A DISCONTINUOUS PETROV-GALERKIN METHOD WITH LAGRANGIAN MULTIPLIERS FOR SECOND ORDER ELLIPTIC PROBLEMS *

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Abstract. We present a Discontinuous Petrov-Galerkin method (DPG) for finite element discretization scheme of second order elliptic boundary value problems. The novel approach emanates from a one-element weak formulation of the differential problem (that is typical of Discontinuous Galerkin methods (DG)) which is based on introducing variables defined in the interior and on the boundary of the element. The interface variables are suitable Lagrangian multipliers that enforce interelement continuity of the solution and of its normal derivative, thus providing the proper connection between neighboring elements. The internal variables can be eliminated in favor of the interface variables using static condensation to end up with a system of reduced size having as unknowns the Lagrangian multipliers. A stability and convergence analysis of the novel formulation is carried out and its connection with mixed-hybrid and DG methods is explored. Numerical tests on several benchmark problems are included to validate the convergence performance and the flux-conservation properties of the DPG method.

Key words. Petrov-Galerkin formulations, mixed and hybrid finite element methods, discontinuous Galerkin methods, elliptic problems.

AMS subject classifications. 65N12, 65N30, 65N15

1. Introduction and motivation. Recent years have seen an always increasing use, development and analysis of *discontinuous methods* in the approximation of boundary value problems. Within this active research area, Discontinuous Galerkin (DG) formulations certainly occupy a prominent position (we refer to [19] for a survey on the state-of-the-art of the literature on DG methods) and their success in the approximation of hyperbolic problems has prompted for their extension to cover the case of parabolic and elliptic equations.

A considerable impulse in the direction of extending the use of DG methods to parabolic and elliptic equations is due to the contributions given in [6, 7], where discontinuous finite elements of high order are used in the numerical solution of the compressible Navier–Stokes equations. Two methodological aspects in [6, 7] are of particular importance as for their influence on later research activity.

The first aspect is the technique used to accommodate the viscous terms arising in the momentum and energy balance equations within the structure of the DG formulations traditionally devoted to hyperbolic problems. The technique consists in introducing a new unknown, related to the gradient of the conservative variables, and then providing a consistent approximation for the new unknown. This strategy is closely related to classical *mixed methods* and is one of the starting motivations of the work conducted, although in different directions, in [2, 3, 16] and in the present article.

The second aspect is the extension of the concept and use of *numerical fluxes* in the treatment of boundary terms arising from integration by parts of the equations at the element level. Numerical fluxes are a key ingredient of any performing DG formulation and must be properly designed to impart stability and accuracy to the approximation. This is usually done by borrowing their expression from finite volume techniques, as discussed in [3] in the case of DG methods applied to the numerical solution of elliptic boundary value problems. The choice of numerical fluxes in DG methods is not trivial since it must be tailored to the problem

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at hand, leading in some cases to an involved implementation of the resulting scheme, a drawback that is quite common to many high-order finite volume formulations.

The motivation of the Discontinuous Petrov-Galerkin (DPG) method proposed in the present article strongly arises from this latter observation. It is indeed a fact that the *values of the variables on the element boundaries* (or an appropriate representation of them) are the ingredients to be used to provide the necessary coupling between neighboring elements. Having this clear in mind, an alternative approach to numerical flux definition may be pursued by introducing *independent interface variables* that are single-valued functions solely defined on element boundaries (*hybrid interface variables*). The hybrid interface variables are suitable Lagrangian multipliers that enforce the continuity of the displacement (the scalar variable of the problem) and of the normal stress (the vector variable of the problem) across the interfaces of the finite element triangulation. By doing so, proper interelement connection can be enforced without needing to exhibit any specific upfront recipe for the numerical flux. Therefore, the DPG method establishes a connection between DG and hybrid methods, connection that is presently object of analogous research activity by many authors in different areas (see for example [21, 22, 20]).

The DPG method was proposed in [10] where a stability and convergence analysis of the formulation was carried out in one spatial dimension. Then, the method has been applied to the numerical solution of scalar advective-diffusive models [12, 11] and of fluid-mechanical problems in both compressible and incompressible regimes [17].

In the present article we carry out the theoretical analysis of the stability and convergence properties of the novel formulation applied to the solution of an elliptic boundary value model problem in two spatial dimensions, aspect that was still lacking. We also discuss the efficient computer implementation of the scheme, this strenghtening the connection between the DPG methodology and classical DG and mixed-hybrid approaches. Numerical results are then shown to demonstrate the convergence and conservation properties of the novel formulation.

The paper is organized as follows: in Sect.2 we introduce the one-element weak formulation that is the starting point of the DPG approach. In Sect.3 we set up the formulation at the continuous level and we carry out its stability analysis. In Sect.4 we introduce the corresponding approximation and in Sect.5 we discuss the construction of appropriate finite element spaces, addressing in particular the case of the element of lowest degree (DPG_0) for which we carry out a stability and error analysis in Sect.6. We address the issue of an efficient implementation of the DPG_0 formulation in Sect.7. In Sect.8 we present some numerical results to validate the convergence performance while in Sect.9 we assess the conservation properties of the DPG method. Finally, in Sect.10 we end with some concluding remarks.

2. One-element formulation of the elliptic model problem. We consider the following elliptic model problem:

$$-\operatorname{div} \nabla u = f \quad \text{in } \Omega, \qquad u = g_D \quad \text{on } \Gamma_D, \qquad \nabla u \cdot \boldsymbol{n} = g_N \quad \text{on } \Gamma_N, \qquad (2.1)$$

where Ω is an open bounded set of $\mathbb{R}^2$ with Lipschitz continuous boundary $\Gamma = \partial \Omega$ such that $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \neq \emptyset$, and where f, g_D and g_N are given functions. Problem (2.1) will be referred to as the *primal formulation* and u as the *primal* unknown. Upon introducing the auxiliary unknown $\boldsymbol{\sigma} = \nabla u$, problem (2.1) may be rewritten as the first order system:

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma} = f \quad \text{in } \Omega, \quad \boldsymbol{\sigma} = \nabla u \quad \text{in } \Omega, \\ u = g_D \quad \text{on } \Gamma_D, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n} = g_N \quad \text{on } \Gamma_N. \end{cases} \qquad (2.2)$$

Problem (2.2) will be referred to as the *mixed formulation* of (2.1). In this latter context we shall refer in a generalized sense to the mixed unknowns u and $\boldsymbol{\sigma}$ as *displacements* and *stresses*, respectively.

Given a triangulation $\mathcal{T}_h$ of Ω made of triangles, we consider the following *one-element weak form* of problem (2.2) (see Sect.3.1 for the notation):

$\forall\, K \in \mathcal{T}_h$, find $(\boldsymbol{\sigma}^K, u^K)$ such that

$$
\begin{cases}
\displaystyle \int_K \boldsymbol{\sigma}^K \cdot \boldsymbol{q}^K \, dx + \int_K u^K \, \mathrm{div}\, \boldsymbol{q}^K \, dx - \int_{\partial K} u_{\partial K}\, \boldsymbol{q}_{\partial K} \cdot \boldsymbol{n}_K \, ds = 0 & \forall\, \boldsymbol{q}^K, \\[2mm]
\displaystyle \int_K \boldsymbol{\sigma}^K \cdot \nabla v^K \, dx - \int_{\partial K} \boldsymbol{\sigma}_{\partial K} \cdot \boldsymbol{n}_K\, v_{\partial K}\, ds = \int_K f^K\, v^K \, dx & \forall\, v^K,
\end{cases}
$$

(2.3)

where $\boldsymbol{\sigma}^K, u^K, \boldsymbol{q}^K$ and v^K belong to spaces of smooth vector and scalar functions defined on K and where the symbols $\boldsymbol{\sigma}_{\partial K}$ and $u_{\partial K}$ represent the traces on ∂K of $\boldsymbol{\sigma}^K$ and u^K, respectively, properly accounting for the boundary conditions. Notice that a formal integration by parts has been performed on both the equations in (2.2).

System (2.3) is a general setting from which both Discontinuous Galerkin and hybrid formulations can be derived, these latter after a suitable use of integration by parts in (2.3)$_1$ or (2.3)$_2$. The common factor shared by DG and hybrid formulations relies on the role played by the variables traced on the element interfaces that are the connectors demanded to preserve the proper coupling between K and its neighbors.

In DG methods the interelement constraints are enforced by defining on ∂K specific expressions for $\boldsymbol{\sigma}_{\partial K}$ and $u_{\partial K}$ as functions of the internal variables, the so-called *numerical fluxes* (see [2, 3, 16]).

In hybrid formulations the variables traced on the element interfaces are instead suitable Lagrange multipliers and are *additional* unknowns of the problem. In particular, primal mixed hybrid methods [34, 36] are obtained integrating by parts (2.3)$_1$, while dual mixed hybrid methods [14] are obtained integrating by parts (2.3)$_2$. System (2.3) is thus in dual-primal mixed hybrid form. In both cases a symmetric Galerkin formulation is obtained from the nonsymmetric formulation (2.3) and only *one* Lagrangian multiplier is introduced, with the conclusion that in hybrid formulations a different numerical treatment is applied to the displacement and stress fields.

The choice of introducing independent interface unknowns as interelement connectors, thus avoiding the need of *ad hoc* definition of the numerical fluxes, while preserving at the same time a completely parithetic (and discontinuous) approximation of u and $\boldsymbol{\sigma}$ on $\mathcal{T}_h$, as in DG formulations, is the main idea underlying the DPG method discussed in the forthcoming sections.

3. The DPG formulation. In the sequel we introduce the DPG formulation and carry out a stability and convergence analysis of the method.

3.1. Notation and functional setting. We let $\overline{\Omega} = \bigcup \overline{K}$ be a regular partition $\mathcal{T}_h$ of the domain Ω into triangular elements K (see [18]), *i.e.* we suppose that there exists a constant $\sigma \geq 1$ such that $(h_K/\rho_K) \leq \sigma$ for all $K \in \mathcal{T}_h$, h_K being the diameter of K and $\rho_K = \sup\{\mathrm{diam}(S)\,|\, S \text{ is a ball contained in } K\}$. We let $\mathcal{E}_h$ be the set of the edges of $\mathcal{T}_h$, and the edge shared by the elements K and K' will be referred to as $e_{K-K'}$. For each element $K \in \mathcal{T}_h$, we denote by ∂K the Lipschitz continuous boundary of K and by $\boldsymbol{n}_K$ the unit outward

normal vector along the boundary ∂K. We also let $\partial K_D = \partial K \cap \Gamma_D, \partial K_N = \partial K \cap \Gamma_N$. Moreover, if v is any function defined in Ω, we denote by v^K its restriction to the element K and by $v_{\partial K}$ its restriction to the element boundary ∂K. Similarly, if η is any function defined on $\mathcal{E}_h$, we denote by $\eta_{\partial K}$ its restriction on $\mathcal{E}_h \cap \partial K$.

