

# Acceleration of a fixed point algorithm for fluid-structure interaction using transpiration conditions

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## Abstract

In this work, we address the numerical solution of fluid-structure interaction problems. This issue is particularly difficult to face when the fluid and the solid densities are of the same order, for instance as it happens in hemodynamic applications, since fully implicit coupling schemes are required to ensure stability of the resulting method. Thus, at each time step, we have to solve a highly non-linear coupled system, since the fluid domain depends on the unknown displacement of the structure. Standard strategies for solving this non-linear problems, are fixed point based methods such as Block-Gauss-Seidel (BGS) iterations. Unfortunately, these methods are very CPU time consuming and usually show slow convergence. We propose a modified fixed-point algorithm which combines the standard BGS iterations with a transpiration formulation. Numerical experiments show the great improvement in computing time with respect to the standard BGS method.

*Keywords:* fluid-structure interaction; Block Gauss-Seidel iterations; transpiration.

## 1 Introduction

Large displacements low speed problems where a flexible elastic structure interacts with the flow of an external or internal fluid occur in many engineering fields: from civil engineering (aeroelasticity) to biomechanics (biomedical flows). One challenge arising in the numerical approximation of these fluid-structure problems, is the definition of fast and accurate coupling algorithms that allow to predict the longterm time evolution and the stability of the overall system. This issue is particularly difficult to face when the fluid and the solid densities are of the same order, for instance as it happens in hemodynamics, since only implicit schemes can ensure stability of the resulting method (see [16, 18, 10]). Thus, at each time step, the rule is to solve a highly coupled non-linear system (the fluid domain depends on the structural motion) using efficient methods that preserve, inside inner loops, the fluid-structure subsystem splitting. Standard strategies to solve this non-linear system are fixed-point based methods as Block-Jacobi or Block-Gauss-Seidel (BGS) iterations, see [3, 16, 18, 15]. Recent advances concerns the use of Block Newton based method [24, 8, 10] for a fast convergence towards the solution of the non-linear coupled problem.

In this paper we focus on accelerating numerical algorithms involving fixed-point iterations. It is well known that the standard Block-Jacobi or Block-Gauss-Seidel iterations are very CPU time consuming. Indeed, we must add, to the often slow convergence of the algorithms, the cost of updating the fluid mesh and the corresponding fluid matrices at each iteration. We propose a modified fixed-point algorithm which combines the Block-Gauss-Seidel iterations (see [3]) with a transpiration formulation (see [11, 22, 23, 5, 7, 14]). The underlying idea of our approach relies on the fact that standard BGS iterations associated to moderate interface deformations can be treated through transpiration techniques. These formulations do not require updating the fluid computational mesh and matrices. They only involve modifications of the interface boundary conditions.

The outline of this paper is as follows. In section 2, we introduce the fluid-structure interaction problem and its mathematical description. We use the classical Arbitrary Lagrange-Euler formulation for the fluid. Section 3 we address the time discretization and the coupling algorithms, focusing on implicit schemes. We describe the BGS iterations. In section 4 we provide the new algorithm, it combines the Block-Gauss-Seidel iterations with a transpiration formulation. Finally, in section 5, we report the numerical results. They point out the great improvement in computing time of the proposed algorithm.

## 2 Mechanical problem

The system under study occupies a moving domain  $\Omega(t)$  in its actual configuration. It is made of a deformable structure  $\Omega^s(t)$  (vessel wall, pipe-line, ...) surrounding a fluid under motion (blood, oil, ...) in the complement  $\Omega^f(t)$  of  $\Omega^s(t)$  in  $\Omega(t)$  (see Figure 1). The problem consists in finding the time evolution of the configuration  $\Omega^f(t)$ , as well as the velocity and Cauchy stress tensor within the fluid and the structure.

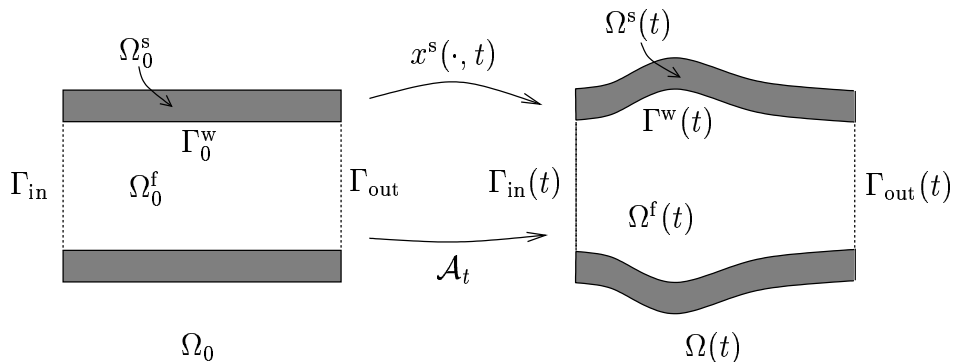


Figure 1: Geometric configurations

We assume the fluid to be Newtonian viscous, homogeneous and incompressible. Its behavior is described by its velocity and pressure. The elastic solid under large displacements is described by its velocity and its stress tensor. The classical conservation laws of the continuum mechanics drive the evolution of these unknowns.

The fluid state satisfies the following incompressible Navier-Stokes equations written in Eulerian conservative formulation:

$$\begin{aligned} \rho \frac{\partial u}{\partial t} + \operatorname{div}[\rho u \otimes u - \sigma(u, p)] &= 0, \quad \text{in } \Omega^f(t), \\ \operatorname{div} u &= 0, \quad \text{in } \Omega^f(t), \end{aligned} \tag{1}$$

where  $u$ ,  $p$  and  $\rho$  stand, respectively, for the fluid velocity, pressure and density. In addition, the fluid stress tensor is given by

$$\sigma(u, p) = -p\mathbf{I} + \mu [\nabla u + (\nabla u)^T],$$

with  $\mu$  the kinetic viscosity of the fluid.

