Anisotropic mesh adaptation in Computational Fluid Dynamics: application to the advection-diffusion-reaction and the Stokes problems*

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Abstract

In this work we develop an anisotropic a posteriori error analysis of the advection-diffusion-reaction and the Stokes problems. This is the first step towards the study of more complex situations, such as the Oseen and Navier-Stokes equations, which are very common in Computational Fluid Dynamic (CFD) applications. The leading idea of our analysis consists in combining the anisotropic interpolation error estimates for affine triangular finite elements provided in [14, 15] with the a posteriori error analysis based on a dual problem associated with the problem at hand [6, 34]. Anisotropic interpolation estimates take into account more in detail the geometry of the triangular elements, i.e. not just their diameter but also their aspect ratio and orientation. On the other hand, the introduction of the dual problem allows us to control suitable functionals of the discretization error, e.g. the lift and drag around bodies in external flows, mean and local values, etc. The combined use of both approaches yields an adaptive algorithm which, via an iterative process, can be used for designing the optimal mesh for the problem at hand.

Keywords Anisotropic mesh adaption, a posteriori error estimators, advection-diffusion reaction problem, Stokes problem, finite element methods, computational fluid dynamics.

1 Introduction and motivation

Many physical problems in CFD are characterized by solutions exhibiting directional features. Navier-Stokes, Stokes and Euler equations are typical examples

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where boundary, internal layers or shocks may develop. In these situations, the effectiveness of finite element procedures can be improved if the mesh is suitably oriented. However, standard a priori and a posteriori procedures do not provide enough information to control mesh orientation. Some anisotropic techniques based on heuristic approaches have been devised in the past (see e.g. [9, 16, 24, 33]). A typical methodology consists in estimating the Hessian and/or the gradient of the numerical solution and then using this information to drive the mesh adaption procedure. Although the results are sometimes impressive, these techniques yet lack a rigorous link with a bound of the discretization error.

More rigorous approaches using theoretically based anisotropic adaptivity have been developed in, e.g. [2, 14, 27, 35] and [11] where the theory of [23] is used in an anisotropic framework. In this paper, we extend the theory of [14, 15] to the advection-diffusion-reaction and the Stokes problems. Precisely, we develop an anisotropic a posteriori error analysis for both these problems, moving from the dual-based approach illustrated in [6, 34]. This technique allows us to adaptively control suitable functionals of the discretization error which might be related to meaningful physical quantities such as e.g. the lift and drag around bodies in external flows, mean and local values, etc. Since we employ stabilized finite elements, we adopt stability coefficients chosen according to the anisotropic analysis provided in [30].

The outline of the paper is as follows. In Section 2 we recall the anisotropic framework of [14, 15] and the anisotropic interpolation estimates needed for the sequel. In Sections 3 and 4 we address the advection-diffusion-reaction and the Stokes problems, respectively, and present their anisotropic a priori and a posteriori error analysis. Finally, in Section 5 we show how a nearly optimal mesh, i.e. a mesh minimizing the number of triangles for a given accuracy, can be obtained from the derived a posteriori error estimates. We assess the quality of our a posteriori analysis on some numerical test cases.

2 Anisotropy: framework and interpolation error estimates

Let us briefly summarize the set-up leading to the anisotropic analysis used in the sequel (see [14] for more details).
Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and let $\{\mathcal{T}_h\}_h$ denote a family of conforming triangulations of $\overline{\Omega}$ into triangles K of diameter $h_K \leq h$, where $h = \max_{K \in \mathcal{T}_h} h_K$. Let $T_K : \widehat{K} \to K$ be the invertible affine mapping from a reference triangle $\widehat{K}$ into the general one K, where $\widehat{K}$ can be indifferently chosen as e.g. the right triangle $(0,0),(1,0),(0,1)$ or the equilateral one $(-1/2,0),(1/2,0),(0,\sqrt{3}/2)$. In either cases, let $M_K \in \mathbb{R}^{2\times 2}$ be the (nonsingular) Jacobian of T_K, i.e. $\vec{x} = (x_1, x_2)^T = T_K(\widehat{\vec{x}}) = M_K \widehat{\vec{x}} + \vec{t}_K$, for any $\widehat{\vec{x}} = (\widehat{x}_1, \widehat{x}_2)^T \in \widehat{K}$, with $\vec{t}_K \in \mathbb{R}^2$.

The distinguishing feature of our anisotropic approach consists in exploiting the spectral properties of the mapping T_K to describe the orientation and the shape of each triangle K. With this aim, let us factorize matrix M_K via the polar decomposition as $M_K = B_K Z_K$, B_K and Z_K being symmetric positive definite and orthogonal matrices, respectively. Furthermore, B_K can be written in terms of its eigenvalues $\lambda_{1,K}$, $\lambda_{2,K}$ (with $\lambda_{1,K} \geq \lambda_{2,K}$) and of its eigenvectors $\vec{r}_{1,K}$, $\vec{r}_{2,K}$ as $B_K = R_K^T \Lambda_K R_K$, with $\Lambda_K = \text{diag}(\lambda_{1,K}, \lambda_{2,K})$ and $R_K^T = [\vec{r}_{1,K}, \vec{r}_{2,K}]$. The shape aspect-ratio of any $K \in \mathcal{T}_h$ with respect to $\widehat{K}$ can be measured by the stretching factor $s_K = \lambda_{1,K}/\lambda_{2,K} (\geq 1)$, $s_{\widehat{K}}$ being equal to 1.

Starting from these decompositions, anisotropic interpolation error estimates have been derived for both the Lagrange and Clément interpolation operators [14, 15]. We state here only the results used in the sequel, while referring to [14, 15, 30] for the detailed derivation.

Let $\mathcal{V}_h$ be the finite element space of continuous affine elements, and $\Pi_h : C^0(\overline{\Omega}) \to \mathcal{V}_h$ and $I_h : L^2(\Omega) \to \mathcal{V}_h$ the standard Lagrange and Clément linear interpolants, respectively. We denote their restrictions to an element $K \in \mathcal{T}_h$ by Π_K and I_K, respectively. In view of the use of the Clément interpolation operator, let Δ_K be the patch of all the elements sharing a vertex with K. In the sequel, we assume the cardinality of any patch Δ_K as well as the diameter of the reference patch $\Delta_{\widehat{K}} = T_K^{-1}(\Delta_K)$ to be uniformly bounded independently of the geometry of the mesh, i.e., there exists a positive integer M and a constant $\widehat{C} > 0$ such that, for any $K \in \mathcal{T}_h$,

$$\text{card}(\Delta_K) \leq M \quad \text{and} \quad \text{diam}(\Delta_{\widehat{K}}) \leq \widehat{C}\,, \tag{1}$$

where $\widehat{C} \geq h_{\widehat{K}}$. In particular, the latter inequality rules out some too distorted reference patches (see Fig. 1.1 in [30]).

Throughout, we use a standard notation to denote the Sobolev spaces of functions with Lebesgue measurable derivatives, and their norms. The following results can now be stated.

Proposition 2.1 *Let $v \in H^2(K)$, for any $K \in \mathcal{T}_h$ and let e denote one of the three edges of K. Then there exist two constants $C_1 = C_1(\widehat{K})$ and $C_2 = C_2(\widehat{K})$ such that*

$$\|v - \Pi_K(v)\|_{L^2(K)} \leq C_1 \left[\sum_{i,j=1}^{2} \lambda_{i,K}^2 \lambda_{j,K}^2 L_K^{i,j}(v) \right]^{1/2}, \tag{2}$$

$$\|v - \Pi_K(v)\|_{L^2(e)} \leq C_2 \left(\frac{\lambda_{1,K}^2 + \lambda_{2,K}^2}{\lambda_{2,K}^3} \right)^{1/2} \left[\sum_{i,j=1}^{2} \lambda_{i,K}^2 \lambda_{j,K}^2 L_K^{i,j}(v) \right]^{1/2}, \tag{3}$$

where

$$L_K^{i,j}(v) = \int_K \left(\vec{r}_{i,K}^{\,T} H_K(v)\, \vec{r}_{j,K} \right)^2 d\vec{x}, \quad \text{with } \; i,\, j = 1,\, 2, \tag{4}$$

and $H_K(v)$ is the Hessian matrix associated with the function v (restricted to K).