Given an integer $m \geq 0$ and the real numbers $p, q \in [1, \infty)$, we define the following local space

$$W^{m,p}(K) = \left\{ v \in L^p(K) \mid D^\alpha v \in L^p(K) \, \forall \alpha, |\alpha| \leq m \right\} \qquad \forall K \in \mathcal{T}_h,$$

provided with the usual norm and seminorm $||v||_{m,p,K}$ and $|v|_{m,p,K}$. When $p = 2, W^{m,2}(K)$ is the usual $H^m(K)$ Sobolev space (see [27]) and the simplified notation $||.||_{m,K}$ and $|.|_{m,K}$ will be used. We also introduce the local space

$$W_q(\mathrm{div}; K) = \left\{ \boldsymbol{\tau} \in (L^q(K))^2 \mid \mathrm{div}\, \boldsymbol{\tau} \in L^q(K) \right\},$$

provided with the usual graph norm $||\boldsymbol{\tau}||_{q,\mathrm{div},K}$. When $q = 2$, the space $W_2(\mathrm{div}; K)$ is the Sobolev space $H(\mathrm{div}; K)$ (see [14]).

From now on, p and q will be chosen to be conjugate numbers, $i.e.$, $1/p + 1/q = 1$. It will be useful in the sequel to consider the space of the traces on ∂K of functions $v \in W^{1,p}(K)$ and $\boldsymbol{\tau} \in W_q(\mathrm{div}; K)$. Notice that the trace $v_{\partial K}$ belongs to the space $W^{1/q,p}(\partial K)$, while the normal trace $\boldsymbol{\tau} \cdot \boldsymbol{n}|_{\partial K}$ belongs to the space $W^{-1/q,q}(\partial K)$; the spaces $W^{1/q,p}(\partial K)$ and $W^{-1/q,q}(\partial K)$ are provided with the following norms

$$||\boldsymbol{\tau} \cdot \boldsymbol{n}||_{-1/q,q,\partial K} = \sup_{v \in W^{1,p}(K)} \frac{\langle \boldsymbol{\tau} \cdot \boldsymbol{n}, v \rangle_{\partial K}}{||v||_{1,p,K}}, \quad \forall \boldsymbol{\tau} \in W_q(\mathrm{div}; K) \qquad (3.1)$$

and

$$||v||_{1/q,p,\partial K} = \sup_{\boldsymbol{\tau} \in W_q(\mathrm{div};K)} \frac{\langle \boldsymbol{\tau} \cdot \boldsymbol{n}, v \rangle_{\partial K}}{||\boldsymbol{\tau}||_{W_q(\mathrm{div};K)}}, \qquad \forall v \in W^{1,p}(K). \qquad (3.2)$$

Note that for $p = q = 2$ the quantity in (3.1) is the standard norm $||\boldsymbol{\tau} \cdot \boldsymbol{n}||_{-1/2,\partial K}$.

Proceeding as in [23, 24], we assume henceforth that

$$4/3 < p < 2. \qquad (3.3)$$

Under this condition, functions belonging to $W^{1/q,p}(\partial K)$ need not be continuous at the vertices of ∂K, unlike in standard dual-hybrid methods where the hybrid variable belongs to the space $W^{1/2,2}(\partial K) \equiv H^{1/2}(\partial K)$ and its approximation *must* be continuous at the vertices. This is the main reason for assuming the limitation (3.3), which allows instead for adopting *discontinuous piecewise finite elements* for the approximation of the hybrid variable on $\mathcal{E}_h$ at the expense of a slightly stronger regularity on functions belonging to $W_q(\mathrm{div}; K)$ (indeed we have $q > 2$). While this latter extra-amount of regularity has no practical limiting consequences on the choice of the finite element spaces for the approximation of functions in $W_q(\mathrm{div}; K)$, the relaxed continuity requirements for the hybrid variable has the advantage of producing an approximation of the normal stresses that is *continuous* on each edge of $\mathcal{T}_h$. This is not the case with standard hybrid methods, where this latter important conservation property is achieved only in an average sense over the patch of elements surrounding each node of the triangulation.

3.2. DPG weak formulation. We introduce the trial function spaces

$$\Sigma = (L^2(\Omega))^2, \qquad U = L^q(\Omega),$$

$$\Lambda = \left\{ \lambda \in \prod_{K \in \mathcal{T}_h} W^{1/q,p}(\partial K), \lambda^K = \lambda^{K'} \text{ on } e_{K-K'}, \forall K, K' \in \mathcal{T}_h, \ \lambda^K = g_D \text{ on } \partial K_D \ \forall K \in \mathcal{T}_h \right\},$$

$$M = \left\{ \mu \in \prod_{K \in \mathcal{T}_h} H^{-1/2}(\partial K), \ \mu^K + \mu^{K'} = 0 \text{ on } e_{K-K'}, \forall K, K' \in \mathcal{T}_h, \ \mu^K = g_N \text{ on } \partial K_N \ \forall K \in \mathcal{T}_h \right\}$$

and the test function spaces $W = \prod_{K \in \mathcal{T}_h} W_q(\operatorname{div}; K)$ and $V = \prod_{K \in \mathcal{T}_h} H^1(K)$. We set
$X = (U \times \Lambda)$, $Y = (\Sigma \times M)$ and we introduce the compact notation $\widetilde{u} = (u; \lambda)$ and
$\widetilde{\sigma} = (\sigma; \mu)$.

The DPG weak formulation of problem (2.1) is obtained from (2.3) by introducing the
hybrid variables λ and μ to represent the values $u_{\partial K}$ and $\sigma_{\partial K}$, respectively, and by summing
up on the triangles, and reads:

find $(\widetilde{u}, \widetilde{\sigma}) \in (X \times Y)$ such that

$$\begin{cases} a(\widetilde{\sigma}, q) & + & b_1(\widetilde{u}, q) & = & 0 & \forall q \in W, \\ b_2(\widetilde{\sigma}, v) & & & = & (f, v) & \forall v \in V, \end{cases} \tag{3.4}$$

where $(\cdot, \cdot)$ is the usual L^2 product and where we have set

$$a(\widetilde{\sigma}, q) = \sum_{K \in \mathcal{T}_h} \int_K \sigma \cdot q \, dx, \qquad b_1(\widetilde{u}, q) = \sum_{K \in \mathcal{T}_h} \left(\int_K u \operatorname{div} q \, dx - \int_{\partial K} \lambda q \cdot n \, ds \right),$$

$$b_2(\widetilde{\sigma}, v) = \sum_{K \in \mathcal{T}_h} \left(\int_K \sigma \cdot \nabla v \, dx - \int_{\partial K} \mu \, v \, ds \right).$$

Due to the *simultaneous* presence of the two Lagrangian multipliers λ and μ, the resulting
scheme lacks the formal symmetry of a standard Galerkin mixed-hybrid formulation and
becomes a Discontinuos Petrov-Galerkin method for the numerical approximation of second
order boundary value problems. It is characterized by a completely equal treatment of the
mixed variables u and σ. Indeed, since the integration by parts has relaxed all the regularity
requirements on u and σ at the expense of more regular test functions q and v, an equal-order
interpolation for these internal fields is allowed in the finite element approximation of (3.4).

Whenever continuous test functions q and v are used in (3.4), we recover the Dual-Primal
method proposed and analyzed in [29]. Therefore, the DPG method can be fully regarded as
an hybridization of the above mentioned scheme. Furthermore, the mixed system obtained
by taking continuous test function spaces on Ω for σ and u in (3.5)$_1$ and (3.7)$_1$ yields, upon
summing over $\mathcal{T}_h$, the Primal-Dual formulation proposed and analyzed in [38].

REMARK 3.1. *Equation (3.4)$_1$ may be thought as derived from the following integral
form*

$$\int_K (\sigma^K - \nabla u^K) \cdot q^K \, dx + \int_{\partial K} (u_{\partial K} - \lambda_{\partial K}) \eta_{\partial K} \, ds = 0 \quad \forall \, q, \eta, \tag{3.5}$$

*where q and η are smooth enough test functions. Choosing $\eta_{\partial K} = q^K \cdot n_K|_{\partial K}$, and inte-
grating by parts, yields*

$$\int_K (\sigma^K \cdot q^K + u^K \operatorname{div} q^K) \, dx - \int_{\partial K} \lambda_{\partial K} \, q_{\partial K} \cdot n_K \, ds = 0 \qquad \forall q. \tag{3.6}$$

Similarly, equation (3.4)$_2$ may be thought as derived from the following integral form

$$\int_K (\operatorname{div} \boldsymbol{\sigma}^K + f^K) v^K \, dx + \int_{\partial K} (\boldsymbol{\sigma}_{\partial K} \cdot \boldsymbol{n}_K - \mu_{\partial K}) \xi_{\partial K} \, ds = 0 \quad \forall \, v, \xi, \qquad (3.7)$$

where v and ξ are smooth enough test functions. Choosing $\xi_{\partial K} = v_{\partial K}$ and integrating by parts yields

$$\int_K \boldsymbol{\sigma}^K \cdot \nabla v^K \, dx - \int_{\partial K} \mu_{\partial K} \, v_{\partial K} \, ds = 0 \qquad \forall v. \qquad (3.8)$$

The DPG weak formulation can then be formally interpreted as a mixed-hybrid virtual work principle where nonvanishing *virtual variations $\delta u = v$ and $\delta(\boldsymbol{\sigma} \cdot \boldsymbol{n}) = \boldsymbol{q} \cdot \boldsymbol{n}$ are allowed on ∂K_D and ∂K_N, respectively (see, e.g., [4] for an extensive discussion of this topic).*

3.3. Existence and uniqueness of the DPG solution. In this section we prove the existence and uniqueness of the solution of problem (3.4). To do so, we make use of the generalized saddle point problem theory introduced in [32] and further developed in [8]. For ease of presentation, we assume henceforth that $\Gamma \equiv \Gamma_D$ with $g_D = 0$.

LEMMA 3.1 (Existence). *Assume that the solution $\overline{u}$ of problem (2.1) is such that $\overline{\sigma} \in H(\operatorname{div}; \Omega)$, where $\overline{\sigma} = \nabla \overline{u}$. Then, we have that $\widetilde{u} = (\overline{u}, \overline{u}|_{\mathcal{E}_h})$ and $\widetilde{\sigma} = (\overline{\sigma}, (\overline{\sigma} \cdot \boldsymbol{n})|_{\mathcal{E}_h})$ is a solution of problem (3.4).*

Proof. Taking $v \in H_0^1(\Omega)$ in equation (3.4)$_2$, yields (see [14], Ch. 3, Prop.1.1).