Because of the large displacements which are involved, the variation in time of the actual configuration can not be neglected. The evolution of the fluid domain  $\Omega^f(t)$  is induced by the structural deformation through the fluid-structure interface  $\Gamma^w(t)$ . Indeed, by definition  $\Omega^f(t) = \Omega(t) - \overline{\Omega}^s(t)$ . It leads us to describe  $\Omega^f(t)$  according to a map  $\mathcal{A}_t$  acting in a fixed reference domain  $\Omega_0^f$ . This approach is usually used for the solid domain  $\Omega^s(t)$ , by means of the Lagrangian formulation [2, 12] as we will precise later. Therefore, to describe the evolution of  $\Omega^f(t)$ , we introduce a family of homeomorphisms  $\{\mathcal{A}_t\}_{t \in \mathbb{R}^+}$  (see figure 1)

$$\begin{aligned} \mathcal{A}_t : \overline{\Omega}_0^f &\longrightarrow \mathbb{R}^3 \\ x_0 &\longmapsto x = \mathcal{A}_t(x_0). \end{aligned}$$

The choice of the configuration  $\Omega_0^f$  and of the map  $\mathcal{A}_t$  may be arbitrary, hence the name of Arbitrary Lagrangian Eulerian (ALE) formulation given to the resulting equations. Given a material reference configuration  $\Omega_0^s$  for the solid (see figure 1) the map

$$x^s : \overline{\Omega}_0^s \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3,$$

stands for the solid motion. For  $x_0 \in \Omega_0^s$ ,  $x^s(x_0, t)$  represents the position at time  $t \geq 0$  of the material point  $x_0$ . This corresponds to the classical Lagrangian flow [2, 12]. The map  $\mathcal{A}_t$  can be defined from  $x^s|_{\Gamma_0^s}$ , as an arbitrary extension over the domain  $\overline{\Omega}_0^f$ . In short, the ALE map  $\mathcal{A}_t$  is given by

$$\mathcal{A}_t(x_0) = \operatorname{Ext}(x^s|_{\Gamma_0^s})(x_0, t), \quad \forall x_0 \in \overline{\Omega}_0^f,$$

where  $\operatorname{Ext}$  represents an extension operator from  $\Gamma_0^w$  to  $\overline{\Omega}_0^f$ .

**Remark 1** In the sequel  $\frac{\partial}{\partial t}|_{x_0}$  stand for the time derivative operator keeping the space variable  $x_0$  fixed. More formally, if  $u : \Omega(t) \times \mathbb{R}^+ \rightarrow \mathbb{R}$ , then

$$\frac{\partial u}{\partial t}|_{x_0} = \left[ \frac{\partial}{\partial t} (u \circ \mathcal{A}_t) \right] \circ \mathcal{A}_t^{-1},$$

where the composition operator applies to the space variables only.

**Remark 2** *The operator Ext is arbitrarily defined inside  $\Omega_0^f$ . It can be any reasonable extension of the material interface deformation:*

$$\mathcal{A}_t = \text{Ext}(x_{|\Gamma_0^w}^s), \quad \frac{\partial \mathcal{A}_t}{\partial t} \Big|_{\Gamma_0^w} = \dot{x}_{|\Gamma_0^w}^s.$$

*For instance, Ext may be defined from an harmonic extension of the interface displacement.*

This map allows us to rewrite the time derivative in (1), yielding to the classical incompressible Navier-Stokes equations written in ALE conservative formulation [16], satisfied by  $u : \Omega_0^f \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$  and  $p : \Omega_0^f \times \mathbb{R}^+ \rightarrow \mathbb{R}$ :

$$\begin{aligned} \frac{\partial J_{\mathcal{A}_t} u}{\partial t} \Big|_{x_0} + \text{div} [u \otimes (u - w) - \sigma(u, p)] &= 0, \quad \text{in } \Omega^f(t), \\ \text{div } u &= 0, \quad \text{in } \Omega^f(t), \end{aligned} \tag{2}$$

where the quantities  $J_{\mathcal{A}_t}$  and  $w$  are defined by:

$$F_{\mathcal{A}_t} = \frac{\partial \mathcal{A}_t}{\partial x_0}, \quad J_{\mathcal{A}_t} = \det(F_{\mathcal{A}_t}) > 0, \quad w = \frac{\partial \mathcal{A}_t}{\partial t}.$$

**Remark 3** *By definition,  $\dot{x}^s$  represents the solid velocity, whereas  $w$  stands for the velocity of the fluid domain, which usually differs from the fluid velocity inside  $\Omega^f(t)$ .*

The solid evolution is given by its displacement  $d$  around the reference material configuration  $\Omega_0^s$ ,

$$x^s(x_0, t) = x_0 + d(x_0, t), \quad \forall x_0 \in \Omega_0^s,$$

and the stress tensor field  $S$  (second Piola-Kirchhoff tensor [2]). The field  $S$  is related to  $d$  through an appropriate constitutive law (see [2, 12]). Then, the pair  $(d, S)$  satisfies the non-linear elastodynamic equations [2]:

$$\ddot{d} - \text{div}_0(FS) = f, \quad \text{in } \Omega_0^s. \tag{3}$$

The coupling between the solid and the fluid is realized through standard boundary conditions at the fluid-structure interface  $\Gamma_0^w$ , namely, the kinematic continuity of the velocity and the kinetic continuity of the stress [16]:

$$\begin{aligned} u &= \omega, \quad \text{on } \Gamma_0^w, \\ FS n_0 &= J \sigma(u, p) F^{-T} n_0, \quad \text{on } \Gamma_0^w, \end{aligned} \tag{4}$$

where  $J$  and  $F$  are defined by

$$F = \frac{\partial x^s}{\partial x_0}, \quad J = \det(F) > 0,$$

and  $n_0$  stands for the unit normal vector on  $\Gamma_0^w$  pointing inside  $\Omega_0^s$ . In summary, the strong coupled problem, with a ALE formulation for the fluid, is given by:

$$\begin{aligned}
\rho \frac{\partial J_{\mathcal{A}_t} u}{\partial t} \Big|_{x_0} + \operatorname{div} [\rho u \otimes (u - w) - \sigma(u, p)] &= 0, & \text{in } \Omega^f(t), \\
\operatorname{div} u &= 0, & \text{in } \Omega^f(t), \\
u &= w, & \text{on } \Gamma_0^w, \\
\sigma(u, p)n &= g & \text{on } \Gamma_{\text{in}}(t) \cup \Gamma_{\text{out}}(t), \\
\rho_0 \ddot{d} - \operatorname{div}_0(FS) &= f, & \text{in } \Omega_0^s, \\
FSn_0 &= J\sigma F^{-T}n_0, & \text{on } \Gamma_0^w, \\
d &= 0, & \text{on } \Gamma_0^D, \\
FSn_0 &= 0, & \text{on } \Gamma_0^N, \\
\mathcal{A}_t &= \operatorname{Ext}(x_{|\Gamma_0^w}^s), \quad w = \frac{\partial \mathcal{A}_t}{\partial t}, & \text{in } \Omega_0^f,
\end{aligned} \tag{5}$$

where  $g$  stands for the external forces acting on the fluid. For simplification purposes, no fluid body forces are considered. The boundary conditions to be imposed on  $\Gamma_{\text{in}}(t) \cup \Gamma_{\text{out}}(t)$  as well as on  $\partial\Omega_0^s - \Gamma_0^w = \Gamma_0^D \cup \Gamma_0^N$  will depend on the problem under consideration.

