Spectral methods

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ABSTRACT

Spectral methods represent a family of methods for the numerical approximation of partial differential equations. Their common denominator is to rely on high order polynomial expansions, notably trigonometric polynomials for periodic problems, orthogonal Jacobi polynomials for non periodic boundary value problems. They have the potential of providing high rate of convergence when applied to problems with regular data. They can be regarded as members of the broad family of (generalized) Galerkin methods with numerical evaluation of integrals based on Gaussian nodes. In a first part we introduce the methods on a computational domain of simple shape, analyze their approximation properties as well as their algorithmic features. Next we address the issue of how these methods can be extended to more complex geometrical domains by retaining their distinctive approximation properties.

1. Introduction

In the past three decades, spectral methods have evolved from their noble ancestor, the Fourier method based on trigonometric expansions, through the more flexible Galerkin method with Gaussian integration all the way maintaining their most distinguished feature: the very high

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rate of convergence.

They are numerical methods for solving boundary-value problems for partial differential equations.

For reader's convenience, we will gradually approach this subject by first addressing the case of periodic problems, where the so-called Fourier methods are used. Then we turn to non-periodic problems and address collocation approximations based on algebraic polynomial expansions. The different concepts are first explained on one-dimensional intervals. Then we address the case of a square or a cube or a simplex, and finally the case of more complex geometrical domains. We illustrate the case of elliptic equations, Stokes and Navier-Stokes equations, then advection equations and conservation laws.

2. Fourier methods

In their early stage, spectral methods were designed to approximate the periodic solution of partial differential equations by a truncated Fourier series. If

$$u(x) = \sum_{k=-\infty}^{+\infty} u_k \varphi_k(x), \qquad \varphi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

is the unknown solution, the numerical solution is sought in the form

$$u_N(x) = \sum_{k=-N/2}^{N/2-1} u_{N,k} \varphi_k(x)$$

where N is an (even) integer which dictates the size of the approximate problem. Note that the unknowns are represented by the Fourier coefficients $\{u_{N,k}\}$. This approximation has a potentially tremendous high quality, since for all $0 \leq k \leq s$, there exists a positive constant $C_{k,s}$ such that

$$\inf_{v_N \in S_N} \|u - v_N\|_{H^k(0,2\pi)} \leq C_{k,s} N^{k-s} \|u\|_{H^s_p(0,2\pi)} \tag{1}$$

provided u belongs to the Sobolev space $H^s_p(0,2\pi)$ of periodic functions having s derivatives in $L^2(0,2\pi)$. Here $S_N = \text{span}\{\varphi_k \mid N/2 \leq k \leq N/2-1\}$ denotes the space of the finite Fourier series of order N. This abstract approximation property can be reflected into a corresponding

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error estimate for the difference $u - u_N$. Actually, in the most classical approach, the spectral Fourier method consists of approximating a given PDE, say

$$Lu(x) = f(x), \quad x \in \Omega \tag{2}$$

with $\Omega = (0, 2\pi)$, where L is a differential operator and f is a given $2\pi-$periodic function, by taking its L^2-projection upon the subspace S_N, i.e.,

$$\text{find } u_N \in S_N \quad s.t. \qquad (Lu_N - f, \varphi) = 0 \qquad \forall \varphi \in S_N. \tag{3}$$

Here $(v, w) = \int_\Omega v\overline{w}$ denotes the $L^2(\Omega)$ inner product. If L is a constant coefficient operator, this yields an embarrassingly simple problem. As a matter of fact, due to the L^2-orthogonality of the functions $\{\varphi_k\}$, i.e., $(\varphi_k, \varphi_m) = \delta_{km}$, $\forall k, m \in \mathbb{Z}$, equations (3) yield, after Fourier-transforming the residual $Lu_N - f$, the following set of explicit equations for the unknowns:

$$\lambda_k u_{N,k} = \hat{f}_k, \quad -N/2 \le k \le N/2 - 1 \tag{4}$$

where $\hat{f}_k = (f, \varphi_k)$ is the $k-th$ Fourier coefficient of f while λ_k is the $k-th$ eigenvalue of L. For instance, if

$$L = -D(\alpha D) + \beta D + \gamma I \qquad (\text{with } \alpha, \beta, \gamma \text{ constants}) \tag{5}$$

where D denotes differentiation with respect to x and I the identity, then $\lambda_k = \alpha k^2 + i\beta k + \gamma$.

Moreover, in this special case u_N coincides indeed with the truncated of order N of the Fourier series of the exact solution u, thus the bound (1) (with $v_N = u_N$) provides an error estimate.

However, the one that we have just described is an overly fortunate circumstance. Should indeed some of the coefficients α, β or γ be functions of x (or, even worse, of u, yielding a nonlinear equation), then convolution sums between the unknown frequency coefficients $\{u_{N,k}\}$ and the Fourier coefficients of α, β, γ will arise, and the diagonal structure of equations (4) would be lost. A variant of the projection approach (3) can be based on evaluating the convolution sums by *discrete Fourier transform*. This requires introducing equally spaced nodes, $x_j = \pi j/N$, $j = 0, \ldots, N-1$, then replacing the exact integrals in (3) by numerical integration; the resulting scheme is

$$\text{find } u_N^c \in S_N \quad s.t. \qquad (Lu_N^c - f, \varphi)_N = 0 \qquad \forall \varphi \in S_N \tag{6}$$

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where $(v,w)_N = \frac{2\pi}{N}\sum_{j=0}^{N-1} v(x_j)\overline{w}(x_j)$ is the Gaussian approximation of the scalar product (v,w). The exactness of the Gaussian approximation on S_N, namely the property that $(v,w)_N = (v,w), \forall v,w \in S_N$, enables us to recover from (6) a *collocation* formulation $L_N u_N^c = f$ at all nodes x_j, where L_N is obtained from L by replacing each derivative by the corresponding so-called *pseudo-spectral derivative*. This means that for any smooth function v, Dv is replaced by $D(I_N v)$, where

$$I_N v \in S_N, \quad I_N v(x) = \sum_{k=-N/2}^{N/2-1} v_k^* \varphi_k(x), \quad v_k^* = (v,\varphi_k)_N, \tag{7}$$

is the interpolant of v at the nodes $\{x_j\}$.

The interpolation error satisfies, for $0 \le k \le s,\ s \ge 1$,

$$\|v - I_N v\|_{H^1(0,2\pi)} \le C_s N^{k-s} \|v\|_{H_p^s(0,2\pi)}. \tag{8}$$

and so does the collocation error $u - u_N^c$. A consequence of (8) (when $k=1$) is that the error on the pseudo-spectral derivative $\|v' - (I_N v)'\|_{L^2(0,2\pi)}$ decreases like a constant time N^{1-s} provided that $v \in H_p^s(0,2\pi)$ for some $s \ge 1$. Indeed, one can even prove that

$$\|Dv - D(I_N v)\|_{L^2(0,2\pi)} \le C(\eta) N e^{-N\eta/2} \quad \forall 0 < \eta < \eta_0$$

provided that v is analytic in the strip $|\mathrm{Im}\, z| < \eta_0$. This exponential rate of convergence is often referred to as *spectral convergence*, as it is a distinguishing feature of spectral methods.

There is, however, a major difference between the collocation approach and the L^2-projection approach (3). In the latter, the unknowns are the frequency coefficients $\{u_{N,k}\}$ of u_N, whereas in the collocation approach one looks for the nodal values $\{u_j = u_N^c(x_j)\}$ of u_N^c. These values may be interpreted as the coefficients of u_N^c with respect to the trigonometric *Lagrange basis* associated with the nodes x_j; indeed, observing that $u_N^c = I_N u_N^c$, using (7) and exchanging summations over k and j, one gets

$$u_N^c(x) = \sum_{j=0}^{N-1} u_j \frac{2\pi}{N} \sum_{k=-N/2}^{N/2-1} \overline{\varphi_k(x_j)}\varphi_k(x) = \sum_{j=0}^{N-1} u_j \psi_j(x)$$

where $\psi_j \in S_N$ satisfies $\psi_j(x_m) = \delta_{j,m},\ 0 \le j,m \le N-1$.

The same approach can be pursued for boundary-value problems set on multidimensional intervals $\Omega = (0,2\pi)^d$, $d = 2,3$, by tensorizing basis functions and collocation nodes.

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3. Algebraic Polynomial Expansion

When a boundary-value problem with non-periodic data (of Dirichlet, Neumann or mixed type) has to be solved numerically, the trigonometric expansion is no longer adequate to guarantee high order of accuracy. Then Jacobi orthogonal polynomials are used to provide orthogonal basis for the approximation space.

The finite dimensional space $\mathbb{P}_N$ is now made of algebraic polynomials of degree less than or equal to N.

The historical approach, inspired by the Fourier method, aimed at expanding the approximate solution with respect to a basis of orthogonal polynomials

$$u_N(x) = \sum_{k=0}^{N} u_{N,k} p_k(x) \tag{9}$$

where $u_{N,k}$ represent now the unknown frequency coefficients.