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu v \, ds = 0 \qquad \forall v \in H_0^1(\Omega).$$

Then equation (3.4)$_2$ becomes

$$\int_\Omega \boldsymbol{\sigma} \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in H_0^1(\Omega),$$

from which it follows that $\overline{\sigma} = \nabla \overline{u}$ is a solution. Similarly, taking $\boldsymbol{q} \in W_q(\operatorname{div}; \Omega)$ in (3.4)$_1$, yields

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \lambda \boldsymbol{q} \cdot \boldsymbol{n} \, ds = 0 \qquad \forall \boldsymbol{q} \in W_q(\operatorname{div}; \Omega).$$

Then equation (3.4)$_1$ becomes

$$\int_\Omega \nabla \overline{u} \cdot \boldsymbol{q} \, dx + \int_\Omega u \operatorname{div} \boldsymbol{q} \, dx = 0 \qquad \forall \boldsymbol{q} \in W_q(\operatorname{div}; \Omega),$$

for which $\overline{u}$ is a solution.
Let us now go back to the hybrid fields; integrating by parts the first term in the bilinear form $b_2(\cdot, \cdot)$ we obtain

$$\sum_{K \in \mathcal{T}_h} \left(\int_{\partial K} v \overline{\sigma} \cdot \boldsymbol{n} \, ds - \int_{\partial K} \mu \, v \, ds \right) = 0 \qquad \forall v \in V,$$

which shows that $\mu|_{\partial K} = \overline{\sigma} \cdot \boldsymbol{n}|_{\partial K}$ is a solution (recall indeed that $\overline{\sigma} \in H(\operatorname{div}; \Omega)$ implies $\overline{\sigma}^K \cdot \boldsymbol{n}_K + \overline{\sigma}^{K'} \cdot \boldsymbol{n}_{K'} = 0 \quad \forall e_{K-K'}$). Proceeding similarly with the bilinear form $b_1(\cdot, \cdot)$, we obtain

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\overline{u} - \lambda) \boldsymbol{q} \cdot \boldsymbol{n} \, ds = 0 \qquad \forall \boldsymbol{q} \in W,$$

which shows that $\lambda|_{\partial K} = \overline{u}|_{\partial K}$ is a solution. The consistency of the continuous DPG formulation with the original problem is thus proved. $\square$

Before dealing with the issue of the uniqueness of the solution of problem (3.4), we state the following useful property that is an extension of the Helmholtz Decomposition Principle to the present functional setting (cf.[14], Ch.VII, Proposition 3.4 and Remark 3.3., and [25]).

PROPOSITION 3.1. *Every function* $\boldsymbol{w} \in (L^q(\Omega))^2$ *admits the orthogonal decomposition*

$$\boldsymbol{w} = \nabla \xi \oplus \operatorname{curl} \phi,$$

where $\xi \in W_0^{1,q}(\Omega), \phi \in W^{1,q}(\Omega) \setminus \mathbb{R}$ *and* $\operatorname{curl} \phi = (\frac{\partial \phi}{\partial x_1}, -\frac{\partial \phi}{\partial x_2})^T$.

Using the above proposition, we can characterize the null spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ associated with the bilinear forms $b_1(\cdot, \cdot)$ and $b_2(\cdot, \cdot)$ as follows:

$$\mathcal{K}_1 = \{\boldsymbol{q} \in W \mid b_1(\widetilde{u}, \boldsymbol{q}) = 0, \forall \widetilde{u} \in Y\} = \{\boldsymbol{q} \in W_q(\operatorname{div}; \Omega) \mid \operatorname{div} \boldsymbol{q} = 0 \text{ in } \Omega\},$$

$$\mathcal{K}_2 = \{\widetilde{\boldsymbol{\sigma}} \in X \mid b_2(\widetilde{\boldsymbol{\sigma}}, v) = 0, \forall v \in V\} = \{\boldsymbol{\sigma} \in \mathcal{K}_1; \mu = \boldsymbol{\sigma} \cdot \boldsymbol{n} \text{ on } \partial K, \forall K \in \mathcal{T}_h\}.$$

The continuous bilinear forms $b_1(\cdot, \cdot)$ and $b_2(\cdot, \cdot)$ induce the following orthogonal decompositions in terms of the closed subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively

$$W = \mathcal{K}_1 \oplus \mathcal{W}_1 \quad \text{and} \quad X = \mathcal{K}_2 \oplus \mathcal{W}_2,$$

where $\mathcal{W}_1 = \mathcal{K}_1^{\perp}$ and $\mathcal{W}_2 = \mathcal{K}_2^{\perp}$.

In order to prove the uniqueness of the solution of the DPG weak formulation, let us check the weak coerciveness of $a(\cdot, \cdot)$ and the inf-sup condition for $b_1(\cdot, \cdot)$ and $b_2(\cdot, \cdot)$.

PROPOSITION 3.2. *There exists a constant* $\delta > 0$ *such that*

$$\sup_{\boldsymbol{q} \in \mathcal{K}_1} a(\boldsymbol{\sigma}, \boldsymbol{q}) \geq \delta \|\boldsymbol{\sigma}\|_{0,\Omega} \|\boldsymbol{q}\|_{0,\Omega}, \forall \boldsymbol{\sigma} \in \mathcal{K}_2 \tag{3.9}$$

$$\sup_{\boldsymbol{\sigma} \in \mathcal{K}_2} a(\boldsymbol{\sigma}, \boldsymbol{q}) > 0, \forall \boldsymbol{q} \in \mathcal{K}_1, \boldsymbol{q} \neq 0. \tag{3.10}$$

Proof. Let $\boldsymbol{\sigma} \in \mathcal{K}_2$ and take $\boldsymbol{q}^* \in \mathcal{K}_1$. Condition (3.9) is immediately verified with $\delta = 1$ by taking $\boldsymbol{q}^* = \boldsymbol{\sigma}$ since $a(\boldsymbol{\sigma}, \boldsymbol{q}^*) = \|\boldsymbol{\sigma}\|_{0,\Omega} \|\boldsymbol{q}^*\|_{0,\Omega} \forall \boldsymbol{\sigma} \in \mathcal{K}_2$. Let now $\boldsymbol{q} \in \mathcal{K}_1$, $\boldsymbol{q} \neq 0$ and $\boldsymbol{\sigma}^* \in \mathcal{K}_2$. Taking $\boldsymbol{\sigma}^* = \boldsymbol{q}$, condition (3.10) is immediately verified. $\square$

PROPOSITION 3.3. *There exists a constant* $\gamma_1 > 0$ *such that*

$$\sup_{\boldsymbol{q} \in W} b_1(\widetilde{u}, \boldsymbol{q}) \geq \gamma_1 \|u\|_{0,\Omega} \|\boldsymbol{q}\|_{0,\Omega}, \forall \widetilde{u} \in X. \tag{3.11}$$

Proof. Let $\widetilde{u} \in X$ and $\boldsymbol{q}^* = \nabla w$, where w is the solution of the Dirichlet problem

$$\triangle w = u \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \Gamma.$$

Since $u \in L^q(\Omega)$, we have $\boldsymbol{q}^* \in W_q(\operatorname{div}; \Omega)$, and there exists a constant C such that $\|\boldsymbol{q}^*\|_{0,\Omega} \leq C\|u\|_{0,\Omega}$. Moreover, it is easy to verify that $b_1(\widetilde{u}, \boldsymbol{q}^*) = \|u\|_{0,\Omega}^2$ from which (3.11) immediately follows. $\square$

PROPOSITION 3.4. *There exists a constant $\gamma_2 > 0$ such that*

$$\sup_{\widetilde{\sigma} \in X} b_2(\widetilde{\sigma}, v) \geq \gamma_2 \|\sigma\|_{0,\Omega} \|v\|_{0,\Omega}, \ \forall v \in V. \tag{3.12}$$

Proof. Let $v \in V$ and $\sigma^* \in W_q(\mathrm{div}; \Omega)$, $\mathrm{div}\sigma^* \neq 0$ and $\mu^* = \sigma^* \cdot \boldsymbol{n}$ on ∂K, $\forall K \in \mathcal{T}_h$. Then, after integrating by parts over each element $K \in \mathcal{T}_h$, we have

$$b_2((\sigma^*; \mu^*), v) = - \sum_{K \in \mathcal{T}_h} \int_K v \, \mathrm{div}\sigma^* \, dx,$$

from which we prove (3.12) following the same lines as in the proof of Prop. 3.3. $\square$

The following theorem is an immediate consequence of the previous results.

THEOREM 3.5. *Under the regularity assumptions stated in Lemma 3.1, problem* (3.4) *admits a unique solution* $(\widetilde{u}, \widetilde{\sigma}) \in (Y \times X)$.

Having proved Prop.3.2, Prop.3.3 and Prop.3.4 we can state the following stability result.

COROLLARY 3.6. *The solution* $(\widetilde{\sigma}, \widetilde{u})$ *of the DPG problem* (3.4) *satisfies the estimate*

$$\|\widetilde{u}\|_{\widetilde{X}} \leq K_1 \|f\|_{0,\Omega}, \qquad \|\widetilde{\sigma}\|_{\widetilde{Y}} \leq K_2 \|f\|_{0,\Omega}$$

where $\|\widetilde{u}\|_{\widetilde{X}}^2 = \|u\|_{0,\Omega}^2 + \|\lambda\|_\Lambda^2$, $\|\widetilde{\sigma}\|_{\widetilde{Y}}^2 = \|\sigma\|_{0,\Omega}^2 + \|\mu\|_M^2$ *and where* $K_1 = (c_1^2 + c_2^2)^{1/2}$, $K_2 = (c_3^2 + c_4^2)^{1/2}$, *with*

$$c_1 = \frac{\delta(2 + \gamma_2) + 2}{\gamma_2 \delta}, \quad c_2 = \frac{2}{\delta}, \quad c_3 = \frac{\delta(2 + \gamma_2) + 2}{\gamma_2 \delta}, \quad c_4 = \frac{2}{\delta}.$$

4. The DPG finite element approximation. Given the finite-dimensional spaces

$$X_h \subset X, \quad Y_h \subset Y, \qquad \text{and} \qquad W_h \subset W, \quad V_h \subset V,$$

the DPG finite element approximation of problem (2.1) reads:

find $(\widetilde{u}_h, \widetilde{\sigma}_h) \in (X_h \times Y_h)$ such that

$$\begin{cases} a(\widetilde{\sigma}_h, \boldsymbol{q}_h) & + & b_1(\widetilde{u}_h, \boldsymbol{q}_h) & = & 0 & \forall \boldsymbol{q}_h \in W_h, \\ b_2(\widetilde{\sigma}_h, v_h) & & & = & (f, v_h) & \forall v_h \in V_h. \end{cases} \tag{4.1}$$

We have to define the spaces X_h, V_h, W_h, V_h and specify their degrees of freedom.

The choice of the finite element spaces is absolutely nontrivial in mixed Petrov-Galerkin formulations. The idea is first to lay down the properties we want the trial finite element spaces to satisfy and then to select accordingly the discrete test finite element spaces in order to end up with a stable and convergent approximate scheme.

4.1. Trial finite element spaces. The objectives we want the discrete approximation to achieve are the highest possible level of discontinuity and an equal-order interpolation for u_h and σ_h and for λ_h and μ_h, respectively. The motivation for adopting equal-order interpolation for both mixed and hybrid variables is that by doing so the numerical performance of a scheme may be significantly enhanced. As a matter of fact, mixed formulations can be interpreted as a *phase-space* approach. Established approaches in dynamics problems applications suggest that an equal-order treatment of the two fields is the right key to achieve correct energy conservation (see [9]).

A natural choice for both internal and interface unknown fields is to consider on each triangle K polynomial finite elements of equal order, respectively in K and on each edge of ∂K. We let henceforth k be a nonnegative integer and we denote by $\mathbb{P}_k(K)$ the space of all polynomials of degree $\leq k$ on K and by $R_k(\partial K)$ the space of all functions defined over the boundary ∂K of K whose restrictions to any side $e \in \partial K$ are polynomials of degree $\leq k$. Notice that functions in $R_k(\partial K)$ need not be continuous at the vertices of K.