Problem (5) can be written in weak variational form as follows: find  $\mathcal{A}_t : \Omega_0^f \rightarrow \mathbb{R}^3$ ,  $u : \Omega^f(t) \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$ ,  $p : \Omega^f(t) \times \mathbb{R}^+ \rightarrow \mathbb{R}$  and  $d : \Omega_0^s \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$  such that

$$\begin{aligned}
\rho \frac{d}{dt} \int_{\Omega^f(t)} u \cdot v^f dx - \rho \int_{\Omega^f(t)} \operatorname{div}[u \otimes (u - w)] \cdot v^f dx + \int_{\Omega^f(t)} \sigma(u, p) : \nabla v^f dx \\
- \int_{\Gamma_{\text{in}}(t) \cup \Gamma_{\text{out}}(t)} g(t) \cdot v^f da + \int_{\Omega^f(t)} q^f \operatorname{div} u dx &= 0, \quad \forall (v^f, q^f) \in V^f \times Q^f, \\
u &= \omega, \quad \text{on } \Gamma_0^w, \\
m^s(\ddot{d}, v^s) + a^s(d, v^s) &= \int_{\Omega_0^s} f \cdot v^s dx - \int_{\Gamma_0^w} J\sigma(u, p)F^{-T}n_0 \cdot v^s dx, \quad \forall v \in V^s = H_{\Gamma_0^D}^1(\Omega_0^s)^3, \\
\mathcal{A}_t &= \operatorname{Ext}(x_{|\Gamma_0^w}^s), \quad w = \frac{\partial \mathcal{A}_t}{\partial t}, \quad \text{in } \Omega_0^f.
\end{aligned} \tag{6}$$

Here, the spaces of fluid test functions are defined, as in [9], by

$$V^f = \{v \circ \mathcal{A}_t^{-1}, \quad v \in H^1(\Omega_0^f)^3\}, \quad Q^f = \{q \circ \mathcal{A}_t^{-1}, \quad q \in L^2(\Omega_0^f)\},$$

and

$$m^s(\ddot{d}, v^s) = \int_{\Omega_0^s} \rho_0 \ddot{d} \cdot v^s dx, \quad a^s(d, v^s) = \int_{\Omega_0^s} FS : \nabla v^s dx,$$

stand for the solid mass and stiffness integrals. Similar integrals appear when dealing with more general structures such as elastic beams or shells in large displacements (see [1, 16]).

### 3 Time discretization: coupling algorithms

Concerning the time discretization of (6) several schemes can be considered. For instance, a first strategy, which leads to the standard *loosely coupled algorithms* [6, 19, 20], consists in taking an explicit scheme for the fluid (respectively for the structure) and an implicit scheme for the structure (respectively for the fluid). Thus, at each time step, the fluid solution is completely determined starting from the solution of the previous time step and, once the fluid load at the interface has been computed, the structure can be advanced on time updating the position of the interface. In short, the geometry and the interface coupling are explicitly treated. This strategy is computationally cheaper and performs well in many practical situations, for example, in aeroelasticity applications [6, 19, 20]. However, numerical experiments and some analysis on simplified models (see [16, 18, 10]) indicate that these staggered algorithms are unstable when the structure is light, in particular when the fluid and structure densities are comparable, as it happens in hemodynamic applications. In these situations, fluid-structure equilibrium must be ensured at each time accurately. In other words, the geometry and the interface coupling have to be implicitly treated, and then implicit schemes must be considered.

For the above mentioned reasons, in this paper we will focus on fully coupled implicit schemes. In the sequel, we consider in (6) an implicit Euler treatment of the velocity derivatives in the fluid domain and a mid-point rule for the structural equation. The resulting time discretized problem writes: find  $u^{n+1} : \Omega^f(t_{n+1}) \rightarrow \mathbb{R}^3$ ,  $p^{n+1} : \Omega^f(t_{n+1}) \rightarrow \mathbb{R}$  and  $d^{n+1} : \Omega_0^s \rightarrow \mathbb{R}^3$  satisfying the following coupled non-linear problem:

$$\begin{aligned}
\mathcal{A}_{t_{n+1}} &= \text{Ext}(x^{s,n+1}|_{\Gamma_0^w}), \quad \omega^{n+1} = \frac{1}{\Delta t} (\mathcal{A}_{t_{n+1}} - \mathcal{A}_{t_n}), \quad \text{in } \Omega_0^f, \\
\frac{\rho}{\Delta t} \int_{\Omega^f(t_{n+1})} u^{n+1} \cdot v^f dx &+ \rho \int_{\Omega^f(t_{n+1})} \text{div} [u^{n+1} \otimes (u^{n+1} - \omega^{n+1})] \cdot v^f dx \\
&+ \int_{\Omega^f(t_{n+1})} \sigma(u^{n+1}, p^{n+1}) : \nabla v^f dx - \int_{\Gamma_{\text{in}}(t_{n+1}) \cup \Gamma_{\text{out}}(t_{n+1})} g(t_{n+1}) \cdot v^f da \\
&+ \int_{\Omega^f(t_{n+1})} q^f \text{div} u^{n+1} dx = \frac{\rho}{\Delta t} \int_{\Omega^f(t_n)} u^n \cdot v^f dx, \quad \forall (v^f, q^f) \in V^f \times Q^f, \\
u^{n+1} &= \omega^{n+1}, \quad \text{on } \Gamma^w(t_{n+1}), \\
\frac{1}{\Delta t} m^s (d^{n+1} - d^n, v^s) &+ \frac{1}{2} (a^s(d^{n+1}, v^s) + a^s(d^n, v^s)) \\
&= - \int_{\Gamma^w(t_{n+1})} \sigma(u^{n+1}, p^{n+1}) n \cdot v^s da, \quad \forall v^s \in V^s, \\
\frac{1}{2} (j^{n+1} + j^n) &= \frac{1}{\Delta t} (d^{n+1} - d^n), \quad \text{in } \Omega_0^s.
\end{aligned} \tag{7}$$

Therefore, at each time step, we have to solve a highly non-linear system, where the fluid domain and the structural displacement are tightly coupled. Assuming that problem (7) has been appropriately discretized in space, for instance by a FEM formulation, we formally

write the resulting non-linear system as a fixed point problem,

$$\begin{aligned}
(\mathbf{x}^f, \mathbf{w}^f) &= \mathcal{M}((\mathbf{x}^f, \mathbf{w}^f), (\mathbf{x}^s, \mathbf{w}^s)), & \text{(fluid mesh update),} \\
(\mathbf{u}, \mathbf{p}) &= \mathcal{F}((\mathbf{u}, \mathbf{p}), (\mathbf{x}^f, \mathbf{w}^f)), & \text{(fluid subproblem),} \\
(\mathbf{x}^s, \mathbf{w}^s) &= \mathcal{S}((\mathbf{x}^s, \mathbf{w}^s), (\mathbf{u}, \mathbf{p})), & \text{(solid subproblem),}
\end{aligned} \tag{8}$$

where  $(\mathbf{u}, \mathbf{p})$ ,  $(\mathbf{x}^s, \mathbf{w}^s)$  and  $(\mathbf{x}^f, \mathbf{w}^f)$  stand, respectively, for the discretized fluid velocity and pressure, the structural motion and velocity, and the fluid domain motion and velocity at a given time step.

