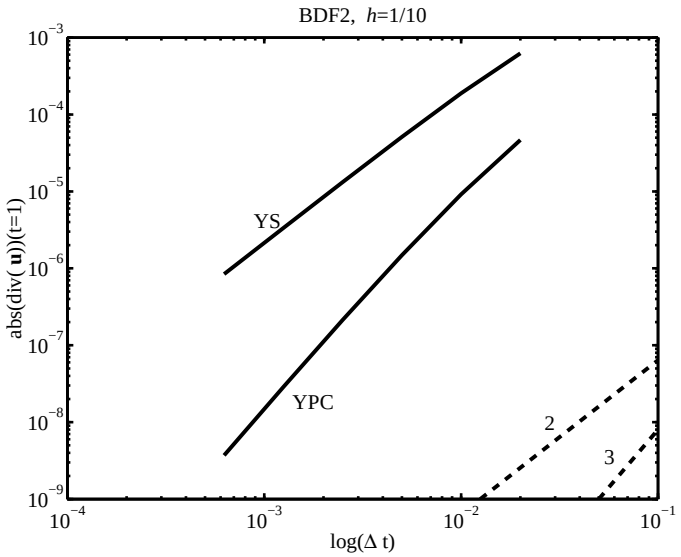

FIG. 6.2. *Divergence of the computed velocity field in $T_f = 1$ for different values of Δt. Comparison between the YS and YPC schemes, with a BDF2 time discretization scheme.*

Mass conservation and Pressure errors (Yosida and Yosida Pressure Correction schemes). We start focusing our attention on the Yosida (denoted by YS) and YPC scheme. In particular, we consider the divergence of the velocity field computed by the two schemes at the final time ($T_f = 1$). From the analysis of Sect. 5.1, the first effect of the pressure correction is to modify the dependence on Δt of the residual of the mass equation, which changes from $\mathcal{O}(\Delta t^2)$ to $\mathcal{O}(\Delta t^3)$, independently of the

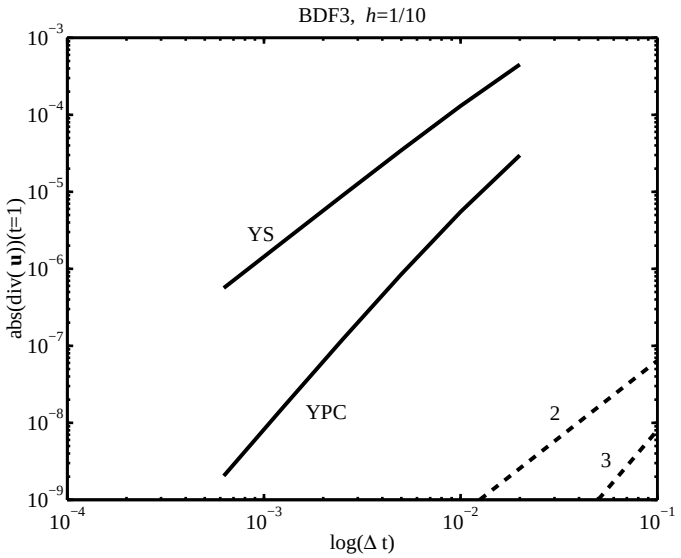

FIG. 6.3. *Divergence of the computed velocity field in $T_f = 1$ for different values of Δt. Comparison between the YS and YPC schemes, with a BDF3 time discretization scheme.*