We take on each triangle $K \in \mathcal{T}_h$

$$X_h^k(K) = \mathbb{P}_k(K) \times R_k(\partial K), \qquad Y_h^k(K) = (\mathbb{P}_k(K))^2 \times R_k(\partial K), \qquad (4.2)$$

and we set

$$X_h^k = \prod_{K \in \mathcal{T}_h} X_h^k(K), \qquad Y_h^k = \prod_{K \in \mathcal{T}_h} Y_h^k(K), \qquad (4.3)$$

where functions belonging to $R_k(\partial K)$ are single-valued on each internal edge and satisfy the appropriate boundary conditions on Γ_D and Γ_N, respectively. For brevity of notation we also set $\mathcal{X}_h^k(K) = X_h^k(K) \times Y_h^k(K)$ and $\mathcal{X}_h^k = X_h^k \times Y_h^k$.

4.2. Test finite element spaces. Let us now address the issue of properly choosing the finite element test spaces for the DPG approximation. We will start by setting up necessary conditions for the dimension of the test finite element spaces in order the linear system arising from (4.1) to be a square one. The stability of the approximation will provide a sufficient criterion for explicitly selecting the discrete test functions.

We start with performing a count of the total degrees of freedom corresponding to the choice (4.2) as a function of the polynomial degree k. Subtracting the total number of constraints enforced by the definition of the hybrid field finite element spaces from the previously obtained amount, provides the total number of equations that must be written to end up with a *square* algebraic linear system for each value of k. Denoting by NE, Ned, Ni and Nb the number of triangles, edges, internal edges and boundary edges respectively, we have

$$\dim(\mathcal{X}_h^k) = \frac{3}{2}(k+1)(k+6)\texttt{NE}, \qquad (4.4)$$

while the total number of constraints is Nc = (k+1)(2 Ni + Nb).

Applying Euler's theorem (Ned = (3 NE + Nb)/2), we can express the total number of constraints as a function of NE as Nc = 3(k+1)NE, from which it follows that the dimension of the global finite element test space

$$\mathcal{V}_h^k(K) = W_h^k(K) \times V_h^k(K), \qquad \mathcal{V}_h^k = W_h^k \times V_h^k$$

that is needed to end up with a square linear system for each value of k is

$$\dim(\mathcal{V}_h^k) = \dim(\mathcal{X}_h^k) - \texttt{Nc} = \frac{3}{2}(k+1)(k+4)\texttt{NE}. \qquad (4.5)$$

Looking at (4.4) and (4.5) it clearly appears that for each k, the degrees of freedom for both trial and test spaces as well as the total number of constraints can all be expressed as function of the sole number of mesh triangles NE. Therefore, the proper design of the finite element test function spaces can be carried out at the single element level. Precisely, denoting by Nc(K) the number of constraints on triangle K, relation (4.5) can be written at the element level as

$$\dim(\mathcal{V}_h^k(K)) = \dim(\mathcal{V}_h^k(K)) - \texttt{Nc}(K) = \frac{3}{2}(k+1)(k+4) \quad \forall K \in \mathcal{T}_h. \qquad (4.6)$$

This equation expresses the balance between degrees of freedom, constraints and number of equations that must be fulfilled independently on each single element K. Based on these constraints, we start in the next section with the construction of the finite element test space $\mathcal{X}_h^k(K)$ in the lowest degree case $k = 0$. The resulting local finite element space will be denoted as $\mathrm{DPG}_0(K) = \mathcal{X}_h^0(K) \times \mathcal{V}_h^0(K), \forall K \in \mathcal{T}_h$.

5. Choice of the finite element spaces. In this section we discuss in detail the lowest order finite element approximation DPG_0 and then we use this procedure as a guideline for the generation of higher-order elements.

5.1. DPG_0 finite element approximation. Setting $k = 0$, relation (4.6) gives

$$\dim(\mathcal{V}_h^k(K)) = \dim(W_h^0(K)) + \dim(V_h^0(K)) = 6, \qquad \forall K \in \mathcal{T}_h.$$

The minimal choice for the scalar finite element test space is $V_h^0(K) = \mathbb{P}_1(K)$. By doing so, 3 degrees of freedom are left for the vector finite element test space, that can be conveniently saturated by setting $W_h^0(K) = \mathbb{RT}_0(K)$, where $\mathbb{RT}_k(K) = (\mathbb{P}_k(K))^2 \oplus \boldsymbol{x}\mathbb{P}_k(K)$, $\forall K \in \mathcal{T}_h$ is the Raviart-Thomas finite element space of degree k [35]. The DPG_0 local finite element space is then defined on each element as

$$\mathrm{DPG}_0(K) = \underbrace{\left(\mathbb{P}_0(K) \times R_0(\partial K) \times (\mathbb{P}_0(K))^2 \times R_0(\partial K)\right)}_{\mathcal{X}_h^0(K)} \times \underbrace{\left(\mathbb{RT}_0(K) \times \mathbb{P}_1(K)\right)}_{\mathcal{V}_h^0(K)}. \quad (5.1)$$

5.2. Higher order DPG methods. Higher-order finite elements will be consistently denoted as $\mathrm{DPG}_k(K) = \mathcal{X}_h^k(K) \times \mathcal{V}_h^k(K), \forall K \in \mathcal{T}_h$. Under the assumption that the local finite element trial space is defined as in (4.2)-(4.3), the question is how to construct a suitable test finite element space such that the following conditions are satisfied:

1. the dimension of the test finite element space is

$$\dim(\mathcal{V}_h^k(K)) = \dim(W_h^k(K)) + \dim(V_h^k(K)) = \frac{3}{2}(k+1)(k+4) \quad \forall K \in \mathcal{T}_h;$$
$$(5.2)$$

2. the following inf-sup condition is verified

$$\mu_h \in R_k(\partial K), \quad \int_{\partial K} \mu_h v_h \, ds = 0 \qquad \forall v \in V_h^k(K) \quad \text{implies} \quad \mu_h = 0. \quad (5.3)$$

The first condition ensures to end up with a square system. The second condition forces a restriction on the minimum order of the polynomial space for v_h. There is no need of such a condition for the term $\int_{\partial K} \lambda_h \, \boldsymbol{q}_h \cdot \boldsymbol{n} \, ds$ since both λ_h and the normal traces of functions in $W_h^k(K)$ are *discontinuous* on ∂K.

In the construction procedure, we must distinguish between the case when k is an even or an odd integer. Indeed, using the results stated by Lemma 4 and Lemma 6 in [34] (where the same compatibility problem occurs), we have that condition (5.3) holds if

$$\begin{cases} v_h \in \mathbb{P}_{k+1}(K) & \text{for } k \text{ even}, \ k \geq 0 \\ v_h \in \widehat{\mathbb{P}}(K), \ \mathbb{P}_{k+1}(K) \subset \widehat{\mathbb{P}}(K) \subset \mathbb{P}_{k+2}(K) & \text{for } k \text{ odd}, \ k \geq 1. \end{cases} \quad (5.4)$$

In the first case (k even), the family of finite element spaces is immediately built by setting $V_h^k(K) = \mathbb{P}_{k+1}(K)$ and then suitably saturating the degrees of freedom implied by (5.2)

$$\dim(W_h^k(K)) = \dim(\mathcal{V}_h^k(K)) - \dim(V_h^k(K)) = k^2 + 5k + 3, \qquad \forall K \in \mathcal{T}_h \quad (5.5)$$

by choosing $W_h^k(K) = \mathbb{BDFM}_{k+1}(K)$, $k \geq 0$ (for the definition of this space and its properties, see [13, 14]).

The situation is more complicated when k is odd. In this case the choice $v_h \in \mathbb{P}_{k+1}(K)$ is not allowed by (5.4), while the choice $v_h \in \mathbb{P}_{k+2}(K)$ is acceptable but unnecessarily expensive. In [34] it has been shown that in order to satisfy condition (5.4) it is sufficient to enrich the space $\mathbb{P}_{k+1}(K)$ with a single additional degree of freedom suitably excerpted from the space $\mathbb{P}_{k+2}(K)$. Setting thus $V_h^k(K) = \widehat{\mathbb{P}}(K)$, relation (5.2) becomes for each $K \in \mathcal{T}_h$

$$\dim(W_h^k(K)) = \dim(\mathcal{V}_h^k(K)) - \dim(V_h^k(K)) = k^2 + 5k + 2. \qquad (5.6)$$

A possible choice is then

$$\mathcal{V}_h^k(K) = \mathbb{RT}_k(K) \oplus B_{k-1}(K) \oplus B_k(K) \oplus ... \oplus B_{2k-3}(K), \qquad k \geq 1$$

where $(k + 1)(k + 3)$ degrees of freedom are saturated by the $\mathbb{RT}_k$ space and the remaining $(k-1)$ degrees of freedom are saturated by adding $(k-1)$ bubble functions B_l defined as (see [37]) $B_l(K) = \{\boldsymbol{q} \,|\, \boldsymbol{q} = \mathrm{curl}\,(b_K w)\}$, with $w \in \mathbb{P}_l(K)$, $b_K = \prod_{i=1}^{3} z_i(\boldsymbol{x})$, z_i, $i = 1, 2, 3$, being the barycentric coordinates in K.

To summarize, the family of DPG_k finite element spaces is defined as

$$\mathcal{X}_h^k(K) = \mathbb{P}_k(K) \times R_k(\partial K), \qquad \mathcal{V}_h^k(K) = (\mathbb{P}_k(K))^2 \times R_k(\partial K), \quad k = 0, 1, 2...$$

and letting $m = 0, 1, 2, ...$, we have

$$\begin{cases} W_h^{2m}(K) = \mathbb{BDFM}_{2m+1}(K), \quad V_h^{2m}(K) = \mathbb{P}_{2m+1}(K), \qquad k = 2m, \\ W_h^{2m+1}(K) = \mathbb{RT}_{2m+1}(K) \oplus B_{2m}(K)... \oplus B_{4m-1}(K), \; V_h^{2m+1}(K) = \widehat{\mathbb{P}}(K), \; k = 2m + 1. \end{cases}$$

Notice that with the above choices the matrix arising from the term $\int_{\partial K} \lambda_h \boldsymbol{q}_h \cdot \boldsymbol{n}\, ds$ is always square and nonsingular since both $\lambda_h|_{\partial K}$ and $\boldsymbol{q}_h \cdot \boldsymbol{n}|_{\partial K}$ belong to the same finite element space due to the properties of the $\mathbb{RT}_k$ and $\mathbb{BDFM}_{k+1}$ spaces (see [14], Chap.3).

6. Stability and convergence analysis of the approximate DPG_0 solution. In this section and in the remainder of the article, we focus our attention on the member of the DPG_k family of lowest degree, the DPG_0 finite element. Numerical results on the convergence performance of higher order elements of the family can be found in [10].