Remark 2.1 *The interpolation estimates* (2) *and* (3) *can be easily extended to the case of vector-valued functions $\vec{v} : \Omega \to \mathbb{R}^2$, where $\vec{v} = (v_1, v_2)^T$. In this case, the above results still hold formally provided that the terms $L_K^{i,j}(v)$ be replaced by*

$$L_K^{i,j}(\vec{v}) = \sum_{l=1,2} \int_K \left(\vec{r}_{i,K}^{\,T} H_K(v_l)\, \vec{r}_{j,K}\right)^2 d\vec{x}, \quad \text{with} \;\; i,\, j = 1,\, 2. \tag{5}$$

Likewise, we can prove the following

Proposition 2.2 *Let $v \in H^1(\Omega)$. Then there exist three constants $C_1 = C_1(M, \widehat{C})$, $C_2 = C_2(M, \widehat{C})$ and $C_3 = C_3(M, \widehat{C})$ such that, for any $K \in \mathcal{T}_h$ and any $e \in \partial K$*

$$\|v - I_K(v)\|_{L^2(K)} \le C_1 \left[\sum_{i=1}^{2} \lambda_{i,K}^2 \left(\vec{r}_{i,K}^{\,T} G_K(v) \vec{r}_{i,K} \right) \right]^{1/2},$$

$$|v - I_K(v)|_{H^1(K)} \le C_2 \frac{1}{\lambda_{2,K}} \left[\sum_{i=1}^{2} \lambda_{i,K}^2 \left(\vec{r}_{i,K}^{\,T} G_K(v) \vec{r}_{i,K} \right) \right]^{1/2},$$

$$\|v - I_K(v)\|_{L^2(e)} \le C_3 \frac{1}{\lambda_{2,K}^{1/2}} \left[\sum_{i=1}^{2} \lambda_{i,K}^2 \left(\vec{r}_{i,K}^{\,T} G_K(v) \vec{r}_{i,K} \right) \right]^{1/2},$$

where $G_K(v) \in \mathbb{R}^{2\times 2}$ is the symmetric positive semi-definite matrix given by

$$G_K(v) = \sum_{T \in \Delta_K} \begin{bmatrix} \displaystyle\int_T \left(\frac{\partial v}{\partial x_1} \right)^2 d\vec{x} & \displaystyle\int_T \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2}\, d\vec{x} \\ \displaystyle\int_T \frac{\partial v}{\partial x_1} \frac{\partial v}{\partial x_2}\, d\vec{x} & \displaystyle\int_T \left(\frac{\partial v}{\partial x_2} \right)^2 d\vec{x} \end{bmatrix}. \tag{6}$$

Notice that the terms $L_K^{i,j}(v)$ and $G_K(v)$ in (4) and (6) have the dimensions of squared $H^2(K)$-seminorm and $H^1(K)$-seminorm, respectively.

The result below turns out to be useful to complete the a posteriori analysis, even if not related to the interpolation error estimates just mentioned (see [29] for the proof).

Proposition 2.3 *For any function $v \in H^1(\Omega)$ and for any $\alpha, \beta > 0$, it holds that*

$$\min(\alpha, \beta) \le \frac{\alpha\, (\vec{r}_{1,K}^{\,T}\, G_K(v)\, \vec{r}_{1,K}) + \beta\, (\vec{r}_{2,K}^{\,T}\, G_K(v)\, \vec{r}_{2,K})}{|v|_{H^1(\Delta_K)}^2} \le \max(\alpha, \beta),$$

$G_K(v)$ being the matrix defined in (6).

Let us start our a posteriori analysis by dealing firstly with the advection-diffusion-reaction model problem.

3 The advection-diffusion-reaction problem

We address the standard scalar advection-diffusion-reaction problem with mixed boundary conditions: find u such that

$$\begin{cases} -\mu\Delta u + \vec{\beta}\cdot\nabla u + \alpha\, u = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \Gamma_D\,, \\ \mu\dfrac{\partial u}{\partial n} = g & \text{on } \Gamma_N\,, \end{cases} \tag{7}$$

where Γ_D and Γ_N are suitable measurable nonoverlapping partitions of the boundary $\partial\Omega$ of Ω with $\Gamma_D \neq \emptyset$ and such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$; the source $f \in L^2(\Omega)$, the diffusivity $\mu \in \mathbb{R}^+$, the advective field $\vec{\beta} \in (W^{1,\infty}(\Omega))^2$, with $\nabla\cdot\vec{\beta} = 0$, the reaction coefficient $\alpha \in L^\infty(\Omega)$ with $\alpha \geq 0$ a.e. in Ω, and $g \in L^2(\Gamma_N)$ are given data, while $\partial u/\partial n = \nabla u \cdot \vec{n}$ is the normal derivative of u, $\vec{n}$ being the unit outward normal to $\partial\Omega$.

The weak form associated with (7) is: find $u \in V \equiv H^1_{\Gamma_D}(\Omega) = \{v \in H^1(\Omega) \text{ s.t. } v|_{\Gamma_D} = 0\}$ such that, for any $v \in V$,

$$\int_\Omega \mu\nabla u\cdot\nabla v\, d\vec{x} + \int_\Omega (\vec{\beta}\cdot\nabla u + \alpha\, u)v\, d\vec{x} = \int_\Omega f\, v\, d\vec{x} + \int_{\Gamma_N} g\, v\, ds\,. \tag{8}$$

The discrete form associated with (8) is obtained by projecting onto the space $V_h \subset V$ of continuous piecewise linear finite elements and stabilizing through the streamline-diffusion method [13] which yield: find $u_h \in V_h$ such that, for any $v_h \in V_h$,

$$\begin{aligned} & \int_\Omega \mu\nabla u_h\cdot\nabla v_h\, d\vec{x} + \int_\Omega (\vec{\beta}\cdot\nabla u_h + \alpha\, u_h)v_h\, d\vec{x} \\ + \quad & \sum_{K\in\mathcal{T}_h}\int_K \tau_K\,(-\mu\Delta u_h + \vec{\beta}\cdot\nabla u_h + \alpha\, u_h)\,(\vec{\beta}\cdot\nabla v_h)\, d\vec{x} \\ = \quad & \int_\Omega f\, v_h\, d\vec{x} + \int_{\Gamma_N} g\, v_h\, ds + \sum_{K\in\mathcal{T}_h}\int_K \tau_K\, f(\vec{\beta}\cdot\nabla v_h)\, d\vec{x}\,, \end{aligned} \tag{9}$$

where we have introduced the element stability coefficients τ_K's. Problem (9) can be cast in the abstract form: find $u_h \in V_h$ such that, for any $v_h \in V_h$,

$$A_\tau(u_h, v_h) = F_\tau(v_h)\,,$$

where the stabilized bilinear form $A_\tau : V\times V \to \mathbb{R}$ and the stabilized linear form

$F_\tau : V \to \mathbb{R}$ for smooth enough functions u and v, are defined as

$$\begin{aligned} A_\tau(u,v) &= \int_\Omega \mu \nabla u \cdot \nabla v \, d\vec{x} + \int_\Omega (\vec{\beta} \cdot \nabla u + \alpha\, u) v \, d\vec{x} \\ &+ \sum_{K \in \mathcal{T}_h} \int_K \tau_K \, (-\mu \Delta u + \vec{\beta} \cdot \nabla u + \alpha\, u)\, (\vec{\beta} \cdot \nabla v)\, d\vec{x}\,, \\ F_\tau(v) &= \int_\Omega f\, v \, d\vec{x} + \int_{\Gamma_N} g\, v\, ds + \sum_{K \in \mathcal{T}_h} \int_K \tau_K\, f(\vec{\beta} \cdot \nabla v)\, d\vec{x}\,. \end{aligned} \tag{10}$$

We also let A_0 and F_0 be the corresponding nonstabilized bilinear and linear form, respectively, obtained by simply taking $\tau_K = 0$, for any $K \in \mathcal{T}_h$, in (10). Notice that the exact solution u to (7) satisfies $A_\tau(u,v) = F_\tau(v)$, for any $v \in V$, provided that u has extra regularity, namely $u \in V$ with $\Delta u|_K \in L^2(K)$, for any $K \in \mathcal{T}_h$. For example, $u \in H^2(\Omega)$ suffices. Whenever this extra regularity is guaranteed, the usual Galerkin orthogonality property with respect to A_τ

$$A_\tau(u - u_h, v_h) = 0\,, \qquad \text{for any } v_h \in V_h\,,$$

follows. Unfortunately, due to the presence of the mixed boundary conditions in (7), this extra regularity is difficult to obtain in practice. In general if $u \in V$ then it satisfies $A_0(u,v) = F_0(v)$, for any $v \in V$, and we have the following (weaker) result

Lemma 3.1 *Let $e_h = u - u_h$, then for any $v_h \in V_h$, it holds that*

$$\begin{aligned} A_0(e_h,\, v_h) &= \int_\Omega \mu \nabla e_h \cdot \nabla v_h \, d\vec{x} + \int_\Omega (\vec{\beta} \cdot \nabla e_h + \alpha\, e_h) v_h \, d\vec{x} \\ &= \sum_{K \in \mathcal{T}_h} \int_K \tau_K \, (-\mu \Delta u_h + \vec{\beta} \cdot \nabla u_h + \alpha\, u_h - f)\, (\vec{\beta} \cdot \nabla v_h)\, d\vec{x}\,. \end{aligned} \tag{11}$$

Proof. The thesis follows on subtracting (9) from (8) tested against $v = v_h$, for any $v_h \in V_h$. □

Notice that relation (11) differs from the standard Galerkin orthogonality property due to the presence of the stabilization terms and to the reduced regularity of the solution u of (8).