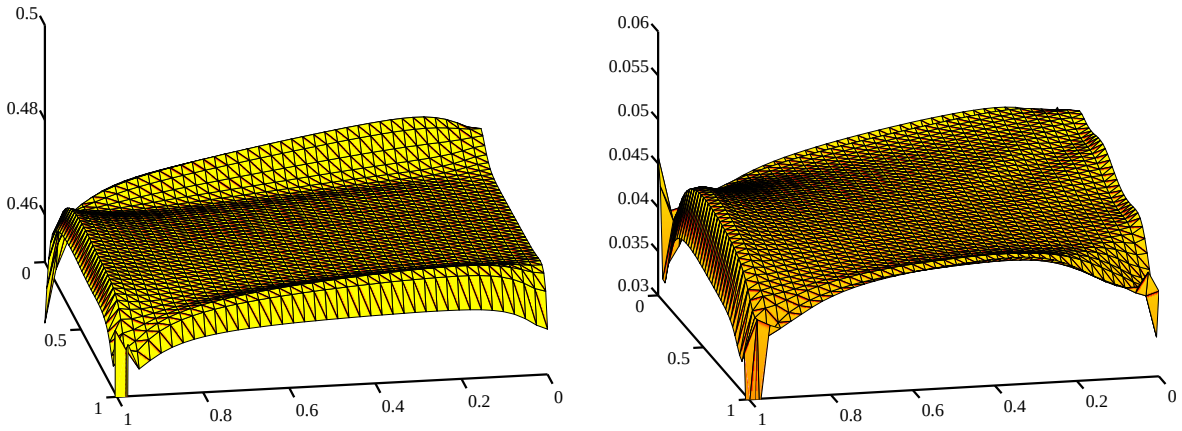

FIG. 6.4. *Space distribution of the pressure error for the YS (left) and the YPC (right) schemes.*

time discretization adopted. This is clearly confirmed by Fig. 6.1, 6.2 and 6.3, where for the three different order of BDF the divergence of the velocity field is computed for several time step sizes. The effect of the pressure correction is evident. The circumstance that the divergence of the computed velocity (independently of the time discretization scheme) is $\mathcal{O}(\Delta t^3)$ is shared also by other schemes proposed in the differential splitting framework, which are an evolution of the Timmermans scheme (see [11] and also [10]).

Another way for investigating the effect of the pressure correction is to check the space distribution of the pressure error (see Fig. 6.4). From this picture we infer that in the algebraic approach we actually do not have significant boundary layers for the

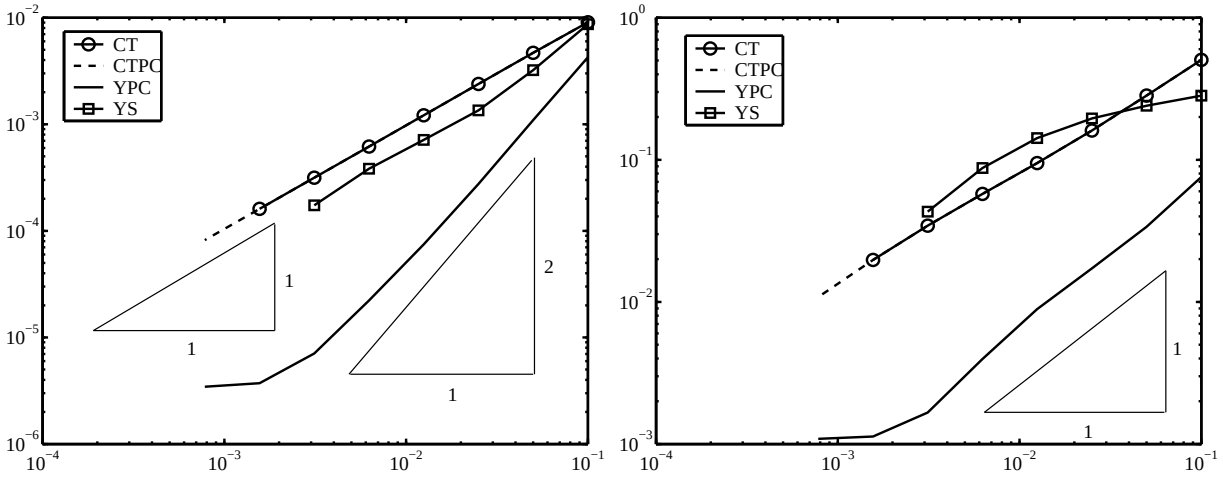

FIG. 6.5. *Errors in the $L^\infty(\mathbf{L}^2)$(left) and $L^2(\mathbf{H}^1)$(right) norms for the velocity, in the case of* BDF2 non incremental *time discretization. Note that in all the figure presented in this Section the error reduction with the time step stops whenever the error is completely due to the space discretization.*