6.1. Existence and uniqueness of the DPG_0 solution. We start proving the uniqueness (and thus the existence) of the solution of problem (4.1). To do so, we characterize the discrete null spaces $\mathcal{K}_1^h$ and $\mathcal{K}_2^h$ as

$$\begin{aligned} \mathcal{K}_1^h &= \{\boldsymbol{q}_h \in W_h \,|\, b_1(\widetilde{u}_h, \boldsymbol{q}_h) = 0, \forall \widetilde{u}_h \in Y_h\} \\ &= \{\boldsymbol{q}_h \in W_h \,|\, \boldsymbol{q}_K \cdot \boldsymbol{n}_K + \boldsymbol{q}_{K'} \cdot \boldsymbol{n}_{K'} = 0, \; \forall e_{K-K'}, \; \mathrm{div}\,\boldsymbol{q}_h = 0 \text{ in } \Omega\}, \\ \mathcal{K}_2^h &= \{\widetilde{\boldsymbol{\sigma}}_h \in X_h \,|\, b_2(\widetilde{\boldsymbol{\sigma}}_h, v_h) = 0, \forall v \in V_h\} \\ &= \{\boldsymbol{\sigma}_h \in \Sigma \,|\, \boldsymbol{\sigma}_K \cdot \boldsymbol{n}_K + \boldsymbol{\sigma}_{K'} \cdot \boldsymbol{n}_{K'} = 0 \; \forall e_{K-K'}; \; \mu_h = \boldsymbol{\sigma}_h \cdot \boldsymbol{n} \text{ on } \partial K, \; \forall K \in \mathcal{T}_h\}. \end{aligned}$$

The continuous bilinear forms $b_1(\cdot, \cdot)$ and $b_2(\cdot, \cdot)$ induce the following orthogonal decompositions in terms of the closed subspaces $\mathcal{K}_1^h$ and $\mathcal{K}_2^h$, respectively

$$W^h = \mathcal{K}_1^h \oplus \mathcal{W}_1^h \quad \text{and} \quad X^h = \mathcal{K}_2^h \oplus \mathcal{W}_2^h, \quad \text{where } \mathcal{W}_1^h = \mathcal{K}_1^{h,\perp}, \mathcal{W}_2^h = \mathcal{K}_2^{h,\perp}.$$

Moreover, the following properties hold.

PROPOSITION 6.1. *There exists a constant $\delta' > 0$ independent of h such that*

$$
\begin{aligned}
&\sup_{\boldsymbol{q}_h \in \mathcal{K}_1^h} a(\boldsymbol{\sigma}_h, \boldsymbol{q}_h) \geq \delta' \|\boldsymbol{\sigma}_h\|_{0,\Omega} \|\boldsymbol{q}_h\|_{0,\Omega}, \ \forall \boldsymbol{\sigma}_h \in \mathcal{K}_2^h, \\
&\sup_{\boldsymbol{\sigma}_h \in \mathcal{K}_2^h} a(\boldsymbol{\sigma}_h, \boldsymbol{q}_h) > 0, \ \forall \boldsymbol{q}_h \in \mathcal{K}_1^h, \boldsymbol{q}_h \neq 0.
\end{aligned}
\tag{6.1}
$$

Proof. Let $\boldsymbol{\sigma}_h \in \mathcal{K}_2^h$ and take $\boldsymbol{q}_h^* \in \mathcal{K}_1^h$. Condition (3.9) is immediately verified with $\delta' = 1$ by taking $\boldsymbol{q}_h^* = \boldsymbol{\sigma}_h$ since $a(\boldsymbol{\sigma}_h, \boldsymbol{q}_h^*) = \|\boldsymbol{\sigma}_h\|_{0,\Omega} \|\boldsymbol{q}_h^*\|_{0,\Omega} \ \forall \boldsymbol{\sigma}_h \in \mathcal{K}_2^h$. Let now $\boldsymbol{q}_h \in \mathcal{K}_1^h$, $\boldsymbol{q}_h \neq 0$ and $\boldsymbol{\sigma}_h^* \in \mathcal{K}_2^h$. Taking $\boldsymbol{\sigma}_h^* = \boldsymbol{q}_h$, condition (3.10) is immediately verified. □

PROPOSITION 6.2. *There exists a constant $\gamma_1' > 0$ independent of h such that*

$$
\sup_{\boldsymbol{q}_h \in W_h} b_1(\widetilde{u}_h, \boldsymbol{q}_h) \geq \gamma_1' \|u_h\|_{0,\Omega} \|\boldsymbol{q}_h\|_{0,\Omega}, \ \forall \widetilde{u}_h \in Y_h.
\tag{6.2}
$$

Proof. Let $\widetilde{u}_h \in Y_h$ and take $\boldsymbol{q}^* \in W_h$ such that $\boldsymbol{q}_K^* \cdot \boldsymbol{n}_K + \boldsymbol{q}_{K'}^* \cdot \boldsymbol{n}_{K'} = 0$ on $e_{K-K'}$ $\forall K, K' \in \mathcal{T}_h$. Using then Lemma 7.2.1 of [33], condition (6.2) immediately follows. □

In order to prove the discrete inf-sup condition for the bilinear form $b_2(\cdot, \cdot)$, we need introduce the space $\widetilde{V} = \prod_{K \in \mathcal{T}_h} (H^1(K) \setminus \mathbb{R})$, equipped with the norm $|||v|||_{\widetilde{V}} = (\sum_{K \in \mathcal{T}_h} |v|_{1,K}^2)^{1/2}$. Notice that this norm is indeed a norm on the space $\widetilde{V}$ and is equivalent to the norm $||\cdot||_V$ for functions $v \in \widetilde{V}$. We also consider a finite dimensional approximation of $\widetilde{V}$, i.e., the space $\widetilde{V}_h \subset \widetilde{V}$ defined as $\widetilde{V}_h = \prod_{K \in \mathcal{T}_h} \mathbb{P}_1(K) \setminus \mathbb{R}$.

PROPOSITION 6.3. *There exists a constant $\gamma_2' > 0$ independent of h such that*

$$
\sup_{\widetilde{\boldsymbol{\sigma}}_h \in X_h} b_2(\widetilde{\boldsymbol{\sigma}}_h, v_h) \geq \gamma_2' \|\boldsymbol{\sigma}_h\|_{0,\Omega} |||v_h|||_{\widetilde{V}}, \ \forall v_h \in \widetilde{V}_h.
\tag{6.3}
$$

Proof. Let $v_h \in \widetilde{V}_h$. Take $\boldsymbol{\sigma}_h^* \in \Sigma_h$ such that $\boldsymbol{\sigma}_h^* = \nabla v_h$ on K, $\forall v_h \in \widetilde{V}_h$, $\forall K \in \mathcal{T}_h$ and set $\mu_h^* \equiv 0$. Then we have $b_2(\widetilde{\boldsymbol{\sigma}}_h^*, v_h) = \|\boldsymbol{\sigma}\|_{0,\Omega} |||v_h|||_{\widetilde{V}}$ and (6.3) immediately follows with $\gamma_2' = 1$. □

REMARK 6.1. *The above proof reveals that the choice $v_h = constant$ on each $K \in \mathcal{T}_h$ does not allow by itself to state condition (6.3). However, taking $v_h = 1$ on K and equal to zero elsewhere, is a possible and significant choice since it provides the local conservation property of the DPG formulation $-\int_{\partial K} \mu_h \, ds = \int_K f \, dx$. Moreover a global conservation property can be shown to hold as well by taking $v_h \equiv 1$ on Ω, and yielding the relation $-\int_\Gamma \mu_h \, ds = \int_\Omega f \, dx$. A detailed discussion of this subject will be carried out in Sect.9.*

REMARK 6.2. *Choosing $\boldsymbol{\sigma}_h^* = 0$, the bilinear form $b_2(\cdot, \cdot)$ yields the familiar relation of primal hybrid formulations $\sum_{K \in \mathcal{T}_h} \int_K \mu_h v_h = 0 \ \forall v_h \in V_h$, that admits the unique solution $\mu_h \equiv 0$.*

The following theorem is an immediate consequence of the previous results.

THEOREM 6.4. *The DPG_0 approximation of problem (2.2) admits a unique solution $(\widetilde{u}_h, \widetilde{\boldsymbol{\sigma}}_h) \in (X_h \times Y_h)$.*

6.2. Error estimates. In the following sections we establish optimal error estimates for the mixed variables u_h and $\boldsymbol{\sigma}_h$ and for the hybrid variables λ_h and μ_h.

6.2.1. Projection operators. In view of the error analysis of the DPG formulation, it is useful to introduce some approximation operators. We denote by P_K the projection operator from $L^2(K)$ onto $\mathbb{P}_0(K)$ satisfying the approximation property

$$||v - P_K v||_{0,K} \leq Ch|v|_{1,K} \qquad \forall v \in H^1(K). \tag{6.4}$$

From the operator P_K, for all $v \in L^2(\Omega)$, we construct the global operator P_h as

$$P_h v|_K = P_K v \qquad \forall K \in \mathcal{T}_h. \tag{6.5}$$

We also need introduce the projection operator ρ_h^0 from $\prod_{K \in \mathcal{T}_h} L^2(\partial K)$ onto $R_0(\partial K)$ such that, for all $\lambda \in \prod_{K \in \mathcal{T}_h} L^2(\partial K)$, we have

$$\int_{\partial K} (\rho_h^0 \lambda - \lambda) r_0 \, ds = 0 \qquad \forall r_0 \in R_0(\partial K), \qquad \forall K \in \mathcal{T}_h. \tag{6.6}$$

REMARK 6.3. *The operator ρ_h^0 is well defined since, by Sobolev's embedding theorem ([27]), we have that $W^{1/q,p}(\partial K) \hookrightarrow L^2(\partial K)$.*

6.2.2. Error estimates for the mixed variables. The following optimal error estimates hold.

THEOREM 6.5. *Let $(u, \boldsymbol{\sigma})$ be the solution of (3.4) and $(u_h, \boldsymbol{\sigma}_h)$ be the solution of (4.1). If $\boldsymbol{\sigma} \in (H^1(\Omega))^2$, then there exists a positive constant C independent of h such that*

$$\begin{aligned} ||u - u_h||_{0,\Omega} &\leq Ch(|u|_{1,\Omega} + |\boldsymbol{\sigma}|_{1,\Omega}), \\ ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\Omega} &\leq Ch|\boldsymbol{\sigma}|_{1,\Omega}. \end{aligned} \tag{6.7}$$

Proof. From (3.4) and (4.1) we have

$$\begin{cases} a(\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h, \boldsymbol{q}_h) & + & b_1(\widetilde{u} - \widetilde{u}_h, \boldsymbol{q}_h) & = & 0 & \forall \boldsymbol{q}_h \in W_h, \\ b_2(\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h, v_h) & & & = & 0 & \forall v_h \in V_h. \end{cases} \tag{6.8}$$

Taking $\boldsymbol{q}_h \in \mathcal{K}_1^h$, the first relation in (6.8) becomes

$$a(\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h, \boldsymbol{q}_h) = 0 \qquad \forall \boldsymbol{q}_h \in \mathcal{K}_1^h. \tag{6.9}$$