The matter of choice were the Chebyshev polynomials, $p_k(x) = T_k(x) = \cos(k\theta)$, $\theta = \cos^{-1}(x)$, $-1 \leq x \leq 1$, owing to their analogy with trigonometric polynomials. Since the Chebyshev basis does not necessarily match the boundary requirement (as $T_k(1) = 1$, $T_k(-1) = (-1)^k$, $\forall k \geq 0$), one device consists of projecting the equation residual on the reduced space $\mathbb{P}_{N-2}$, enforcing the boundary conditions afterward. For instance, for a Dirichlet boundary-value problem like (2), where now $\Omega = (-1, 1)$, and Dirichlet boundary conditions $u(-1) = u_-$, $u(+1) = u_+$, the solution (9) is required to satisfy:

$$(Lu_N - f, T_k)_\omega = 0, \quad 0 \leq k \leq N-2, \quad u_N(-1) = u_-,\ u_N(+1) = u_+ \,. \tag{10}$$

This approach was termed the *Lanczos-Tau method*. The symbol $(u, v)_\omega = \int_{-1}^{1} uv\,\omega dx$, is the so-called *weighted scalar product* with respect to the Chebyshev weight function $\omega(x) = (1 - x^2)^{-1/2}$, $-1 < x < 1$. The weighted scalar product is used, instead of the more traditional one $(\cdot, \cdot)$, in order to take advantage (to the highest possible extent) of the Chebyshev orthogonality,

$$(T_k, T_m)_\omega = 0 \quad \text{if} \quad k \neq m,\ (T_0, T_0)_\omega = \pi,\ (T_k, T_k)_\omega = \frac{\pi}{2} \quad \forall k \geq 1.$$

When L has constant coefficients, the Lanczos-Tau problem (10) yields an algebraic system for the frequency coefficients $\{u_{N,k}\}$ with a structured matrix for which efficient diagonalization

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algorithms can be devised, a circumstance which is also featured by the multidimensional problems which are generated by tensorization.

However, this is not general enough as this structure gets lost for more general kind of differential operators. A more flexible approach (in analogy with what was done in the Fourier case) consists of adopting a nodal representation of u_N at selected Gauss-Lobatto nodes $x_j = \cos\frac{\pi j}{N}$, $j = 0, \ldots, N$, then looking for a standard Galerkin approximation with integrals replaced by Gauss-Lobatto integration:

$$(u, v)_N = \sum_{j=0}^{N} \alpha_j u(x_j) v(x_j) \tag{11}$$

where $\alpha_j = \frac{\pi}{N}$ for $j = 1, \ldots, N-1$, $\alpha_0 = \alpha_N = \frac{\pi}{2N}$ are the quadrature coefficients.

Should we still consider the baby Dirichlet boundary-value problem for the operator L introduced in (5), the corresponding discrete problem would read:

$$\begin{aligned} &\text{find } u_N \in \mathbb{P}_N, \quad u_N(-1) = u_-,\ u_N(1) = u_+,\ s.t. \\ &\qquad (\alpha u_N', v_N')_N + (\beta u_N', v_N)_N + (\gamma u_N, v_N)_N = (f, v_N)_N, \quad \forall v_N \in \mathbb{P}_N^0, \end{aligned} \tag{12}$$

where now $\mathbb{P}_N^0 = \{v_N \in \mathbb{P}_N |\, v_N(-1) = v_N(1) = 0\}$. This time, however, the expansion is made in terms of the nodal Lagrangian basis at Gauss-Lobatto nodes, that is using instead of (9)

$$u_N(x) = \sum_{k=0}^{N} u_j \psi_j(x)$$

where ψ_j is the unique algebraic polynomial of degree N s.t. $\psi_j(x_i) = \delta_{ij}$, $\forall i, j = 0, \ldots, N$.

One may show that

$$\psi_j(x) = \frac{-1}{N(N+1)} \cdot \frac{(1-x^2)}{(x-x_j)} \cdot \frac{L_N'(x)}{L_N(x_j)}, \quad j = 0, \ldots, N. \tag{13}$$

The same approximation framework can be set up by replacing the Chebyshev polynomials with the Legendre polynomials $\{L_k,\ k = 0, 1, \ldots\}$, which are orthogonal with respect to the traditional L^2-scalar product (otherwise said with respect to the weight function $\omega = 1$).

The approximate problem still reads like (12), however this time the nodes $\{x_j\}$ and the coefficients $\{\alpha_j\}$ are those of the (Legendre) Gauss-Lobatto integration.

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A similar approach can be undertaken in several dimensions. For instance, consider a second order elliptic boundary-value problem

$$\begin{cases} Lu = f & \text{in } \Omega = (-1,1)^d, \quad d = 2,3 \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{14}$$

together with its weak form

$$\text{find } u \in V = H_0^1(\Omega): \quad a(u,v) = (f,v) \quad \forall v \in V. \tag{15}$$

The bilinear form $a : V \times V \to \mathbb{R}$ is associated with the operator L; for instance, if

$$Lu = -\text{div}(\alpha \nabla u) + \boldsymbol{\beta} \cdot \nabla u + \gamma u, \qquad \text{with } \alpha \geq \alpha_0 > 0 \tag{16}$$

then $a(u,v) = \int_\Omega (\alpha \nabla u \cdot \nabla v + \boldsymbol{\beta} \cdot \nabla u \, v + \gamma u v)$.

Upon introducing the tensorized Legendre Gauss-Lobatto quadrature nodes and coefficients $\mathbf{x_k} = (x_{k_1}, \ldots, x_{k_d})$ and $\alpha_\mathbf{k} = \alpha_{k_1} \ldots \alpha_{k_d}$, $(k_i = 0, \ldots, N)$, the Legendre Galerkin approximation of (15) with numerical integration (GNI) becomes

$$\text{find } u_N \in V_N = \mathbb{P}_N^0: \quad a_N(u_N, v_N) = (f, v_N)_N \quad \forall v_N \in V_N \tag{17}$$

where $\mathbb{P}_N$ is now the set of polynomials of degree $\leq N$ with respect to each of the independent variables, and $\mathbb{P}_N^0$ is its subspace made of those polynomials vanishing at $\partial\Omega$. Moreover

$$(u,v)_N = \sum_\mathbf{k} \alpha_\mathbf{k} u(\mathbf{x_k}) v(\mathbf{x_k}) \tag{18}$$

is the Gauss-Lobatto quadrature formula which approximates the scalar product (u,v), while a_N is the discrete bilinear form which is obtained from a by replacing each scalar product $(\cdot,\cdot)$ with $(\cdot,\cdot)_N$. Owing to the property that the quadrature formula (18) has degree of exactness $2N-1$, the Galerkin numerical integrated problem (17) can still be interpreted as a collocation method. Indeed, it follows from (17) that $L_N u_N = f$ at all internal nodes $\mathbf{x_k}$, where L_N is the approximation of L obtained by replacing each exact derivative by the derivative of the interpolant I_N at the Gauss-Lobatto nodes. The interpolation operator I_N is defined as follows: $I_N v(\mathbf{x_k}) = v(\mathbf{x_k})$, $I_N v \in (\mathbb{P}_N)^d$, for all $v \in C^0(\overline{\Omega})$. Then, the operator approximating (16) is

$$L_N u_N = -\text{div}(I_N(\alpha \nabla u_N)) + \boldsymbol{\beta} \cdot \nabla u_N + \gamma u_N.$$

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Existence and uniqueness of the solution of (18) follow from the assumption that $a_N(\cdot,\cdot)$ is a uniformly coercive form on the space $V \times V$, i.e.,

$$\exists \alpha^* > 0 \text{ independent of } N \text{ s.t.} \quad a_N(v_N, v_N) \geq \alpha^* \|v_N\|^2_{H^1(\Omega)}, \qquad \forall v_N \in V_N. \tag{19}$$

This is the case for the problem at hand if, e.g., $\boldsymbol{\beta}$ is constant and γ non negative.

The convergence analysis of the GNI approximation can be carried out by invoking the Strang Lemma for generalized Galerkin approximation. Precisely, the following error estimate holds:

$$\begin{aligned} \|u - u_N\| \quad \leq \quad & \inf_{w_N \in V_N} \left[\left(1 + \frac{M}{\alpha^*}\right) \|u - w_N\| + \frac{1}{\alpha^*} \sup_{v_N \in V_N \setminus \{0\}} \frac{a(w_N, v_N) - a_N(w_N, v_N)}{\|v_N\|} \right] \\ & + \frac{1}{\alpha^*} \sup_{v_N \in V_N \setminus \{0\}} \frac{(f, v_N) - (f, v_N)_N}{\|v_N\|} \end{aligned}$$

where $\|\cdot\|$ is the norm of $H^1(\Omega)$ and M is the constant of continuity of the bilinear form $a(\cdot,\cdot)$.

Three sources contribute to the approximation error:

- the best approximation error, which can be immediately bounded by taking $w_N = I_{N-1}u$:

$$\inf_{w_N \in V_N} \|u - w_N\| \leq \|u - I_{N-1}u\|$$

- the error on the numerical quadrature, which can be bounded as follows

$$\sup_{v_N \in V_N \setminus \{0\}} \frac{(f, v_N) - (f, v_N)_N}{\|v_N\|} \leq C_2 \left(\|f - I_N f\|_{L^2(\Omega)} + \|f - P_{N-1} f\|_{L^2(\Omega)} \right)$$

  where $P_{N-1}f$ is the truncated Legendre series of f;

- the error on the bilinear form, on its hand, is less immediate to estimate. However, having chosen $w_N = I_{N-1}u$ which is a polynomial of degree $N-1$, using the degree of exactness of the quadrature formula and assuming that the coefficients of the operator are constant, one easily checks that $a(w_N, v_N) - a_N(w_N, v_N) = 0$, i.e., this error is actually null. If the coefficients are non-constant, one can control it in terms of the interpolation error measured in $H^1(\Omega)$.