Standard strategies to solve (8) are Block-Jacobi or Block-Gauss-Seidel iterations, see [3]. For instance, the Block-Gauss-Seidel method consists in sub-iterating between the three subproblems. For  $k \geq 0$  we set

$$\begin{aligned}
(\mathbf{x}_{k+1}^f, \mathbf{w}_{k+1}^f) &= \mathcal{M}((\mathbf{x}_k^f, \mathbf{w}_k^f), (\mathbf{x}_k^s, \mathbf{w}_k^s)), & \text{(update the fluid mesh),} \\
(\mathbf{u}_{k+1}, \mathbf{p}_{k+1}) &= \mathcal{F}((\mathbf{u}_k, \mathbf{p}_k), (\mathbf{x}_{k+1}^f, \mathbf{w}_{k+1}^f)), & \text{(solve the fluid subproblem),} \\
(\mathbf{x}_{k+1}^s, \mathbf{w}_{k+1}^s) &= \mathcal{S}((\mathbf{x}_k^s, \mathbf{w}_k^s), (\mathbf{u}_{k+1}, \mathbf{p}_{k+1})), & \text{(solve the solid subproblem),}
\end{aligned} \tag{9}$$

with  $\mathbf{u}_0$ ,  $\mathbf{p}_0$ ,  $\mathbf{x}_0^s, \mathbf{w}_0^s$ ,  $\mathbf{x}_0^f$  and  $\mathbf{w}_0^f$  given from fluid-structure state computed at the previous time step. We stop the iterations when the difference between two successive interface displacement falls below a fixed tolerance (see also subsection 4). Eventually, we can add a relaxation step on the structure displacement, see [16, 18, 10, 15].

It is well known that this strategy is very CPU time consuming. Indeed, we must add to the often slow convergence of the algorithm the cost of updating the fluid mesh and the corresponding fluid matrices at each iteration. In this paper, we propose a method to improve the performance of the standard BGS iteration (9).

## 4 New algorithm: BGS with transpiration

Each iteration of the Block-Gauss-Seidel method (9) involves the numerical solution of the following non-linear problem: for  $k \geq 0$  find  $\mathcal{A}_{t_{n+1}}^{k+1} : \Omega_0^f \longrightarrow \mathbb{R}^3$ ,  $u_{k+1}^{n+1} : \Omega_{k+1}^f \longrightarrow \mathbb{R}^3$ ,

$p_{k+1}^{n+1} : \Omega_{k+1}^f \longrightarrow \mathbb{R}$  and  $d_{k+1}^{n+1} : \Omega_0^s \longrightarrow \mathbb{R}^3$  satisfying :

$$\begin{aligned}
\mathcal{A}_{t_{n+1}}^{k+1} &= \text{Ext}(x_k^{s,n+1} |_{\Gamma_0^w}), \quad w_{k+1}^{n+1} = \frac{1}{\Delta t} (\mathcal{A}_{t_{n+1}}^{k+1} - \mathcal{A}_{t_n}), \quad \text{in } \Omega_0^f, \\
\frac{\rho}{\Delta t} \int_{\Omega_{k+1}^f} u_{k+1}^{n+1} \cdot v^f dx + \rho \int_{\Omega_{k+1}^f} \text{div} [u_{k+1}^{n+1} \otimes (u_k^{n+1} - w_{k+1}^{n+1})] \cdot v^f dx \\
&+ \int_{\Omega_{k+1}^f} \sigma(u_{k+1}^{n+1}, p_{k+1}^{n+1}) : \nabla v^f dx - \int_{\Gamma_{\text{in}} \cup \Gamma_{\text{out}}} g(t_{n+1}) \cdot v^f da \\
&+ \int_{\Omega_{k+1}^f} q^f \text{div} u_{k+1}^{n+1} dx = \frac{\rho}{\Delta t} \int_{\Omega^f(t_n)} u^n \cdot v^f dx, \quad \forall (v^f, q^f) \in V^f \times Q^f, \quad (10) \\
u_{k+1}^{n+1} &= w_{k+1}^{n+1}, \quad \text{on } \Gamma_k^w, \\
\frac{1}{\Delta t} m^s (d_{k+1}^{n+1} - d^n, v^s) + \frac{1}{2} (a^s (d_k^{n+1}, v^s) + a^s (d^n, v^s)) \\
&= - \int_{\Gamma_{k+1}^w} \sigma(u_{k+1}^{n+1}, p_{k+1}^{n+1}) n \cdot v^s da, \quad \forall v^s \in V^s, \\
\frac{1}{2} (d_{k+1}^{n+1} + d^n) &= \frac{1}{\Delta t} (d_{k+1}^{n+1} - d^n), \quad \text{in } \Omega_0^s.
\end{aligned}$$

Here  $\mathcal{A}_{t_{n+1}}^{k+1}$  represents the ALE map computed at the  $(k+1)$ -th iteration of the BGS method, and  $\Omega_{k+1}^f$  the corresponding fluid domain, namely  $\Omega_{k+1}^f = \mathcal{A}_{t_{n+1}}^{k+1}(\Omega_0^f)$ .

Then main disadvantage of the standard BGS method lies on the fact that its iterations are very costly. Indeed, each iteration (10) involves an update of the fluid domain  $\Omega_{k+1}^f(t_{n+1})$  and of its velocity  $w_{k+1}^{n+1}$ . Consequently, the fluid matrices have to be recomputed on this new configuration. This feature arises because we are using an ALE formulation for the fluid (large displacements are involved in the whole fluid-structure problem). However, between to successive BGS iterations the fluid-structure interface frequently exhibits moderate variations.

In order to be able to solve a low cost fluid-structure problems featuring moderate deformation, aeronautical engineers have developed transpiration techniques, from an idea of Lighthill [13]. These formulations do not require to update the computational grid, but only involve modifications of the interface boundary conditions (see for instance [11, 22, 23, 5, 7, 14]). Consequently, whenever the fluid-structure interface shows a “small” variation (with respect to a specified tolerance  $\text{TOL}_{\text{trans}}$ ) between steps  $k$  and  $k+1$  of (10), the interface motion can be taken into account for the fluid problem through transpiration boundary conditions, without the need of updating the mesh and, consequently, the system matrices.