Let us introduce the decomposition $\widetilde{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_h, \mu_h) = \widetilde{\boldsymbol{\sigma}}_h^0 + \widetilde{\boldsymbol{\sigma}}_h^\perp$, where $\widetilde{\boldsymbol{\sigma}}_h^0 = (\boldsymbol{\sigma}_h^0, \mu_h^0) \in \mathcal{K}_2^h$ and $\widetilde{\boldsymbol{\sigma}}_h^\perp = (\boldsymbol{\sigma}_h^\perp, \mu_h^\perp) \in \mathcal{W}_2^h$. Introducing the projection operator $\widetilde{\Pi}_h = ((P_h)^2, \rho_h^0)$ where P_h and ρ_h^0 have been defined in (6.5) and (6.6), respectively, and using the decomposition $\boldsymbol{\sigma}_h = (\boldsymbol{\sigma}_h^0 + \boldsymbol{\sigma}_h^\perp)$, equation (6.9) reads

$$a((\widetilde{\Pi}_h \boldsymbol{\sigma})^0 - \boldsymbol{\sigma}_h^0, \boldsymbol{q}_h) = a(\widetilde{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{q}_h) + a(\boldsymbol{\sigma}_h^\perp - (\widetilde{\Pi}_h \boldsymbol{\sigma})^\perp, \boldsymbol{q}_h) \qquad \forall \boldsymbol{q}_h \in \mathcal{K}_1^h,$$

that, using the coercivity and continuity of the bilinear form $a(\cdot, \cdot)$, yields

$$||(\widetilde{\Pi}_h \boldsymbol{\sigma})^0 - \boldsymbol{\sigma}_h^0||_{0,\Omega} \leq (||\widetilde{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}||_{0,\Omega} + ||(\widetilde{\Pi}_h \boldsymbol{\sigma})^\perp - \boldsymbol{\sigma}_h^\perp||_{0,\Omega}). \tag{6.10}$$

We need now bound the quantity $||(\widetilde{\Pi}_h \boldsymbol{\sigma})^\perp - \boldsymbol{\sigma}_h^\perp||_{0,\Omega}$. Using (6.5) into (6.8)$_2$, we get

$$b_2(\widetilde{\Pi}_h \widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h, v_h) = b_2(\widetilde{\Pi}_h \widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}, v_h) = - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\rho_h^0 \mu - \mu) v_h \, ds \qquad \forall v_h \in V_h. \tag{6.11}$$

Recalling Lemma 9 in [34], we have that

$$\int_{\partial K} (\boldsymbol{\sigma} \cdot \boldsymbol{n} - \rho_h^0 \mu) v \, ds \leq C \frac{h_K}{\rho_K} |\boldsymbol{\sigma}|_{1,K} |v|_{1,K} \qquad \forall v \in H^1(K).$$

Using this latter relation in (6.11) and the discrete inf-sup condition for $b_2(\cdot, \cdot)$, we get the estimate

$$||(\widetilde{\Pi}_h \boldsymbol{\sigma})^\perp - \boldsymbol{\sigma}_h^\perp||_{0,\Omega} \leq C h |\boldsymbol{\sigma}|_{1,\Omega}. \tag{6.12}$$

Now, gathering (6.10) and (6.12) and using the triangle inequality, we end up with $(6.7)_2$. Let us now prove $(6.7)_1$. Taking $\boldsymbol{q}_h \in \mathcal{W}_1^h$ in the first equation of (6.8) we get

$$b_1(\widetilde{u} - \widetilde{u}_h, \boldsymbol{q}_h) = a(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{q}_h) \qquad \forall \boldsymbol{q}_h \in \mathcal{W}_1^h.$$

Introducing the projection operator $\widetilde{P}_h = (P_h, \rho_h^0)$, we write the latter relation as

$$b_1(\widetilde{P}_h \widetilde{u} - \widetilde{u}_h, \boldsymbol{q}_h) = a(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{q}_h) \qquad \forall \boldsymbol{q}_h \in \mathcal{W}_1^h.$$

Then using the discrete inf-sup condition for $b_1(\cdot, \cdot)$, we get

$$||P_h u - u_h||_{0,\Omega} \leq C ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\Omega} \leq C h |\boldsymbol{\sigma}|_{1,\Omega}. \tag{6.13}$$

Eventually, using $(6.7)_2$, (6.4) and the triangle inequality, we get estimate $(6.7)_1$. □

6.3. Error estimates for the hybrid variables. In this section, we derive a priori error estimates for the discretization errors associated with the hybrid variables λ_h and μ_h. In doing this, we shall prove some equivalence results between the DPG$_0$ method and hybrid formulations, both of primal and dual type.

To start with, we let $W_{h,0}^{NC}$ denote the set of nonconforming functions in V_h that are affine on each $K \in \mathcal{T}_h$ and are continuous at the midpoint of each edge and vanish at the midpoint of each edge of Γ. Then, we define $u_h^* \in W_{h,0}^{NC}$ as the piecewise linear nonconforming function such that

$$\int_{\partial K} u_h^* \eta_h \, ds = \int_{\partial K} \lambda_h \eta_h \, ds \qquad \forall \eta_h \in R_0(\partial K) \qquad \forall K \in \mathcal{T}_h \tag{6.14}$$

which implies in particular that $u_h^*(x_{MP,i}) = \lambda_{h,e_i}$ for each edge $e_i \in \partial K$, $x_{MP,i}$ being the coordinate vector of the midpoint of e_i. Using the fact that $\boldsymbol{q}_h \cdot \boldsymbol{n}_K \in R_0(\partial K)$ and (6.14), we get

$$\int_{\partial K} \lambda_h \boldsymbol{q}_h \cdot \boldsymbol{n}_K \, ds = \int_{\partial K} u_h^* \boldsymbol{q}_h \cdot \boldsymbol{n}_K \, ds = \int_K u_h^* \mathrm{div} \boldsymbol{q}_h \, dx + \int_K \boldsymbol{q}_h \cdot \nabla u_h^* \, dx \qquad \forall \boldsymbol{q}_h \in W_h(K).$$

Substituting this latter expression in $(4.1)_1$, we obtain

$$\int_K (\boldsymbol{\sigma}_h - \nabla u_h^*) \cdot q_h \, dx + \int_K (u_h - u_h^*) \mathrm{div} \boldsymbol{q}_h \, dx = 0 \qquad \forall \boldsymbol{q}_h \in W_h(K). \tag{6.15}$$

Taking $q_h \in (\mathbb{P}_0(K))^2$ in (6.15) yields

$$\boldsymbol{\sigma}_h^K = \nabla u_h^* \qquad \forall K \in \mathcal{T}_h, \tag{6.16}$$

while taking $q_h = (x, y)^T$ in (6.15) and using (6.14) yields

$$u_h^K = \frac{\int_K u_h^* \, dx}{|K|} = P_K u_h^* = \frac{1}{3} \sum_{i=1}^3 \lambda_{h,e_i} \qquad \forall K \in \mathcal{T}_h. \tag{6.17}$$

Relation (6.17) shows that u_h^K is the *average value* of u_h^* on K and thus the average value of the hybrid variables λ_h on the edges of the element. Let us now consider equation $(4.1)_2$ and take $v_h \in W_{h,0}^{NC}$. Equation $(4.1)_2$ becomes

$$\sum_{K \in \mathcal{T}_h} \int_K \nabla u_h^* \cdot \nabla v_h \, dx = \sum_{K \in \mathcal{T}_h} \int_K f v_h \, dx \qquad \forall v_h \in W_{h,0}^{NC}. \tag{6.18}$$

Relation (6.18) shows that u_h^* actually *coincides* with the solution $u_h^{NC} \in W_{h,0}^{NC}$ of problem (2.1) obtained with the nonconforming finite element approximation (see [34]).

Then, the following error estimate can be proved.

THEOREM 6.6. *Let u be the solution of problem* (2.1) *such that $u \in H^2(\Omega) \cap H_0^1(\Omega)$, and u_h^* be the solution of problem* (6.18) *such that* (6.14) *holds. Then, under the assumption that the polygonal domain Ω is convex, we have (see [34])*

$$||u - u_h^*||_{0,\Omega} \le C h^2 |u|_{2,\Omega}. \tag{6.19}$$

The above theorem is a superconvergence result for the piecewise linear nonconforming extension over Ω of the hybrid variable λ_h computed by the DPG$_0$ formulation. Moreover, the estimate (6.19) can be regarded as the counterpart for the DPG formulation of Theorem 2.2 in [1] valid for the dual-mixed method with hybridization.

Considering again equation $(4.1)_2$ and taking this time $v_h \in V_h$, we obtain

$$\int_{\partial K} \mu_h v_h \, ds = \int_K \nabla u_h^* \cdot \nabla v_h \, dx - \int_K f v_h \, dx \qquad \forall v_h \in V_h(K) \quad \forall K \in \mathcal{T}_h, \tag{6.20}$$

which coincides with the post-processing procedure discussed in [36] Sect.19 for the primal-hybrid formulation. This result actually demonstrates that the hybrid field μ_h computed by the DPG$_0$ approximation coincides with the field $\mathbf{p}_h \cdot \boldsymbol{n}$ computed by the primal-hybrid formulation. We have then the following result.

THEOREM 6.7. *Under the assumptions of Theorem 6.6 and the condition stated in Remark 6.2, we have*

$$||\boldsymbol{\sigma} \cdot \boldsymbol{n} - \mu_h||_{-1/2,h} \le C h |\boldsymbol{\sigma}|_{1,\Omega}$$

where $\forall \xi \in R_0(\partial K)$ we define the norm $||\xi||_{-1/2,h} = (\sum_{e \in \mathcal{E}_h} |e| ||\xi||_{0,e}^2)^{1/2}$ (see [1]).

Proceeding along the above guideline, it is possible to further explore the connection existing between the DPG$_0$ formulation and the dual mixed method. In view of establishing this connection, we assume henceforth f to be piecewise constant over $\mathcal{T}_h$. Under this hypothesis, we can use the following result proved in [28]

$$\begin{aligned}
u_h^{DM} - P_K u_h^* &= u_h^{DM} - u_h^K = -\frac{f^K}{4} \left(|x_{CG,K}^2| - \frac{1}{|K|} \int_K |x|^2 \, dx \right) \\
&= \frac{1}{144} f^K \sum_{i=1}^3 |e_i|^2 = \mathcal{O}(h_K^2) \, \forall K \in \mathcal{T}_h,
\end{aligned} \tag{6.21}$$

where $x_{CG,K}$ is the coordinate vector of the center of gravity of K and $u_h^{DM} \in \mathbb{P}_0(K)$ is the solution computed by the dual-mixed method. Using the result (6.21) and recalling the standard estimates for the dual-mixed approximation (see [23],[14]), gives by the triangle inequality the following result.