We can conclude by taking advantage of the optimality of the truncation error in the L^2-norm and that of the interpolation error in both the L^2- and H^1- norm:

$$\forall f \in H^r(\Omega),\ r \geq 0,\ \|f - P_N f\|_{L^2(\Omega)} \leq C_3 N^{-r} \|f\|_{H^r(\Omega)}$$

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$$\forall g \in H^s(\Omega),\ s \geq 1,\ N\|g - I_N g\|_{L^2(\Omega)} + \|g - I_N g\| \leq C_4 N^{1-s}\|g\|_{H^s(\Omega)}.$$

Thus we obtain that

$$\|u - u_N\| \leq C_5 \left(N^{-r}\|f\|_{H^r(\Omega)} + N^{1-s}\|u\|_{H^s(\Omega)}\right)$$

provided u and f have the requested regularity.

A few comments on the implementation of the method are in order. The algebraic system associated with (17) reads $A\mathbf{u} = \mathbf{f}$, where $a_{ij} = a_N(\psi_j, \psi_i)$, $f_i = (f, \psi_i)_N$, $\mathbf{u} = (u_i)$, $u_i = u_N(\mathbf{x}_i)$ and $\{\psi_j\}$ denote the Lagrangian basis functions of S_N^d associated with all the nodal points $\{\mathbf{x}_j\}$. The matrix A, which is non-singular whenever (19) is fulfilled, is ill conditioned: indeed, there exist two constants C_1, C_2 s.t.

$$C_1 N^3 \leq \mathrm{cond}(A) \leq C_2 N^3$$

where $\mathrm{cond}(A)$ is the (spectral) condition number of A. The use of a preconditioned iterative procedure (e.g., the conjugate gradient when $\boldsymbol{\beta} = \mathbf{0}$, or a Krylov iteration otherwise) is mandatory. A possible preconditioner is given by the diagonal of A. This yields a preconditioned system whose condition number behaves like a constant times N^2. A more drastic improvement would be achieved by taking as a preconditioner the matrix associated with the (piecewise-linear) finite element discretization of the operator (16) at the same Legendre Gauss-Lobatto nodes. This is an optimal preconditioner as the condition number of the preconditioned system becomes independent of N.

4. Algebraic Expansions on Triangles

Spectral methods for multi-dimensional problems rely their efficiency on the tensor product structure of the expansions they use. This feature naturally suggests to set the methods on patches of Cartesian products of intervals, such as squares or cubes, possibly after applying a smooth mapping. On the other hand, triangles, tetrahedra, prisms and similar figures allow one to handle complex geometries in a more flexible way. So, a natural question arises: can one match the advantages of a tensor product structure with those of a triangular geometry?

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A positive answer to this question was given by M. Dubiner, who introduced the concepts of collapsed Cartesian coordinate systems and warped tensor products. We now describe this approach in 2D, pointing to the book by Karniadakis and Sherwin (1999) for the 3D extensions. Let us introduce the reference triangle $\mathcal{T} = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ -1 < x_1, x_2; \ x_1 + x_2 < 0\}$, as well as the reference square $\mathcal{Q} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \ : \ -1 < \xi_1, \xi_2 < 1\}$. The mapping

$$(x_1, x_2) \mapsto (\xi_1, \xi_2), \qquad \xi_1 = 2\frac{1+x_1}{1-x_2} - 1, \quad \xi_2 = x_2 \tag{20}$$

is a bijection between $\mathcal{T}$ and $\mathcal{Q}$. Its inverse is given by

$$(\xi_1, \xi_2) \mapsto (x_1, x_2), \qquad x_1 = \frac{1}{2}(1+\xi_1)(1-\xi_2), \quad x_2 = \xi_2.$$

Note that the mapping $(x_1, x_2) \mapsto (\xi_1, \xi_2)$ sends the ray in $\mathcal{T}$ issuing from the upper vertex $(-1, 1)$ and passing through the point $(x_1, -1)$ into the vertical segment in $\mathcal{Q}$ of equation $\xi_1 = x_1$. Consequently, the transformation becomes singular at the upper vertex, although it stays bounded therein. The Jacobian of the inverse transformation is given by $\dfrac{\partial(x_1, x_2)}{\partial(\xi_1, \xi_2)} = \dfrac{1}{2}(1 - \xi_2)$. We term (ξ_1, ξ_2) the *collapsed Cartesian coordinates* of the point on the triangle whose regular Cartesian coordinates are (x_1, x_2).

Denote by $\{P_k^{(\alpha,\beta)}(\xi)\}$ the family of Jacobi polynomials of increasing degree $k \geq 0$, which form an orthogonal system with respect to the measure $(1-\xi)^\alpha(1+\xi)^\beta d\xi$ in $(-1, 1)$ (note that $P_k^{(0,0)}(\xi)$ is the Legendre polynomial $L_k(\xi)$ introduced in the previous section). For $\mathbf{k} = (k_1, k_2)$, define the *warped tensor product* function on $\mathcal{Q}$

$$\Phi_{\mathbf{k}}(\xi_1, \xi_2) := \Psi_{k_1}(\xi_1)\Psi_{k_1,k_2}(\xi_2), \tag{21}$$

$$\text{where} \qquad \Psi_{k_1}(\xi_1) := P_{k_1}^{(0,0)}(\xi_1), \qquad \Psi_{k_1,k_2}(\xi_2) := (1-\xi_2)^{k_1} P_{k_2}^{(2k_1+1,0)}(\xi_2) \tag{22}$$

which is a polynomial of degree k_1 in ξ_1 and $k_1 + k_2$ in ξ_2. By applying the mapping (20), one obtains the function defined on $\mathcal{T}$

$$\varphi_{\mathbf{k}}(x_1, x_2) := \Phi_{\mathbf{k}}(\xi_1, \xi_2) = P_{k_1}^{(0,0)}(2\frac{1+x_1}{1-x_2} - 1)(1-x_2)^{k_1} P_{k_2}^{(2k_1+1,0)}(x_2). \tag{23}$$

It is easily seen that $\varphi_{\mathbf{k}}$ is a polynomial of global degree $k_1 + k_2$ in the variables x_1, x_2.

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Furthermore, thanks to the orthogonality of Jacobi polynomials, one has for $\mathbf{k} \neq \mathbf{h}$

$$\int_{\mathcal{T}} \varphi_{\mathbf{k}}(x_1, x_2)\varphi_{\mathbf{h}}(x_1, x_2)\, dx_1 dx_2 =$$
$$= \frac{1}{2}\int_{-1}^{1} P_{k_1}^{(0,0)}(\xi_1)P_{h_1}^{(0,0)}(\xi_1)\, d\xi_1 \int_{-1}^{1} P_{k_2}^{(2k_1+1,0)}(\xi_2)P_{h_2}^{(2h_1+1,0)}(\xi_2)(1-\xi_2)^{k_1+h_1+1}\, d\xi_2 = 0.$$

We conclude that the set $\{\varphi_{\mathbf{k}} \ : \ 0 \leq k_1, k_2;\ k_1 + k_2 \leq N\}$ is an orthogonal basis of the space $\mathcal{P}_N(\mathcal{T})$ of the polynomials of global degree $\leq N$ in the variables x_1, x_2.

While orthogonality simplifies the structure of mass and stiffness matrices, it makes the enforcement of boundary conditions, or matching conditions between elements, uneasy. To overcome this difficulty, it is possible to modify the previous construction, by building a new basis, say $\{\varphi_{\mathbf{k}}^m\}$, made of boundary functions (3 vertex functions plus $3(N-1)$ edge functions) and internal functions (bubbles). Each basis function retains the same 'warped tensor product' structure as above. Indeed, it is enough to replace in one dimension the Jacobi basis $P_k^{(\alpha,0)}(\xi)$ (with $\alpha = 0$ or $2k+1$) by the modified basis given by the two boundary functions $\dfrac{1+\xi}{2}$ and $\dfrac{1-\xi}{2}$ and the $N-1$ bubbles $\dfrac{1+\xi}{2}\dfrac{1-\xi}{2}P_{k-1}^{(\alpha,1)}(\xi)$, $k = 1, \ldots, N-1$. These univariate functions are then combined as in (21) to form the two-dimensional basis.