A transpiration interface condition on  $\Gamma_k^w$  can be derived, in a heuristic way (see [11, 22, 23, 14] and refer to [5, 7, 17] for a more rigorous justification), from a truncated Taylor expansion of the fluid velocity in the neighborhood of the reference fluid-structure interface  $\Gamma_k^w$ , see figure 2:

$$u_{k+1}^{n+1}(x_{k+1}^f) = u_{k+1}^{n+1}(x_k^f) - \nabla u_{k+1}^{n+1}(x_k^f)(x_{k+1}^f - x_k^f) + o(\|x_{k+1}^f - x_k^f\|), \quad (11)$$



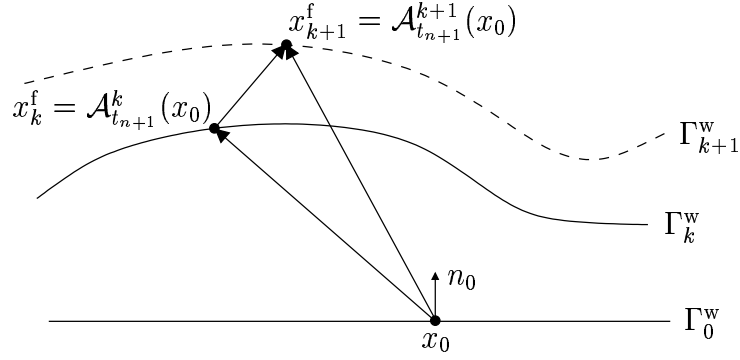


Figure 2: Taylor expansion of the fluid velocity.

on  $\Gamma_0^w$ . Thus, from the kinematic condition

$$u_{k+1}^{n+1} = w_{k+1}^{n+1}, \quad \text{on } \Gamma_{k+1}^w,$$

we get the following transpiration condition on the known interface  $\Gamma_k^w$

$$u_{k+1}^{n+1} = w_{k+1}^{n+1} - \nabla u_{k+1}^{n+1}(x_{k+1}^f - x_k^f), \quad \text{on } \Gamma_k^w.$$

The implicit dependence on  $u_{k+1}^{n+1}$  can be made explicit by modifying the relation into

$$u_{k+1}^{n+1} = w_{k+1}^{n+1} - \nabla u_{k+1}^{n+1}(x_{k+1}^f - x_k^f). \quad (12)$$

This latter condition can now be used to approximate the fluid subproblem in (10). Indeed, by setting

$$\Omega_{\text{trans}}^f = \Omega_k^f, \quad \Gamma_{\text{trans}}^w = \Gamma_k^w, \quad u_{\text{trans}}^{n+1} = u_k^{n+1}, \quad w_{\text{trans}}^{n+1} = w_k^{n+1},$$

we introduce the following transpired fluid subproblem

$$\begin{aligned} w_{k+1}^{n+1} &= \frac{1}{\Delta t} (d_k^{n+1} - d^n), \quad \text{on } \Gamma_0^w, \\ \frac{\rho}{\Delta t} \int_{\Omega_{\text{trans}}^f} u_{k+1}^{n+1} \cdot v^f dx + \rho \int_{\Omega_{\text{trans}}^f} \text{div} [u_{k+1}^{n+1} \otimes (u_{\text{trans}}^{n+1} - w_{\text{trans}}^{n+1})] \cdot v^f dx \\ &+ \int_{\Omega_{\text{trans}}^f} \sigma(u_{k+1}^{n+1}, p_{k+1}^{n+1}) : \nabla v^f dx - \int_{\Gamma_{\text{in}}(t_{n+1}) \cup \Gamma_{\text{out}}(t_{n+1})} g(t_{n+1}) \cdot v^f da \\ &+ \int_{\Omega_{\text{trans}}^f} q^f \text{div} u_{k+1}^{n+1} dx = \frac{\rho}{\Delta t} \int_{\Omega^f(t_n)} u^n \cdot v^f dx, \quad \forall (v^f, q^f) \in V^f \times Q^f, \\ u_{k+1}^{n+1} &= w_{k+1}^{n+1} - \nabla u_k^{n+1}(d_{k+1}^{n+1} - d_k^{n+1}), \quad \text{on } \Gamma_{\text{trans}}^w, \end{aligned} \quad (13)$$

The obtained fluid-subproblem allow us to take into account the interface motion, while keeping a fixed fluid domain. This is achieved by using non-standard boundary conditions on the fixed reference interface  $\Gamma_{\text{trans}}^w$ , without the need of updating the mesh and, consequently, the system matrices.

**Remark 4** In (13) the velocity  $w_{k+1}^{n+1}$  is no longer needed inside the fluid domain. Therefore, we just compute it at the fluid structure interface  $\Gamma_0^w$ .

In the same way, the fluid load at the moving interface can be recovered from a similar Taylor expansion:

$$\sigma_{k+1}(x_{k+1}^f) = \sigma_{k+1}(x_k^f) + \nabla \sigma_{k+1}(x_k^f)(x_{k+1}^f - x_k^f) + o(\|x_{k+1}^f - x_k^f\|), \quad (14)$$

on  $\Gamma_0^w$ . Thus, in (10), the solid subproblem can be now replaced by

$$\begin{aligned} & \frac{1}{\Delta t} m^s (\dot{d}_{k+1}^{n+1} - \dot{d}^n, v^s) + \frac{1}{2} (a^s(d_k^{n+1}, v^s) + a^s(d^n, v^s)) \\ &= - \int_{\Gamma_0^w} J_{\mathcal{A}_{t_{n+1}}^{k+1}} (\sigma_{k+1} + \nabla \sigma_{k+1}(x_{k+1}^f - x_k^f)) F_{\mathcal{A}_{t_{n+1}}^{k+1}}^{-T} n_0 \cdot v^s \, da, \quad \forall v^s \in V^s, \quad (15) \\ & \frac{1}{2} (\dot{d}_{k+1}^{n+1} + \dot{d}^n) = \frac{1}{\Delta t} (d_{k+1}^{n+1} - d^n), \quad \text{in } \Omega_0^s. \end{aligned}$$

**Remark 5** A simpler approximation, see [14, 5], can be obtained by replacing the one order Taylor expansions (11) and (14) by zeroth order expressions. Hence, the interface transpiration condition in (13) reduces to

$$u_{k+1}^{n+1} = w_{k+1}^{n+1}, \quad \text{on } \Gamma_{\text{trans}}^w,$$

and the fluid interface load in (15) to

$$\int_{\Gamma_k^w} \sigma_{k+1} n \cdot v^s \, da. \quad (16)$$

By exploiting the previous considerations we have derived the modified BGS algorithm reported in figure 3. The boxes on the right of figure 3 represents the transpiration loop. Here, instead of updating fluid mesh and matrices, we just enforce the transpiration velocity  $w_f^{k+1}$  at the interface. Tolerances  $\text{TOL}_{\text{trans}}^{\text{in}}$  and  $\text{TOL}_{\text{trans}}^{\text{out}}$  define the range of relative interface displacements where the transpiration formulation will be used. The convergence test of the whole algorithm is always made after two standard BGS iterations, (2x in the figure), in order to ensure the convergence of the original coupled problem. This also implies that the algorithm terminates with standard BGS iterations and with an updated mesh.