THEOREM 6.8. *Let $(u, \boldsymbol{\sigma})$ be the solution of (2.1) and $(u_h, \boldsymbol{\sigma}_h)$ be the solution of (4.1). If the triangulation $\mathcal{T}_h$ is uniformly regular and $\boldsymbol{\sigma} \in (H^1(\Omega))^2$, $\mathrm{div}\boldsymbol{\sigma} \in H^1(\Omega)$, then*

$$||P_h u - u_h||_{0,\Omega} \le Ch^2(|\boldsymbol{\sigma}|_{1,\Omega} + |\mathrm{div}\boldsymbol{\sigma}|_{1,\Omega}). \qquad (6.22)$$

Relation (6.22) can be interpreted as a superconvergence result for u_h at the center of gravity of each triangle K. This latter result also allows us to derive an optimal estimate for the quantity $||\rho_h^0\lambda - \lambda_h||_{1/q,p,\partial K}$. To proceed, we first need recall the following result [23]:

LEMMA 6.9. *For all $T \in W^{-1/q,q}(\partial K)$ there exists a unique $\boldsymbol{q}_h \in \mathbb{RT}_0(K)$ such that $\forall K \in \mathcal{T}_h$ we have*

$$\int_{\partial K} (\boldsymbol{q}_h \cdot \boldsymbol{n} - T)\, r_0\, ds = 0 \qquad \forall r_0 \in R_0(\partial K). \qquad (6.23)$$

Furthermore, if $\mathcal{T}_h$ is uniformly regular, then there is a constant C independent of K such that

$$||\boldsymbol{q}_h||_{0,K} \le Ch^{2/p-1}||T||_{-1/q,q,\partial K}, \quad ||\mathrm{div}\boldsymbol{q}_h||_{0,K} \le Ch^{2/p-2}||T||_{-1/q,q,\partial K}, \quad (6.24)$$

where p satisfies (3.3), q is its conjugate and the norm $||\cdot||_{-1/q,q,\partial K}$ has been defined in (3.1). For the definition of a uniformly regular triangulation, see [18].

We are now in a position to state the following result.

THEOREM 6.10. *Under the assumptions of Theorem 6.8, we have*

$$||\rho_h^0\lambda - \lambda_h||_{1/q,p,\partial K} \le Ch^{2/p}\left(|\boldsymbol{\sigma}|_{1,\Omega} + |\mathrm{div}\boldsymbol{\sigma}|_{1,\Omega}\right) \qquad \forall K \in \mathcal{T}_h. \qquad (6.25)$$

Proof. Let $\boldsymbol{q}$ be any element of $W_q(\mathrm{div}; K)$ and $\overline{\boldsymbol{q}}_h \in \mathbb{RT}_0(K)$ be defined by (6.23) with $T = \boldsymbol{q} \cdot \boldsymbol{n}|_{\partial K}$ and such that $\boldsymbol{q}_h|_K = \overline{\boldsymbol{q}}_h$, $\boldsymbol{q}_h|_{K'} = 0 \,\forall\, K' \ne K$, $\forall K \in \mathcal{T}_h$. Subtracting the first equation of (4.1) from the first equation of (3.4), we get

$$\int_{\partial K} \overline{\boldsymbol{q}}_h \cdot \boldsymbol{n}(\lambda - \lambda_h)\, ds = \int_K (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \overline{\boldsymbol{q}}_h\, dx + \int_K (u - u_h)\, \mathrm{div}\overline{\boldsymbol{q}}_h\, dx,$$

which can be written as

$$\int_{\partial K} \overline{\boldsymbol{q}}_h \cdot \boldsymbol{n}(\rho_h^0\lambda - \lambda_h)\, ds = \int_K (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \overline{\boldsymbol{q}}_h\, dx + \int_K (P_K u - u_h)\, \mathrm{div}\overline{\boldsymbol{q}}_h\, dx.$$

Owing to the definition of $\overline{\boldsymbol{q}}_h$, using (6.24), (6.22) and the definition (3.2), we eventually get the estimate (6.25). $\square$

Since $p < 2$, estimate (6.25) can be regarded as a superconvergence property for λ_h.

To conclude our equivalence analysis, we show that

$$\mu_h^K = \boldsymbol{\sigma}_h^{DM} \cdot \boldsymbol{n}_K \qquad \forall K \in \mathcal{T}_h, \qquad (6.26)$$

where $\boldsymbol{\sigma}_h^{DM} \in \mathbb{RT}_0(K)$ is the solution computed by the dual-mixed method. We recall the following result proved in [28]

$$\boldsymbol{\sigma}_h \equiv \nabla u_h^* = \boldsymbol{\sigma}_h^{DM} + (x - x_{CG})\frac{f^K}{2} \qquad \forall K \in \mathcal{T}_h.$$

Substituting the above relation into $(4.1)_2$, integrating by parts and observing that $\operatorname{div}\boldsymbol{\sigma}_h^{DM} + f^K = 0$, we obtain

$$\int_{\partial K}(\mu_h^K - \boldsymbol{\sigma}_h^{DM}\cdot\boldsymbol{n}_K)v_h\,dx = 0 \qquad \forall v_h\in\mathbb{P}_1(K),$$

which clearly implies $\mu_h = \boldsymbol{\sigma}_h^{DM}\cdot\boldsymbol{n}_K$ on ∂K. This result shows that the values μ_h are actually the degrees of freedom of the variable $\boldsymbol{\sigma}_h^{DM}$, and, as such, they provide a simple procedure to recover a self-equilibrated stress field within each element satisfying interelement traction reciprocity.

6.4. Elliptic problem with variable coefficients. We come now to briefly addressing the extension of the DPG_0 method to the case of an elliptic model problem with variable coefficients. With this aim, we consider the Poisson problem

$$-\operatorname{div}(a(x)\nabla u) = f \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma, \tag{6.27}$$

where $a = a(x)$ is a symmetric positive definite matrix-valued function. The mixed form of (6.27) reads

$$-\operatorname{div}\boldsymbol{\sigma} = f \quad \text{in} \quad \Omega, \qquad \boldsymbol{\sigma} = a(x)\nabla u \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \Gamma. \tag{6.28}$$

In this case, the discrete formulation (4.1) becomes:

find $(u_h,\lambda_h;\boldsymbol{\sigma}_h,\mu_h)\in(X_h\times Y_h)$ such that

$$\begin{cases} \displaystyle\sum_{K\in\mathcal{T}_h}\left(\int_K a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{q}_h\,dx + \int_K u_h\operatorname{div}\boldsymbol{q}_h\,dx - \int_{\partial K}\lambda_h\boldsymbol{q}\cdot\boldsymbol{n}\,ds\right) = 0 & \forall\boldsymbol{q}_h\in W_h, \\[2ex] \displaystyle\sum_{K\in\mathcal{T}_h}\left(\int_K\boldsymbol{\sigma}_h\cdot\nabla v_h\,dx - \int_{\partial K}\mu_h\,v_h\,ds\right) = \int_\Omega fv_h\,dx & \forall v_h\in V_h. \end{cases}$$
$$\tag{6.29}$$

We let again u_h^* be a nonconforming approximation of the solution u of problem (6.27) satisfying (6.14). Taking $\boldsymbol{q}_h\in(\mathbb{P}_0(K))^2$ in $(6.29)_1$ and integrating by parts as done in the case $a(x)\equiv 1$, we obtain

$$\int_K\left(a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{q}_h - \nabla u_h^*\cdot\boldsymbol{q}_h\right)dx = 0 \qquad \forall K\in\mathcal{T}_h.$$

Upon introducing the *harmonic average* of $a(x)$ defined as $\widetilde{\alpha}_K := (\frac{1}{|K|}\int_K(a(x))^{-1}\,dx)^{-1}$, we immediately get the equivalence

$$\boldsymbol{\sigma}_h = \widetilde{\alpha}^{-1}\nabla u_h^* \qquad \forall K\in\mathcal{T}_h.$$

Taking $v_h\in W_{h,0}^{NC}$ in $(6.29)_2$ and using the previous relation yields

$$\sum_{K\in\mathcal{T}_h}\int_K\widetilde{\alpha}_K^{-1}\nabla u_h^*\cdot\nabla v_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K fv_h\,dx \qquad \forall v_h\in W_{h,0}^{NC}, \tag{6.30}$$

demonstrating that u_h^* turns out to be the nonconforming approximation of problem (6.28) *with harmonic averaging of the coefficient* $a(x)$. It is relevant to observe that u_h^* actually differs from the solution u_h^{NC} of the standard nonconforming approximation of problem (6.28), which would simply read

$$\sum_{K\in\mathcal{T}_h}\int_K\overline{a}_K\nabla u_h^{NC}\cdot\nabla v_h\,dx = \sum_{K\in\mathcal{T}_h}\int_K fv_h\,dx \qquad \forall v_h\in W_{h,0}^{NC},$$

where $\overline{a}_K := \frac{1}{|K|} \int_K a(x)\, dx$ is the usual average of $a(x)$ on K. In presence of strong variations of the coefficient a, the harmonic average is well known to provide superior accuracy and stability than the standard average (see [5, 1, 15, 30]).

7. Computer implementation of the DPG method. The object of the present section is to discuss an efficient computer implementation of the DPG_0 method. The main issue is to reduce the dimension of the algebraic linear system arising from (4.1). To start with, we consider the following system of 6 equations in 9 unknowns that arises from the contribution of each triangle in (4.1)

$$
\begin{bmatrix} A & B & C & \emptyset \\ D & \emptyset & \emptyset & E \end{bmatrix}
\begin{pmatrix} \boldsymbol{\sigma} \\ \mathbf{u} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix}
= \begin{pmatrix} \mathbf{0} \\ \mathbf{f} \end{pmatrix},
\tag{7.1}
$$

where the bold symbols represent the vectors of unknowns and given data, and $\mathbf{f}$ is the right hand side integral in $(4.1)_2$.

One the one hand, one can exploit the nature of hybrid formulations of the DPG method performing a static condensation of the internal variables in favor of the hybrid variables. Defining the new variable $\widehat{\boldsymbol{\sigma}} = [\boldsymbol{\sigma}, \mathbf{u}]^T$, system (7.1) can be rewritten as

$$
\begin{bmatrix} \mathcal{A} & C & \emptyset \\ \mathcal{D} & \emptyset & E \end{bmatrix}
\begin{pmatrix} \widehat{\boldsymbol{\sigma}} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix}
= \begin{pmatrix} \mathbf{0} \\ \mathbf{f} \end{pmatrix},
\qquad \text{with} \quad \mathcal{A} = [A,\, B], \qquad \mathcal{D} = [D,\, \emptyset].
\tag{7.2}
$$

The (3×3) matrix $\mathcal{A}$ is nonsingular, so that $\widehat{\boldsymbol{\sigma}}$ can be eliminated in favor of the sole edge variable $\boldsymbol{\lambda}$, obtaining the following reduced system of 3 equations in 6 interface unknowns

$$
-\mathcal{D}\mathcal{A}^{-1}C\boldsymbol{\lambda} + E\boldsymbol{\mu} = \mathbf{f} \qquad \forall K \in \mathcal{T}_h,
\tag{7.3}
$$

that is the algebraic form of (6.20). The matrix E is square and nonsingular because it emanates from the bilinear form $\int_{\partial K} \mu_h v_h\, ds$, $\mu_h \in R_0(\partial K)$, $v_h \in \mathbb{P}_1(K)$, which satisfies the discrete inf-sup condition. Therefore, we can eliminate $\boldsymbol{\mu}$ in favor of the sole unknown $\boldsymbol{\lambda}$

$$
\boldsymbol{\mu} = E^{-1}\mathcal{D}\mathcal{A}^{-1}C\boldsymbol{\lambda} + E^{-1}\mathbf{f} \qquad \forall K \in \mathcal{T}_h.
\tag{7.4}
$$

Enforcing the condition that the hybrid variable μ_h is single-valued on each internal edge yields a square symmetric and positive definite linear system for the sole unknown $\boldsymbol{\lambda}$ of dimension `Ni`.