With such basis at hand, one can discretize a boundary value problem by the Galerkin method with numerical integration (GNI). To this end, one needs a high precision quadrature formula on $\mathcal{T}$. Since

$$\int_{\mathcal{T}} f(x_1, x_2)\, dx_1 dx_2 = \frac{1}{2}\int_{-1}^{1} d\xi_1 \int_{-1}^{1} F(\xi_1, \xi_2)(1-\xi_2)\, d\xi_2,$$

it is natural to use a tensor product Gaussian formula in $\mathcal{Q}$ for the measure $d\xi_1(1-\xi_2)d\xi_2$. This is obtained by tensorizing a $(N+1)$-point Gauss-Lobatto formula for the measure $d\xi_1$ with a N-point Gauss-Radau formula for the measure $(1-\xi_2)d\xi_2$ with $\xi_2 = -1$ as integration knot (excluding the singular point $\xi_2 = 1$ from the integration knots makes life easier in the construction of the matrices). The resulting formula is exact for all polynomials in $\mathcal{Q}$ of degree $\leq 2N-1$ in each variable ξ_1, ξ_2; hence, in particular, it is exact for all polynomials in $\mathcal{T}$ of global degree $\leq 2N-1$ in the variables x_1, x_2. Note, however, that the number of quadrature nodes in $\mathcal{T}$ is $N(N+1)$, whereas the dimension of $\mathcal{P}_N(\mathcal{T})$ is $\frac{1}{2}(N+1)(N+2)$; thus, no basis in $\mathcal{P}_N(\mathcal{T})$ can be the Lagrange basis associated with the quadrature nodes. This means that

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the GNI method based on the quadrature formula described above cannot be equivalent to a collocation method at the quadrature points.

Finally, we observe that the GNI mass and stiffness matrices on $\mathcal{T}$ can be efficiently built by exploiting the tensor product structure of both the basis functions and the quadrature points, through the sum-factorization technique.

5. Stokes and Navier-Stokes equations

Spectral methods are very popular among the community of fluid-dynamicists. Owing to their excellent approximation properties, spectral methods can in fact provide very accurate simulations of complex flow patterns. However, special care is needed for the treatment of the incompressibility constraint. With the aim of simplification, let us first address the linear Stokes equations

$$\begin{cases} -\nu\triangle\mathbf{u} + \mathrm{grad}\, p = \mathbf{f} & \text{in } \Omega\,(\subset \mathbb{R}^d,\ d = 2,3) \\ \mathrm{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \end{cases} \tag{24}$$

where $\nu > 0$ is the kinematic fluid viscosity, $\mathbf{u}$ the fluid velocity, p the fluid pressure and $\mathbf{f}$ the vector of volumic forces.

A natural spectral GNI discretization reads: find $\mathbf{u}_N \in V_N$, $p_N \in Q_N$ s.t.

$$\begin{cases} ((\nu\nabla\mathbf{u}_N, \nabla\mathbf{v}_N))_N - (p_N, \mathrm{div}\mathbf{v}_N)_N = ((\mathbf{f}, \mathbf{v}_N))_N & \forall \mathbf{v}_N \in V_N \\ -(q_N, \mathrm{div}\mathbf{v}_N)_N = 0 & \forall q_N \in Q_N \end{cases} \tag{25}$$

where $(\cdot,\cdot)_N$ is the discrete Gauss-Lobatto scalar product (18) while $((\cdot,\cdot))_N$ denotes its generalization to the case of vector functions. Moreover, $V_N = (\mathbb{P}_N^0)^d$ while Q_N is a polynomial space that needs to be chosen conveniently in order that the following Brezzi condition be satisfied:

$$\exists \beta_N > 0 :\ \forall q_N \in Q_N,\ \exists \mathbf{v}_N \in V_N \ s.t.\ (q_N, \mathrm{div}\mathbf{v}_N)_N \geq \beta_N \|q_N\|_{L^2(\Omega)} \|\mathbf{v}_N\|_{H^1(\Omega)}. \tag{26}$$

The lack of this condition, i.e., the existence of non-constant pressures $q_N \in Q_N$ such that $(q_N, \mathrm{div}\mathbf{v}_N)_N = 0, \forall \mathbf{v}_N \in V_N$, implies the existence of *spurious pressure modes* which pollute the computed pressure p_N.

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The constant β_N, called the *inf-sup constant*, depends on the way Q_N is chosen, and has a special role in the analysis of the spectral approximation (25). Two choices are commonly proposed in practice. The first one is $Q_N = \mathbb{P}_{N-2} \cap L^2_0(\Omega)$, i.e. the space of polynomials of degree $N-2$ with zero average. In that case $\beta_N \simeq CN^{(1-d)/2}$.

An alternative approach consists of choosing $Q_N = \mathbb{P}_{[\lambda N]} \cap \mathbb{P}_{N-2} \cap L^2_0(\Omega)$, for some $\lambda :\ 0 < \lambda < 1$, where $[\lambda N]$ denotes the largest integer $\leq \lambda N$; in this case, $\beta_N \geq \beta > 0$.

The latter approach allows one to derive uniform stability and optimal error bounds for the approximate solution. In general, the approximation is optimal when β_N is uniformly bounded from below as N increases. If this is not the case, the pressure approximation undergoes a loss of accuracy of order β_N^{-1}. For instance, in the case where $Q_N = \mathbb{P}_{N-2} \cap L^2_0(\Omega)$, the following error bound can be proven, provided the assumed regularity for the exact solution $\mathbf{u}$, p and the forcing term $\mathbf{f}$ holds for suitable values of $s \geq 1$ and $t \geq 0$:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_N\|_{(H^1(\Omega))^d} + N^{(1-d)/2}\|p - p_N\|_{L^2(\Omega)} \quad &\leq \quad CN^{1-s}\left(\|\mathbf{u}\|_{(H^s(\Omega))^d} + \|p\|_{H^{s-1}(\Omega)}\right) \\ &\quad + N^{-t}\|\mathbf{f}\|_{(H^1(\Omega))^d}. \end{aligned}$$

Note that the $(N+1)^2$ Gauss-Lobatto nodes are used to interpolate the discrete velocity components, while the subset made of the $(N-1)^2$ interior Gauss-Lobatto nodes can be used to interpolate the discrete pressure. Alternatively, one could use a staggered grid made of the $(N-1)^2$ Gauss nodes for the pressure (and change in (25) the discrete integrals $(p_N, \mathrm{div}\mathbf{v}_N)_N$ and $(q_N, \mathrm{div}\mathbf{u}_N)_N$ accordingly). This, however, would require interpolation between meshes as in this case velocity and pressure feature nodal representations with respect to different sets of nodes.

The algebraic formulation of the discrete Stokes problem (25) yields the classical block structure matrix form

$$\begin{bmatrix} A & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{27}$$

where we have used test functions $\mathbf{v}_N$ based on the Lagrangian polynomials of degree N of the velocity approximation, and test functions q_N based on the Lagrangian polynomials of degree $N-2$ (at the interior nodes) of the pressure approximation.

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Upon eliminating (although only formally!) the $\mathbf{u}$ vector, one obtains from (27) the reduced pressure system

$$S\mathbf{p} = \mathbf{g}, \qquad \text{with} \quad S = DA^{-1}D^T \text{ and } \quad \mathbf{g} = DA^{-1}\mathbf{f} \; . \tag{28}$$

The pressure matrix S has $(N-1)^2$ rows and columns. It is symmetric; moreover it is positive definite iff $\mathrm{Ker}\, D^T = 0$, a condition which is equivalent to (26).

If we consider the generalized eigenvalue problem $S\mathbf{w} = \lambda M\mathbf{w}$, where M is the pressure mass matrix $(\psi_j, \psi_i)_N$, ($\{\psi_j\}$ being the Lagrangian polynomials (of degree $\leq N-1$) associated with the interior Gauss-Lobatto nodes), then the maximum generalized eigenvalue λ_{max} is uniformly bounded (from above) by the coercivity constant α of the discrete bilinear form $((\nabla \mathbf{u}_N, \nabla \mathbf{v}_N))_N$ (we can assume $\alpha = 1$ in the case at hand), whereas the minimum one λ_{min} is given by β_N^2. As a consequence, the condition number of the matrix $M^{-1}S$ is $\mathrm{cond}(M^{-1}S) \sim \beta_N^{-2}$, thus $\sim N^{d-1}$ in the case of the $\mathbb{P}_N - \mathbb{P}_{N-2}$ discretization.

Since S is close to M (the discrete variational equivalent of the identity operator), M can serve as preconditioner of (28). The corresponding PCG (Preconditioned Conjugate Gradient) method will converge in $O(N^{1/2})$ iterations for 2D problems and in $O(N)$ for 3D ones. In practice, however, the convergence is faster, as the previous estimate on the asymptotic behaviour of β_N is too pessimistic.

Several kind of generalizations are in order.

First of all, we mention that the Stokes system (24) could be reduced to a single (vector) equation by L^2-projection upon the divergence free subspace $V_{div} = \{\mathbf{v} \in (H_0^1(\Omega))^d | \, \mathrm{div}\, \mathbf{v} = 0\}$:

$$\text{find } \mathbf{u} \in V_{div} : \quad \int_\Omega \nu \nabla \mathbf{u} \cdot \nabla \mathbf{v} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; , \qquad \forall \mathbf{v} \in V_{div} \; .$$

Since this is a well-posed elliptic problem, a unique velocity field can be obtained and, afterward, a unique pressure p can be recovered in $L_0^2(\Omega)$.

The simple structure of the reduced problem prompts to a Galerkin (or GNI) discretization. However, a computer implementation is far from trivial, as one should elaborate a set of polynomial basis functions which are inherently divergence-free. This task has been successfully accomplished only for some specific boundary value problems, for instance when Ω is a cylindrical domain and Fourier expansion in the angular direction is combined with an

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expansion in terms of Chebyshev polynomials in both the longitudinal and the radial direction. A similar idea is behind the approach by Batcho and Karniadakis to generate eigenfunctions of a generalized Stokes operator and use them as polynomial divergence-free functions.