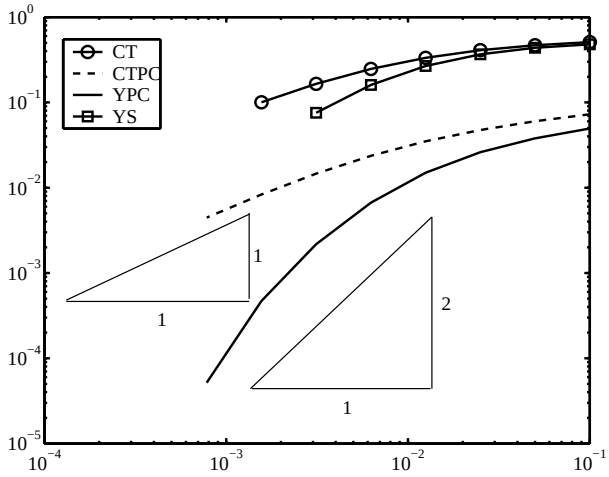

FIG. 6.6. *Errors in the $L^2(L^2)$norm for the pressure, in the case of* BDF2 non incremental *time discretization.*

pressure error (which is the case of "standard" differential splitting techniques) and this is particularly true for the pressure corrected scheme, whose error associated is significantly smaller than in the uncorrected case.

Accuracy tests (BDF2 and BDF3). We compare the numerical results obtained for $h = 1/40$ for different sizes of the time step. We consider in particular BDF time discretization schemes of order 2 and 3. The errors have been computed with respect to the norms $L^\infty(0,T,L^2(\Omega)\times L^2(\Omega)$ and $L^2(0,T,H^1(\Omega)\times H^1(\Omega))$ for the velocity and $L^2(0,T,L^2(\Omega))$ for the pressure (in the sequel, these norms will be denoted $L^\infty(\mathbf{L}^2)$, $L^2(\mathbf{H}^1)$ and $L^2(L^2)$ respectively).

In Fig. 6.5 and 6.6 we illustrate the results for the BDF2 *non incremental* schemes.

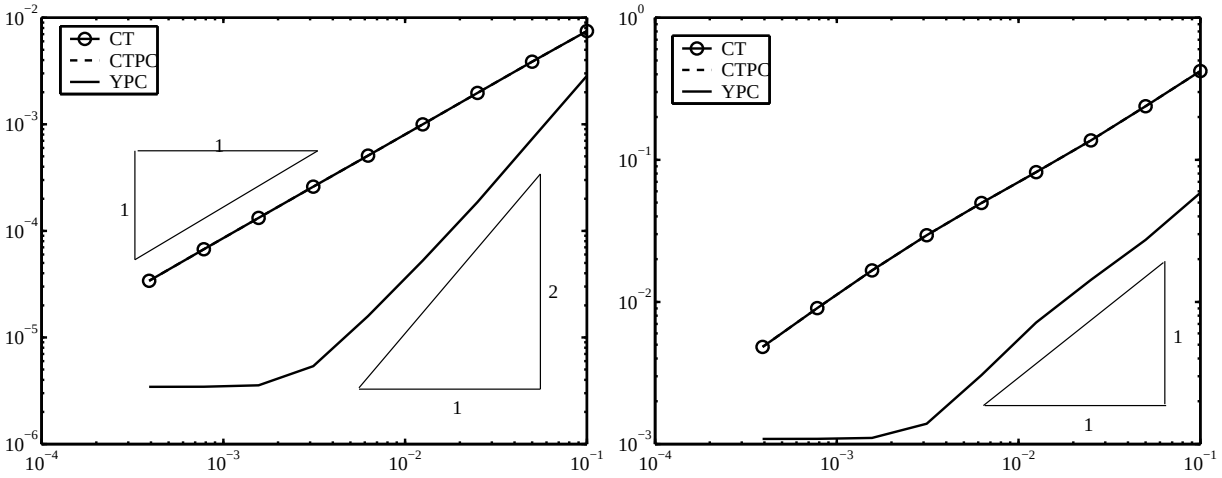

FIG. 6.7. *Errors in the* $L^\infty(\mathbf{L}^2)$*(left) and* $L^2(\mathbf{H}^1)$*(right) norms for the velocity, in the case of* BDF3 non incremental *time discretization.*