The convergence test can be done on the relative error

$$\frac{\|d_{k+1}^{n+1} - d_k^{n+1}\|}{\|d_{k+1}^{n+1}\|} < \text{TOL},$$

in a suitable norm, for example the discrete  $L^\infty$  norm. The main advantage of the relative error is that it is adimensional. However, when the structure is almost at rest, this stopping

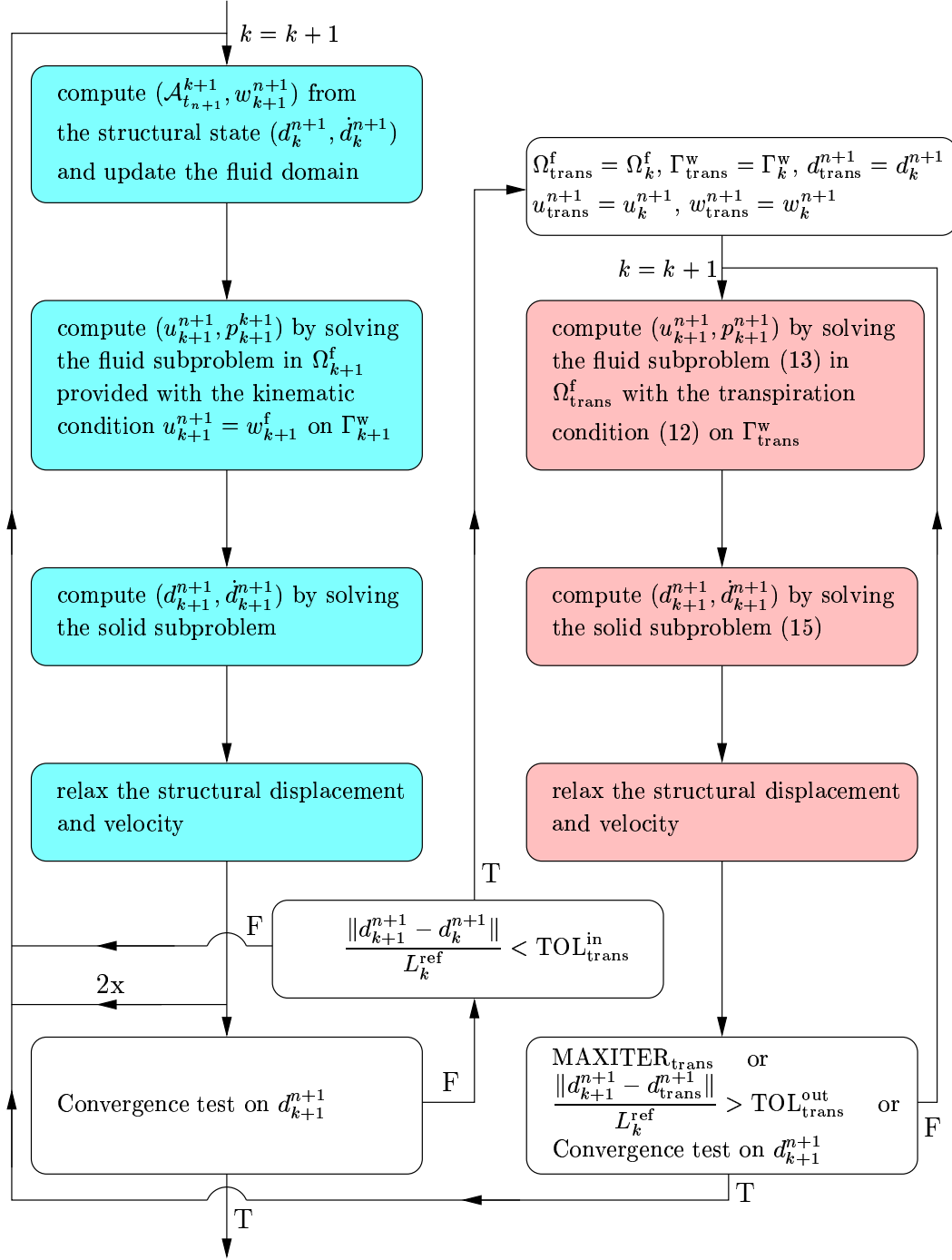


Figure 3: Diagram of the proposed algorithm. In blue the BGS part, in red the transpiration steps.

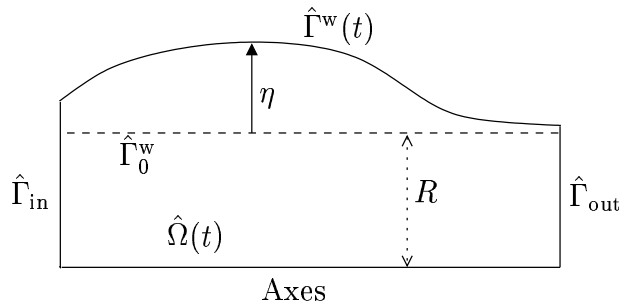


Figure 4: The computational domain.

criteria becomes too restrictive. This is not the case when using adimensional absolute errors. Therefore, we have chosen here the following expression

$$\frac{\|d_{k+1}^{n+1} - d_k^{n+1}\|}{d^{\text{ref}}} < \text{TOL},$$

where  $d^{\text{ref}}$  is a reference displacement of the domain. For example, for the study of blood flow in an artery it could be the mean radius of the vessel.

In order to test whether to activate the transpiration part of the algorithm, the relative error described above is useless. Indeed, what we have to measure in this case is how much the computational fluid domain is distant from the actual fluid domain. The transpiration may be adopted only when this distance is small. Hence, the condition that has to be satisfied is

$$\frac{\|d_{k+1}^{n+1} - d_{\text{trans}}^{n+1}\|}{L_k^{\text{ref}}} < \text{TOL}_{\text{trans}},$$

where  $L_k^{\text{ref}}$  is a characteristic length of the fluid domain at the  $k$ -th iteration. In blood fluid dynamics  $L_k^{\text{ref}}$  can be taken as the minimum of  $\|R + d_k^{n+1}\|$  over the interface points.

## 5 Numerical experiment

We have applied the above algorithm to a fluid-structure problem arising in the modeling of blood flow on large arteries. Namely, a thin elastic tube conveying an incompressible viscous fluid. In order to simplify the problem we considered the axisymmetric incompressible Navier-Stokes equations (see [4]) combined with a generalized string model (see [18]) for the structure. The initial domain is a cylinder of radius  $R = 0.5$  cm and length  $L = 12$  cm. The cylinder wall may deform only along the radial direction. The fluid and the structure are initially at rest.