3.1 An anisotropic a priori error analysis

In [30] we re-addressed the question of a careful design for the element stability coefficients for a scalar advective-diffusive problem in the framework of anisotropic meshes. We limited ourselves to problem (7), with $\alpha = 0$, provided with homogeneous Dirichlet boundary conditions and to the case of affine finite elements. Starting from the analysis in [19] we studied the convergence of the stabilized method (9) in a mesh dependent norm taking into account the effect of the stability terms as functions of the coefficients τ_K's. The new values of the latter are then obtained through error analysis considerations by requiring that the convergence rate be of maximal order in both the advective and diffusive dominated regimes. The main result of this analysis is:

Theorem 3.1 *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of (8) with $\Gamma_N = \emptyset$ and let u_h be the corresponding linear finite element approximation. Then the expression for the stability coefficients and of the local Péclet number $\mathbb{P}\mathrm{e}_K$ as functions of the anisotropic spacings is given by $\tau_K = \lambda_{2,K}\, \xi(\mathbb{P}\mathrm{e}_K)/(2\, \|\vec{\beta}\|_{L^\infty(K)})$, $\mathbb{P}\mathrm{e}_K = \lambda_{2,K} \|\vec{\beta}\|_{L^\infty(K)}/(6\,\mu)$, respectively, where $\xi(\mathbb{P}\mathrm{e}_K) = \mathbb{P}\mathrm{e}_K$ if $\mathbb{P}\mathrm{e}_K < 1$ and $\xi(\mathbb{P}\mathrm{e}_K) = 1$ if $\mathbb{P}\mathrm{e}_K \geq 1$. For this choice there exists a constant $C = C(\widehat{K})$ such that*

$$\|u - u_h\|_h^2 \leq C \sum_{K \in \mathcal{T}_h} \left\{ \lambda_{2,K}^2 \Big(\lambda_{2,K}\, \|\vec{\beta}\|_{L^\infty(K)}\, \mathcal{H}(\mathbb{P}\mathrm{e}_K - 1) + \mu\, \mathcal{H}(1 - \mathbb{P}\mathrm{e}_K) \Big) \left[s_K^4\, L_K^{1,1}(u) + L_K^{2,2}(u) + 2\, s_K^2\, L_K^{1,2}(u) \right] \right\},$$

where the quantities $L_K^{i,j}(u)$ are defined as in (4), $\mathcal{H}(\cdot)$ is the Heaviside function and the discrete norm $\|\cdot\|_h$ is defined by

$$\|w\|_h^2 = \mu\, \|\nabla w\|_{L^2(\Omega)}^2 + \sum_{K \in \mathcal{T}_h} \|\tau_K^{1/2}\, \vec{\beta} \cdot \nabla w\|_{L^2(K)}^2$$

for any $w \in H_0^1(\Omega)$.

The choice for the τ_K's suggested in the theorem above will be numerically validated in Section 5. We refer to [30] for more details and to [3, 4, 5, 7, 28] for alternative approaches.

3.2 An anisotropic a posteriori error analysis

In this section, we extend the a posteriori error analysis carried out in [18] dealing with an advection-diffusion problem associated with the transport of a solute (like oxygen or lipids) by the blood stream in a large artery. In particular, we follow the method of Dual-Weighted Residuals [6, 34] which allows to compute optimally economical meshes as well as reliable and efficient error bounds for arbitrary functionals of the error.

Let us introduce the following dual problem associated with (8): find $z \in V$ such that

$$A_0^*(z, v) = J(v) \qquad \text{for any } v \in V, \tag{12}$$

with $A_0^*(\cdot,\cdot)$ the adjoint form to A_0 and where $J(\cdot) : V \to \mathbb{R}$ is a linear functional to be suitably chosen according to the physical quantity to control. Notice that if $J(v) = \int_\Omega \varphi v \, d\vec{x}$ with $\varphi \in L^2(\Omega)$, then the choice of the space V for the adjoint problem (12) is consistent with the theory in [23].
We can prove the following result

Theorem 3.2 *Let u, u_h and z be the solutions to (8), (9) and (12), respectively, and $e_h = u - u_h$. Then for any $z_h \in V_h$, we have*

$$J(e_h) = \sum_{K\in\mathcal{T}_h} \left\{ \int_K \rho_K(u_h)(z - z_h - \tau_K(\vec{\beta}\cdot\nabla z_h))\, d\vec{x} + \frac{1}{2}\int_{\partial K} j_e(z - z_h)\, ds \right\} \quad (13)$$

where, for any $K \in \mathcal{T}_h$ and $e \in \partial K$, $\rho_K(u_h) = (f + \mu\Delta u_h - \vec{\beta}\cdot\nabla u_h - \alpha\, u_h)|_K$ and

$$j_e = \begin{cases} 0 & \text{if } e \in \Gamma_D\,, \\ -2\left(\mu\,\dfrac{\partial u_h}{\partial n_K} - g\right) & \text{if } e \in \Gamma_N\,, \\ -\mu\left[\dfrac{\partial u_h}{\partial n_K}\right]_e & \text{if } e \in \mathcal{E}_h^{\rm int}\,, \end{cases}$$

are the element interior and boundary residuals, respectively, associated with the finite element solution u_h. Here $\vec{n}_K$ is the unit outward normal to ∂K, $\mathcal{E}_h^{\rm int}$ denotes the set of the internal edges of the skeleton $\mathcal{E}_h$ of the triangulation $\mathcal{T}_h$, and $[\partial u_h/\partial n_K]_e = \partial u_h/\partial n_K + \partial u_h/\partial n_{K'}$ stands for the jump of the normal derivative $\partial u_h/\partial n_K$ over the edge e, K' being the triangle sharing the edge e with K.

Proof. By choosing $v = e_h$ in (12) and exploiting the identity $A_0^*(z,v) = A_0(v,z)$, for any $v \in V$, we obtain

$$J(e_h) = A_0^*(z, e_h) = A_0(e_h, z) = \int_\Omega \mu\nabla e_h\cdot\nabla z\, d\vec{x} + \int_\Omega (\vec{\beta}\cdot\nabla e_h + \alpha\, e_h) z\, d\vec{x}. \quad (14)$$

Using Lemma 3.1 in (14), we have, for any $z_h \in V_h$,

$$\begin{aligned} J(e_h) &= \int_\Omega \mu\nabla e_h\cdot\nabla(z - z_h)\, d\vec{x} + \int_\Omega (\vec{\beta}\cdot\nabla e_h + \alpha\, e_h)(z - z_h)\, d\vec{x} \\ &+ \sum_{K\in\mathcal{T}_h} \int_K \tau_K\,(-\mu\Delta u_h + \vec{\beta}\cdot\nabla u_h + \alpha\, u_h - f)(\vec{\beta}\cdot\nabla z_h)\, d\vec{x}\,. \end{aligned}$$

The definition of e_h and the weak form (8) yield

$$\begin{aligned}
J(e_h) &= \int_\Omega f(z-z_h)\,d\vec{x} + \int_{\Gamma_N} g\,(z-z_h)\,ds - \int_\Omega \mu\nabla u_h\cdot\nabla(z-z_h)\,d\vec{x} \\
&- \int_\Omega (\vec{\beta}\cdot\nabla u_h + \alpha\, u_h)(z-z_h)\,d\vec{x} \\
&+ \sum_{K\in\mathcal{T}_h}\int_K \tau_K(-\mu\Delta u_h + \vec{\beta}\cdot\nabla u_h + \alpha\, u_h - f)(\vec{\beta}\cdot\nabla z_h)\,d\vec{x}\,.
\end{aligned}$$

We now express the integrals on Ω as sum of integrals over the elements of the triangulation and we integrate by parts where needed to obtain

$$\begin{aligned}
J(e_h) &= \sum_{K\in\mathcal{T}_h}\int_K (f+\mu\Delta u_h - \vec{\beta}\cdot\nabla u_h - \alpha\, u_h)(z-z_h)\,d\vec{x} \\
&- \sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\mathcal{E}_h^{\rm int}} \mu\frac{\partial u_h}{\partial n_K}(z-z_h)\,ds - \sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\Gamma_N}\left(\mu\frac{\partial u_h}{\partial n_K} - g\right)(z-z_h)\,ds \\
&- \sum_{K\in\mathcal{T}_h}\int_K \tau_K\,(f+\mu\Delta u_h - \vec{\beta}\cdot\nabla u_h - \alpha\, u_h)(\vec{\beta}\cdot\nabla z_h)\,d\vec{x}\,.
\end{aligned}$$

The result (13) follows on recalling the definitions of $\rho_K(u_h)$ and j_e. □

Let us state the following theorem, which is the desired anisotropic a posteriori estimate for problem (7).

Theorem 3.3 *Let u, u_h and z be the solutions to* (8), (9) *and* (12), *respectively, and $e_h = u - u_h$. Then there exists a constant $C = C(M,\widehat{C})$ such that*

$$\begin{aligned}
|J(e_h)| \le C\sum_{K\in\mathcal{T}_h} & \left[\|\rho_K(u_h)\|_{L^2(K)}\left(1+\frac{\tau_K}{\lambda_{2,K}}\|\vec{\beta}\|_{L^\infty(K)}\right) + \frac{1}{2\lambda_{2,K}^{1/2}}\|j_e\|_{L^2(\partial K)}\right] \\
&\left(\sum_{i=1}^{2}\lambda_{i,K}^2\left(\vec{r}_{i,K}^{\,T} G_K(z)\vec{r}_{i,K}\right)\right)^{1/2}, \qquad (15)
\end{aligned}$$

where $\rho_K(u_h), j_e$ are defined as in Theorem 3.2 and $G_K(z)$ is given in (6).