A different kind of generalization consists of using equal-order interpolation $\mathbb{P}_N - \mathbb{P}_N$ for both discrete velocity and pressure fields. However, this choice would give rise to a couple of subspaces V_N and Q_N which violate the Brezzi condition (26), yielding spurious pressure modes that swamp the physically relevant pressure. In the line of what is nowadays common practice in the finite element community, Canuto et al. have proposed and analyzed a stabilization by bubble functions. The idea consists in adding to $(\mathbb{P}_N^0)^d$ a supplementary space spanned by local polynomial functions having support in one small element called cell. In 2D, a cell is a quadrilateral whose four vertices are four neighboring Gauss-Lobatto points, whereas in 3D it is a brick whose eight vertices are eight such points. The new velocity space is now given by $V_N = (\mathbb{P}_N^0)^d \oplus B_N^d$, where B_N^d denotes the space of bubble functions, while the pressure space is simply $Q_N = \mathbb{P}_N \cap L_0^2(\Omega)$.

After a careful analysis on the effect of the interaction of the local bubble functions with the global polynomials, and upon eliminating the bubble-functions contribution by static condensation, it is proven that the new stabilized discrete problem can be regarded as a Galerkin problem like (25), however the continuity equation is modified by the presence of the additional term

$$\sum_C \tau_C (J_h \mathbf{r}_N, J_h(\nabla(q_h)))_C$$

which plays the role of stabilizing term to damp the oscillatory pressure modes. Here C is a generic cell, and $(\cdot,\cdot)_C$ is the $L^2(C)$ scalar product. Moreover q_h is the (piecewise-linear) finite element interpolant of the test function q_N at the Gauss-Lobatto nodes, $\mathbf{r}_N := -\nu\triangle\mathbf{u}_N + \nabla p_N - \mathbf{f}$ is the residual, J_h is the L^2-projection operator into the space of piecewise constant functions on the cells. Finally, τ_C is the cell stabilization parameter which can be expressed in terms of the cell size h_C, the magnitude of the velocity field on C and the fluid viscosity. Several expressions for τ_C are actually available based on alternative approaches that are residual-free.

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The Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{u} - \nu \triangle \mathbf{u} + C(\mathbf{u}) + \operatorname{grad} p = \mathbf{f} & \text{in } \Omega\,(\subset \mathbb{R}^d,\ d = 2,3) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \end{cases} \tag{29}$$

differ from (24) due to the presence of the acceleration term $\partial_t \mathbf{u}$ and the convective term $C(\mathbf{u})$. The latter can take the standard *convective form* $\mathbf{u} \cdot \nabla \mathbf{u}$, however other expressions are used as well, such as the *conservative* form $\operatorname{div}(\mathbf{u}\mathbf{u})$ or the *skew-symmetric form* $\frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{u} + \operatorname{div}(\mathbf{u}\mathbf{u}))$. The three forms are all equivalent for the continuous equations (with homogeneous Dirichlet boundary conditions) because of the incompressibility condition. However, this is no longer true at the discrete level. Indeed, the GNI spectral discretization of the Navier-Stokes equations has different stability properties depending upon which form of $C(\mathbf{u})$ is employed.

For the time discretization of (29) fully implicit methods would produce a non-symmetric nonlinear system. To avoid that, the convective term must be treated explicitly. One way is to combine backward-difference (BDF) discretization of linear terms with Adams-Bashforth (AB) discretization of the convective one. A classical recipe is the so-called BDF2/AB3, i.e. the combination of the 2nd order backward difference discretization with the third order Adams-Bashforth discretization:

$$\begin{aligned} \left(\frac{3}{2\Delta t} M + A\right) \mathbf{u}^{n+1} + D^T \mathbf{p}^{n+1} &= \frac{1}{\Delta t} M \left(2\mathbf{u}^n - \frac{1}{2}\mathbf{u}^{n-1}\right) + M\mathbf{f}^{n+1} \\ &\quad -2\left(\frac{23}{12} C(\mathbf{u}^n) - \frac{4}{3} C(\mathbf{u}^{n-1}) + \frac{5}{12} C(\mathbf{u}^{n-2})\right) \\ D\mathbf{u}^{n+1} &= 0 \end{aligned}$$

where M is the velocity mass matrix, while A, D^T and D are the matrices introduced before. To increase time-accuracy, a BDF3 discretization is coupled with an extrapolation of the nonlinear term. This gives (Karniadakis, Israeli and Orszag)

$$\begin{aligned} \left(\frac{11}{6\Delta t} M + A\right) \mathbf{u}^{n+1} + D^T \mathbf{p}^{n+1} &= \frac{1}{\Delta t} M \left(3\mathbf{u}^n - \frac{3}{2}\mathbf{u}^{n-1} + \frac{1}{3}\mathbf{u}^{n-2}\right) + M\mathbf{f}^{n+1} \\ &\quad - \left(3C(\mathbf{u}^n) - 3C(\mathbf{u}^{n-1}) + C(\mathbf{u}^{n-2})\right) \\ D\mathbf{u}^{n+1} &= 0 \end{aligned}$$

This scheme is third order accurate with respect to Δt.

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6. Advection equations and conservation laws

In order to illustrate spectral approximations to hyperbolic problems, we consider the linear and non-linear 1D model equations $u_t + au_x = 0$ and $u_t + f(u)_x = 0$, supplemented by initial and appropriate boundary conditions. In addition to the standard issues related to spectral discretizations (efficient implementation, imposition of boundary conditions, stability, accuracy for smooth solutions), here we face a new problem. Indeed, the equation may propagate singularities along characteristics, or even (in the non-linear case) generate singularities from smooth initial data. So, the question arises: what is the interest of using high-order methods in such cases? We will answer this question in the second part of the present section.

For periodic problems, say in $(0, 2\pi)$, the Fourier-Galerkin method is the conceptually simplest choice: find $u_N = u_N(t) \in S_N$ such that

$$(u_{N,t} + au_{N,x}, \varphi) = 0 \qquad \text{or} \qquad (u_{N,t} + f(u_N)_x, \varphi) = 0, \qquad \forall \varphi \in S_N.$$

Taking $\varphi = u_N$, integrating by parts and using periodicity, one obtains $\frac{d}{dt}\|u_N(t)\|^2_{L^2(0,2\pi)} \leq K\|u_N(t)\|^2_{L^2(0,2\pi)}$ (with $K = \max_{[0,2\pi]} a_x$) for the linear advection equation. This proves the L^2-stability of the approximation.

In terms of Fourier coefficients, the Galerkin method for the advection equation is equivalent to the set of ordinary differential equations

$$(u_{N,k})' + (au_{N,x})_k = 0, \qquad -N/2 \leq k \leq N/2 - 1.$$

Setting for simplicity $b = u_{N,x}$, we have $(ab)_k = \sum_{h=-N/2}^{N/2-1} a_{k-h} b_h$. This is a family of convolution sums, which can be computed in $O(N^2)$ operations. A more efficient action consists of transforming back a and b in physical space, taking the pointwise product at the nodes $x_j = \pi j/N,\ j = 0, \ldots, N-1$, and returning to Fourier space. Using the FFT, the full process costs $O(N \log N)$ operations. This is the *pseudospectral evaluation* of convolutions sums. There is an error involved, since one replaces the exact projection $P_N(ab)$ of ab upon S_N by its interpolant $I_N(ab)$ at the nodes. Such error, termed the *aliasing error*, is negligible if N is so large that the essential features of u are resolved. Otherwise, appropriate de-aliasing techniques can be applied, such as increasing the number of interpolation nodes.

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This process applies for the conservation law as well, provided the nonlinearity is polynomial (as for Burgers's equation, $f(u_N) = \frac{1}{2}u_N^2$, or the convective term $\mathbf{u}_N \nabla \mathbf{u}_N$ in the Navier-Stokes equations). It can be extended to the non-periodic case, by using the Chebyshev nodes $x_j = \cos \pi j/N,\ j = 0, \ldots, N$.

The Fourier-Galerkin method with the pseudospectral evaluation of convolutions sums is nothing else than the Galerkin method with numerical integration described in (6), or equivalently, the collocation method at the quadrature points

$$u_{N,t}^c(x_j) + a(x_j)u_{N,x}^c(x_j) = 0, \qquad j = 0, \ldots, N-1.$$

Unless $a(x) \geq \alpha > 0$ for all x, this scheme is (weakly) unstable, due to the aliasing error. Writing the convective term in the *skew-symmetric* form

$$au_x = \frac{1}{2}(au)_x + \frac{1}{2}au_x - \frac{1}{2}a_x u \tag{30}$$

and applying pseudo-spectral derivatives, one recovers the same stability estimates as for the pure Galerkin method (in practice, such a more expensive form in rarely necessary). Again, similar considerations apply in the non-linear case as well.