The weak axisymmetric formulation of the Navier-Stokes equations reads

$$\begin{aligned} \rho \frac{d}{dt} \int_{\hat{\Omega}(t)} r u \cdot v + \rho \int_{\hat{\Omega}(t)} r ((u - w) \cdot \nabla_a) u \cdot v + \rho \int_{\hat{\Omega}(t)} r \operatorname{div}_a(w) uv + \rho \int_{\hat{\Omega}(t)} w_r uv + \\ + 2\mu \int_{\hat{\Omega}(t)} r (\varepsilon_a(u) : \varepsilon_a(v)) + 2\mu \int_{\hat{\Omega}(t)} \frac{1}{r} u_r v_r - \int_{\hat{\Omega}(t)} r p \operatorname{div}_a v - \int_{\hat{\Omega}(t)} p v_r \\ = \int_{\hat{\Omega}(t)} r f \cdot v + \int_{\hat{\Gamma}^N(t)} r g \cdot v \end{aligned}$$

$$\int_{\hat{\Omega}(t)} r q \operatorname{div}_a u + \int_{\hat{\Omega}(t)} q u_r = 0,$$

where  $\varepsilon_a(u) = \frac{1}{2} (\nabla_a u + (\nabla_a u)^T)$ ,  $\nabla_a = \begin{pmatrix} \partial_r \\ \partial_z \end{pmatrix}$  and  $\operatorname{div}_a u = \partial_r u_r + \partial_z u_z$  stand for differential operators in the cylindrical coordinates,  $\hat{\Omega}(t)$  for the half section of the fluid domain  $\Omega^f(t)$  at time  $t$  and  $\hat{\Gamma}^N(t) = \hat{\Gamma}_{\text{in}}(t) \cup \hat{\Gamma}_{\text{out}}(t)$  the inlet and outlet Neumann boundary sections.

The wall displacement  $\eta$  is modeled by a *vibrating string* model (see [21]) governed by the equation

$$\rho_w h \frac{\partial^2 \eta}{\partial t^2} - S_z \frac{\partial^2 \eta}{\partial z^2} + \frac{Eh}{(1 - \nu^2) R^2} \eta = H,$$

where the forcing term  $H$  is equal to  $(\sigma(u, p - p_0)n) \cdot e_r$  (the full contribution of the fluid Cauchy tensor to the normal stress is taken into account). The external pressure  $p_0$  and the initial pressure of the fluid are both set equal to zero. The wall density is  $\rho_w = 1.1 \text{g/cm}^3$ , its thickness  $h = 0.1 \text{cm}$ , Young Modulus  $E = 8.25 \cdot 10^4 \text{dyne/cm}^2$ , Poisson coefficient  $\nu = 0.5$  and longitudinal stress  $S_z = 2750 \text{g/cm}^2$ .

On the outlet we impose  $\sigma(u, p)n = 0$  and on the inlet one “pressure wave” of a period of 5 ms, i.e.,

$$\sigma(u, p)n = \begin{cases} -\frac{P_{\text{in}}}{2} [1 - \cos(\frac{2\pi t}{5})] n & , t \leq 5 \text{ ms}, \\ 0 & , t > 5 \text{ ms}. \end{cases}$$

with  $P_{\text{in}} = 4 \cdot 10^4 \text{ dyne/cm}^2$ . The fluid density is  $\rho = 1 \text{ g/cm}^3$  and the viscosity  $\mu = 0.035 \text{poise}$ . We have adopted axisymmetric  $\mathbb{P}_1 \text{iso}\mathbb{P}_2 / \mathbb{P}_1$  finite elements for the fluid and  $\mathbb{P}_1$  for the structure. The time is discretized by a mid-point scheme for the structure and implicit Euler for the fluid equations (see [18]), with a time step of  $\Delta t = 0.1 \text{ ms}$ .

We have used the simplified form (16) for the forcing term and the following values for the tolerances in the proposed numerical scheme:  $\text{TOL} = 10^{-6}$  for the fixpoint (absolute) convergence test with reference displacement equal to 10% of the initial radius of the artery,  $\text{TOL}_{\text{trans}}^{\text{in}} = 0.05$ ,  $\text{TOL}_{\text{trans}}^{\text{out}} = 0.1$  and  $\text{MAXITER}_{\text{trans}} = 50$ . We approximate the characteristic length of the domain by the initial radius of the artery,  $L_k^{\text{ref}} = R$ .

In figure 5 we report the number of sub-iterations per time step required by the standard BGS method compared with the one obtained using the modified BGS scheme with

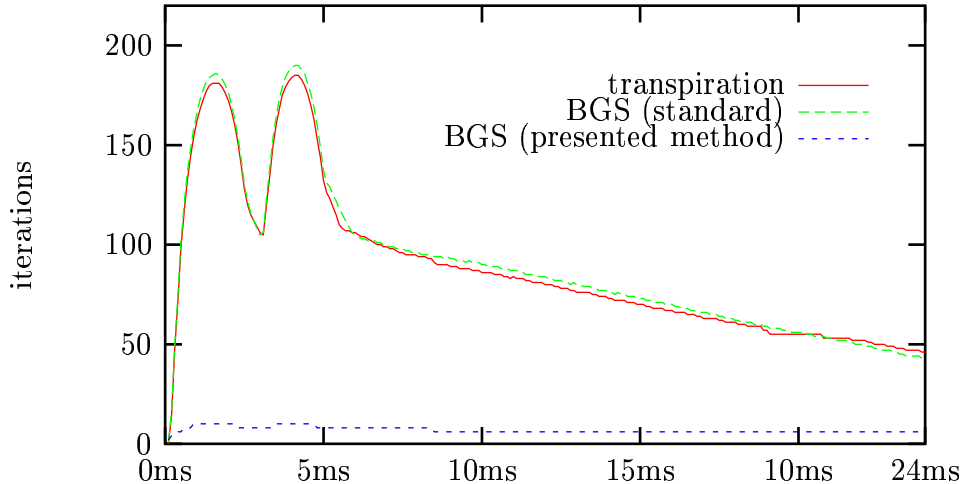


Figure 5: Iterations history.

MAXITER <sub>trans</sub>	50	200
Standard	157min	
Order 0	99min	101min
Order 1	102 min	106 min

Table 1: CPU time w.r.t. standard BGS or first order transpiration scheme.

transpiration. The number of BGS iterations is strongly reduced in the transpiration version. Let us notice that at each time step, the number of outer iterations is almost equal in the two schemes. However, the computing time is greatly reduced: a gain of 35% over 240 time steps.

We have also tested the zeroth order formulation with (12) substituted by  $u_{k+1}^{n+1} = w_{k+1}^{n+1}$ . The CPU time and the number of iterations are of the same order. The slight difference in CPU time derives from the computation of the fluid velocity gradients. The fact that the convergence obtained with the two alternatives is similar is due to the limited contribution of the velocity gradients for this test case. Indeed, the additional contribution given by the first order scheme is only  $10^{-7}$  times the zeroth order term. This hangs on from the very little variations in the wall displacement between the first two BGS iterations and the following transpiration ones.