Proof. The thesis follows from (13), by firstly adding and subtracting z in the term involving the advective field,

$$J(e_h) = \sum_{K\in\mathcal{T}_h}\left\{\int_K \rho_K(u_h)\big(z - z_h - \tau_K\left[\vec{\beta}\cdot\nabla((z_h - z) + z)\right]\big)\,d\vec{x} + \frac{1}{2}\int_{\partial K} j_e(z-z_h)\,ds\right\},$$

then using the Cauchy-Schwarz inequality on all terms involving $z - z_h$ together with relation

$$\|\vec{\beta}\cdot\nabla z\|_{L^2(K)} \le \|\vec{\beta}\|_{L^\infty(K)}\|\nabla z\|_{L_2(K)}$$

and finally using Proposition 2.3 (with $\alpha = s_K$ and $\beta = 1/s_K$) to bound $\|\nabla z\|_{L_2(K)}$ and Proposition 2.2 after choosing $z_h|_K = I_K(z)$. □

Remark 3.1 *We point out that* (15) *has a classical structure, according to the theory in [6, 34], as it can be cast in the form*

$$|J(e_h)| \le C \sum_{K\in\mathcal{T}_h} \alpha_K \, R_K(u_h)\, \omega_K(z)\,, \tag{16}$$

where $\alpha_K = (\lambda_{1,K}\,\lambda_{2,K})^{3/2}$,

$$R_K(u_h) = \frac{1}{(\lambda_{1,K}\lambda_{2,K})^{1/2}}\left(\|\rho_K(u_h)\|_{L^2(K)} \left(1 + \frac{\tau_K}{\lambda_{2,K}}\|\vec{\beta}\|_{L^\infty(K)}\right) + \frac{1}{2\lambda_{2,K}^{1/2}}\|j_e\|_{L^2(\partial K)}\right)$$

and

$$\omega_K(z) = \frac{1}{(\lambda_{1,K}\,\lambda_{2,K})^{1/2}}\left(s_K\left(\vec{r}_{1,K}^{\,T} G_K(z)\vec{r}_{1,K}\right) + \frac{1}{s_K}\left(\vec{r}_{2,K}^{\,T} G_K(z)\vec{r}_{2,K}\right)\right)^{1/2}.$$

The target functionals of the discretization error are bounded in terms of residuals and interpolation errors associated with the primal and dual problems, respectively. Thus, choosing different control functionals implies changing the dual problem only. In particular, the residual terms are related to the source of errors when approximating the solution of the primal problem, while the solution of the dual problem determines the way these errors accumulate and propagate according to the chosen controlled functionals.
Expression (16) *will be used in Section 5 to develop an adaptive technique.*

We are now in a position to analyze the Stokes problem.

4 The Stokes problem

In this section we extend the a posteriori analysis provided in Section 3 to the Stokes problem. We seek the velocity $\vec{u}$ and the pressure p of an incompressible fluid, subject to mixed boundary conditions: find $(\vec{u}, p)$ such that

$$\begin{cases} -\mu\Delta\vec{u} + \nabla p = \vec{f} & \text{in } \Omega\,, \\ \nabla\cdot\vec{u} = 0 & \text{in } \Omega\,, \\ \mu\,(\nabla\vec{u})\,\vec{n} - p\,\vec{n} = \vec{g} & \text{on } \Gamma_N\,, \\ \vec{u} = \vec{0} & \text{on } \Gamma_D\,, \end{cases} \tag{17}$$

where Γ_D and Γ_N are suitable measurable nonoverlapping partitions of the boundary $\partial\Omega$ of Ω with $\Gamma_D \ne \emptyset$ and such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$; the source term $\vec{f} \in [L^2(\Omega)]^2$, the viscosity $\mu \in \mathbb{R}^+$, $\vec{g} \in [L^2(\Gamma_N)]^2$ are given data, and $\vec{n}$ is again the unit outward normal to $\partial\Omega$. The weak form associated with (17) is: find $(\vec{u}, p)$ in $W \times Q$ such that, for any $(\vec{v}, q) \in W \times Q$,

$$\begin{cases} \displaystyle\int_\Omega \mu \nabla \vec{u} : \nabla \vec{v}\, d\vec{x} - \int_\Omega p \nabla\cdot\vec{v}\, d\vec{x} = \int_\Omega \vec{f} \cdot \vec{v}\, d\vec{x} + \int_{\Gamma_N} \vec{g} \cdot \vec{v}\, ds, \\ \displaystyle -\int_\Omega q\, \nabla\cdot\vec{u}\, d\vec{x} = 0\,, \end{cases} \tag{18}$$

where $W \equiv [H^1_{\Gamma_D}(\Omega)]^2 = \{\vec{v} \in [H^1(\Omega)]^2 \text{ s.t. } \vec{v}|_{\Gamma_D} = \vec{0}\}$ and $Q \equiv L^2(\Omega)$. If $\Gamma_N = \emptyset$ then $Q = L^2_0(\Omega) = \{v \in L^2(\Omega) \text{ s.t. } \int_\Omega v\, d\vec{x} = 0\}$.

We discretize (18) by using the GLS method [21, 26]. The discrete problem is: find $(\vec{u}_h, p_h)$ in $W_h \times Q_h$ such that, for any $(\vec{v}_h, q_h) \in W_h \times Q_h$,

$$\begin{cases} \displaystyle\int_\Omega \mu \nabla \vec{u}_h : \nabla \vec{v}_h\, d\vec{x} - \int_\Omega p_h \nabla\cdot\vec{v}_h\, d\vec{x} = \int_\Omega \vec{f} \cdot \vec{v}_h\, d\vec{x} + \int_{\Gamma_N} \vec{g} \cdot \vec{v}_h\, ds\,, \\ \displaystyle -\int_\Omega q_h\, \nabla\cdot\vec{u}_h\, d\vec{x} - \sum_{K \in \mathcal{T}_h} \tau_K \int_K \nabla p_h \cdot \nabla q_h\, d\vec{x} = -\sum_{K \in \mathcal{T}_h} \tau_K \int_K \vec{f} \cdot \nabla q_h\, d\vec{x}\,, \end{cases} \tag{19}$$

where both the spaces $W_h \subset W$ and $Q_h \subset Q$ are formed by continuous piecewise linear finite element functions over $\mathcal{T}_h$, and the τ_K's are the stabilization coefficients whose expression will be specified in the next subsection. Due to this choice of the finite element space, the stabilized methods, such as GLS, SUPG [8, 20, 26] and the method proposed in [12], do coincide with each other.

4.1 An anisotropic a priori error analysis

As in the case of the advection-diffusion-reaction problem, when using stabilized finite elements the design of the stability coefficients τ_K's in (19) is a critical issue, in particular in the case of strongly anisotropic grids. In [32] numerical experiments show that good results can be obtained using the minimum edge length. In [30] we provide a theoretical construction of the stability coefficients for problem (17) with $\Gamma_N = \emptyset$. We recall here the main result of this analysis.

Theorem 4.1 *Let $(\vec{u}, p) \in ([H^1_0(\Omega)]^2 \cap [H^2(\Omega)]^2) \times (L^2_0(\Omega) \cap H^1(\Omega))$ be the solution to (18) with $\Gamma_N = \emptyset$ and let $(\vec{u}_h, p_h)$ be the corresponding affine finite element approximation. Then an anisotropic expression for the stability coefficients is given by $\tau_K = \alpha\, \lambda^2_{2,K}/\mu$, where $\alpha \simeq O(1)$ is a positive constant. With this choice there exists a constant $C = C(M, \widehat{C}, \widehat{K})$ such that*

$$\|(\vec{u} - \vec{u}_h, p - p_h)\|^2_h \le C \sum_{K \in \mathcal{T}_h} \left\{ \frac{\mu}{\lambda^2_{2,K}} \sum_{i,j=1}^{2} \lambda^2_{i,K} \lambda^2_{j,K} L^{i,j}_K(\vec{u}) + \frac{1}{\mu} \sum_{i=1}^{2} \lambda^2_{i,K} \big(\vec{r}^{\,T}_{i,K} G_K(p) \vec{r}_{i,K}\big) \right\},$$

where the quantities $L_K^{i,j}(\vec{u})$ and $G_K(p)$ are defined as in (5) *and* (6)*, respectively, M and $\widehat{C}$ are given through relations* (1)*, and the discrete norm $\|(\cdot,\cdot)\|_h$ is defined, for any $(\vec{\varphi},\psi) \in [H_0^1(\Omega)]^2 \times (L_0^2(\Omega) \cap H^1(\Omega))$ by*

$$\|(\vec{\varphi},\psi)\|_h = \Big(\mu\|\nabla\vec{\varphi}\|_{L^2(\Omega)}^2 + \sum_{K\in\mathcal{T}_h} \|\tau_K^{1/2}\nabla\psi\|_{L^2(K)}^2\Big)^{1/2}.$$

The definition in Theorem 4.1 for the stability coefficients τ_K's has been obtained by error analysis considerations by demanding the maximal order for the convergence rate of $\|(\vec{u}-\vec{u}_h, p-p_h)\|_h$. Numerical experiments confirm that this choice gives better results than the naïve choice $\tau_K \simeq h_K^2/\mu$ [21], in particular when the mesh is "aligned" with the solution, numerical diffusion being reduced (see [30] for the details). In a loose sense, alignment means that, on any element of the mesh, the largest directional (first or second) derivatives of the numerical solution are along the direction $\vec{r}_{2,K}$ of the minimum size of the element, and consequently, the smallest derivatives are along the direction $\vec{r}_{1,K}$ of maximum spacing of the element. A theoretical investigation of the constant α in Theorem 4.1 is carried out in [31].

4.2 An anisotropic a posteriori error analysis

Here we the extend the a posteriori error analysis of Section 3.2 to the Stokes problem [17]. Again, we go through a dual problem associated with the primal one according to the theory in [6, 34]. In this case, the dual problem related to (18) is: find $(\vec{w},r)$ in $W\times Q$ such that, for any $(\vec{v},q)\in W\times Q$,

$$\left\{\begin{array}{l} \displaystyle\int_\Omega \mu\nabla\vec{w} : \nabla\vec{v}\,d\vec{x} - \int_\Omega r\nabla\cdot\vec{v}\,d\vec{x} = J_1(\vec{v}), \\ \displaystyle -\int_\Omega q\,\nabla\cdot\vec{w}\,d\vec{x} = J_2(q), \end{array}\right. \tag{20}$$

where $J_1(\cdot) : [H^1(\Omega)]^2 \to \mathbb{R}$ and $J_2(\cdot) : L^2(\Omega) \to \mathbb{R}$ are suitable linear functionals chosen according to the quantities one wishes to control. Notice that the dual Stokes problem coincides with the primal one, except for the boundary conditions and the data, due to the symmetry of (18). Moreover, if $J_1(\vec{v}) = \int_\Omega \vec{\varphi}\cdot\vec{v}\,d\vec{x}$ and $J_2(q) = \int_\Omega \psi\, q\, d\vec{x}$ for any $\vec{\varphi} \in [L^(\Omega)]^2$ and $\psi \in L^2(\Omega)$, then the boundary conditions involved in (20) are consistent with the theory in [23].