We now turn to the discretization of non-periodic problems, in the framework of Legendre methods. The advection equation is well-posed provided we prescribe the solution, say $u(x_b) = g_b$, at the *inflow points* $x_b \in B_-$, where $B_\pm = \{x_b \in \{-1, 1\} : (\pm 1)a(x_b)n_b > 0\}$ with $n_b = x_b$. The most obvious way to account for the boundary conditions is to enforce them exactly (or strongly) in the discrete solution: $u_N \in \mathbb{P}_N$ satisfies $u_N(x_b) = g_b,\ \forall x_b \in B_-$. The corresponding Galerkin method is L^2-stable. Indeed, assuming for simplicity $g_b = 0$, we take u_N itself as test function and after integration by parts we get

$$\frac{1}{2}\frac{d}{dt}\|u_N\|^2_{L^2(-1,1)} - \frac{1}{2}K\|u_N\|^2_{L^2(-1,1)} + \sum_{x_b \in B_+} a(x_b)n_b u_N^2(x_b) \leq 0$$

whence stability easily follows. A similar result holds for the Galerkin method with numerical integration (GNI) at the Legendre-Gauss-Lobatto points, provided we use the skew-symmetric form (30) of the convective term. The GNI scheme is equivalent to enforce the equation at the internal nodes and at the non-inflow boundary points ($x_b \notin B_-$).

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A more flexible way to handle the boundary conditions, useful e.g. in domain decomposition and for systems of equations, is to enforce them in a weak sense. The rationale is that, if stability holds, then accuracy is assured provided the boundary conditions are matched to within the same consistency error as for the equation at the interior. Thus, we seek $u_N = u_N(t) \in \mathbb{P}_N$ satisfying, for all $v_N \in \mathbb{P}_N$,

$$(u_{N,t}, v_N)_N - (u_N, a v_{N,x})_N + \sum_{x_b \in B_+} a(x_b) n_b u_N(x_b) v_N(x_b) = \sum_{x_b \in B_-} |a(x_b) n_b| g_b v_N(x_b). \quad (31)$$

This GNI formulation follows by integrating by parts the convective term (for simplicity, we assume a constant, otherwise we use the skew-symmetric form (30)). Choosing as v_N the polynomial δ-function at each quadrature node, we see that the advection equation is enforced at all internal and non-inflow nodes, whereas at the inflow nodes we have

$$u_{N,t}(x_b) + a u_{N,x}(x_b) + \frac{1}{w_b} |a(x_b) n_b| (u_N(x_b) - g_b) = 0.$$

Since $1/w_b \sim cN^2$ as $N \to +\infty$, this shows that the boundary condition is indeed enforced by a *penalty* method. The stability of the scheme (31) immediately follows by taking $v_N = u_N$. Stability is actually guaranteed even if we multiply each boundary term in (31) by any constant $\tau_b \geq 1/2$, thus enhancing the flexibility of the penalty method.

Let us now consider the non-linear conservation law $u_t + f(u)_x = 0$. The stability (and convergence) of spectral discretizations is a much more delicate issue than for the linear advection equation. Indeed, the equation may develop singular solutions at a finite time, which correspond to the accumulation of energy in the high frequency modes or, equivalently, to the onset of oscillations around discontinuities (Gibbs phenomenon). The nonlinear mechanism may amplify the high frequency components, leading to destructive instabilities (in stronger norms than L^2). On the other hand, oscillations should not be brutally suppressed: they are inherent to the high order representation of discontinuous functions, and they may hide the correct information which allows the reconstruction of the exact solution. Thus, the good spectral discretization should guarantee enough stability while preserving enough accuracy. Furthermore, it should allow the discrete solution to converge to the physically relevant exact solution, by fulfilling an appropriate entropy condition.

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The mathematically most rigorous discretization which matches these requirements is the *spectral viscosity method*. In the Fourier-Galerkin context, it amounts to considering the modified equation

$$u_{N,t} + (P_N f(u_N))_x = \varepsilon_N (-1)^s D_x^s (Q_m D_x^s u_N)$$

where $\varepsilon_N \sim cN^{1-2s}$, $m = m_N \sim N^\vartheta$ for some $\vartheta < 1 - 1/(2s)$, and the Fourier coefficients of Q_m satisfy $Q_{m,k} = 0$ if $|k| \leq m$, $Q_{m,k} = 1 - (m/|k|)^{(2s-1)/\vartheta}$ if $|k| > m$. Thus, the s-th order artificial viscosity is applied only to sufficiently high frequency modes. For $s = 1$, one can prove that the solution is bounded in $L^\infty(0, 2\pi)$, it satisfies the estimate $\|u_N(t)\|_{L^2(0,2\pi)} + \sqrt{\varepsilon_N}\|u_{N,x}(t)\|_{L^2(0,2\pi)} \leq C\|u_N(0)\|_{L^2(0,2\pi)}$, and it converges to the correct entropy solution.

A computationally simpler and widely used road to stabilization consists of *filtering* the spectral solution during time advancing,

$$u_N(t) \quad \mapsto \quad \mathcal{F}_N u_N(t) = \sum_{k=-N/2}^{N/2} \sigma(2k/N) u_{N,k}(t) \exp(ikx)$$

where $\sigma = \sigma(\eta)$ is a smooth, even function satisfying $\sigma(0) = 1$, $\sigma^{(j)}(0) = 0$ for all j with $1 \leq j \leq$ some s, monotonically decreasing for $\eta > 0$ and vanishing (or being exponentially small) for $\eta > 1$. A popular choice is the exponential filter $\sigma(\eta) = \exp(-\alpha\eta^{2s})$. Interestingly, the effect of the spectral viscosity correction described above can be closely mimicked by applying the exponential filter with $\sigma(2k/N) = \exp(-\varepsilon_N Q_{m,k} k^2)$.

If the solution of the conservation law is piecewise analytic but discontinuous, its truncation $P_N u$ or its interpolation $I_N u$ are highly oscillatory around the singularities, and converge slowly ($O(N^{-1})$) to u away from them. However, they contain enough information to allow the reconstruction of the exact solution with exponential accuracy, away from the singularities, by a post-processing as described below. It follows that the crucial feature of the discretization scheme is the capability of producing an approximation u_N which is *spectrally close to $P_N u$ or to $I_N u$*. This is precisely what is obtained by the spectral viscosity method or by the equivalent filtering procedure.

Given $P_N u$ (similar considerations apply to $I_N u$), the post processing reconstruction may be *local* or *global*. In the former case, a spectrally accurate approximation of u at a point

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x_0 of analyticity is given by $u_N^*(x_0) = \int_{\mathbb{R}} K_\nu(x_0, y)\varrho(x_0 - y)P_N u(y)dy$, where $\nu = [\beta N]$ for some $\beta \in (0, 1)$, $K_\nu(x, y)$ is, for each x, a ν-degree polynomial approximation of the delta at x (e.g., for Fourier, $K_\nu(x, y) = 1 + \sum_{k=0}^{\nu/2} \cos(x - y)$ is the Dirichlet kernel), whereas $\varrho(\eta)$ is a C^∞-localizer around $\eta = 0$. In the latter case, a spectrally accurate approximation of u on an interval $[a, b]$ of analyticity is given (Gottlieb and Shu, 1997) by the orthogonal projection of $P_N u$ upon $\mathbb{P}_\nu([a, b])$ (again $\nu = [\beta N]$) with respect to the weighted inner product $\int_a^b u(x)v(x)\omega_\nu(x)dx$, with $\omega_\nu(x) = ((x - a)(b - x))^{\nu-1/2}$, which varies with N. The projection is computed via the Gegenbauer polynomials (i.e., the Jacobi polynomials $\{P_k^{(\nu-1/2,\nu-1/2)}\}$) translated and scaled to $[a, b]$.

7. The Spectral Element Method

The spectral element method (SEM) represents another example of Galerkin method. However, the finite dimensional space is now made of piecewise algebraic polynomials of high degree on each element of a fixed partition of the computational domain. For a one-dimensional problem, such as e.g. (2), we split $\Omega = (a, b)$ into a set of M disjoint intervals Ω_e, $e = 1, \ldots, M$, whose end points are $a = \overline{x}_0 < \overline{x}_1 < \ldots \overline{x}_M = b$. Then we set

$$V_{N,M} = \{v \in C^0(\overline{\Omega}) |\, v_{|\Omega_e} \in \mathbb{P}_N, \forall e = 1, \ldots, M,\ v(a) = v(b) = 0\}.$$

The approximation of (2) by SEM reads:

$$\text{find } u_{N,M} \in V_{N,M} :\ a(u_{N,M}, v) = (f, v) \quad \forall v \in V_{N,M}\ . \tag{32}$$

This approach shares the same structure than the p-version of the finite element method. As in the latter, the number M of subintervals is frozen, while the local polynomial degree (that we indicate by N in the SEM context and by p in the FEM context) is increased to improve accuracy. More precisely, if $h = (b - a)/M$ denotes the constant length of each subinterval, one has

$$\|u' - (\Pi_{N,M}u)'\|_{L^2(a,b)} + \frac{N}{h}\|u - \Pi_{N,M}u\|_{L^2(a,b)} \leq C(s)h^{\min(N,s)}N^{-s}\|u'\|_{H^s(a,b)}\ ,\ s \geq 0 \tag{33}$$

where $\Pi_{N,M}$ is the SEM interpolant.

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If u is arbitrarily smooth (s large) it is advantageous to keep h fixed and let $N \to \infty$.

Should the different degree of smoothness suggest the use of a non-uniform polynomial degree, another upper bound for the left hand side of (33) is

$$\sum_{e=1}^{M} C_e N_e^{-s_e} \|u'\|_{H^{s_e}(\Omega_e)}, \quad s_e \geq 1, \quad \forall e = 1, \ldots, M$$

where N_e is the polynomial degree used in the e-th element Ω_e and $H^{s_e+1}(\Omega_e)$ is the local smoothness of u in Ω_e.