## 6 Conclusion

In this work we dealt with the numerical solution of fluid-structure interaction problems in which the fluid density is comparable to that of the structure (for instance in hemodynamics applications). Thus, at each time step, we had to solve a highly non-linear coupled

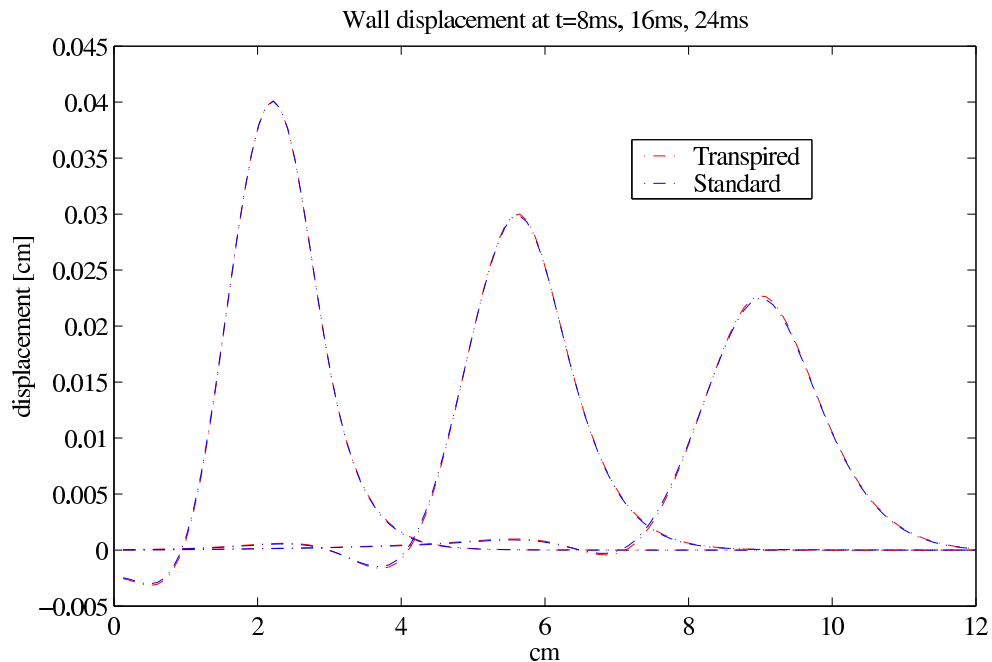


Figure 6: Wall displacement.

system, since the fluid domain depends on the unknown structural motion. We proposed a modified fixed-point algorithm which combines the Block-Gauss-Seidel iterations with a transpiration formulation. Numerical experiments pointed out the relevant improvement of computing time with respect to the standard method.

The sensibility analysis of the different parameters ( $\text{MAXITER}_{\text{trans}}$ , TOL, use of zeroth or first order transpiration formulation) as well as variants of the algorithm and other tests will be reported in a forthcoming work (see [4]).

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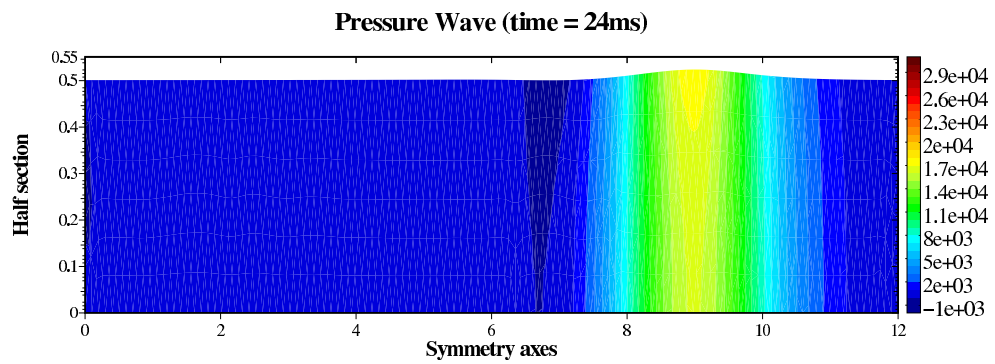
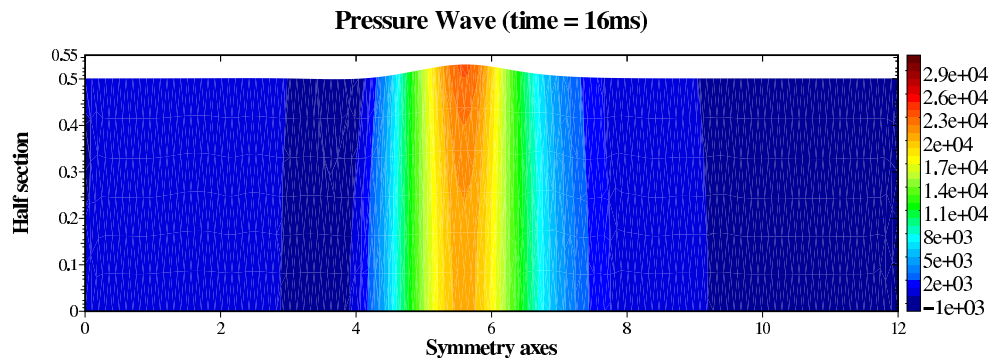
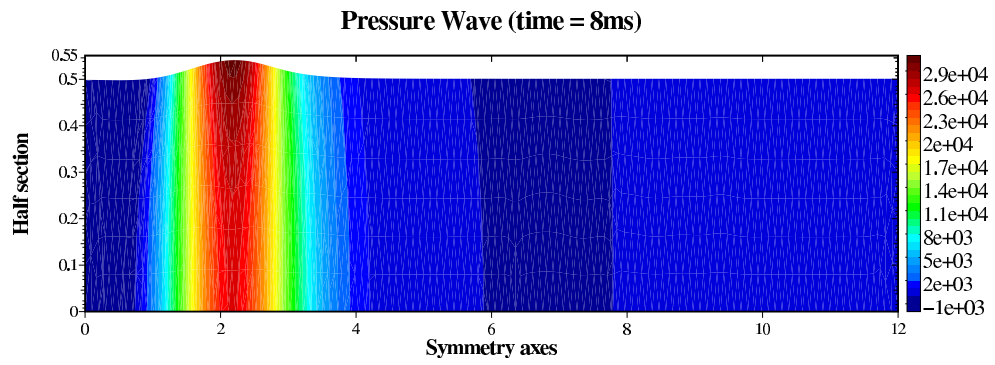


Figure 7: Pressure Wave.



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