Let us start with the following result

Lemma 4.1 *Let $\vec{e}_u = \vec{u}-\vec{u}_h$ and $e_p = p-p_h$. Then for any $(\vec{v}_h,q_h)\in W_h\times Q_h$, it holds that*

$$\int_\Omega \mu\nabla\vec{e}_u : \nabla\vec{v}_h\,d\vec{x} - \int_\Omega e_p\nabla\cdot\vec{v}_h\,d\vec{x} - \int_\Omega q_h\nabla\cdot\vec{e}_u\,d\vec{x} = -\sum_{K\in\mathcal{T}_h}\tau_K\int_K (\nabla p_h - \vec{f})\cdot\nabla q_h\,d\vec{x}\,.$$

Proof. The proof follows on subtracting (19) from (18) tested against discrete functions $\vec{v}_h$ and q_h. $\square$

Let us now state the following theorem, which is the starting point of our anisotropic a posteriori error analysis for the Stokes problem.

Theorem 4.2 *Let $(\vec{u}, p)$ and $(\vec{u}_h, p_h)$ be the solutions to* (18) *and* (19), *respectively, and $\vec{e}_u$, e_p their corresponding discretization errors. Then for any $(\vec{v}_h, q_h) \in W_h \times Q_h$, we have*

$$J_1(\vec{e}_u) + J_2(e_p) = \sum_{K \in \mathcal{T}_h} \left\{ \int_K \vec{\rho}_K^1(\vec{u}_h, p_h) \cdot (\vec{w} - \vec{v}_h)\, d\vec{x} + \int_K \rho_K^2(\vec{u}_h)(r - q_h)\, d\vec{x} \right.$$
$$\left. + \ \tau_K \int_K \vec{\rho}_K^1(\vec{u}_h, p_h) \cdot \nabla q_h\, d\vec{x} + \frac{1}{2} \int_{\partial K} \vec{j}_e \cdot (\vec{w} - \vec{v}_h)\, ds \right\}, \tag{21}$$

where, for any $K \in \mathcal{T}_h$ and any edge $e \in \partial K$, $\vec{\rho}_K^1(\vec{u}_h, p_h) = (\vec{f} + \mu \Delta \vec{u}_h - \nabla p_h)|_K$ and $\rho_K^2(\vec{u}_h) = (\nabla \cdot \vec{u}_h)|_K$ are the element internal residuals associated with the momentum and continuity equations, respectively whereas

$$\vec{j}_e = \begin{cases} \vec{0} & \text{if } e \in \Gamma_D\,, \\ 2\,(\vec{g} - (\mu(\nabla \vec{u}_h\, \vec{n}_K) - p_h\, \vec{n}_K)) & \text{if } e \in \Gamma_N\,, \\ -\left[(\mu(\nabla \vec{u}_h\, \vec{n}_K) - p_h\, \vec{n}_K)\right]_e & \text{if } e \in \mathcal{E}_h^{\mathrm{int}}\,, \end{cases}$$

is the edge residual, proportional to the jump of the normal component of the Cauchy stresses. The quantities $\vec{n}_K, \mathcal{E}_h^{\mathrm{int}}$ and $[\cdot]_e$ are defined as in Theorem 3.2.

Proof. Let us choose $\vec{v} = \vec{e}_u \in W$ and $q = e_p \in Q$ in (20). Notice that the same choice for the space W for the primal and dual problems is motivated by the fact that $(\vec{u} - \vec{u}_h)|_{\Gamma_D} = \vec{0}$, i.e. there is no approximation error associated with the Dirichlet boundary condition. Using Lemma 4.1, we have

$$J_1(\vec{e}_u) + J_2(e_p) = \int_\Omega \mu \nabla(\vec{w} - \vec{v}_h) : \nabla \vec{u}\, d\vec{x} - \int_\Omega (r - q_h) \nabla \cdot \vec{u}\, d\vec{x}$$
$$- \int_\Omega p \nabla \cdot (\vec{w} - \vec{v}_h)\, d\vec{x} - \left[\int_\Omega \mu \nabla(\vec{w} - \vec{v}_h) : \nabla \vec{u}_h\, d\vec{x} - \int_\Omega (r - q_h) \nabla \cdot \vec{u}_h\, d\vec{x} \right.$$
$$\left. - \int_\Omega p_h \nabla \cdot (\vec{w} - \vec{v}_h)\, d\vec{x} \right] - \sum_{K \in \mathcal{T}_h} \tau_K \int_K (\nabla p_h - \vec{f}) \cdot \nabla q_h\, d\vec{x}\,. \tag{22}$$

From (18) where we take $\vec{v} = \vec{w} - \vec{v}_h \in W$ and $q = r - q_h \in Q$ and integrating by parts the fourth and sixth term in (22), we have

$$
\begin{aligned}
& J_1(\vec{e}_u)+J_2(e_p)=\int_\Omega \vec{f}\cdot(\vec{w}-\vec{v}_h)\,d\vec{x}+\int_{\Gamma_N}\vec{g}\cdot(\vec{w}-\vec{v}_h)\,ds \\
& -\sum_{K\in\mathcal{T}_h}\Big[-\int_K \mu(\vec{w}-\vec{v}_h)\cdot\Delta\vec{u}_h\,d\vec{x}+\int_{\partial K}\mu(\nabla\vec{u}_h\,\vec{n}_K)\cdot(\vec{w}-\vec{v}_h)\,ds \\
& -\int_K (r-q_h)\nabla\cdot\vec{u}_h\,d\vec{x}+\int_K(\vec{w}-\vec{v}_h)\cdot\nabla p_h\,d\vec{x}-\int_{\partial K}(\vec{w}-\vec{v}_h)\cdot\vec{n}_K\,p_h\,ds \\
& +\tau_K\int_K(\nabla p_h-\vec{f})\cdot\nabla q_h\,d\vec{x}\Big],
\end{aligned}
$$

where some of the integrals have been split over the elements of the triangulation $\mathcal{T}_h$. Result (21) follows on rewriting the above terms and using the definitions of the residuals $\vec{\rho}_K^1(\vec{u}_h,p_h),\rho_K^2(\vec{u}_h)$ and $\vec{j}_e$. □

Finally, we complete our a posteriori error analysis by proving the following

Theorem 4.3 *Let $(\vec{w},r)$ be the solution to the dual problem (20) with $\vec{w}\in[H^2(\Omega)]^2\cap W$ and let $\vec{e}_u,e_p$ as in Lemma 4.1. Then there exists a constant $C=C(M,\widehat{C},\widehat{K})$ such that*

$$
\begin{aligned}
& |J_1(\vec{e}_u)+J_2(e_p)|\le C\sum_{K\in\mathcal{T}_h}\Bigg\{ \\
& \left[||\vec{\rho}_K^1(\vec{u}_h,p_h)||_{L^2(K)}+\frac{1}{2}||\vec{j}_e||_{L^2(\partial K)}\left(\frac{\lambda_{1,K}^2+\lambda_{2,K}^2}{\lambda_{2,K}^3}\right)^{1/2}\right]\left[\sum_{i,j=1}^{2}\lambda_{i,K}^2\lambda_{j,K}^2L_K^{i,j}(\vec{w})\right]^{1/2} \\
& +\left[||\rho_K^2(\vec{u}_h)||_{L^2(K)}+\frac{\tau_K}{\lambda_{2,K}}||\vec{\rho}_K^1(\vec{u}_h,p_h)||_{L^2(K)}\right]\left[\sum_{i=1}^{2}\lambda_{i,K}^2\left(\vec{r}_{i,K}^{\,T}G_K(r)\vec{r}_{i,K}\right)\right]^{1/2}\Bigg\},
\end{aligned}
\tag{23}
$$

where $\vec{\rho}_K^1(\vec{u}_h,p_h),\rho_K^2(\vec{u}_h),\vec{j}_e$ are defined as in Theorem 4.2, the quantities $L_K^{i,j}(\vec{w})$ and $G_K(r)$ are given in (5) and in (6), respectively, and the constants M and $\widehat{C}$ are defined through (1).