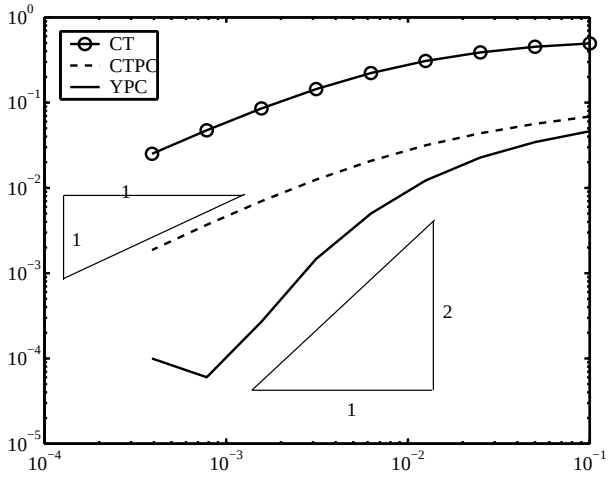

FIG. 6.8. *Errors in the* $L^2(L^2)$*(left) norm for the pressure, in the case of* BDF3 non incremental *time discretization.*

Results suggest that the pressure correction has a relevant effect for the Yosida scheme, both for the velocity and the pressure. For the (Algebraic) Chorin-Temam scheme (denoted by CT in the figures), the pressure correction gives a significant improvement only on the pressure. In particular, in the $L^\infty(\mathbf{L}^2)$norm, YPC exhibits a second order of accuracy which is not shared by the other schemes (in particular CTPC). In the $L^2(\mathbf{H}^1)$norm all the schemes are first order accurate, even if YPC features an error significantly lower than the others. For the pressure, results suggest that YPC is asymptotically second order accurate, while CTPC seems to be first order, even if there is an evident error reduction with respect to the uncorrected CT scheme.

In Fig. 6.7 and 6.8 numerical results for the BDF3 *non incremental* schemes are reported. For what concerns the convergence order, we observe that it is substantially unchanged with respect to the case of a BDF2 time discretization. In particular,

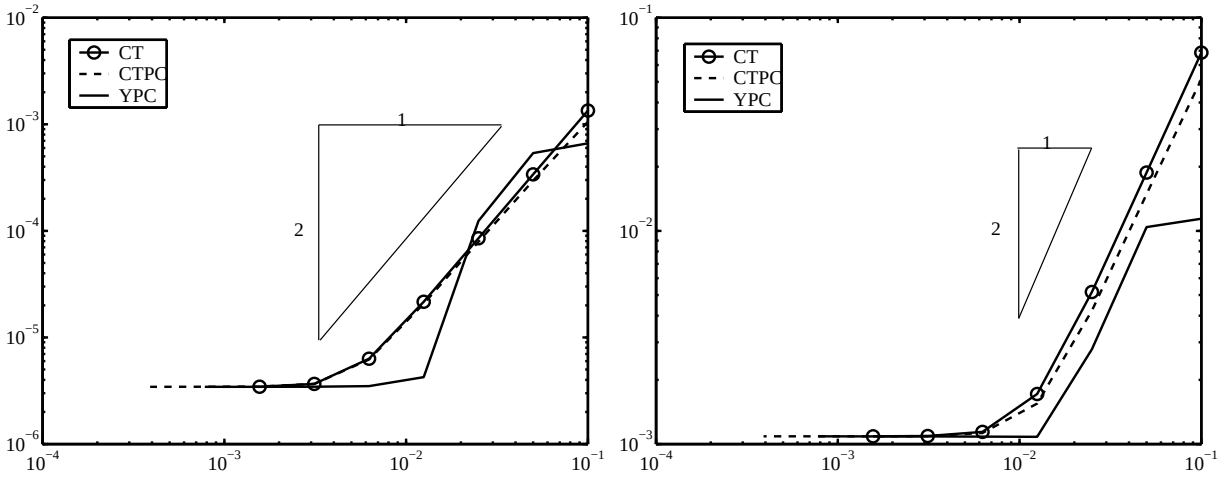

FIG. 6.9. *Errors in the* $L^\infty(\mathbf{L}^2)$*(left) and* $L^2(\mathbf{H}^1)$*(right) norms for the velocity, in the case of* BDF2 incremental *time discretization.*