On the other hand, one can instead exploit the DG nature of the DPG method eliminating the hybrid variables (counterpart of the numerical fluxes) in favor of the internal variables. Since both E and C are square nonsingular matrices, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ can be eliminated in favor of $\widehat{\boldsymbol{\sigma}}$, obtaining on each element $K \in \mathcal{T}_h$

$$
\boldsymbol{\lambda} = \mathcal{A}\widehat{\boldsymbol{\sigma}}, \qquad \boldsymbol{\mu} = E^{-1}(\mathbf{f} - \mathcal{D}\widehat{\boldsymbol{\sigma}}).
\tag{7.5}
$$

After some algebra, one sees that the first relation in (7.5) expresses λ_h on each element as a discrete Taylor expansion of u about the center of gravity of the element, while the second relation in (7.5) represents a *conservative finite volume-like* discretization of the equilibrium equation. Enforcing now the hybrid variables to be single-valued on each internal edge of the triangulation, we end up with a square nonsingular linear algebraic system of dimension `3NE` in the sole internal (and fully discontinuous over $\mathcal{T}_h$) unknown vector $\widehat{\boldsymbol{\sigma}}$.

8. Numerical results. In this section we present the results obtained applying the DPG_0 formulation to the numerical solution of two elliptic model problems.

8.1. Elliptic model problem no.1. We consider problem (2.1) on the unit square with $\Gamma \equiv \Gamma_D$, such that the exact solution is the "bubble function" $u = x(1-x)y(1-y)$, with the right hand side f computed accordingly. In Fig.8.1 (left) we show the computed convergence rates using four different unstructured meshes for the quantities: $\|u - u_h\|_{0,\Omega}$, ' $\circ$ ', $\|P_h u - u_h\|_{0,\Omega}$, '$\nabla$', $\|u - u_h^*\|_{0,\Omega}$, '$\square$' and $\|\rho_h^0 \lambda - \lambda_h\|_{-1/2,h}$, '$*$', while in Fig.8.1 (right) we show the computed convergence rates for the quantities: $\|\sigma - \sigma_h\|_{0,\Omega}$, '$\square$' and $\|\rho_h^0 \mu - \mu_h\|_{-1/2,h}$, '$\circ$'. The computed errors are in agreement with the theoretical estimates of Sect.6.2.
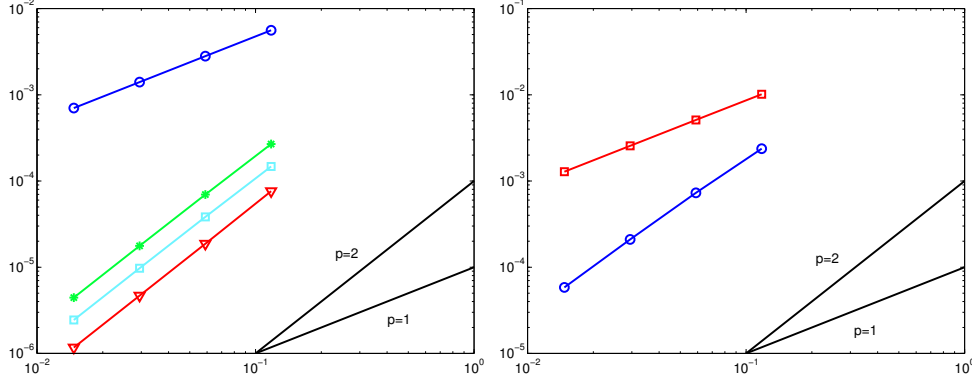

FIGURE 8.1. *Error norms for the elliptic model problem no.1.*

8.2. Elliptic model problem no.2. We study the problem of a 2D steady flow system in a porous medium modeled by the Darcy's law [31]: find the hydraulic potential P and the associated velocity field $q = \kappa \nabla P$, where κ is the hydraulic conductivity tensor such that

$$\begin{cases} -\operatorname{div} q = 0 \quad \text{in} \quad \Omega, \qquad q = \kappa \nabla P \quad \text{in} \quad \Omega, \\ P = P_D \quad \text{on} \quad \Gamma_D, \qquad q \cdot n = q_N \quad \text{on} \quad \Gamma_N. \end{cases} \tag{8.1}$$

In Fig.8.2 (left) we show the computational domain, the boundary conditions and the piecewise constant values of κ which are seen to attain strong variations on Ω. In Fig.8.2 (right), we show the velocity field represented as a $\mathbb{RT}_0$ finite element function reconstructed over $\overline{\Omega}$ from the computed values μ_h as in (6.20). The continuity of the normal component of the velocity field across interelement edges is a crucial property when computing the flow streamlines (see [31] for a discussion of this issue).

9. Conservation properties of the DPG method. The present section is aimed at enlightening through a numerical example the conservation properties of the DPG method. We observe that:

1. integral global conservation is achieved by taking in (4.1) $\tau_h \equiv [1,1]^T$ and $v_h \equiv 1$ in Ω, respectively;
2. integral local conservation is achieved by taking in (4.1) $\tau_h = [1,1]^T$ and $v_h = 1$, in any subdomain $\mathcal{S} \subseteq \Omega$ and zero elsewhere respectively.

In the standard Continuous Galerkin (CG) method neither the first nor the second choice for the test function are admissible [26]. Recovering fluxes that enjoy the desired conservation properties requires a post-processing procedure, thus adding additional computational cost to
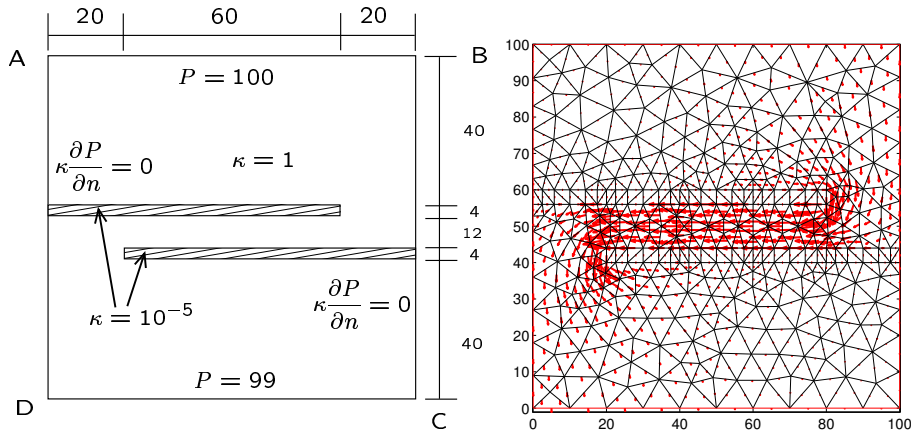
FIGURE 8.2. *Problem setting for the flow in a porous medium (left) and associated velocity field (right).*

the basic CG discretization. Moreover, if for example a nodal flux approach is used as in [26], overshoots and undershoots appear when a node coincide with an endpoint of the interface, since there the flux is artificially enforced to be continuous.

To numerically assess these concepts, we solve the Poisson equation on the domain $\Omega = [0, \pi] \times [0, \pi]$ with $u = 0$ on $\Gamma = \partial \Omega$ and $f = 1$. To test local conservation properties, we split Ω into the subdomains $\Omega_1 = [0, \frac{3}{4}\pi] \times [0, \pi]$, $\Omega_2 = [\frac{3}{4}\pi, \pi] \times [0, \pi]$ such that $\Omega = \Omega_1 \cup \Omega_2$ and with boundaries Γ_1 and Γ_2, respectively. From the exact solution of the problem (see [39]), we compute the fluxes $\boldsymbol{\sigma} \cdot \boldsymbol{n} = \nabla u \cdot \boldsymbol{n}$ on Γ (Fig.9.1), Γ_1 (Fig.9.2, left), and Γ_2 (Fig.9.2, right) and we compare them with the numerical fluxes obtained from the displacement field u_{CG} of a piecewise linear *CG* approximation and with the field μ_h obtained from the *DPG*$_0$ (using the same grid). The *DPG* fluxes are accurate and do not exhibit spurious oscillations at the endpoints of the boundaries. Moreover, the global equilibrium $\int_\Omega f \, dx + \int_\Gamma \boldsymbol{\sigma} \cdot \boldsymbol{n} \, ds = 0$ is verified to machine precision by the *DPG* approximation.

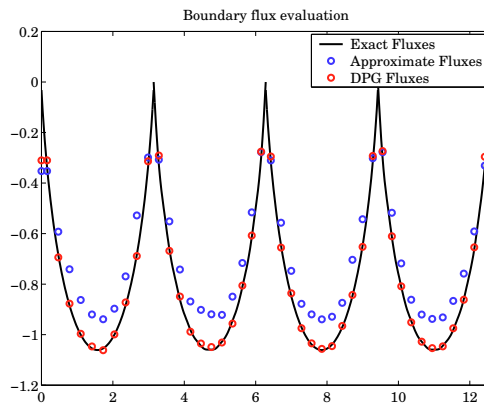FIGURE 9.1. *Exact fluxes on* Γ *compared with the fluxes computed by the* CG *method with no post-processing and with the interface field* μ_h *computed by the* DPG$_0$ *method.*

10. Conclusions. In this article we have presented the Discontinuous Petrov-Galerkin method (DPG) for the finite element discretization scheme of second order elliptic boundary
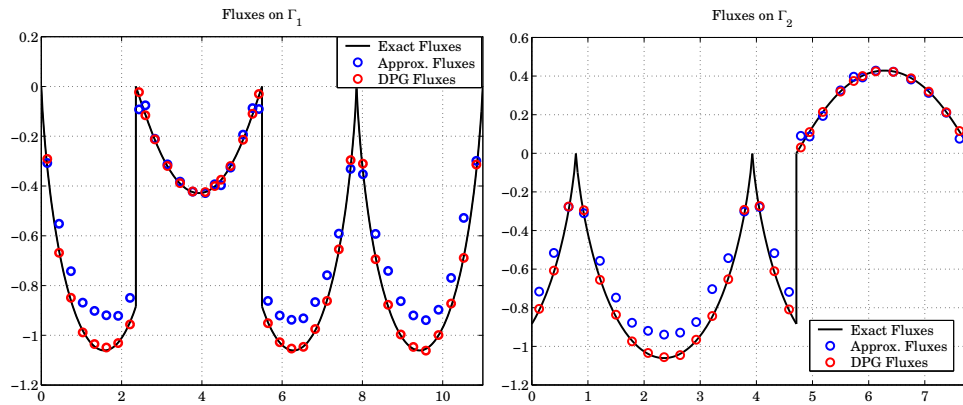
FIGURE 9.2. *Exact fluxes on the boundaries* Γ_1 *(left) and* Γ_2 *(right) compared with the fluxes computed by the* CG *method with no post-processing and interface fields computed by the* DPG_0 *method.*

value problems. A stability and convergence analysis of the novel formulation has been carried out and numerical results have been shown to validate the computational performance of the novel formulation. Introducing the DPG formulation has established a clear connection between mixed-hybrid and Discontinuous Galerkin methods. This result is the motivation and starting point for future investigations and applications of the novel scheme to deal with more general problems.

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