Proof. Since in (21) the functions $\vec{v}_h$ and q_h are arbitrary, we choose them as suitable interpolants of $\vec{w}$ and r, respectively, i.e. $\vec{v}_h|_K=\Pi_K(\vec{w})$ and $q_h|_K=I_K(r)$. Using the Cauchy-Schwarz inequality in (21), we obtain

$$
\begin{aligned}
& |J_1(\vec{e}_u)+J_2(e_p)|\le\sum_{K\in\mathcal{T}_h}\Big\{||\vec{\rho}_K^1(\vec{u}_h,p_h)||_{L^2(K)}||\vec{w}-\Pi_K(\vec{w})||_{L^2(K)} \\
& +||\rho_K^2(\vec{u}_h)||_{L^2(K)}||r-I_K(r)||_{L^2(K)}+\tau_K||\vec{\rho}_K^1(\vec{u}_h,p_h)||_{L^2(K)}||\nabla I_K(r)||_{L^2(K)} \\
& +\frac{1}{2}||\vec{j}_e||_{L^2(\partial K)}||\vec{w}-\Pi_K(\vec{w})||_{L^2(\partial K)}\Big\}.
\end{aligned}
\tag{24}
$$

For the purpose of obtaining a computable bound, it suffices to estimate the interpolation errors in (24) together with the term $||\nabla I_K(r)||_{L^2(K)}$. With this aim, we use Propositions 2.2 and 2.3, and the vectorial extension of Proposition 2.1. We have

$$
\begin{aligned}
|J_1(\vec{e}_u) + J_2(e_p)| \leq C \sum_{K\in\mathcal{T}_h} \Bigg\{ & ||\vec{\rho}^{\,1}_K(\vec{u}_h, p_h)||_{L^2(K)} \left[\sum_{i,j=1}^{2} \lambda_{i,K}^2 \lambda_{j,K}^2 L_K^{i,j}(\vec{w}) \right]^{1/2} \\
& + ||\rho^2_K(\vec{u}_h)||_{L^2(K)} \left[\sum_{i=1}^{2} \lambda_{i,K}^2 \left(\vec{r}_{i,K}^{\,T} G_K(r) \vec{r}_{i,K} \right) \right]^{1/2} \\
& + \frac{\tau_K}{\lambda_{2,K}} ||\vec{\rho}^{\,1}_K(\vec{u}_h, p_h)||_{L^2(K)} \left[\sum_{i=1}^{2} \lambda_{i,K}^2 \left(\vec{r}_{i,K}^{\,T} G_K(r) \vec{r}_{i,K} \right) \right]^{1/2} \\
& + \frac{1}{2} ||\vec{j}_e||_{L^2(\partial K)} \left(\frac{\lambda_{1,K}^2 + \lambda_{2,K}^2}{\lambda_{2,K}^3} \right)^{1/2} \left[\sum_{i,j=1}^{2} \lambda_{i,K}^2 \lambda_{j,K}^2 L_K^{i,j}(\vec{w}) \right]^{1/2}.
\end{aligned}
$$

We remark that the term $||\nabla I_K(r)||_{L^2(K)}$ in (24) has been dealt with combining suitably the triangle inequality together with Proposition 2.2 and Proposition 2.3 with $\alpha = s_K$ and $\beta = 1/s_K$, to give

$$
\begin{aligned}
||\nabla I_K(r)||_{L^2(K)} & \leq ||\nabla (I_K(r) - r)||_{L^2(K)} + ||\nabla r||_{L^2(K)} \\
& \leq C \frac{1}{\lambda_{2,K}} \left[\sum_{i=1}^{2} \lambda_{i,K}^2 \left(\vec{r}_{i,K}^{\,T} G_K(r) \vec{r}_{i,K} \right) \right]^{1/2}.
\end{aligned}
$$

Result (23) follows by suitably rewriting the anisotropic terms coming from the interpolation estimates. □

Expression (23) has a classical structure according to the theory in [6, 34], analogously as what already said in Remark 3.1, as it can be cast in the form

$$
|J_1(\vec{e}_u) + J_2(e_p)| \leq C \sum_{K\in\mathcal{T}_h} \left\{ \alpha_K^1 \, R_K^1(\vec{u}_h, p_h) \, \omega_K^1(\vec{w}) \; + \; \alpha_K^2 \, R_K^2(\vec{u}_h, p_h) \, \omega_K^2(r) \right\}, \tag{25}
$$

where $\alpha_K^1 = (\lambda_{1,K}\,\lambda_{2,K})^2$, $\alpha_K^2 = (\lambda_{1,K}\,\lambda_{2,K})^{3/2}$,

$$
\begin{aligned}
R_K^1(\vec{u}_h, p_h) &= \frac{1}{(\lambda_{1,K}\lambda_{2,K})^{1/2}} \left(||\vec{\rho}^{\,1}_K(\vec{u}_h, p_h)||_{L^2(K)} + \frac{1}{2} ||\vec{j}_e||_{L^2(\partial K)} \left(\frac{\lambda_{1,K}^2 + \lambda_{2,K}^2}{\lambda_{2,K}^3} \right)^{1/2} \right), \\
R_K^2(\vec{u}_h, p_h) &= \frac{1}{(\lambda_{1,K}\lambda_{2,K})^{1/2}} \left(||\rho^2_K(\vec{u}_h)||_{L^2(K)} + \frac{\tau_K}{\lambda_{2,K}} \, ||\vec{\rho}^{\,1}_K(\vec{u}_h, p_h)||_{L^2(K)} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega_K^1(\vec{w}) &= \frac{1}{(\lambda_{1,K}\lambda_{2,K})^{1/2}} \left(s_K^2 L_K^{1,1}(\vec{w}) + 2 L_K^{1,2}(\vec{w}) + \frac{1}{s_K^2} L_K^{2,2}(\vec{w}) \right)^{1/2} \\
\omega_K^2(r) &= \frac{1}{(\lambda_{1,K}\lambda_{2,K})^{1/2}} \left(s_K \, \left(\vec{r}_{1,K}^{\,T} G_K(r) \vec{r}_{1,K} \right) + \frac{1}{s_K} \, \left(\vec{r}_{2,K}^{\,T} G_K(r) \vec{r}_{2,K} \right) \right)^{1/2}.
\end{aligned}
$$

This formulation will be the starting point to drive the adaptive procedure in the following section.

5 Numerical assessment

Aim of this section is to explain how we can get suitable information from the derived a posteriori error estimates to construct the new anisotropic adapted mesh. Then we test such an adaptive procedure on some numerical test cases for both the advection-diffusion-reaction and the Stokes problems.

5.1 Adaptive procedure

The anisotropic information provided by estimates (15) and (23) has been employed within the mesh adaptation procedures in a predictive fashion in order to develop a *metric based algorithm*. Indeed a standard way of generating an anisotropic mesh on a domain Ω is to endow it with a metric represented by a symmetric positive definite tensor $M : \Omega \to \mathbb{R}^{2\times 2}$ [10, 22]. An iterative process is thus carried out and the information concerning the metric is updated by employing the data stemming from the actual mesh. Within this iterative procedure, by analyzing the solutions of the primal and dual problems on a grid $\mathcal{T}_h^k$ we seek for an optimal metric $\widetilde{M}$ to drive the generation of a better, adapted grid $\mathcal{T}_h^{k+1}$. The metric $\widetilde{M}$ is approximated in a piecewise constant fashion over $\mathcal{T}_h^k$, i.e. $\widetilde{M}|_K \in \mathbb{R}^{2\times 2}$, for any $K \in \mathcal{T}_h^k$.
We have that $\widetilde{M} = \widetilde{R}\,\widetilde{\Lambda}^{-2}\widetilde{R}^{\mathrm{T}}$ where $\widetilde{\Lambda} = \mathrm{diag}(\widetilde{\lambda}_1, \widetilde{\lambda}_2)$ and $\widetilde{R}^{\mathrm{T}} = (\widetilde{\vec{r}}_1, \widetilde{\vec{r}}_2)$ are a positive diagonal and an orthogonal tensor, respectively. The quantities $\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\vec{r}}_1, \widetilde{\vec{r}}_2$ are approximated by piecewise constant functions over the triangulation $\mathcal{T}_h^k$, i.e. $\widetilde{\vec{r}}_i|_K = \widetilde{\vec{r}}_{i,K}, \widetilde{\lambda}_i|_K = \widetilde{\lambda}_{i,K}$ for any $K \in \mathcal{T}_h^k$ and for $i = 1, 2$. Because of the definition of the mesh metric (see [22]) these quantities are directly related to the spacing and stretching directions on the adapted grid.

We are then interested in choosing matrix $\widetilde{M}$ so that: *i)* the discretization error in the adapted grid be equidistributed on all the mesh elements while *ii)* maximizing the area of the elements. This latter requirement improves the efficiency of the solution of the linear system associated with the discretization of the partial differential equation at hand, because it tries to reduce the number of degrees of freedom, i.e., the size of the stiffness matrix.

In order to satisfy the constraints *i)* and *ii)*, let us move from relations (16) and (25) by rewriting them as

$$|J(e_h)| \le C \sum_{K\in\mathcal{T}_h} \eta_K \quad \text{and} \quad |J_1(\vec{e}_u) + J_2(e_p)| \le C \sum_{K\in\mathcal{T}_h} \eta_K\,,$$

respectively, where the local estimator $\eta_K = \alpha_K\, R_K(u_h)\, \omega_K(z)$ in the advective-diffusive-reactive case while $\eta_K = \alpha_K^1\, R_K^1(\vec{u}_h, p_h)\, \omega_K^1(\vec{w}) + \alpha_K^2\, R_K^2(\vec{u}_h, p_h)\, \omega_K^2(r)$ for the Stokes problem. Notice that, except for the multiplicative constants α_K, α_K^1, α_K^2, the terms involved in the definition of the local estimators η_K are quantities independent of the measure of the triangle K at least asymptotically, i.e. when the mesh is sufficiently fine.

Since we can only act on the local estimators, we impose on the one hand that $\eta_K = \tau$ for any $K \in \mathcal{T}_h^k$, where τ is a given tolerance, and on the other hand that $|K|$ be as large as possible. This amounts to solving a minimization problem involving the L_K's and the G_K's terms. Let us analyze separately the two problems.