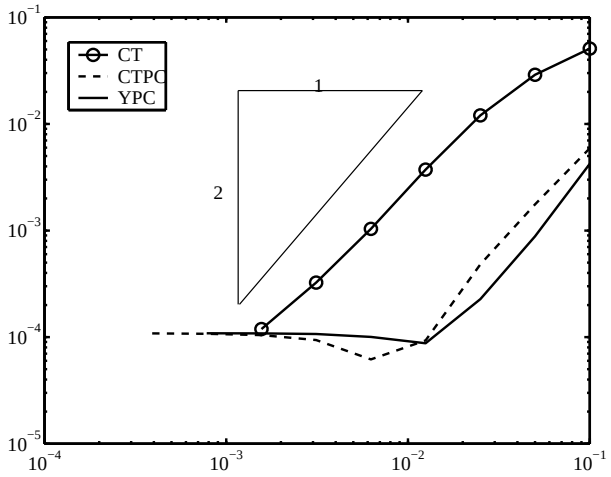

FIG. 6.10. *Errors in the* $L^2(L^2)$*norm for the pressure, in the case of* BDF2 incremental *time discretization.*

CTC is first order accurate with respect to all the monitored norms (even if the pressure is by far more accurate with respect to the CT scheme), while YPC is second order accurate in the $L^\infty(\mathbf{L}^2)$(velocity) and $L^2(L^2)$(pressure) norms. It is first order accurate in $L^2(\mathbf{H}^1)$norm of the velocity error. This means that the pressure correction by itself yields a splitting error $\mathcal{O}(\Delta t^2)$ which therefore does not affect the accuracy of a BDF2 time discretization, while reduces the accuracy of the BDF3 one. It is however worthy pointing out that the errors in the BDF3 case are slightly lower than the corresponding ones of the BDF2 case.

Now, let us consider the errors in the *incremental* case. In Fig. 6.9 and 6.10 we present the results of an BDF2 incremental scheme with a first order pressure extrapolation. Numerical results suggest that the method is second order accurate with respect all the norms considered here. Actually, the improvements given by

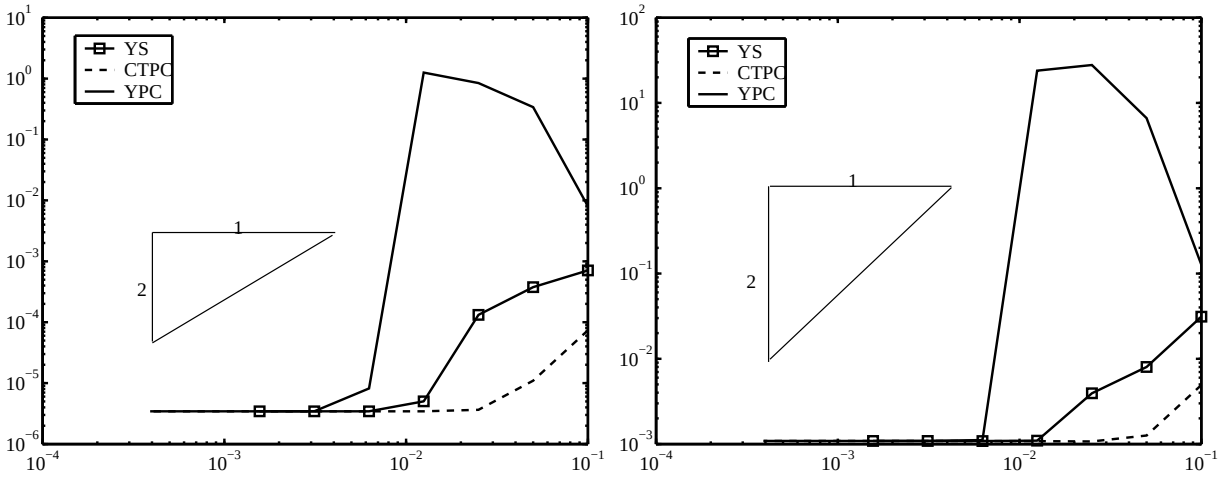

FIG. 6.11. *Errors in the* $L^\infty(\mathbf{L}^2)$*(left) and* $L^2(\mathbf{H}^1)$*(right) norms for the velocity, in the case of* BDF3 incremental *time discretization.*