SEM was first introduced by A. Patera for Chebyshev expansion, then generalized to the Legendre case by Y. Maday and A. Patera.

Both approaches (SEM and p-version of FEM) make use of a parental element, say $\hat{\Omega} = (-1, 1)$, on which the basis functions are constructed. However, the main difference consists in the way the basis functions are chosen (and therefore on the structure of the corresponding stiffness matrix).

FEMs of p-type are defined in terms of the Legendre polynomials $L_k(\xi)$ of degree k ($k = 2, \ldots, p$), $\xi \in \hat{\Omega}$. Precisely, the $p+1$ basis functions on $\hat{\Omega}$ are defined by:

$$\varphi_1(\xi) = \frac{1-\xi}{2}, \qquad \varphi_p(\xi) = \frac{1+\xi}{2},$$

$$\varphi_k(\xi) = \sqrt{\frac{2k-1}{2}} \int_{-1}^{\xi} L_{k-1}(s)ds = \frac{1}{\sqrt{2(2k-1)}}(L_k(\xi) - L_{k-2}(\xi)), \qquad k = 2, \ldots, p.$$

The first two terms ensure C^0 continuity of the trial functions.

For the algebraic realization of SEM, nodal basis functions are those introduced in (13). Being associated to the special set of Legendre Gauss-Lobatto nodes, once they are mapped on the current element $\{\Omega_e,\ e = 1, \ldots, M\}$, they can be used to generate shape functions, then allow us to use LGL quadrature formulas for the evaluation of the entries of the stiffness and other matrices and the right hand side. This is reflected by replacing (32) with the more interesting GNI-SEM version:

$$\text{find } u_{N,M} \in V_{N,M} : \quad \sum_{e=1}^{M} a_{N,\Omega_e}(u_{N,M}, v) = \sum_{e=1}^{M} (f, v)_{N,\Omega_e} \qquad \forall v \in V_{N,M} \tag{34}$$

where $(u, v)_{N,M}$ is the correspondent in Ω_e of the Legendre Gauss-Lobatto inner product (11), $(u, v)_{N,\Omega_e} = \sum_{j=0}^{N} \alpha_j^e u(x_j^e) v(x_j^e)$, with $\alpha_j^e = \alpha_j \frac{b-a}{2}$, x_j^e is the correspondent of x_j in Ω_e. Moreover, $a_{N,\Omega_e}(u, v)$ is the elemental bilinear form.

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Still considering the case of the differential operator (5) as an instance, we end up with the following form:

$$a_{N,\Omega_e}(u,v) = (\alpha u', v')_{N,\Omega_e} + (\beta u', v)_{N,\Omega_e} + (\gamma u, v)_{N,\Omega_e}$$

in analogy with the left hand side of (12).

The multidimensional case can be addressed by first introducing the tensorized basis functions on the parental element $\hat{\Omega} = (-1,1)^d$ $(d = 2,3)$, then mapping basis functions and nodal points on every current element Ω_e (now a quadrilateral or parallelepipedal structure, possibly with curved edges or surfaces). The functional structure of our problem remains formally the same as in (34), and similar is the kind of error estimate that can be achieved. Obviously, this time $V_{N,M}$ is made of globally continuous functions that satisfy homogeneous Dirichlet boundary data (if any). They are patchwork of elemental functions which are the mapping of the nodal basis functions according to the transformation $T_e : \hat{\Omega} \to \Omega_e$ that maps the parental element $\hat{\Omega}$ into the current element Ω_e.

8. The mortar method

This method has been introduced by Bernardi, Maday and Patera with the aim of allowing spectral elements having different polynomial degrees or being geometrically non-conforming, but also to allow the coupling of spectral (element) method with the finite element method. Its generality, however, goes beyond these two specific examples. Consider, for the sake of illustration, the Poisson problem with homogeneous Dirichlet conditions. The idea is to approximate its weak form (13) by the following discrete problem:

$$\text{find } u_\delta \in V_\delta \ : \ \sum_{i=1}^{M} \int_{\Omega_i} \nabla u_\delta \cdot \nabla v_\delta = \sum_{i=1}^{M} \int_{\Omega_i} f v_\delta \quad \forall\ v_\delta \in V_\delta. \tag{35}$$

Here, $\delta > 0$ is a parameter describing the quality of the discretisation, and V_δ is a finite dimensional space that approximates $H_0^1(\Omega)$ without being contained into $C^0(\overline{\Omega})$. More precisely, V_δ is a subspace of the following space:

$$Y_\delta := \{v_\delta \in L^2(\Omega) \,|\, v_{\delta|\Omega_i} \in Y_{i,\delta},\ i = 1, \ldots, M\} \tag{36}$$

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where, for each $i = 1, \ldots, M$, $Y_{i,\delta}$ is a finite dimensional subspace of $H^1(\Omega_i)$: it can be either a finite element space, or a polynomial spectral (elements) space. In any case, no requirement of compatibility is made for the restriction of the functions of Y_δ on the element interface Γ.

Heuristically, the space V_δ will be made up of functions belonging to Y_δ that satisfy some kind of matching across Γ. Precisely, assuming for simplicity that there are only two elements, if $v_\delta \in V_\delta$ and $v_\delta^{(1)} \in Y_{1,\delta}$, $v_\delta^{(2)} \in Y_{2,\delta}$ denotes its restriction to Ω_1 and Ω_2, respectively, for a certain fixed index i the following integral matching conditions should be satisfied:

$$\int_\Gamma (v_\delta^{(1)} - v_\delta^{(2)}) \mu_\delta^{(i)} = 0 \qquad \forall\ \mu_\delta^{(i)} \in \Lambda_\delta^{(i)} \tag{37}$$

where $\Lambda_\delta^{(i)}$ denotes the restriction to Γ of the functions of $Y_{i,\delta}$.

If we take $i = 2$ in (37), this amounts to letting Ω_1 play the role of *master* and Ω_2 that of *slave*, and (37) has to be intended as the way of generating the value of $v_{\delta|\Gamma}^{(2)}$ once $v_{\delta|\Gamma}^{(1)}$ is available. The alternative way, i.e. taking $i = 1$ in (37), is also admissible. Depending upon the choice of index i made in (37), the method will produce different solutions.

The mathematical rationale behind the choice of the matching condition (37) (rather than a more 'natural' condition of pointwise continuity at one set of grid nodes on Γ) becomes clear from the convergence analysis on problem (35).

With this aim we introduce

$$||v||_* := (||v||_{0,\Omega}^2 + ||\nabla v_{|\Omega_1}||_{0,\Omega_1}^2 + ||\nabla v_{|\Omega_2}||_{0,\Omega_2}^2)^{1/2} \tag{38}$$

which is a norm (the 'graph' norm) on the Hilbert space

$$H_* := \{v \in L^2(\Omega) \,|\, v_{|\Omega_1} \in H^1(\Omega_1),\ v_{|\Omega_2} \in H^1(\Omega_2)\}. \tag{39}$$

Owing to the Poincaré inequality, we have that

$$\sum_{i=1}^{2} \int_{\Omega_i} |\nabla v_\delta|^2 \geq \alpha_* ||v_\delta||_*^2 \qquad \forall\ v_\delta \in V_\delta \tag{40}$$

whence the discrete problem (35) admits a unique solution by a straightforward application of the Lax–Milgram lemma.

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For any $v_\delta \in V_\delta$ we now have

$$\begin{aligned}\alpha_* ||u_\delta - v_\delta||_*^2 &\leq \sum_{i=1}^{2} \int_{\Omega_i} |\nabla(u_\delta - v_\delta)|^2 \\ &\leq \sum_{i=1}^{2} \int_{\Omega_i} \nabla u_\delta \cdot \nabla(u_\delta - v_\delta) - \sum_{i=1}^{2} \int_{\Omega_i} \nabla v_\delta \cdot \nabla(u_\delta - v_\delta) \\ &= \sum_{i=1}^{2} \int_{\Omega_i} f(u_\delta - v_\delta) - \sum_{i=1}^{2} \int_{\Omega_i} \nabla v_\delta \cdot \nabla(u_\delta - v_\delta).\end{aligned} \tag{41}$$

Replacing f by $-\Delta u$ and integrating by parts on each Ω_i we obtain:

$$\begin{aligned}\sum_{i=1}^{2} \int_{\Omega_i} f(u_\delta - v_\delta) &= \sum_{i=1}^{2} \int_{\Omega_i} \nabla u \cdot \nabla(u_\delta - v_\delta) \\ &\quad - \int_\Gamma \frac{\partial u}{\partial n} \left[(u_\delta - v_\delta)^{(1)} - (u_\delta - v_\delta)^{(2)} \right]\end{aligned} \tag{42}$$

(here, $\frac{\partial}{\partial n}$ is the normal derivative on Γ pointing into Ω_2).