The advection-diffusion-reaction problem
In this case requirement *ii)* amounts to solving the problem: find $\tilde{s}_K$ and $\widetilde{\vec{r}}_{1,K}$ such that

$$\tilde{s}_K \left(\widetilde{\vec{r}}_{1,K}^{\,T} \, G_K(z) \, \widetilde{\vec{r}}_{1,K} \right) + \frac{1}{\tilde{s}_K} \left(\widetilde{\vec{r}}_{2,K}^{\,T} \, G_K(z) \, \widetilde{\vec{r}}_{2,K} \right)$$

be minimized subject to $\tilde{s}_K \geq 1$, $\widetilde{\vec{r}}_{1,K}, \widetilde{\vec{r}}_{2,K} \in \mathbb{R}^2$, $\|\widetilde{\vec{r}}_{1,K}\| = \|\widetilde{\vec{r}}_{2,K}\| = 1$, $\widetilde{\vec{r}}_{1,K} \cdot \widetilde{\vec{r}}_{2,K} = 0$ where $\| \cdot \|$ is the Euclidean norm. The solution of this minimization problem (see [29]) identifies $\widetilde{\vec{r}}_{1,K}$ with a unitary vector parallel to the eigenvector associated with the minimum eigenvalue of $G_K(z)$ while $\tilde{s}_K = \widetilde{\lambda}_{1,K}/\widetilde{\lambda}_{2,K} = (\max(\mathrm{eig}(G_K(z)))/\min(\mathrm{eig}(G_K(z))))^{1/2}$. Then $\widetilde{\vec{r}}_1|_K = \widetilde{\vec{r}}_{1,K}$ and $\widetilde{\lambda}_1/\widetilde{\lambda}_2|_K = \tilde{s}_K$. Finally, requirement *i)* allows us to obtain the specific values for $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$.

The Stokes problem
In this case constraint *ii)* leads to solving a minimization problem involving both the L_K's and the G_K's terms. We proceed by solving two decoupled subproblems for $\widetilde{s}_K$ and $\widetilde{\vec{r}}_{1,K}$, the first one associated with the terms L_K's and the second one with the terms G_K's.
Let us denote with $h_{1,K}$, $h_{2,K}$, (with $|h_{1,K}| \geq |h_{2,K}|$) and $g_{1,K}$, $g_{2,K}$, (with $g_{1,K} \geq g_{2,K}$) the eigenvalues of $H_K(\vec{w})$ and $G_K(r)$, respectively. The solution of the first minimization problem implies $\widetilde{\vec{r}}_{1,K}$ to be parallel to the eigenvector associated with the eigenvalue $h_{2,K}$ and $\widetilde{s}_K = (|h_{1,K}|/|h_{2,K}|)^{1/2}$. On the other hand the solution of the second problem requires $\widetilde{\vec{r}}_{1,K}$ to be parallel to the eigenvector associated with the eigenvalue $g_{2,K}$ and $\widetilde{s}_K = (g_{1,K}/g_{2,K})^{1/2}$. Finally, requirement *i)* plus demanding that each minima of the two subproblems be equal to $\tau/2$ allow us to obtain two couples of values for $\widetilde{\lambda}_{1,K}$ and $\widetilde{\lambda}_{2,K}$. The metric $\widetilde{M}$ is thus obtained by summing the metrics coming from the two subproblems, weighting them with the corresponding residuals [22].

5.2 Some test cases

In this section we assess the performance of the adaptive procedure just described on some numerical test cases.
We highlight that here we are not interested in discussing how the solution z of the dual problem is actually computed. In other words, we are not concerned with the discretization issue of the dual problem: we just assume that we have some accurate enough approximation of z to compute the quantities $G_K(z)$ in (16) and $G_K(r)$, $L_K^{i,j}(\vec{w})$ in (25), respectively (see also [1]).

Throughout, all the anisotropic meshes have been obtained by the mesh generator *BAMG* [25].

5.2.1 The advection-diffusion-reaction problem

We consider in the following two test cases concerning pure advection-diffusion problems ($\alpha = 0$ in (7)).

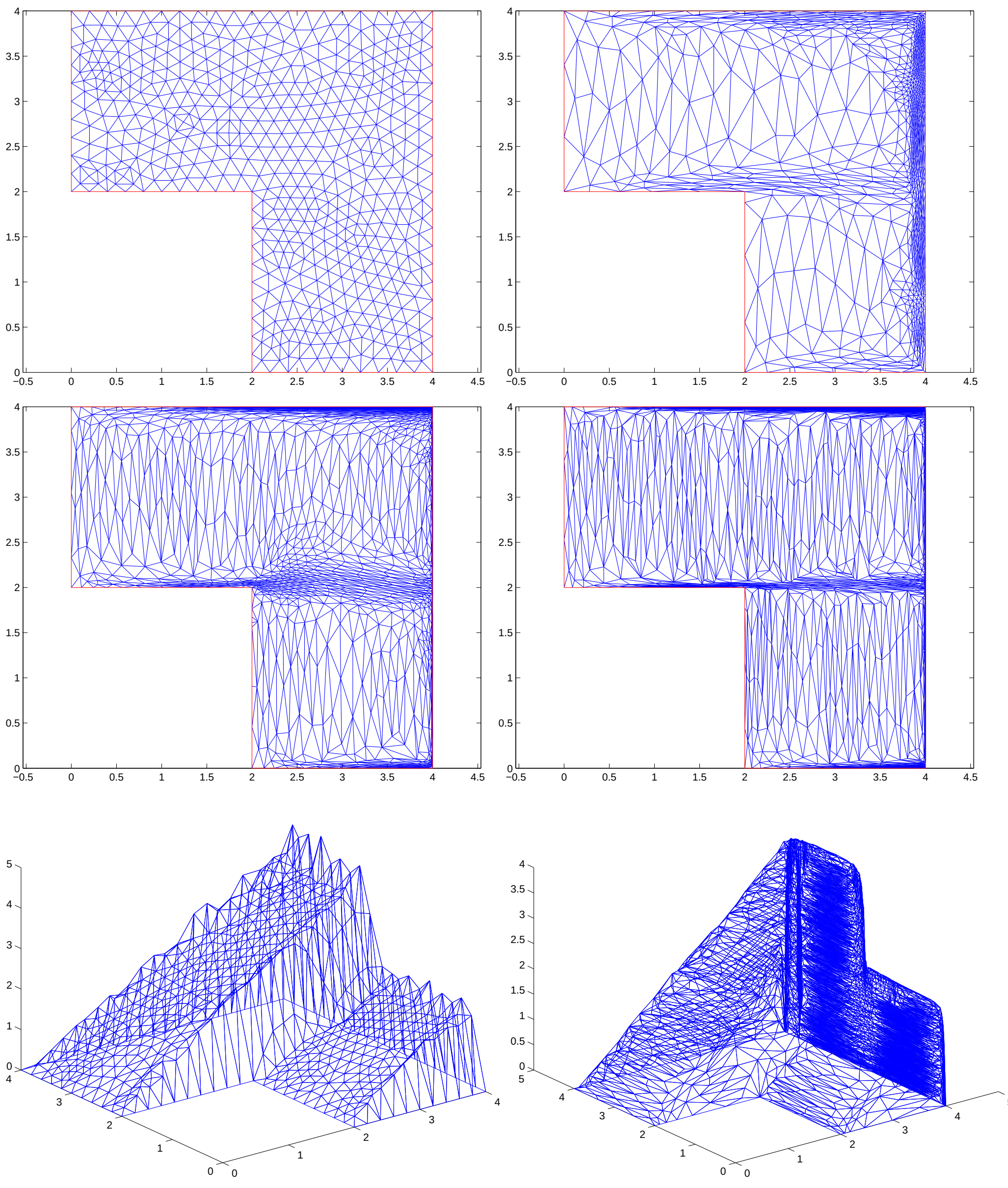

Figure 1: Double ramp test case: sequence of adapted meshes and numerical solutions on the first and last meshes.

The "double ramp" example
For this test case let us choose in (7) $\mu = 10^{-3}$, $f = 1$, $\vec{\beta} = (1, 0)^T$ and homogeneous Dirichlet boundary conditions, i.e. $\Gamma_N = \emptyset$. The domain Ω coincides with an L-shaped region contained in a square of edge length equal to 4 (see Fig. 1). The solution u exhibits a strong boundary layer along $x = 4$, two crosswind boundary layers along $y = 0$ and $y = 4$ and a crosswind internal layer along $y = 2$. With reference to Fig. 1, the adaptive process starts from a uniform mesh (top-left) with 1024 triangles and the first adapted mesh (top-right) consists of 1411 triangles. The fourth and fifth adapted meshes (center-left and center-right), reach 3805 and 4534 elements, respectively. The numerical solution on the initial mesh and on the one obtained after five iterations are displayed in the bottom-left and bottom-right positions, respectively. The functional J has been chosen as $J(v) = A_0(v, u)$ for any $v \in V$. This choice allows us to control the energy norm of the discretization error, as

$$J(u - u_h) = A_0(u - u_h, u) = A_0(u - u_h, u - u_h).$$

Notice that we have used the Galerkin orthogonality property which holds provided that $\Delta u|_K \in L^2(K)$, for any $K \in \mathcal{T}_h$. All the four layers characterizing the solution u are well-captured by the anisotropic error estimate (16) as the mesh elements are stretched along the direction of the layers themselves.

The channel test case
Let us consider the same L-shaped domain Ω as in the "double ramp" test case. We choose in (7) $\mu = 10^{-3}$, $f = 0$ and an advective field $\vec{\beta} = (y, -x)^T$. Homogeneous Dirichlet boundary conditions are assigned on all the boundary $\partial\Omega$ of the domain except for the edge $x = 0$ where $u = 1$. The solution u exhibits an outflow boundary layer at y=0 and two circular shaped internal layers. In Fig. 2 we provide the initial mesh consisting of 4096 elements and the first adapted one with 2189 triangles. The functional J has been chosen in order to control the energy norm of the discretization error.