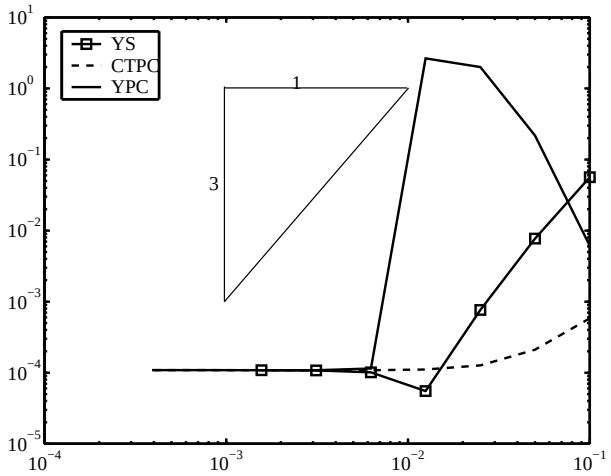

FIG. 6.12. *Errors in the* $L^2(L^2)$*norm for the pressure, in the case of* BDF3 incremental *time discretization.*

the pressure correction are minimal on the velocity for the CTPC scheme and more significant, if Δt is sufficiently small, for the YPC scheme. Pressure correction yields a relevant improvement of the solution on the pressure solution for both CTPC and YPC. For large values of Δt YPC exhibits some strange behavior which is probably due to the poor stability properties of the method.

Specific considerations have to be deserved to the case of BDF3 incremental version. From the numerical results presented in Fig. 6.11 and 6.12 we observe that:

1. the uncorrected YS method in fact is second order accurate for the velocity and third order accurate for the pressure;
2. the CTPC scheme features very good results also for large values of the time step, in such a way that it is difficult to draw an order of accuracy;
3. the YPC method has a strange behavior when Δt is large, which is probably

induced by numerical instability. Surprisingly enough, under a threshold on the time step (which in our simulations is about 10^{-2}) the scheme is extremely accurate, reducing immediately the error to the contribution of the space discretization solely.

REMARK The Pressure Correction yields improvements in the accuracy of the solution in particular on the pressure field. However, the computational cost is clearly increasing, since we need to solve two linear systems for S, rather than one. This increment can be strongly reduced (at least in 2D problems) by resorting to a direct method of solution. Since H is a s.p.d. matrix, we exploit the QR factorization $\mathrm{H}^{1/2}\mathrm{D}^T = \mathrm{QR}$, where Q is an orthogonal square ($N_{\mathbf{u}} \times N_{\mathbf{u}}$) matrix and, if the inf-sup condition holds, R is a triangular full-rank $N_{\mathbf{u}} \times N_p$ matrix such that: $\mathrm{R} = \left[\mathrm{R}_0^T \ \mathbf{0}\right]^T$, where R_0 is a non-singular $N_p \times N_p$ triangular matrix. In this way $\mathrm{S} = \mathrm{R}_0^T\mathrm{R}_0$, yielding the Cholesky factorization of S. In this way, the solution for the system in S reduces to the solution of two triangular systems, whose computational cost is really low (for more details, see [25] and [17]).

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7. Conclusions and Future Developments. In this paper we introduce a new family of methods for the Navier Stokes equations, based on a pressure correction step. The idea of pressure correction has been already introduced in the framework of differential schemes (Timmermans scheme), but it is new in the field of algebraic splitting. We give a mathematical basis to this approach and numerically verify that it actually improves solutions, yielding a reduction of the errors or even, in some cases, an increment of the accuracy order. The latter conclusion holds true in particular for the YPC scheme. The accuracy improvement seems however limited to the second order, at least in the non incremental approach.

The pressure correction with a BDF3 incremental time advancing gives interesting results in the case of the CTPC scheme. While in all the other cases this method was usually worse than the YPC, in this case it exhibits good results that need to be investigated furtherly. YPC features very good results in the non incremental approach. On the other hand, whenever it is coupled with the incremental approach can be unstable. In fact, we proven that YPC is only conditional stable, but with the non incremental formulation we actually never observed numerical instabilities in a reasonable range for the time step sizes. Actually, it seems that the incremental approach can be somehow less stable. Moreover, it is worthy to mention that from preliminar numerical results the stability bound on Δt in the BDF3 incremental time advancing is proportional to the inverse of the viscosity. For low values of the viscosity (which means for high Reynolds numbers) our conjecture is that the stability bound becomes less restrictive. This observation is in agreement with the circumstance that the use of YPC as a preconditioner of the Navier Stokes solver is well suited in particular for low viscosity problems (see [7] and also [6]). A more specific stability analysis for this scheme and, in particular, the role of the incremental formulation will be however the subject of a future development of this work.

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