Denoting by

$$[v_\delta]_\Gamma := v_{\delta|\Gamma}^{(1)} - v_{\delta|\Gamma}^{(2)}$$

the jump across Γ of a function $v_\delta \in V_\delta$, from (41) and (42) we have that

$$\alpha_* ||u_\delta - v_\delta||_*^2 \leq ||u - v_\delta||_* ||u_\delta - v_\delta||_* + \left| \int_\Gamma \frac{\partial u}{\partial n} [u_\delta - v_\delta]_\Gamma \right|$$

and also

$$||u_\delta - v_\delta||_* \leq \frac{1}{\alpha_*} \left(||u - v_\delta||_* + \sup_{w_\delta \in V_\delta} \frac{\left| \int_\Gamma \frac{\partial u}{\partial n} [w_\delta]_\Gamma \right|}{||w_\delta||_*} \right).$$

By the triangle inequality

$$||u - u_\delta||_* \leq ||u - v_\delta||_* + ||u_\delta - v_\delta||_*$$

we then obtain the following inequality for the error $u - u_\delta$:

$$||u - u_\delta||_* \leq \left(1 + \frac{1}{\alpha_*}\right) \inf_{v_\delta \in V_\delta} ||u - v_\delta||_* + \frac{1}{\alpha_*} \sup_{w_\delta \in V_\delta} \frac{\left| \int_\Gamma \frac{\partial u}{\partial n} [w_\delta]_\Gamma \right|}{||w_\delta||_*}. \tag{43}$$

The approximation error of (35) is therefore bounded (up to a multiplicative constant) by the best approximation error (that is, the distance between the exact solution u and the finite dimensional space V_δ) plus an extra error involving interface jumps. The latter would not appear in the framework of classical Galerkin approximation (like the SEM), and is the price to pay for the violation of the conforming property; that is, for the fact that $V_\delta \not\subset H_0^1(\Omega)$.

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The error estimate (43) is *optimal* if each one of the two terms on the right can be bounded by the norm of *local* errors arising from the approximations in Ω_1 and Ω_2, without the presence of terms that combine them in a multiplicative fashion. In this way, we can take advantage of the local regularity of the exact solution as well as the approximation properties enjoyed by the local subspaces $Y_{i,\delta}$ of $H^1(\Omega_i)$.

To generate a basis for the finite dimensional space V_δ, we can proceed as follows. For $i = 1, 2$, let us denote by $\mathcal{N}_i$ the set of nodes in the interior of Ω_i, and by $\mathcal{N}_\Gamma^{(i)}$ the set of nodes on Γ, whose cardinality will be indicated by N_i and $N_\Gamma^{(i)}$, respectively. Note that, in general, $\mathcal{N}_\Gamma^{(1)}$ and $\mathcal{N}_\Gamma^{(2)}$ can be totally unrelated.

Now, denote by $\{\varphi_{k'}^{(1)}\}$, $k' = 1, \ldots, N_1$, the Lagrange functions associated with the nodes of $\mathcal{N}_1$; since they vanish on Γ, they can be extended by 0 in $\overline{\Omega}_2$. These extended functions are denoted by $\{\widetilde{\varphi}_{k'}^{(1)}\}$, and can be taken as a first set of basis functions for V_δ.

Symmetrically, we can generate as many basis functions for V_δ as the number of nodes of $\mathcal{N}_2$ by extending by 0 in $\overline{\Omega}_1$ the Lagrange functions associated with these nodes. These new functions are denoted by $\{\widetilde{\varphi}_{k''}^{(2)}\}$, $k'' = 1, \ldots, N_2$.

Finally, always supposing that Ω_1 is the master domain and Ω_2 its slave, for every Lagrange function $\{\varphi_{m,\Gamma}^{(1)}\}$ in $\overline{\Omega}_1$, $m = 1, \ldots, N_\Gamma^{(1)}$, we obtain a basis function $\{\widetilde{\varphi}_{m,\Gamma}\}$ as follows

$$\widetilde{\varphi}_{m,\Gamma} := \begin{cases} \varphi_{m,\Gamma}^{(1)} & \text{in } \overline{\Omega}_1 \\ \widetilde{\varphi}_{m,\Gamma}^{(2)} & \text{in } \overline{\Omega}_2 \end{cases}$$

where

$$\widetilde{\varphi}_{m,\Gamma}^{(2)} := \sum_{j=1}^{N_\Gamma^{(2)}} \xi_j \varphi_{j,\Gamma}^{(2)},$$

$\varphi_{j,\Gamma}^{(2)}$ are the Lagrange functions in $\overline{\Omega}_2$ associated with the nodes of $\mathcal{N}_\Gamma^{(2)}$, and ξ_j are unknown coefficients that should be determined through the fulfilment of the matching equations (37). Precisely, they must satisfy

$$\int_\Gamma \left(\sum_{j=1}^{N_\Gamma^{(2)}} \xi_j \varphi_{j,\Gamma}^{(2)} - \varphi_{m,\Gamma}^{(1)} \right) \varphi_{l,\Gamma}^{(2)} = 0 \qquad \forall\, l = 1, \ldots, N_\Gamma^{(2)}. \tag{44}$$

A basis for V_δ is therefore provided by the set of all functions $\{\widetilde{\varphi}_{k'}^{(1)}\}$, $k' = 1, \ldots, N_1$, $\{\widetilde{\varphi}_{k''}^{(2)}\}$, $k'' = 1, \ldots, N_2$, and $\{\widetilde{\varphi}_{m,\Gamma}\}$, $m = 1, \ldots, N_\Gamma^{(1)}$.

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Remark In the mortar method the interface matching is achieved through a L^2-interface projection, or, equivalently, by equating first-order moments, thus involving computation of interface integrals. In particular, from equations (37) we have two different kinds of integrals to evaluate (take, for instance, $i = 2$):

$$I_{12} := \int_\Gamma v_\delta^{(1)} \mu_\delta^{(2)}, \quad I_{22} := \int_\Gamma v_\delta^{(2)} \mu_\delta^{(2)}.$$

The computation of I_{22} raises no special difficulties, because both functions $v_\delta^{(2)}$ and $\mu_\delta^{(2)}$ live on the same mesh, the one inherited from Ω_2. On the contrary, $v_\delta^{(1)}$ and $\mu_\delta^{(2)}$ are functions defined on different domains, and the computation of integrals like I_{12} requires proper quadrature rules. This process needs to be done with special care, especially for three-dimensional problems, for which subdomain interfaces are made up of faces, edges and vertices, otherwise the overall accuracy of the mortar approximation could be compromised.

The discrete problem (35) can also be reformulated as a saddle point problem of the following form:

find $u_\delta \in Y_\delta$, $\lambda_\delta \in \Lambda_\delta^{(2)}$ s.t.

$$\begin{cases} a(u_\delta, v_\delta) + b(v_\delta, \lambda_\delta) = \displaystyle\sum_{i=1}^{2} (f, v_\delta^{(i)})_{\Omega_i} & \forall\ v_\delta \in Y_\delta \\ b(u_\delta, \mu_\delta) = 0 & \forall\ \mu_\delta \in \Lambda_\delta^{(2)} \end{cases}$$

where

$$a(w_\delta, v_\delta) := \sum_{i=1}^{2} \int_{\Omega_i} \nabla w_\delta^{(i)} \cdot \nabla v_\delta^{(i)}, \quad b(v_\delta, \mu_\delta) := \int_\Gamma (v_\delta^{(1)} - v_\delta^{(2)}) \mu_\delta.$$

In this system, λ_δ plays the role of the Lagrange multiplier associated with the 'constraint' (37).

Denoting by φ_j, $j = 1, \ldots, N_1 + N_2 + N_\Gamma^{(1)} + N_\Gamma^{(2)}$, a basis of Y_δ and by ψ_l, $l = 1, \ldots, N_\Gamma^{(2)}$, a basis of $\Lambda_\delta^{(2)}$, we introduce the matrices

$$A_{sj} := a(\varphi_j, \varphi_s), \quad B_{ls} := b(\varphi_s, \psi_l).$$

Defining by $\mathbf{u}$ and $\boldsymbol{\lambda}$ the vectors of the nodal values of u_δ and λ_δ, respectively, and by $\mathbf{f}$ the vector whose components are given by $\sum_{i=1}^{2} (f, \varphi_s^{(i)})_{\Omega_i}$, $s = 1, \ldots, N_1 + N_2 + N_\Gamma^{(1)} + N_\Gamma^{(2)}$, we

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have the linear system

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}.$$

The matrix A is block-diagonal (with one block per subdomain Ω_i), each block corresponding to a problem for the Laplace operator with a Dirichlet boundary condition on $\partial\Omega_i \cap \partial\Omega$ and a Neumann boundary condition on $\partial\Omega_i \setminus \partial\Omega$.

After elimination of the degrees of freedom internal to the subdomains, the method leads to the reduced linear system (still of a saddle point type)

$$\begin{bmatrix} S & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_\Gamma \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_\Gamma \\ \mathbf{0} \end{bmatrix}$$

where the matrix S is block-diagonal, C is a jump operator, $\mathbf{u}_\Gamma$ is the set of all nodal values at subdomain interfaces, and $\mathbf{g}_\Gamma$ is a suitable right-hand side.

This system can be regarded as an extension of the Schur complement system to non-conforming approximation (the Lagrange multiplier $\boldsymbol{\lambda}$ indeed accounts for non-matching discretisation at subdomain interfaces). In fact, the ith block of S is the analogue of $\Sigma_{i,h}$, and corresponds to a discretised Steklov–Poincaré operator on the subdomain Ω_i.

Remark All results cited in this note can be recovered from the books and general articles which are quoted in the References.

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