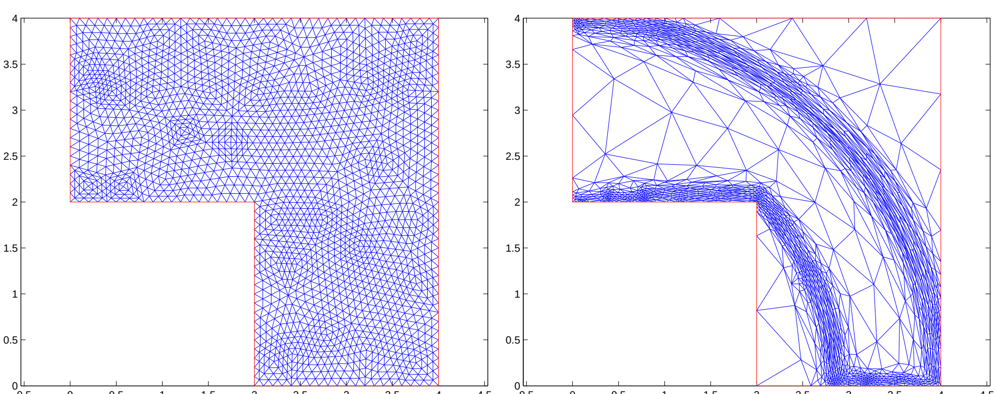

Figure 2: Channel test case: initial and first adapted mesh.

As Fig. 2 shows the adapted mesh suitably follows the directional features of the solution u, the elements being stretched along the three layers.

5.2.2 The Stokes problem

We deal with two test cases.

The driven cavity flow test case
This test case shows the motion of a flow inside a plane square domain $\Omega = (0,1)^2$ with velocity $\vec{u} = (1,0)^T$ prescribed on the top boundary. A no-slip boundary condition is imposed on the vertical sides as well as on the bottom horizontal side and $\mu = 10^{-1}$. In Fig. 3 (top) we show the initial uniform mesh together with

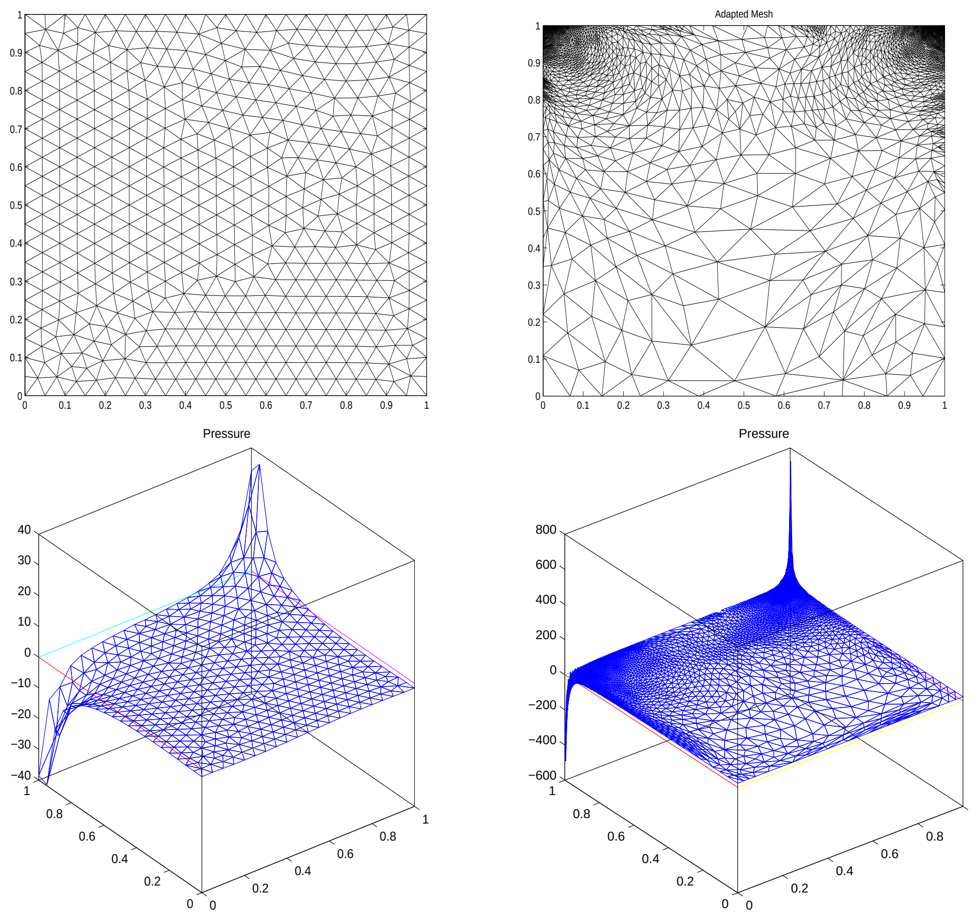

Figure 3: Driven cavity flow test case: initial and second adapted meshes (top) along with the pressure field computed on the initial and the first adapted mesh (bottom).

the anisotropic adapted mesh provided by estimate (23) after two iterations. The pressure fields computed on the initial and the first adapted mesh are shown in the bottom of Fig. 3. The two spikes at the points (0, 1) and (1, 1) are not well captured on the initial mesh while the anisotropic adapted mesh (6136 triangles) turns out to be better to capture these features as the pressure field in Fig. 3 (bottom-right) highlights (compare also the vertical scales of the two pressure plots). The target functionals are $J_1(\vec{v}) = 0$ and $J_2(q) = \int_\Omega 2\, p\, q\, d\vec{x}$. This choice aims at controlling the L^2-norm of the pressure through the linearized functional $J_2(q)$.

Kim and Moin example
Let us consider the domain $\Omega = (0.25, 1.25) \times (0.5, 1.5)$. We have solved on this region the Stokes problem such that the exact solution coincides with the solution of the time-dependent Navier-Stokes equations for

$$\vec{u} = (-\cos(2\pi x)\, \sin(2\pi y)\, \exp(-8\pi^2 \mu t),\ \sin(2\pi x)\, \cos(2\pi y)\, \exp(-8\pi^2 \mu t))^T,$$
$$p = -0.25(\cos(4\pi x) + \cos(4\pi y))\, \exp(-16\pi^2 \mu t),$$

$\mu = 10^{-2}$ and $t = 0.5$.

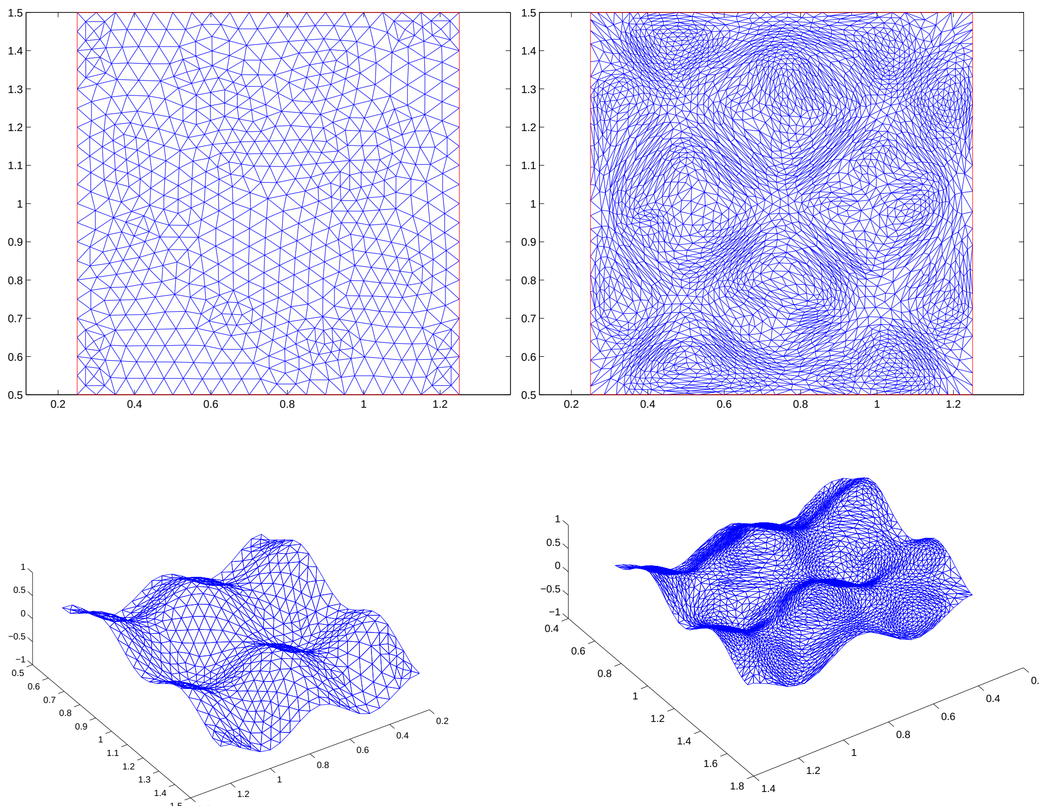

Figure 4: Kim and Moin test case: initial and first adapted mesh (top) and the corresponding approximated pressures (bottom).

All Dirichlet boundary conditions are assumed by restricting the above values for

$\vec{u}$ on the boundary $\partial\Omega$. Figure 4 shows the initial uniform (1336 triangles) and the first adapted mesh (5054 elements) (top) together with the corresponding approximated pressures (bottom). The functional J has been chosen in order to control the L^2-norm of the pressure as in the driven cavity flow test case. Notice how the adapted mesh matches the details of the pressure field.

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